Journal of Geometry and Physics 231 (2027) 106007

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Lagrange theorem and isomorphism theorems in BiHom-groups

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ARTICLE INFO

Article history:
Received 11 October 2025
Received in revised form 11 August 2026
Accepted 10 September 2026
Available online 14 September 2026

MSC:
17D30
20N99
17A99

Keywords:
BiHom-associative group
BiHom-subgroup
Twisting
Homomorphism

ABSTRACT

BiHom-groups are a nonassociative generalization of groups with the associativity law replaced by a twisted associativity law: $\alpha(a)(bc) = (ab)\beta(c)$, where α and β are two multiplicative mappings. The objective of this paper is to introduce and study BiHom-groups. We define the concept and provide examples of BiHom-groups. Additionally, we establish fundamental properties such as normal BiHom-subgroups, BiHom-quotient groups, and homomorphisms of BiHom-groups, and present the first, second, and third isomorphism theorems for BiHom-groups.

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1. Introduction

The paradigm of Hom-algebraic structures originated in deformation settings in geometry and mathematical physics arising from q-difference calculi, twisted derivations, and discretization schemes. The theory of Hom-algebraic structures was initiated by Hartwig, Larsson, and Silvestrov in 2003 in [26], where they developed a general framework for quasi-deformations and discretizations of Lie algebras of vector fields using σ-derivations instead of classical derivations, introduced Hom-Lie and quasi-Hom-Lie algebras as the natural abstract algebraic structures underlying such quasi-deformations and discretizations, and initiated the study of central extensions of Hom-Lie and of the more general quasi-Hom-Lie algebras developed further in [38]. Initial partial motivation for [26,38] came from a desire to develop a systematic approach to the deformations of Witt- and Virasoro-type algebras and q-deformed Jacobi identities in conformal structures and models of deformed symmetries in quantum field theory and integrable systems that appeared, for example, in [2,14–17,23,31,42–45].

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https://doi.org/10.1016/j.geomphys.2026.106007
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In 2004, Larsson and Silvestrov [41] applied the quasi-deformation method by twisted derivations to representations of sl_2 and, in [39], introduced the broader classes of quasi-Lie and quasi-Leibniz algebras and their Hom-type subclasses, including quasi-Hom-Lie, Hom-Lie, quasi-Hom-Leibniz, and Hom-Leibniz algebras. In 2005, graded color generalizations were introduced in [40], including color quasi-Lie, color quasi-Hom-Lie, color Hom-Lie, color quasi-Leibniz, color quasi-Hom-Leibniz, and color Hom-Leibniz algebras, together with the corresponding superalgebra subclasses, while color quasi-Lie algebras of the Witt type were constructed in [57]. Subsequently, Makhlouf and Silvestrov introduced in 2006 Hom-Lie admissible algebras, including Hom-associative algebras and the more general G-Hom-associative algebras encompassing Hom-Vinberg (Hom-left symmetric), Hom-pre-Lie (Hom-right symmetric) algebras and some other Hom-algebras, established their Hom-Lie admissibility, introduced flexible Hom-algebras, and pointed out connections with Hom-type generalizations of derivations and adjoint maps [48].

Since the pioneering works [26,38–41,48], Hom-algebra theory has developed into a broad and active area encompassing Lie theory, representation theory, homology and cohomology, Hopf-type structures, deformation theory, Jordan and alternative algebras, Yang–Baxter type structures, Rota–Baxter operators, and numerous applications in geometry and mathematical physics (see, for example, [1,3–7,9–11,13,18,19,24,25,27,30,32,34–38,41,46,47,49–56,58–67]). The Hom-algebra paradigm encompasses algebraic structures whose defining identities are deformed by one or more twisting maps. These structures provide natural algebraic frameworks for the deformed symmetries arising in mathematical physics, discretization, quantization, and noncommutative geometry. Within this paradigm, Hom-Hopf and monoidal Hom-Hopf algebras extend classical Hopf-theoretic symmetry frameworks, while Hom-Lie algebras and related structures provide twisted analogs of many geometric and representation-theoretic constructions [9,13,22,27,33,49,50].

Recently, the notion of Hom-groups was introduced as a systematic extension of groups, providing a deeper understanding of group-like elements and properties of Hom-Hopf algebras [13,33,49,50], and of Hom-group substructures in many other Hom-structures. Hom-groups generalize groups to Hom-associative settings with the twisting map modifying associativity, unity, invertibility, constructions, substructures, and morphisms, with many examples, classifications, and fundamental properties differing from their classical group-theoretic counterparts. In [28], the notions of Hom-subgroups and cosets were introduced, and Lagrange's theorem for finite Hom-groups was proven. Furthermore, modules over a Hom-group G were introduced, and Hom-group (co)homology with coefficients in them was developed [29]. In [21], new examples of Hom-groups were provided, the first, second, and third isomorphism theorems for Hom-groups were established, and the concept of a Hom-group action was introduced. In [12], the notion of ordered groups was extended to Hom-groups by defining the concept of a Hom-compatible order. Furthermore, in [20], a new type of Hom-group action was introduced, and the class equation for Hom-groups was derived. Additionally, the study in [8] explored the generalization of Hom-type rings.

BiHom-structures constitute an important class of Hom-type structures in which two twisting maps govern twisted identities. This double-twist mechanism is well-suited for modeling anisotropic discretizations and multiscale phenomena in geometric analysis and mathematical physics. Recent advances in BiHom-associative, BiHom-Lie, dendriform, and dialgebraic structures highlight the role of these algebras as natural foundations for noncommutative and quasi-geometric models with twisted observables and nonclassical symmetries [25,32,34,35,37]. BiHom-groups arise as group-like symmetry objects within the BiHom framework, governed by two compatible twisting maps that modify associativity. They are closely related to BiHom-associative and BiHom-Hopf structures and provide algebraic models of twisted symmetries occurring in deformation, discretization, and quantization settings [25,49,50]. From a geometric perspective, BiHom-groups provide an algebraic foundation for twisted actions, orbit-like constructions, quotient structures, and generalized symmetry spaces associated with transformations of geometric spaces evolving under internal twistings, renormalization-type scaling, discretized flows, or endomorphism-driven evolution. Their connection with Hom-Hopf and BiHom-Hopf theories, where group-like elements, comodules, and integrals appear in twisted forms, supports their role as symmetry carriers in generalized geometric settings, including twisted fiber bundles and deformed gauge-type symmetries [49,50]. Moreover, BiHom-groups interact naturally with the Hom–Yang–Baxter equation and Hom-Lie bialgebra frameworks for integrable systems, where twisted R-matrices and braidings occur [61,62]. More broadly, the BiHom framework, including BiHom-associative, BiHom-alternative, and BiHom-Jordan structures, together with Rota–Baxter and pre-Lie operators, underlies perturbative and discretization techniques, such as splitting methods, averaging operators, and renormalization-type constructions [1,18,19]. Together, these developments position BiHom-groups as useful algebraic structures for a wide range of geometric and physical applications.

