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Research Paper

A skew generalization of the Amitsur–Small Theorem

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ABSTRACT

We introduce the notion of rigid extensions to generalize a key lemma underlying the Amitsur–Small Theorem to the skew setting. Building on this framework, we establish a skew version of the Amitsur–Small Theorem.

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1. Introduction

In [1], Amitsur and Small established the following result:

Theorem 1.1 *(Amitsur–Small Theorem). If D is a division ring, then simple $D[x_1, \ldots, x_n]$-modules are finite-dimensional as vector spaces over D.*

Here, the notation $D[x_1, \ldots, x_n]$ denotes the polynomial ring over D in n central indeterminates. Their proof relies on three lemmas, one of which states:

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Lemma 1.2 *(Lemma C in [1]). If L is a maximal left ideal of $D[x_1, \ldots, x_n]$, then $L \cap D[x_k] \neq 0$ for all k.*

Our primary goal is to extend this lemma to the skew setting and thereby obtain a skew version of the Amitsur–Small Theorem. In doing so, we introduce a special class of ring extensions, which we term *rigid extensions* (see Definition 3.1). Our definition of rigid extensions depends on the notion of *skew polynomial rings in several variables*, which serves as a suitable generalization of polynomial rings for our purposes (see Definition 2.1). It is worth noting that for a maximal left ideal L in $D[x_1, \ldots, x_n]$, the left ideal $L \cap D[x_k]$ need not be maximal in $D[x_k]$. A concrete example illustrating this phenomenon was recently constructed by Chapman and Paran in [4], where they introduced and studied the concept of an Amitsur–Small ring.

The paper is organized as follows. Section 2 introduces the notion of skew polynomial rings in several variables and explores their relationship to iterated skew polynomial rings and σ-PBW extensions. In Section 3, we define the concept of a rigid extension and prove that if $R \subset S$ is a rigid extension, then for any maximal left ideal L of S, we have $L \cap R \neq 0$ (see Proposition 3.1). The final section presents some examples of rigid extensions, and in particular, establishes the following skew version of the Amitsur–Small Theorem (see Theorem 4.4):

Theorem 1.3 *(Skew Amitsur–Small Theorem). Let D be a division ring. Let $\sigma : D \to D$ be an automorphism and $\delta : D \to D$ be a σ-derivation such that $\sigma\delta = \delta\sigma$. Consider the ring $D[x_1, \ldots, x_n; \sigma, \delta]$, generated over D by $x_1, \ldots, x_n$, subject to the relations $x_i x_j = x_j x_i$, $x_i a = \sigma(a)x_i + \delta(a)$ for all i, j and $a \in D$. If the center of the skew polynomial ring $D[x; \sigma, \delta]$ contains a nonconstant polynomial, then any simple $D[x_1, \ldots, x_n; \sigma, \delta]$-module is finite-dimensional as a vector space over D.*

The final section also presents a generalization of Lemma 1.2 (see Proposition 4.2):

Proposition 1.4. *Let R be a principal (left and right) ideal domain with infinitely many distinct maximal two-sided ideals. Then for every maximal left ideal L of the polynomial ring $R[x_1, \ldots, x_n]$ over R in central indeterminates, we have $L \cap R \neq 0$.*

Throughout this paper, all rings are assumed to be unital and associative, though not necessarily commutative. We use the term "ideal" to denote a two-sided ideal; one-sided ideals will be explicitly called left or right ideals.

2. Skew polynomial rings in several variables

To generalize the Amitsur–Small theorem, we must also consider polynomial rings whose indeterminates are not central. This section introduces the class of *skew polynomial rings in several variables*, which serves as a suitable framework for such generalizations.

Let R be a subring of a ring S. An (ordered) sequence of elements $x_1, \dots, x_n$ in S is called *(left) algebraically independent over* R if the power products $\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_n^{i_n}$, where $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{N}^n$, are left R-linearly independent. We emphasize that the definition depends on the ordering of the elements $x_1, \dots, x_n$. Let $x_1, \dots, x_n \in S$ be algebraically independent over R. Recall that a *term order* is a well-order $<$ on $\mathbb{N}^n$ such that (1) $(0, \dots, 0) < \mathbf{i}$ for all $(0, \dots, 0) \neq \mathbf{i} \in \mathbb{N}^n$, and (2) if $\mathbf{i} < \mathbf{j}$, then $\mathbf{i} + \mathbf{k} < \mathbf{j} + \mathbf{k}$ for all $\mathbf{k} \in \mathbb{N}^n$. Standard examples of term orders include the lexicographic order, the degree-lexicographic order, and the graded reverse lexicographic order (see Examples 2.1 and 2.2).