This perspective motivates the development of a structural theory of BiHom-groups that parallels the classical group theory. In geometric settings, group actions, orbits, stabilizers, homogeneous spaces, and quotient constructions play a fundamental role. The corresponding theory of BiHom-groups requires compatible notions of BiHom-actions, BiHom-cosets, BiHom-normal subgroups, and BiHom-quotient groups. Structural results, such as Lagrange's theorem and the first, second, and third isomorphism theorems, provide an algebraic foundation for studying twisted symmetry spaces, orbit-like constructions, and quotient geometries arising in deformation, discretization, and quantization frameworks [12,20,21,25,28,29,61,62].

The main objective of this article is to introduce and systematically study BiHom-groups. We begin by defining BiHom-groups, highlighting their key characteristics and providing illustrative examples. We then investigate the fundamental properties of BiHom-groups, such as normal BiHom-subgroups, BiHom-quotient groups, and homomorphisms of BiHom-groups. In light of the geometric and physical motivations, we proceed in the subsequent sections to develop the fundamental structural theory of BiHom-groups, including Lagrange's theorem and the first, second, and third isomorphism theorems, ensuring

that twisted symmetry objects retain the structural properties required for the geometric applications of BiHom-groups and related BiHom-associative structures.

2. Main concepts and properties

For the identity mapping, we use the notation id. For the composition of two mappings f and g, we write either $g \circ f$ or simply gf. The action of binary operations on elements is usually conveniently denoted by juxtaposition if it is clear which operation is meant, or with the operation symbol placed between arguments ($\mu(a, b) = ab = a \cdot b$).

An associative algebraic structure is a set equipped with an operation that satisfies the associativity property, which states that for any elements a, b, and c in the set, the binary operation applied between them is independent of the grouping, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. If this property is not satisfied, it is a nonassociative algebraic structure. Groups are associative binary structures.

2.1. Definitions and examples

Definition 2.1. A (regular) BiHom-group (G, μ, α, β) consists of a set G together with distinguished element e of G, a pair of (invertible) mappings $\alpha, \beta : G \to G$ and a binary operation $\mu : G \times G \to G$ (called a product or a multiplication) satisfying the following axioms.

1) The product mapping $\mu : G \times G \to G$ satisfies the BiHom-associativity for all $x, y, z \in G$ (see [25]),

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), (\beta(z))). \qquad \text{(BiHom-associativity)} \tag{1}$$

2) The maps α, β are multiplicative (that is are group endomorphisms,

$$\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)), \quad \beta(\mu(x, y)) = \mu(\beta(x), \beta(y)). \tag{2}$$

3) There is an element e, called BiHom-unity (or simply just "unity" if no confusion arises), satisfying the BiHom-unitality condition

$$\mu(x, e) = \alpha(x) \quad \text{(right unitality)}, \quad \mu(e, x) = \beta(x) \quad \text{(left unitality)}. \tag{3}$$

4) For every element $x \in G$, there exist elements $x_r^{-1}, x_l^{-1} \in G$, called left inverse and right inverses of x, satisfying the following BiHom-invertibility conditions:

$$\mu(x, x_r^{-1}) = e \quad \text{(right invertibility)}, \quad \mu(x_l^{-1}, x) = e \quad \text{(left invertibility)}. \tag{4}$$

Remark 2.1. BiHom-groups generalize Hom-groups, which are obtained when $\alpha = \beta$, and hence also the groups obtained when $\alpha = \beta = id$, where $\mathrm{id}(x) = x$ for all $x \in G$.

Remark 2.2. The extra assumption of bijectivity, that is, the invertibility of the mappings α, β singles out the class of *regular* BiHom-groups. The multiplicativity of α and β means that the binary BiHom-structure is *multiplicative*.

Example 2.1. The binary operation on $\mathbb{R}_{>0} = \{t \in \mathbb{R} \mid t > 0\}$, defined by $x \otimes y = x^u y^v$ for some $u, v \in \mathbb{R}$ and all $x, y \in \mathbb{R}_{>0}$, is non-associative if and only if $(u, v) \notin \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Moreover, if $(u, v) = (1, 1)$, then $(\mathbb{R}_{>0}, \otimes)$ is a group, while for $(u, v) \in \{(0, 0), (0, 1), (1, 0)\}$ is not a group. The mappings $\alpha, \beta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, defined by $\alpha(x) = x^a, \beta(x) = x^b$ for some $a, b \in \mathbb{R}$ and all $x \in \mathbb{R}_{>0}$, are multiplicative on $\mathbb{R}_{>0}$ for any $a, b \in \mathbb{R}$ and bijective for any $a, b \in \mathbb{R} \setminus \{0\}$.

Let us find all $u, v, a, b \in \mathbb{R}_{>0}$ for which the BiHom-associativity holds. For all $x, y, z \in \mathbb{R}_{>0}$,

$$(x \otimes y) \otimes \beta(z) - \alpha(x) \otimes (y \otimes z) = (x^u y^v)^u (z^b)^v - (x^a)^u (y^u z^v)^v = x^{(u^2)} y^{vu} z^{bv} - x^{au} y^{uv} z^{(v^2)}. \tag{5}$$

Thus, BiHom-associativity holds if and only if $u^2 = au, vu = uv, bv = v^2$. All the solutions of this system of equations in $\mathbb{R}_{>0}$, and the corresponding $\otimes, \alpha, \beta$, are

$$\begin{aligned}
&u = v = 0,\ a, b \in \mathbb{R}: && x \otimes y = 1,\ \alpha(x) = x^a,\ \beta(x) = x^b;\\
&u = 0, b = v,\ a \in \mathbb{R}: && x \otimes y = y^v,\ \alpha(x) = x^a,\ \beta(x) = x^v;\\
&a = u, v = 0,\ b \in \mathbb{R}: && x \otimes y = x^u,\ \alpha(x) = x^u,\ \beta(x) = x^b;\\
&a = u, b = v,\ u, v \in \mathbb{R}: && x \otimes y = x^u y^v,\ \alpha(x) = x^u,\ \beta(x) = x^v.
\end{aligned}$$

The element $h \in \mathbb{R}_{>0}$ is a BiHom-unity in $(\mathbb{R}_{>0}, \otimes, \alpha, \beta)$ if and only if

$$\underbrace{x^u h^v}_{x\otimes h} = \underbrace{x^a}_{\alpha(x)}, \quad \underbrace{h^u x^v}_{h\otimes x} = \underbrace{x^b}_{\beta(x)}, \ \forall x \in \mathbb{R}_{>0},$$

which is satisfied if and only if one of the following cases holds:

$$h \neq 1, (u,v) = (a,b) = (0,0): \quad x \otimes y = 1, \alpha(x) = \beta(x) = 1;$$
$$h = 1, (a,b) = (u,v): \quad x \otimes y = x^u y^v, \ \alpha(x) = x^u, \ \beta(x) = x^v.$$