We fix a term order $<$ on $\mathbb{N}^n$. The notation $\mathbf{x}^{\mathbf{i}} < \mathbf{x}^{\mathbf{j}}$ is sometimes used in place of $\mathbf{i} < \mathbf{j}$. Given a nonzero element

$$f = r_1 \mathbf{x}^{\mathbf{i}_1} + \cdots + r_m \mathbf{x}^{\mathbf{i}_m} \in S,$$

where $0 \neq r_i \in R$ and $\mathbf{i}_1 < \cdots < \mathbf{i}_m$, we define: $\mathrm{lt}(f) := r_m \mathbf{x}^{\mathbf{i}_m}$, called the *leading term* of f; $\mathrm{lp}(f) := \mathbf{x}^{\mathbf{i}_m}$, called the *leading power product* of f; $\mathrm{le}(f) := \mathbf{i}_m$, called the *leading exponent* of f. We use the convention $\mathrm{le}(0) = (-\infty, \dots, -\infty)$.

Definition 2.1. Let S be a ring and R be a subring of S. We say that S is a *skew polynomial ring over* R *in variables* $x_1, \dots, x_n$ if the following conditions hold:

1. S is a free left R-module with basis $\mathbf{x}^{\mathbf{i}}$, where $\mathbf{i} \in \mathbb{N}^n$;
2. There exists a term order $<$ on $\mathbb{N}^n$ with respect to which

$$\mathrm{le}(fg) = \mathrm{le}(f) + \mathrm{le}(g)$$

for all $f, g \in S \setminus \{0\}$.

Under these conditions, we write $S = R[x_1, \dots, x_n; <]$.

Remark 2.1. The order $<$ in the notation $R[x_1, \dots, x_n; <]$ does not affect the underlying ring structure; it is included only to specify the term order throughout the paper.

Before we present some elementary properties of skew polynomial rings in several variables, let us give some examples of such rings.

Example 2.1. Let R be an integral domain and

$$S = R[x_1; \sigma_1, \delta_1] \dots [x_n; \sigma_n, \delta_n]$$

be an iterated skew polynomial ring over R such that every σ_i is injective. Assume furthermore that $\sigma_i(R) \subseteq R$, $\delta_i(R) \subseteq R$, and $\sigma_i(x_j) \in (R \setminus \{0\})x_j + R$ for all $i > j$.

Clearly, $x_1, \ldots, x_n$ are (left) algebraically independent over R. Let $<_l$ be the lexicographic order on $\mathbb{N}^n$ with $x_1 < \cdots < x_n$, i.e.,

$$x_1^{i_1} \cdots x_n^{i_n} <_l x_1^{j_1} \cdots x_n^{j_n} \iff i_n = j_n, \ldots, i_{p+1} = j_{p+1}, i_p < j_p, \text{ for some } p.$$

Then $S = R[x_1, \ldots, x_n; <_l]$ is a skew polynomial ring over R in $x_1, \ldots, x_n$. In the case where $n = 1$, every skew polynomial ring $S = R[x; <_l]$ over R in x is a skew polynomial ring in the usual sense, that is, $S = R[x; \sigma, \delta]$ for a monomorphism $\sigma : R \to R$ and a σ-derivation $\delta : R \to R$. This follows from the fact that the only term order on $\mathbb{N}$ is the order $0 < 1 < 2 < 3 < \cdots$.

The next example shows that every σ-PBW extension is a skew polynomial ring in several variables. We first recall the definition of a σ-PBW extension.

Definition 2.2. Let S be a ring and R be a subring of S. The ring S is said to be a *σ-Poincaré–Birkhoff–Witt extension of R*, or *σ-PBW extension*, if the following conditions hold:

1. S is a free left R-module with basis $\mathbf{x}^{\mathbf{i}}$, where $\mathbf{i} \in \mathbb{N}^n$;
2. For every $1 \leq i \leq n$ and $r \in R \backslash \{0\}$, there exists $a_{i,r} \in R \backslash \{0\}$ such that $x_i r - a_{i,r} x_i \in R$;
3. For every $1 \leq i < j \leq n$, there exists $a_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - a_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n.$$

For a detailed study of σ-PBW extensions, we refer the reader to [6].

Example 2.2. Let R be an integral domain and $S = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a σ-PBW extension of R. Let $<_d$ be the degree-lexicographic order on the set of all power products $x_1^{i_1} \cdots x_n^{i_n}$ with $x_1 < \cdots < x_n$, i.e., $x_1^{i_1} \cdots x_n^{i_n} <_d x_1^{j_1} \cdots x_n^{j_n}$ if and only if $i_1 + \cdots + i_n < j_1 + \cdots + j_n$, or $i_1 + \cdots + i_n = j_1 + \cdots + j_n$ and $x_1^{i_1} \cdots x_n^{i_n} <_l x_1^{j_1} \cdots x_n^{j_n}$. By the second condition in the Definition 2.2, we have $\mathrm{le}(x_i r) = \mathrm{le}(x_i) + \mathrm{le}(r)$ for all $1 \leq i \leq n$ and $r \in R \setminus \{0\}$. It follows from the third property in the definition that $\mathrm{le}(x_j x_i) = \mathrm{le}(x_j) + \mathrm{le}(x_i)$ for all $1 \leq i < j \leq n$. Now a simple induction can be used to show that $\mathrm{le}(fg) = \mathrm{le}(f) + \mathrm{le}(g)$ for all nonzero $f, g \in S$. Since the first conditions in the Definitions 2.1 and 2.2 coincide, we conclude that $S = R[x_1, \ldots, x_n; <_d]$ is a skew polynomial ring over R in $x_1, \ldots, x_n$.

In the following proposition, we collect some properties of skew polynomial rings in several variables.

Proposition 2.1. *Let $S = R[x_1, \ldots, x_n; <]$ be a skew polynomial ring over R in $x_1, \ldots, x_n$. Then the following statements hold:*
1) S is an integral domain.

2) For every $i = 1, \dots, n$, there exists a unique endomorphism $\sigma_i : R \to R$ such that $\mathrm{lt}(x_i r) = \sigma_i(r) x_i$ for all $r \in R$.

Proof. 1) This is a simple consequence of the equality $\mathrm{le}(fg) = \mathrm{le}(f) + \mathrm{le}(g)$, where $f, g \in S \setminus \{0\}$.
2) For every nonzero $r \in R$, we have $\mathrm{lt}(x_i r) = r' x_i$ for a unique $r' \in R \setminus \{0\}$ because $\mathrm{le}(x_i r) = \mathrm{le}(x_i) + \mathrm{le}(r)$ and $\mathrm{le}(r) = (0, \dots, 0)$. This gives a map $\sigma_i : R \to R$ with $\sigma_i(0) = 0$ such that $\mathrm{lt}(x_i r) = \sigma_i(r) x_i$ for all $r \in R$. It is clear that σ_i is additive. Comparing the leading terms of $x_i(r_1 r_2)$ and $(x_i r_1) r_2$ shows that σ_i is an endomorphism. □

The endomorphisms σ_i whose existence is established in the proposition will be called the *structural endomorphisms* of S.

3. Rigid extensions

This section introduces the concept of rigid extensions and presents a class of such extensions.

3.1. Rigid extensions

In this part, we introduce the class of rigid extensions, which is a subclass of skew polynomial rings in several variables. Let us fix some notation. Let R be a ring and $S = R[x_1, \dots, x_n; <]$ be a skew polynomial ring over R in $x_1, \dots, x_n$. Let $\sigma_i : R \to R$ be the endomorphism corresponding to x_i (see Proposition 2.1). For $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{N}^n$, we denote the map $\sigma_1^{i_1} \circ \cdots \circ \sigma_n^{i_n} : R \to R$ by $\sigma^{\mathbf{i}}$. It follows from Part 2 of Proposition 2.1 that

$$\mathrm{lt}(x^{\mathbf{i}} r) = \sigma^{\mathbf{i}}(r) x^{\mathbf{i}} \tag{3.1}$$

for all $\mathbf{i} \in \mathbb{N}^n$ and $r \in R$. For a subset A of R and $r \in R$, we set

$$(A : r) := \{s \in R \mid sr \in A\}.$$

Definition 3.1. The skew polynomial ring $S = R[x_1, \dots, x_n; <]$ is said to be a *rigid extension* of R if it satisfies $\mathrm{lt}(\mathbf{x}^{\mathbf{i}} \mathbf{x}^{\mathbf{j}}) = \mathbf{x}^{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j}$, and the following property: For any collection $\{I_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^n}$ of left ideals of R satisfying $\sigma^{\mathbf{j}}(I_{\mathbf{i}}) \subseteq I_{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{N}^n$, there exists a nonzero element $r \in R$ such that r is not left-invertible, $rS \subseteq Sr$, and $(I_{\mathbf{i}} : \sigma^{\mathbf{i}}(r)) \subseteq (rI_{\mathbf{i}} : \sigma^{\mathbf{i}}(r))$ for all $\mathbf{i} \in \mathbb{N}^n$.