Finally, every $x \in \mathbb{R}_{>0}$ is BiHom-invertible with $x_l^{-1} = x^{-\frac{v}{u}}$, $x_r^{-1} = x^{-\frac{u}{v}}$, if $u \neq 0, v \neq 0$, since then $x \otimes x^{-\frac{u}{v}} = x^u (x^{-\frac{u}{v}})^v = 1, x^{-\frac{v}{u}} \otimes x = (x^{-\frac{v}{u}})^u x^v = 1$. If $u = 0$, then $x \otimes y = y^v$, and any possible BiHom-unit $h \in \mathbb{R}_{>0}$ would need to satisfy $x^a = x \otimes h = h^v$ and $x^b = h \otimes x = x^v$ on $\mathbb{R}_{>0}$ which is possible if and only if $h = 1$, $a = 0$ and $b = v$. In this case, $1 = x_l^{-1} \otimes x = x^v$ means that for $v \neq 0$, there is no left inverse for $x \neq 1$. This means that $h = 1$, $a = 0$ and $b = v \neq 0$ do not give a BiHom-group. If $v = 0$, then $x \otimes y = x^u$, and any possible BiHom-unit $h \in \mathbb{R}_{>0}$ would need to satisfy $x^a = x \otimes h = x^u$ and $x^b = h \otimes x = h^u$ on $\mathbb{R}_{>0}$ which is possible if and only if $h = 1$, $b = 0$ and $a = u$. In this case, $1 = x \otimes x_r^{-1} = x^u$ means that for $u \neq 0$, there is no right inverse for $x \neq 1$. This means that $h = 1$, $b = 0$ and $a = u \neq 0$ do not give a BiHom-group. Therefore, $(\mathbb{R}_{>0}, \otimes, \alpha, \beta)$ is a BiHom-group if and only if either $a = b = u = v = 0$, that is $x \otimes y = 1, \alpha(x) = \beta(x) = 1$ for all $x, y \in \mathbb{R}_{>0}$ (degenerate case) or

$$x \otimes y = x^u y^v, \ \alpha(x) = x^u, \ \beta(x) = x^v, u \neq 0, v \neq 0, \forall x, y \in \mathbb{R}_{>0}. \tag{6}$$

It is a regular BiHom-group since α and β are bijective because $a = u \neq 0$ and $b = v \neq 0$. In the degenerate case $a = b = u = v = 0$, $x \otimes y = 1, \alpha(x) = \beta(x) = 1$ for all $x, y \in \mathbb{R}_{>0}$, and the twisting maps α and β are not bijective, 1 is a BiHom-unit, and any element is the inverse of any other element. Thus, the inverse is not unique. This does not contradict Proposition 2.8 because neither α nor β is bijective in this case.

Example 2.2 *(Construction of BiHom-groups from abelian subgroups of groups using power maps).* In this example, we systematically develop the construction of BiHom-groups from any abelian subgroup H of a group G using arbitrary integer power maps as twisting maps α and β. Let $H \subseteq G, H \neq \{e_G\}$ be a nontrivial abelian subgroup of a group G, and for some integers $u, v, a, b \in \mathbb{Z}$ and all $x, y \in G$,

$$\mu(x,y) = x^u y^v, \quad \alpha(x) = x^a, \quad \beta(x) = x^b.$$

Note that $\mu(H,H) \subseteq H, \alpha(H) \subseteq H, \beta(H) \subseteq H$ yield the algebraic structure (H, μ, α, β) with the same operations μ, α, β restricted from G to H.

The mappings α, β are multiplicative on H for any $a, b \in \mathbb{Z}$, and if the subgroup H is torsion-free, then they are bijective for any $a, b \in \mathbb{Z} \setminus \{0\}$.

The multiplicativity in Definition 2.1(2) holds automatically in abelian subgroups, since the commutativity of H yields $(x^u y^v)^a = x^{ua} y^{va}$, the commutativity of multiplication in $\mathbb{Z}$ implies $ua = au, va = av, ub = bu, vb = bv$, and thus $\alpha(\mu(x,y)) = \alpha(x^u y^v) = (x^u y^v)^a = x^{ua} y^{va} = x^{au} y^{av} = \alpha(x)^u \alpha(y)^v = \mu(\alpha(x), \alpha(y)), \ \beta(\mu(x,y)) = \beta(x^u y^v) = x^{ub} y^{vb} = x^{bu} y^{bv} = \beta(x)^u \beta(y)^v = \mu(\beta(x), \beta(y))$.

Due to commutativity of H, for any $x, y, z \in H$,

$$\mu(\alpha(x), \mu(y,z)) = \mu(x^a, y^u z^v) = (x^a)^u (y^u z^v)^v = x^{au} y^{uv} z^{v^2},$$
$$\mu(\mu(x,y), \beta(z)) = \mu(x^u y^v, z^b) = (x^u y^v)^u (z^b)^v = x^{u^2} y^{uv} z^{bv}.$$

Assume that H is a torsion-free abelian subgroup. The subgroup is torsion-free, which means that the unity element is the only element with finite order; that is, all elements in $H \setminus \{e_H\}$ have infinite order, that is, $t^n \neq e_H$ for any non-zero $n \in \mathbb{Z}$ and any $t \in H \setminus \{e_H\}$. The BiHom-associativity (1) holds for all $x, y, z \in H$ if and only if $u^2 = au, vu = uv, bv = v^2$, which holds if and only if $u^2 = au, bv = v^2$ since $uv = uv$ is always satisfied in $\mathbb{Z}$. Thus, all the cases for which the BiHom-associativity holds in the case of a torsion free H are

$$u = v = 0, \ a, b \in \mathbb{Z}: \quad \mu(x,y) = e, \ \alpha(x) = x^a, \ \beta(x) = x^b; \tag{7}$$
$$u = 0, b = v, \ a \in \mathbb{Z}: \quad \mu(x,y) = y^v, \ \alpha(x) = x^a, \ \beta(x) = x^v; \tag{8}$$
$$a = u, v = 0, \ b \in \mathbb{Z}: \quad \mu(x,y) = x^u, \ \alpha(x) = x^u, \ \beta(x) = x^b; \tag{9}$$
$$a = u, b = v, \ u, v \in \mathbb{Z}: \mu(x,y) = x^u y^v, \ \alpha(x) = x^u, \ \beta(x) = x^v. \tag{10}$$

The element $h \in H$ is a BiHom-unity in $(H, \otimes, \alpha, \beta)$ if and only if

$$\underbrace{x^u h^v}_{\mu(x,h)} = \underbrace{x^a}_{\alpha(x)}, \quad \underbrace{h^u x^v}_{\mu(h,x)} = \underbrace{x^b}_{\beta(x)}, \ \forall x \in H,$$

which is satisfied if and only if one of the following cases holds:

$$h \neq e, (u, v) = (a, b) = (0, 0): \quad \mu(x, y) = e, \alpha(x) = \beta(x) = e; \tag{11}$$

$$h = e, (a, b) = (u, v): \qquad \mu(x, y) = x^u y^v, \ \alpha(x) = x^u, \ \beta(x) = x^v, \tag{12}$$

since $x^a = \alpha(x) = \mu(x, h) = x^u h^v = x^u$ and $x^b = \beta(x) = \mu(h, x) = h^u x^v = x^v$, which are equivalent to $a = u$ and $b = v$, respectively, in the case of torsion-free H.