As an example of a rigid extension, let D be a division ring and x be a central indeterminate. Then the polynomial ring $D[x][t_1, \dots, t_n]$ over $D[x]$ in n central indeterminates is a rigid extension of $D[x]$. This result is a consequence of Lemma A and Lemma B in Amitsur and Small's paper [1]. Our main objective is to give a generalization of this fact

(see Theorem 3.3). We note that an arbitrary Ore extension of $D[x]$ may not be a rigid extension of $D[x]$. As an example, the Weyl algebra $k[x][t; id, \frac{d}{dx}]$ over a commutative field k of characteristic zero is not a rigid extension of $k[x]$ (see Example 3.1).

The concept of rigid extensions is justified by the following result, whose proof is similar to that of Lemma C in [1].

Proposition 3.1. *Let $S = R[x_1, \dots, x_n; <]$ be a rigid extension of R. Then for every maximal left ideal I of S, we have $I \cap R \neq 0$.*

Proof. For $\mathbf{i} \in \mathbb{N}^n$, we set

$$I_{\mathbf{i}} = \{0\} \cup \{r \in R \mid \mathrm{lt}(f) = r\mathbf{x}^{\mathbf{i}} \text{ for some } f \in I\}.$$

It is easy to see that each $I_{\mathbf{i}}$ is a left ideal of R. We claim that $\sigma^{\mathbf{j}}(I_{\mathbf{i}}) \subseteq I_{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j}$. To prove the claim, let $r \in I_{\mathbf{i}}$ and $\mathbf{j} \in \mathbb{N}^n$. There exists $f \in I$ such that $\mathrm{lt}(f) = r\mathbf{x}^{\mathbf{i}}$. We have $\mathrm{le}(\mathbf{x}^{\mathbf{j}} f) = \mathbf{j} + \mathbf{i}$. By Equation (3.1), we have $\mathrm{lt}(\mathbf{x}^{\mathbf{j}} r) = \sigma^{\mathbf{j}}(r)\mathbf{x}^{\mathbf{j}}$, and therefore,

$$\mathrm{lt}(\mathbf{x}^{\mathbf{j}} f) = \mathrm{lt}(\mathbf{x}^{\mathbf{j}} r \mathbf{x}^{\mathbf{i}}) = \mathrm{lt}(\sigma^{\mathbf{j}}(r)\mathbf{x}^{\mathbf{j}}\mathbf{x}^{\mathbf{i}}) = \sigma^{\mathbf{j}}(r)\mathbf{x}^{\mathbf{i}+\mathbf{j}}.$$

Now $\mathbf{x}^{\mathbf{j}} f \in I$ and the claim is proved.

Since S is a rigid extension of R, there exists $r_0 \in R \setminus \{0\}$ such that r_0 is not left-invertible, $r_0 S \subseteq S r_0$, and

$$(I_{\mathbf{i}} : \sigma^{\mathbf{i}}(r_0)) \subseteq (r_0 I_{\mathbf{i}} : \sigma^{\mathbf{i}}(r_0))$$

for all $\mathbf{i}$. If $r_0 \in I$, we are done. Assume that $r_0 \notin I$. Since I is maximal as a left ideal and $r_0 \notin I$, we have $I + S r_0 = S$. In particular, there exists $T \in S$ such that $T r_0 - 1 \in I$. Choose

$$Q = a\mathbf{x}^{\mathbf{m}} + \sum_{\mathbf{i}<\mathbf{m}} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} \in S$$

with $\mathrm{le}(Q) = \mathbf{m}$ as small as possible such that the element

$$Q r_0 - 1 = a\sigma^{\mathbf{m}}(r_0)\mathbf{x}^{\mathbf{m}} + \sum_{\mathbf{i}<\mathbf{m}} a'_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$$

belongs to I. We claim that $\mathbf{m} = (0, \dots, 0)$. Assume, on the contrary, that $\mathbf{m} \neq (0, \dots, 0)$. Since $a\sigma^{\mathbf{m}}(r_0) \in I_{\mathbf{m}}$, that is, $a \in (I_{\mathbf{m}} : \sigma^{\mathbf{m}}(r_0)) = (r_0 I_{\mathbf{m}} : \sigma^{\mathbf{m}}(r_0))$, we see that $a\sigma^{\mathbf{m}}(r_0) = r_0 b$ for some $b \in I_{\mathbf{m}}$. There exists an element $P \in I$ such that

$$P = b\mathbf{x}^{\mathbf{m}} + \sum_{\mathbf{i}<\mathbf{m}} b_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}.$$

Since $r_0S \subseteq Sr_0$, we can find $P_1 \in S$ such that $r_0P = P_1r_0 \in I$. Since $\mathrm{le}(P_1) = \mathrm{le}(P)$, we can write

$$P_1 = c\mathbf{x}^{\mathbf{m}} + \sum_{\mathbf{i}<\mathbf{m}} c'_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}.$$

Then

$$P_1r_0 = r_0P \implies c\mathbf{x}^{\mathbf{m}}r_0 + \sum_{\mathbf{i}<\mathbf{m}} c'_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}r_0 = r_0b\mathbf{x}^{\mathbf{m}} + \sum_{\mathbf{i}<\mathbf{m}} r_0b_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}.$$

Comparing the leading terms and using Equation (3.1), we see that $c\sigma^{\mathbf{m}}(r_0) = r_0b = a\sigma^{\mathbf{m}}(r_0)$. It follows that

$$(Q - P_1)r_0 - 1 \in I.$$

We have $\mathrm{le}(Q - P_1) < \mathrm{le}(Q)$, contradicting the choice of Q. This completes the proof of the claim. Therefore, $Q = a \in R$. Since r_0 is not left-invertible in R, the element $ar_0 - 1$ is a nonzero element in $I \cap R$, completing the proof. □

We now give an example of a skew polynomial ring which is not a rigid extension.

Example 3.1. Consider the Weyl algebra $S = k[x][t; id, \frac{d}{dx}]$ over a commutative field k of characteristic zero as a skew polynomial ring over $R = k[x]$. The left ideal I of S generated by t is a maximal left ideal. Since $I \cap R = 0$, by Proposition 3.1, S is not a rigid extension of R.

3.2. A class of rigid extensions

In this part, we present a family of rigid extensions, which requires a technical lemma. We begin by reviewing some concepts and refer the reader to Cohn's book [5] for further details. Let R be an integral domain and $r \in R$. The element r is called *invariant* if $Rr = rR$. It is called an *Inv-atom* if r is invariant and cannot be written as $r = r_1r_2$, where $r_1, r_2 \in R$ are non-invertible and invariant. The element r is called the *right bound* of an element $s \in R$ if r is invariant and $Rr = rR$ is the largest ideal contained in the right ideal sR. It is easy to see that r is the right bound of s if and only if

$$rR = ann(R/sR).$$

Lemma 3.2. *Let R be a principal (right and left) ideal domain (PID), and let $0 \neq r_0 \in R$. If $(Rr_0 : r) \nsubseteq (rRr_0 : r)$ for some Inv-atom $r \in R$, then r is the right bound of some irreducible left factor of r_0. In particular, there exist at most finitely many such elements r up to right associates for any $0 \neq r_0 \in R$.*

Proof. Let an Inv-atom $r \in R$ satisfy $(Rr_0 : r) \nsubseteq (rRr_0 : r)$. Then we have $r_1 r = r_2 r_0$ for some $r_1 \in R$ and $r_2 \in R \setminus rR$. Since the set $\{s \in R \mid r_2 s \in rR\}$ is a right ideal of R, we have

$$\{s \in R \mid r_2 s \in rR\} = r_3 R$$

for some $r_3 \in R$. Note that r_3 cannot be a unit since otherwise we would have $r_2 \in rR$. Both r and r_0 belong to $r_3 R$ because $Rr = rR$ and $r_2 r_0 = r_1 r \in rR$. Therefore, there exists an irreducible element r' that is a left factor of both r and r_0. In particular, $rR \subset r'R$. Since r is an Inv-atom, we see that r is the right bound of the left irreducible factor r' of r_0, i.e., $rR = ann(R/r'R)$, completing the proof. □

Remark 3.1. Lemma 3.2 can be regarded as a generalization of Lemma A in Amitsur and Small's paper [1].

We are now ready to prove the main result of this section.

Theorem 3.3. *Let R be a PID, and let $S = R[x_1, \dots, x_n; <]$ be a skew polynomial ring over R in $x_1, \dots, x_n$ such that all structural endomorphisms of S are automorphisms, and $\mathrm{lt}(\mathbf{x}^{\mathbf{i}}\mathbf{x}^{\mathbf{j}}) = \mathbf{x}^{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{N}^n$. Assume that R has infinitely many distinct maximal ideals $Rr = rR$ such that $rS \subseteq Sr$. Then S is a rigid extension of R, and consequently, for every maximal left ideal M of S, we have $M \cap R \neq 0$.*