If $(u, v) \neq 0$, then if the left and right BiHom-inverses x_l^{-1}, x_r^{-1} exist, then they satisfy

$$\mu(x_l^{-1}, x) = (x_l^{-1})^u x^v = e \Rightarrow (x_l^{-1})^u = x^{-v} \Rightarrow (x_l^{-1} = x^{-v/u} \text{if } u|v),$$
$$\mu(x, x_r^{-1}) = x^u (x_r^{-1})^v = e \Rightarrow (x_r^{-1})^v = x^{-u} \Rightarrow (x_r^{-1} = x^{-u/v} \text{if } v|u).$$

Note that for torsion free H, the left and right inverses coincide only if $u = v$, and in that case $\alpha(z) = \beta(z) = z^u = z^v$ and the BiHom-group becomes a Hom-group.

In any (not necessarily torsion-free) abelian subgroup H of G, for any $k \in \mathbb{Z}$, $\mu(x, y) = x^k y^k, \alpha(x) = x^k, \beta(x) = x^k$ yield a BiHom-group $(H, \mu, e, \alpha, \beta)$ which is a Hom-group with $x_l^{-1} = x_r^{-1} = x^{-1}$. Since by commutativity of the subgroup H,

$$\alpha(xy) = (xy)^k = x^k y^k = \alpha(x)\alpha(y), \text{ and similarly for } \beta;$$
$$\mu(\alpha(x), \mu(y, z)) = x^k (y^k z^k) = x^k y^k z^k = (x^k y^k) z^k = \mu(\mu(x, y), \beta(z));$$
$$\mu(x, e) = x^k = \alpha(x), \mu(e, x) = x^k = \beta(x), \ \mu(x, x^{-1}) = x^k x^{-k} = e = x^{-k} x^k = \mu(x^{-1}, x).$$

The following examples build on Example 2.2 and can be viewed as matrix extensions of the scalar Example 2.1. These examples often arise in physics and in geometry.

Example 2.3 *(Construction of BiHom-groups from abelian subgroups of $SL_2(\mathbb{R})$).* Consider the group $GL_2(\mathbb{R})$ of 2×2 real matrices with nonzero determinants. Its center is $Z(GL_2(\mathbb{R})) = \mathbb{R}I_2$. The torsion free abelian subgroup $H \subseteq GL_2(\mathbb{R})$ consisting of all positive diagonal matrices, $H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x > 0, y > 0 \right\}$, with the binary operation $\mu(X, Y) = X^u Y^v$ and $\alpha, \beta : H \to H, \alpha(X) = X^u, \beta(X) = X^v$ defined for fixed integers $u, v \geq 1$, the identity matrix $e = I_2$ as a BiHom-unity, and the left and right BiHom-inverses $X_l^{-1} = \begin{pmatrix} a^{-\frac{v}{u}} & 0 \\ 0 & b^{\frac{v}{u}} \end{pmatrix}$, $X_r^{-1} = \begin{pmatrix} a^{-\frac{u}{v}} & 0 \\ 0 & b^{\frac{u}{v}} \end{pmatrix}$, yield a BiHom-group $(H, \mu, I_2, \alpha, \beta)$, which becomes a Hom-group if $u = v$.

Example 2.4 *(Construction of BiHom-groups from abelian subgroups of $SL_2(\mathbb{R})$).* Consider the group $SL_2(\mathbb{R})$ of 2×2 real matrices with determinant 1. Its center is $Z(SL_2(\mathbb{R})) = \{\pm I_2 = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\}$. The torsion free abelian subgroup $H \subseteq SL_2(\mathbb{R})$ consisting of all positive diagonal matrices with determinant 1, $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\}$, with the binary operation $\mu(X, Y) = X^u Y^v$ and $\alpha, \beta : H \to H, \alpha(X) = X^u, \quad \beta(X) = X^v$ defined for fixed integers $u, v \geq 1$, the identity matrix $e = I_2$ as a BiHom-unity, and the left and right BiHom-inverses $X_l^{-1} = \begin{pmatrix} a^{-\frac{v}{u}} & 0 \\ 0 & a^{\frac{v}{u}} \end{pmatrix}$, $X_r^{-1} = \begin{pmatrix} a^{-\frac{u}{v}} & 0 \\ 0 & a^{\frac{u}{v}} \end{pmatrix}$, yield a BiHom-group $(H, \mu, I_2, \alpha, \beta)$, which becomes a Hom-group if $u = v$.

Example 2.5. For all $a, b \in \mathbb{R}$, the additive group of real numbers $(\mathbb{R}, +)$ have a BiHom-group structure with the operation $\oplus$ and multiplicative mappings $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ defined for all $x, y \in \mathbb{R}$ by $x \oplus y = ax + by, \qquad \alpha(x) = ax, \qquad \beta(x) = bx.$ The BiHom-unitality with $e = 0$, and the BiHom-invertibility with BiHom-inverse $x_\oplus^{-1} = -\frac{a}{b}x$ clearly hold, and the BiHom-associativity holds since, for all $x, y, z \in \mathbb{R}$,

$$\begin{aligned} \alpha(x) \oplus (y \oplus z) &= a(ax) + b(ay + bz) = a^2 x + aby + b^2 z \\ &= a(ax + by) + b(bz) = (x \oplus y) \oplus \beta(z). \end{aligned}$$

To simplify the notation of BiHom-groups, we occasionally omit the product symbol in quadruple notation and denote multiplication by juxtaposition.

Proposition 2.6. *Let (G, α, β) be a BiHom-group.*

1. *If β is bijective, then $xy = xz \Rightarrow y = z$ in G.* (Left cancellation)

2. *If α is bijective, then $yx = zx \Rightarrow y = z$ in G.* (Right cancellation)

If the BiHom-group is regular, then both left and right cancellation properties hold in G.

Proof. Let $x, y, z \in G$, and suppose that x_l^{-1} is the left inverse of x. Then,

$$\begin{aligned} xy &= xz, \\ \alpha(x_l^{-1})(xy) &= \alpha(x_l^{-1})(xz), \\ (x_l^{-1}x)\beta(y) &\overset{\text{D } 2.1(1)}{=} (x_l^{-1}x)\beta(z), \\ e\beta(y) &\overset{\text{D } 2.1(4)}{=} e\beta(z), \\ \beta^2(y) &\overset{\text{D } 2.1(3)}{=} \beta^2(z). \end{aligned}$$

Then, y=z since β is invertible. The proof of the statement 2 is similar, but using the right inverse and its image under β in the first step and invertibility of α in the last step,

$$\begin{aligned} yx &= zx, \\ (yx)\beta(x_r^{-1}) &= (zx)\beta(x_r^{-1}), \\ \alpha(y)(xx_r^{-1}) &\overset{\text{D } 2.1(1)}{=} \alpha(z)(xx_r^{-1}), \\ \alpha(y)e &\overset{\text{D } 2.1(4)}{=} \alpha(z)e, \\ \alpha^2(y) &\overset{\text{D } 2.1(3)}{=} \alpha^2(z), \end{aligned}$$

which yields $y = z$ because α is invertible. Finally, in regular BiHom-groups, both α and β are bijective, and hence both the left and right cancellation properties hold. □

By Definition 2.1 we can deduce the following remark and corollary, which provide an important characterization for the BiHom-groups.

Remark 2.3. If e is the BiHom-unity in a BiHom-group, then if at least one of the maps α or β is bijective, then $\alpha(e) = \beta(e) = e$. Indeed, $\alpha(e) \overset{\text{D } 2.1(3)}{=} ee$, and $\alpha(e)\alpha(e) \overset{\text{D } 2.1(2)}{=} \alpha(ee) = \alpha(\alpha(e)) \overset{\text{D } 2.1(3)}{=} \alpha(e)e$, and analogously, $\beta(e)\beta(e) \overset{\text{D } 2.1(2)}{=} e\beta(e)$ Then, if α is bijective, then by the left cancellation property from Proposition 2.6:1, we get $\alpha(e) = e$, and hence $\beta(e) \overset{\text{D } 2.1(3)}{=} ee \overset{\text{D } 2.1(3)}{=} \alpha(e) = e$. Similarly, if β is bijective, then by the right cancellation property from Proposition 2.6:1, we have $\beta(e) = e$, and hence $\alpha(e) \overset{\text{D } 2.1(3)}{=} ee \overset{\text{D } 2.1(3)}{=} \beta(e) = e$. In particular, if the BiHom-group is regular, then both α and β are bijective, and $\alpha(e) = \beta(e) = e$.

Corollary 2.7. *In a BiHom-group (G, α, β), the bijectivity of at least one of the mappings α or β implies commutativity of α or β, that is, $\alpha\beta = \beta\alpha$. In particular, in regular BiHom-groups, the twisting mappings α and β commute.*

Proof. For all $x \in G$, the bijectivity of at least one of the mappings α and β implies $\alpha(e) = \beta(e) = e$ according to Remark 2.3. This and the BiHom-associativity yields

$$\alpha\beta(x) \overset{\text{D } 2.1(3)}{=} \beta(x)e \overset{\text{D } 2.1(3)}{=} (ex)e \overset{\text{R } 2.3}{=} (ex)\beta(e) \overset{\text{D } 2.1(1)}{=} \alpha(e)(xe) \overset{\text{R } 2.3}{=} e(xe) \overset{\text{D } 2.1(3)}{=} \beta\alpha(x). \quad \square$$

The following proposition concerns the uniqueness of unity and inverses.

Proposition 2.8. *Let (G, α, β) be a BiHom-group with either α or β being bijective.*

1. *The BiHom-unity is unique in a BiHom-group.*
2. *If α is bijective, then for all $x \in G$, the left inverse is unique.*
3. *If β is bijective, then for all $x \in G$, the right inverse is unique.*

In regular BiHom-groups, the left and right inverses are unique.

Proof. 1. Let e' be another unity in G. Then, $e' \overset{R\ 2.3}{=} \alpha(e') \overset{D\ 2.1(3)}{=} e'e \overset{D\ 2.1(3)}{=} \beta(e) \overset{R\ 2.3}{=} e$.
2. Let $a \in G$ be another left inverse of x then, $ax = e = x_l^{-1}x$, with the law of right cancellation in the statement 2 of Proposition 2.6, we get $a = x_l^{-1}$, for all $x \in G$.
3. Let $a \in G$ be another right inverse of x, then $xa = e = xx_r^{-1}$, with the law of left cancellation in the statement 1 of Proposition 2.6, we get $a = x_r^{-1}$, for all $x \in G$. □

For the remainder of this article, unless stated otherwise, all BiHom-groups are considered regular, that is, the twisting maps are assumed to be bijective.

Proposition 2.9. *Let (G, μ, α, β) be a BiHom-group, then we have*

1. *The inverse maps α^{-1}, β^{-1} are multiplicative.*
2. *The left inverse of $\alpha(x)$ is the image under α of the left inverse, $\big(\alpha(x)\big)_l^{-1} = \alpha(x_l^{-1})$.*
3. *The right inverse of $\alpha(x)$ is the image under α of the right inverse, $\big(\alpha(x)\big)_r^{-1} = \alpha(x_r^{-1})$.*
4. *The left inverse of $\beta(x)$ is the image under β of the left inverse, $\big(\beta(x)\big)_l^{-1} = \beta(x_l^{-1})$.*
5. *The right inverse of $\beta(x)$ is the image under β of the right inverse, $\big(\beta(x)\big)_r^{-1} = \beta(x_r^{-1})$.*

Proof. 1. For all $x, y \in G$, such that α is invertible by Definition 2.1 and multiplicativity in Definition 2.1,(2), the following equalities hold:

$$\alpha(\alpha^{-1}(xy)) = xy, \quad \alpha\big(\alpha^{-1}(x)\alpha^{-1}(y)\big) \overset{D\ 2.1(2)}{=} \alpha\alpha^{-1}(x)\alpha\alpha^{-1}(y) = xy.$$

Then, $\alpha^{-1}(xy) = \alpha^{-1}(x)\alpha^{-1}(y)$.
2. For all $x \in G$, $\alpha(x_l^{-1})\alpha(x) \overset{D\ 2.1(2)}{=} \alpha(x_l^{-1}x) \overset{D\ 2.1(4)}{=} \alpha(e) \overset{R\ 2.3}{=} e$.
3. For all $x \in G$, $\alpha(x)\alpha(x_r^{-1}) \overset{D\ 2.1(2)}{=} \alpha(xx_r^{-1}) \overset{D\ 2.1(4)}{=} \alpha(e) \overset{R\ 2.3}{=} e$.
4. For all $x \in G$, $\beta(x_l^{-1})\beta(x) \overset{D\ 2.1\ (2)}{=} \beta(x_l^{-1}x) \overset{D\ 2.1\ (4)}{=} \beta(e) \overset{R\ 2.3}{=} e$.
5. For all $x \in G$, $\beta(x)\beta(x_r^{-1}) \overset{D\ 2.1(2)}{=} \beta(xx_r^{-1}) = \beta(e) \overset{R\ 2.3}{=} e$. □

2.2. Construction of BiHom-groups

One of the methods for constructing BiHom-groups is to derive them from a given group or a Hom-group using the twisting principle, that is, twisting the binary operation by a morphism for the Hom-groups or by two morphisms for the groups. First, the construction of a BiHom-group from a given group is presented in the following statement.