Proof. Let us first note that the assumption $rS \subseteq Sr$, where $r \in R$, implies that $r \in R\sigma_i(r)$ for all i. Since σ_i is an automorphism, it follows that

$$\sigma_i(r)R = R\sigma_i(r) = Rr = rR \text{ for all } i, \tag{3.2}$$

where Rr is a maximal ideal satisfying $rS \subseteq Sr$. Let $I_{\mathbf{i}} = Ra_{\mathbf{i}}$, where $\mathbf{i} \in \mathbb{N}^n$, be a family of left ideals of R satisfying $\sigma^{\mathbf{j}}(I_{\mathbf{i}}) \subseteq I_{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{N}^n$. We need to show that there exists $r \in R \setminus \{0\}$ such that r is not left-invertible, $rS \subseteq Sr$, and

$$(I_{\mathbf{i}} : \sigma^{\mathbf{i}}(r)) \subseteq (rI_{\mathbf{i}} : \sigma^{\mathbf{i}}(r))$$

for all $\mathbf{i} \in \mathbb{N}^n$. Set $J_{\mathbf{i}} = \sigma^{-\mathbf{i}}(I_{\mathbf{i}})$. Note that $J_{\mathbf{i}} \subseteq J_{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{N}^n$. Since R is a PID, hence Noetherian, the set of $J_{\mathbf{i}}$s contains only finitely many distinct ideals (see Lemma B in [1]). It follows that we can find finitely many $\mathbf{i}_1, \dots, \mathbf{i}_m$ with the property that for every $\mathbf{i}$, there exists some $1 \leq l \leq m$ such that $\mathbf{i} - \mathbf{i}_l \in \mathbb{N}^n$ and $I_{\mathbf{i}} = \sigma^{\mathbf{i}-\mathbf{i}_l}(I_{\mathbf{i}_l})$. It follows from the assumption and Lemma 3.2 that there exists a maximal ideal $Rr_0 = r_0 R$ in R such that $r_0 S \subseteq Sr_0$ and $(I_{\mathbf{i}} : r_0) \subseteq (r_0 I_{\mathbf{i}} : r_0)$ for all $\mathbf{i} = \mathbf{i}_1, \dots, \mathbf{i}_m$. Moreover, by Lemma 3.2, r_0 can be chosen such that r_0 is not the right bound of any irreducible left factor of $a_{\mathbf{i}}$ for all $\mathbf{i} = \mathbf{i}_1, \dots, \mathbf{i}_m$. We claim that $(I_{\mathbf{i}} : r_0) \subseteq (r_0 I_{\mathbf{i}} : r_0)$ for all $\mathbf{i}$. Assume, on the contrary, that there exists some $I_{\mathbf{i}}$ which does not satisfy the condition. Then there

exists some $1 \leq l \leq m$ such that $\mathbf{i} - \mathbf{i}_l \in \mathbb{N}^n$ and $I_{\mathbf{i}} = \sigma^{\mathbf{i}-\mathbf{i}_l}(I_{\mathbf{i}_l})$. By Lemma 3.2, we have $r_0R = ann(R/r'R)$ for an irreducible left factor r' of $\sigma^{\mathbf{i}-\mathbf{i}_l}(a_{\mathbf{i}_l})$. Applying $\sigma^{-\mathbf{i}+\mathbf{i}_l}$ to the identity $r_0R = ann(R/r'R)$, we obtain

$$\sigma^{-\mathbf{i}+\mathbf{i}_l}(r_0)R = ann(R/\sigma^{-\mathbf{i}+\mathbf{i}_l}(r')R).$$

Equation (3.2) yields

$$r_0R = \sigma^{-\mathbf{i}+\mathbf{i}_l}(r_0)R.$$

It follows from

$$r_0R = \sigma^{-\mathbf{i}+\mathbf{i}_l}(r_0)R = ann(R/\sigma^{-\mathbf{i}+\mathbf{i}_l}(r')R)$$

that r_0 is the right bound of the irreducible left factor $\sigma^{-\mathbf{i}+\mathbf{i}_l}(r')$ of $a_{\mathbf{i}_l}$. This contradicts the choice of r_0. Therefore, $(I_{\mathbf{i}} : r_0) \subseteq (r_0I_{\mathbf{i}} : r_0)$ for all $\mathbf{i}$. Since $R\sigma^{\mathbf{i}}(r_0) = Rr_0$, we see that

$$(I_{\mathbf{i}} : \sigma^{\mathbf{i}}(r_0)) \subseteq (r_0I_{\mathbf{i}} : \sigma^{\mathbf{i}}(r_0))$$

for all $\mathbf{i}$. This completes the proof of the first statement. The second statement follows from Proposition 3.1. □

4. Examples of rigid extensions

This section presents some examples of rigid extensions and establishes a skew Amitsur–Small Theorem. We begin with the one-variable case.

Proposition 4.1. *Let R be a PID, $\sigma : R \to R$ be an automorphism, and $\delta : R \to R$ be a σ-derivation. If R has infinitely many distinct maximal ideals invariant under σ and δ, then the Ore extension $R[x;\sigma,\delta]$ is a rigid extension of R, and consequently, for every maximal left ideal M of $R[x;\sigma,\delta]$, we have $M \cap R \neq 0$.*

Proof. By Theorem 3.3, we only need to show that R has infinitely many maximal ideals $Rr = rR$ such that $rR[x;\sigma,\delta] \subseteq R[x;\sigma,\delta]r$. We claim that $rR[x;\sigma,\delta] \subseteq R[x;\sigma,\delta]r$ for every $r \in R$ such that $Rr = rR$ is a maximal ideal invariant under σ and δ. Since $R[x;\sigma,\delta]$, as a ring, is generated by R and x, it is enough to show that $rx \in R[x;\sigma,\delta]r$. Since $R\sigma(r) = Rr = R\sigma^{-1}(r)$ and $\delta(Rr) \subseteq Rr$, we have

$$rx = x\sigma^{-1}(r) - \delta(\sigma^{-1}(r)) \in R[x;\sigma,\delta]r.$$

This completes the proof of the claim. The result now follows from the assumption that $\sigma(Rr) \subseteq Rr$ and $\delta(Rr) \subseteq Rr$ for infinitely many distinct maximal ideals Rr in R. □

Remark 4.1. In the case where R is commutative, the second conclusion of the proposition holds under the weaker assumption that R is a Dedekind domain, as proved by Bavula [2, Theorem 1.2].

Turning to the several-variable case, we begin with the following result.

Proposition 4.2. *Let R be a PID with infinitely many distinct maximal ideals. Then the polynomial ring $R[x_1, \ldots, x_n]$ in central indeterminates is a rigid extension of R, and consequently, for every maximal left ideal M of $R[x_1, \ldots, x_n]$, we have $M \cap R \neq 0$.*

Proof. For any term order $<$, $R[x_1, \ldots, x_n] = R[x_1, \ldots, x_n; <]$ is a skew polynomial ring in $x_1, \ldots, x_n$ with respect to $<$. The result now follows from Theorem 3.3. □

Remark 4.2. The converse of the proposition holds true for commutative rings. More precisely, if $M \cap R \neq 0$ for every maximal ideal M of the polynomial ring $R[x_1, \ldots, x_n]$ over a commutative PID R in central indeterminates, then R has infinitely many maximal ideals. One can prove this result using the language of G-rings. For more details, see [7, Theorem 27 and Theorem 147].

To present a skew version of the Amitsur–Small Theorem, we need the following result.

Proposition 4.3. *Let D be a division ring, $\sigma : D \to D$ an automorphism and $\delta : D \to D$ a σ-derivation. Then the following statements are equivalent:*
(1) The center of the Ore extension $D[x; \sigma, \delta]$ contains a nonconstant polynomial.
(2) $D[x; \sigma, \delta]$ has infinitely many distinct maximal ideals.

Proof. (1) $\Longrightarrow$ (2): Let $h_0(x)$ be a nonconstant polynomial of the least degree in the center of $D[x; \sigma, \delta]$. Let $q(x)$ be a polynomial in $D[x; \sigma, \delta]$ of the least degree such that $D[x; \sigma, \delta]q(x) = q(x)D[x; \sigma, \delta]$. By Cauchon's description of the ideal structure of $D[x; \sigma, \delta]$ (see [3, Theorem 1.4]), every ideal of $D[x; \sigma, \delta]$ is generated by a nonconstant polynomial $h(x)q(x)^n$, where $h(x) \in C(D)_{\sigma,\delta}[h_0(x)]$ and $n \geq 0$. Here, $C(D)_{\sigma,\delta}$ is the field

$$C(D)_{\sigma,\delta} = \{a \in D : \sigma(a) = a, \delta(a) = 0, \forall b \in D \quad ab = ba\}.$$

It follows that for every irreducible polynomial $g(t) \in C(D)_{\sigma,\delta}[t]$, the ideal of $D[x; \sigma, \delta]$ generated by $g(h_0(x))$ is maximal. It is well known that the polynomial ring $F[t]$ over a commutative field F has infinitely many monic irreducible polynomials, from which (2) follows.
(2) $\Longrightarrow$ (1): If the center of $D[x; \sigma, \delta]$ does not contain a nonconstant polynomial, then by Cauchon's result, every ideal of $D[x; \sigma, \delta]$ is generated by a polynomial $q(x)^n$ for some $n \geq 1$, in which case $D[x; \sigma, \delta]$ would have a single maximal ideal. This proves the implication (2) $\Longrightarrow$ (1). □