Proposition 2.10. *For a group $(G, \times)$ and two commuting automorphisms α and β of G, define a new operation $\otimes$ on G, for all elements x and y in G, by*

$$x \otimes y = \alpha(x) \times \beta(y). \tag{13}$$

Then, the new structure $(G, \otimes, \alpha, \beta)$ is a BiHom-group.

Proof. Let us verify the properties of this BiHom-group. For any $x, y, z \in G$,

BiHom-associativity :

$$\alpha(x) \otimes (y \otimes z) \overset{(13)}{=} \alpha(x) \otimes (\alpha(y) \times \beta(z)) \overset{(13)}{=} \alpha^2(x) \times \beta\alpha(y) \times \beta^2(z)$$
$$= \big(\alpha^2(x) \times \beta\alpha(y)\big) \times \beta^2(z) = \alpha\big(\alpha(x) \times \beta(y)\big) \times \beta\beta(z)$$
$$\overset{(13)}{=} \alpha(x \otimes y) \times \beta\beta(z) = (x \otimes y) \otimes \beta(z);$$

BiHom-unitality :

$$x \otimes e \overset{(13)}{=} \alpha(x) \times \beta(e) = \alpha(x) \times e \overset{D\ 2.1(3)}{=} \alpha(x);$$
$$e \otimes x \overset{(13)}{=} \alpha(e) \times \beta(x) = e \times \beta(x) \overset{D\ 2.1(3)}{=} \beta(x);$$

BiHom-invertibility :

$$x \otimes \alpha\beta^{-1}(x^{-1}) \overset{(13)}{=} \alpha(x) \times \beta\alpha\beta^{-1}(x^{-1}) = \alpha(x) \times \alpha(x^{-1}) = e;$$
$$\beta\alpha^{-1}(x^{-1}) \otimes x \overset{(13)}{=} \alpha\beta\alpha^{-1}(x^{-1}) \otimes \beta(x) = \beta(x^{-1}) \times \beta(x) = e.$$

Thus, if x^{-1} is the inverse of x in $(G, \times)$, then $\beta\alpha^{-1}(x^{-1})$ and $\alpha\beta^{-1}(x^{-1})$ are the left and right inverses of x in the BiHom-group $(G, \otimes, \alpha, \beta)$.

Therefore, $(G, \otimes, \alpha, \beta)$ is a BiHom-group obtained by twisting the group $(G, \times)$ by the commuting homomorphisms α and β. □

Example 2.11. Let us consider $G = (\mathbb{Z}_5, +)$, with α and β automorphisms of $\mathbb{Z}_5$ such that $\alpha(x) = 2x$ and $\beta(x) = 4x$. Then, the twisting operation is given by

$$x \oplus y = \alpha(x) + \beta(y) = 2x + 4y, \quad \forall x, y \in \mathbb{Z}_5.$$

Let us verify the BiHom-associativity. For all $x, y, z \in \mathbb{Z}_5$, we get

$$\begin{aligned}\alpha(x) \oplus (y \oplus z) &= 2(2x) + 4(2y + 4z) = 2^2x + 4 \cdot 2y + 4^2z \\ &= 2(2x + 4y) + 4(4z) = (x \oplus y) \oplus \beta(z).\end{aligned}$$

We obtain the following multiplication table for the additive BiHom-group $G_{\alpha,\beta}$:

$G_{\alpha,\beta}$	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

Next, we present the construction of BiHom-groups from Hom-groups.

Proposition 2.12. *Consider a Hom-group $(G, \times, \alpha)$ and an automorphism β of G that commutes with α. We define a new operation on G, denoted by $\otimes$, for any elements $x, y \in G$, by*

$$x \otimes y = x \times \beta\alpha^{-1}(y). \tag{14}$$

This yields a new structure $(G, \otimes, \alpha, \beta)$ that forms a BiHom-group.

Proof. Recall that $(G, \times, \alpha)$ is a Hom-group, which means it is not associative in the traditional sense. Instead, it is Hom-associative, that is, for all $x, y, z \in G$, we have

$$\alpha(x) \times (y \times z) = (x \times y) \times \alpha(z). \tag{15}$$

Now, let us verify that $(G, \otimes, \alpha, \beta)$ is a BiHom-group. In each step, we indicate exactly what has been used, (mlp) indicates the use of multiplicativity, and (com) indicates the use of commutativity of $\alpha, \alpha^{-1}, \beta$.
BiHom-associativity:

$$\begin{aligned}\alpha(x) \otimes (y \otimes z) &\overset{(14)}{=} \alpha(x) \otimes \big(y \times \beta\alpha^{-1}(z)\big) \overset{(14)}{=} \alpha(x) \times \beta\alpha^{-1}\big(y \times \beta\alpha^{-1}(z)\big) \\ &\overset{(\text{mlp})}{=} \alpha(x) \times (\beta\alpha^{-1}(y) \times \beta^2\alpha^{-2}(z)) \overset{(15)}{=} \big(x \times \beta\alpha^{-1}(y)\big) \times \alpha\beta^2\alpha^{-2}(z) \\ &\overset{(14)(\text{com})}{=} \big(x \otimes y\big) \times \beta\alpha^{-1}\beta(z) \overset{(14)}{=} (x \otimes y) \otimes \beta(z).\end{aligned}$$

BiHom-unity: Note that there exists an element e in the Hom-group $(G, \times, \alpha)$ that satisfies the Hom-unitality property: $x \times e = e \times x = \alpha(x)$, for all x in G. Then, e is a BiHom-unity for the new twisting structure since

$$x \otimes e \overset{(14)}{=} x \times \beta\alpha^{-1}(e) = x \times e = \alpha(x), \quad e \otimes x \overset{(14)}{=} e \times \beta\alpha^{-1}(x) = \alpha\beta\alpha^{-1}(x) = \beta(x).$$

BiHom-invertibility:

$$x \otimes \alpha\beta^{-1}(x^{-1}) \overset{(14)}{=} x \times \beta\alpha^{-1}\alpha\beta^{-1}(x^{-1}) = x \times x^{-1} = e;$$
$$\beta\alpha^{-1}(x^{-1}) \otimes x \overset{(14)}{=} \beta\alpha^{-1}(x^{-1}) \times \beta\alpha^{-1}(x) = \beta\alpha^{-1}(x^{-1} \times x) = \beta\alpha^{-1}(e) = e.$$