Remark 4.3. It is easy to prove that if the center of $D[x;\sigma,\delta]$ contains a nonconstant polynomial, then σ has a finite inner order and δ is quasi-algebraic. Recall that σ is said to have a *finite inner order* if some power σ^n of σ has the form $\sigma^n(r) = ara^{-1}$ for some $a \in D$; δ is called *quasi-algebraic* if δ satisfies an operator equation

$$a_n\delta^n + \cdots + a_1\delta + a_0\delta_{b,\alpha} = 0,$$

where $b, a_i \in D$, $a_n \neq 0$, α is an endomorphism of D, and $\delta_{b,\alpha}$ denotes the map $x \mapsto bx - \alpha(x)b$. For more details, see [8] and the references therein.

Let D be a division ring. Let $\sigma : D \to D$ be an automorphism and $\delta : D \to D$ be a σ-derivation such that $\sigma\delta = \delta\sigma$. Consider the ring $D[x_1, \ldots, x_n; \sigma, \delta]$, generated over D by $x_1, \ldots, x_n$, subject to the relations $x_ix_j = x_jx_i$, $x_ia = \sigma(a)x_i + \delta(a)$ for all i, j and $a \in D$. It can be shown that $D[x_1, \ldots, x_n; \sigma, \delta]$ is an iterated skew polynomial ring. More precisely, we have

$$D[x_1, \ldots, x_n; \sigma, \delta] = D[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n],$$

where $\sigma_i|_D = \sigma, \delta_i|_D = \delta, \sigma_j(x_i) = x_i, \delta_j(x_i) = 0$ for all $1 \leq i < j \leq n$. See also [9, Theorem 4.2].

Theorem 4.4. *Let D be a division ring. Let $\sigma : D \to D$ be an automorphism and $\delta : D \to D$ be a σ-derivation such that $\sigma\delta = \delta\sigma$. If the center of $D[x;\sigma,\delta]$ contains a nonconstant polynomial, then any simple $D[x_1, \ldots, x_n; \sigma, \delta]$-module is finite-dimensional as a vector space over D.*

Proof. The discussion preceding the theorem shows that

$$S = D[x_1, \ldots, x_n; \sigma, \delta] = D[x_1; \sigma, \delta][x_2, \ldots, x_n; <_l]$$

is a skew polynomial ring over $D[x_1; \sigma, \delta]$ in the variables $x_2, \ldots, x_n$, where $<_l$ denotes the lexicographic order with $x_2 < \cdots < x_n$ (see Example 2.1). By Proposition 4.3, $D[x_1; \sigma, \delta]$ has infinitely many maximal ideals. As shown in the proof of the proposition, every maximal ideal of $D[x_1; \sigma, \delta]$ is generated by a central polynomial. Therefore, there are infinitely many maximal ideals $D[x_1; \sigma, \delta]f$ in $D[x_1; \sigma, \delta]$ such that $fS \subseteq Sf$. It follows from Theorem 3.3 that S is a rigid extension of $D[x_1; \sigma, \delta]$. In particular, for every maximal left ideal M in S, we have $M \cap D[x_1; \sigma, \delta] \neq 0$. By symmetry, for every maximal left ideal M in S, we have $M \cap D[x_i; \sigma, \delta] \neq 0$ for all i. To prove the theorem, let V be a simple left S-module, let $0 \neq v \in V$, and write

$$V \cong S/M,$$

where $M = \mathrm{Ann}_S(v)$ is a maximal left ideal. For each i, the nonzero intersection $M \cap D[x_i; \sigma, \delta]$ provides a nonzero polynomial $p_i(x_i)$ satisfying $p_i(x_i)v = 0$. After normalizing

p_i to be monic, this relation permits sufficiently high powers of x_i to be expressed as a D-linear combination of lower powers. Consequently, using the commutation relations among the variables, we see that as a left D-module, V is spanned by finitely many vectors

$$x_1^{e_1} \cdots x_n^{e_n} v, \qquad 0 \leq e_i < \deg p_i.$$

It follows that $\dim_D V < \infty$, completing the proof. □

As a special case, we record the following result.

Corollary 4.5. *Let F be a finite field and $\sigma : F \to F$ be an automorphism. Then every simple $F[x_1, \ldots, x_n; \sigma]$-module is finite-dimensional as a vector space over F.*

Proof. It is clear that σ has a finite order, say m. Then x^m belongs to the center of $F[x; \sigma]$. By Theorem 4.4, every simple $F[x_1, \ldots, x_n; \sigma]$-module is finite-dimensional as a vector space over F. □

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Data availability

No data was used for the research described in the article.

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