Thus, if x^{-1} is the Hom-inverse of x in $(G, \times, \alpha)$, then $\beta(\alpha^{-1}(x^{-1}))$ and $\alpha(\beta^{-1}(x^{-1}))$ are the left and right inverses of x in the BiHom-group $(G, \otimes, \alpha, \beta)$. Therefore, $(G, \otimes, \alpha, \beta)$ is a BiHom-group obtained by twisting the Hom-group $(G, \times, \alpha)$ by the morphism β, which commutes with the mapping α. □

2.3. Properties of BiHom-groups

Proposition 2.13. *Let G be a BiHom-group. If there exists an element $x \in G$ such that $xx = \alpha(x)$ or $xx = \beta(x)$, then in both cases, $x = e$.*

Proof. Suppose $x \in G$ satisfies $xx = \alpha(x)$ or $xx = \beta(x)$. Then,

$$xx = \alpha(x) \overset{\text{D 2.1(3)}}{=} xe \quad \Rightarrow \quad x \overset{\text{Prop.2.6:1}}{=} e \qquad \text{(by left cancellation)},$$

$$xx = \beta(x) \overset{\text{D 2.1(3)}}{=} ex \quad \Rightarrow \quad x \overset{\text{Prop.2.6:2}}{=} e \qquad \text{(by right cancellation)}. \quad \square$$

In the following proposition, we provide a formula for the inverse of the product xy:

Proposition 2.14. *Let (G, α, β) be a BiHom-group. Then, it has the following properties:*

1. *The left inverse of the product xy is $(xy)_l^{-1} = \alpha^{-1}\beta(y_l^{-1})\beta^{-1}\alpha(x_l^{-1})$.*
2. *The right inverse of the product xy is $(xy)_r^{-1} = \alpha^{-1}\beta(y_r^{-1})\beta^{-1}\alpha(x_r^{-1})$.*

Proof. 1. For all $x, y \in G$,

$$\begin{aligned}\left(\alpha^{-1}\beta(y_l^{-1})\beta^{-1}\alpha(x_l^{-1})\right)(xy) &\overset{\text{D 2.1(1)}}{=} \beta(y_l^{-1})\left(\beta^{-1}\alpha(x_l^{-1})\beta^{-1}(xy)\right)\\ &\overset{\text{D 2.1(1)}}{=} \beta(y_l^{-1})\left(\beta^{-1}(x_l^{-1}x)y\right) \overset{\text{D 2.1(4)R 2.3}}{=} \beta(y_l^{-1})(e.y) \overset{\text{D 2.1(3)}}{=} \beta(y_l^{-1})\beta(y)\\ &\overset{\text{D 2.1(3)(2)}}{=} \beta(y_l^{-1}y) \overset{\text{D 2.1(4)}}{=} \beta(e) \overset{\text{R (2.3)}}{=} e.\end{aligned}$$

2. For all $x, y \in G$,

$$\begin{aligned}(xy)\left(\alpha^{-1}\beta(y_r^{-1})\beta^{-1}\alpha(x_r^{-1})\right) &\overset{\text{D 2.1(1)}}{=} \left(\alpha^{-1}(xy)\alpha^{-1}\beta(y_r^{-1})\right)\alpha(x_r^{-1})\\ &\overset{\text{D 2.1(1)Prop.2.9:1}}{=} \left(x\alpha^{-1}(yy_r^{-1})\right)\alpha(x_r^{-1}) \overset{\text{D 2.1(4)R 2.3}}{=} (xe)\alpha(x_r^{-1}) \overset{\text{D 2.1(3)}}{=} \alpha(x)\alpha(x_r^{-1})\\ &\overset{\text{D 2.1(2)}}{=} \alpha(xx_r^{-1}) \overset{\text{D 2.1(4)}}{=} \alpha(e) \overset{\text{R 2.3}}{=} e. \quad \square\end{aligned}$$

Remark 2.4. In group theory, the left and right inverses of an element are the same, which implies that the inverse of the inverse of an element is the element itself, that is, $(x^{-1})^{-1} = x$. However, this is not the case in a BiHom-group, where the left and right inverses are not necessarily the same. What about the inverse of the inverse of an element? is it the element itself? Indeed, for all $x \in G$ we have $xx_r^{-1} = e$, which means that the left inverse of the right inverse is the element itself, that is, $(x_r^{-1})_l^{-1} = x$. Similarly, by $x_l^{-1}x = e$, we get $(x_l^{-1})_r^{-1} = x$. We now show that the left inverse of the left inverse of an element and the right inverse of the right inverse is not itself. In fact, in Example 2.11, we have $1 \oplus 2 = 0$, but $2 \oplus 1 = 3 \neq 0$ and $2 \oplus 4 = 0$. Thus, $\left((1)_r^{-1}\right)_r^{-1} = (2)_r^{-1} = 4 \neq 1$ and $\left((4)_l^{-1}\right)_l^{-1} = (2)_l^{-1} = 1 \neq 4$. This means, in general, $\left((x_r^{-1})\right)_r^{-1} \neq x$ and $\left((x_l^{-1})\right)_l^{-1} \neq x$. However, they have a specific relations, as shown in the following proposition.

Proposition 2.15. *Let (G, α, β) be a BiHom-group, for all $x \in G$, if we note x_l^{-1} and x_r^{-1} respectively left and right inverses of x, then we have the following relation between them*

$$\alpha^2(x_l^{-1}) = \beta^2(x_r^{-1}). \tag{16}$$

Proof. For all $x \in G$ we have

$$\begin{aligned}\alpha^2(x_l^{-1}) &\overset{\text{D 2.1(3)}}{=} \alpha(x_l^{-1})e \overset{\text{D 2.1(4)}}{=} \alpha(x_l^{-1})(xx_r^{-1}) \overset{\text{D 2.1(1)}}{=} (x_l^{-1}x)\beta(x_r^{-1})\\ &\overset{\text{D 2.1(4)}}{=} e\beta(x_r^{-1}) \overset{\text{D 2.1(3)}}{=} \beta^2(x_r^{-1}). \quad \square\end{aligned}$$

Corollary 2.16. *For all $x \in G$, we have*

$$\alpha^2(x) = \alpha^2\left((x_r^{-1})_l^{-1}\right) = \beta^2\left((x_r^{-1})_r^{-1}\right) \quad \text{and} \quad \alpha^2\left((x_l^{-1})_l^{-1}\right) = \beta^2\left((x_l^{-1})_r^{-1}\right) = \beta^2(x).$$

Proof. If we set $y = x_r^{-1}$, then by Proposition 2.15 and Remark 2.4, we get

$$\alpha^2(y_l^{-1}) = \alpha^2\left((x_r^{-1})_l^{-1}\right) \overset{R\ 2.4(3)}{=} \alpha^2(x) \overset{(16)}{=} \beta^2(y_r^{-1}) = \beta^2\left((x_r^{-1})_r^{-1}\right).$$

Similarly, by setting $y = x_l^{-1}$, we obtain

$$\alpha^2(y_l^{-1}) = \alpha^2\left((x_l^{-1})_l^{-1}\right) \overset{(16)}{=} \beta^2(y_r^{-1}) = \beta^2\left((x_l^{-1})_r^{-1}\right) \overset{R\ 2.4(3)}{=} \beta^2(x). \quad \square$$

Next, we prove that the BiHom-groups are a class of quasigroups.

Definition 2.2. A quasigroup $(Q, *)$ is a set Q equipped with a binary operation $* : Q \times Q \to Q$, where for all $a, b \in Q$, there exist unique elements $x, y \in Q$ such that $a * x = b$, and $y * a = b$.

Theorem 2.17. *Every BiHom-group (G, α, β) is a quasigroup.*

Proof. Let (G, α, β) be a BiHom-group, Then, for every $a, b \in G$, we choose $x = \alpha^{-1}(a_r^{-1})\beta^{-2}(b)$ *and* $y = \alpha^{-2}(b)\beta^{-1}(a_l^{-1})$. Then,

$$ax = a\big(\alpha^{-1}(a_r^{-1})\beta^{-2}(b)\big) \overset{D\ 2.1(1)}{=} \big(\alpha^{-1}(a)\alpha^{-1}(a_r^{-1})\big)\beta^{-1}(b) \overset{P\ 2.9:1}{=} \big(\alpha^{-1}(aa_r^{-1})\big)\beta^{-1}(b)$$
$$\overset{D\ 2.1(4)}{=} \big(\alpha^{-1}(e)\big)\beta^{-1}(b) \overset{D\ 2.1(4)R\ 2.3}{=} e\beta^{-1}(b) \overset{D\ 2.1(3)}{=} b.$$

The laws of cancellation from Proposition 2.6 in a BiHom-group guarantee the uniqueness of the choice of x. Indeed, if there exists another solution $x' \in G$ that satisfies the equation $ax' = b$, then $ax = ax'$ which implies by the left cancellation from Proposition 2.6:1 that $x = x'$. Similarly, $y = \alpha^{-2}(b)\beta^{-1}(a_l^{-1})$ is a unique element that satisfies the equation $ya = b$. Thus, (G, α, β) forms a quasigroup. $\square$

3. BiHom-subgroup and characterization

In this section, we define BiHom-subgroups and provide some of their properties.

Definition 3.1. Let (G, α, β) be a BiHom-group, and let H be a nonempty subset that is closed under the product in G. If (H, α, β) itself forms a BiHom-group under the product in G, then H is said to be a BiHom-subgroup of G, denoted by $H \leq_{\alpha,\beta} G$.

Proposition 3.1. *Let H be a BiHom-subgroup of G then*

1) *The identity element of H coincides with the identity element of G.*
2) *The (left or right) inverse in H is unique and equal to the (left or right, respectively) inverse in G, that is, $(x_l^{-1})_G = (x_l^{-1})_H$ and $(x_r^{-1})_G = (x_r^{-1})_H$.*

Proof. 1. If $e_H e_H = \alpha(e_H) = e_H e_G$, then $e_G \overset{P\ 2.6:1}{=} e_H$.
2. For each $x \in H$, we have $(x_l^{-1})_H x = e_H = e_G = x_l^{-1}x$. Then $x_l^{-1} = (x_l^{-1})_H$ and also, $(x_r^{-1})_H x = e_H = e_G = x_r^{-1}x$. Then $x_r^{-1} \overset{P\ 2.6:2}{=} (x_r^{-1})_H$. $\square$

Proposition 3.2. *A nonempty subset H of a BiHom-group G is a BiHom-subgroup if and only if all following conditions are satisfied*

1) *H is closed under the product, that is, $\forall x, y \in H, xy \in H$;*
2) *For all $x \in H$, x_l^{-1} and x_r^{-1} are in H;*
3) *$\alpha(H) = H$ and $\beta(H) = H$.*

Proof. ($\Rightarrow$) Let H be a BiHom-subgroup. Then, by Definition 3.1, (H, α, β) is a BiHom-group, meaning it is closed under the product. By the uniqueness of inverses, x_l^{-1} and x_r^{-1} are in H for all $x \in H$. Since α and β are bijections on H, it follows that $\alpha(H) = H$ and $\beta(H) = H$.
($\Leftarrow$) If a non-empty subset H satisfies hypotheses 1-3. Then, for any element $x \in H$, we can deduce that it possesses a right inverse $x_r^{-1} \in H$. This implies that $xx_r^{-1} = e \in H$, and therefore, (H, α, β) is a BiHom-group in itself. $\square$

Remark 3.1. We can combine conditions 1 and 2 of Proposition 3.2 into one as follows

$$x_l^{-1}y \in H \quad and \quad yx_r^{-1} \in H, \quad \forall x, y \in H.$$

The following proposition provides another perspective to affirm that a subset of a BiHom-group is also a BiHom-subgroup.

Proposition 3.3. *Let G be a group, and let α and β be two commuting automorphisms of G. Suppose H is a subgroup of G such that $\alpha(H) = H$ and $\beta(H) = H$. Consider the BiHom-group (G, α, β) obtained by twisting the group G using α and β such that the operation is given in Proposition 2.10, (13). Then, (H, α, β) forms a BiHom-subgroup of the BiHom-group (G, α, β).*

Proof. For all $x, y \in H$, $x \otimes y = \alpha(x) \cdot \beta(y)$ (where $\cdot$ is the group product) is an element of H, because $\alpha(x), \beta(y) \in H$ and their product is also in H (H is a subgroup), and then H is closed under the BiHom-multiplication.

For all $x \in H$, since H is a subgroup, we have $x^{-1} \in H$. The (left and right) inverses of x in the BiHom-structure, as determined in Proposition 2.10, are given by

$$x_l^{-1} = \alpha\beta^{-1}(x^{-1}), \quad x_r^{-1} = \beta\alpha^{-1}(x^{-1}).$$

Since H is stable under α and β, it follows that $x_l^{-1}, x_r^{-1} \in H$. Therefore, the left and right inverses of any element of H remain in H, confirming that H is a BiHom-subgroup. □

Example 3.4. In Example 2.11, if we take $a, b \in \mathbb{Q}^*$, then $(\mathbb{Q}, \oplus, \alpha, \beta)$ is a BiHom-subgroup of the BiHom-group $(\mathbb{R}, \oplus, \alpha, \beta)$ because $\mathbb{Q}$ is a subgroup of the group $(\mathbb{R}, +)$ and $\alpha(\mathbb{Q}) = \beta(\mathbb{Q}) = \mathbb{Q}$.

Theorem 3.5. *Let H and K be two BiHom-subgroups of G. Then,*

1) *$H \cap K$ is a BiHom-subgroup of G.*
2) *$H \cup K$ is a BiHom-subgroup if and only if $H \subseteq K$ or $K \subseteq H$.*

Proof. Proof of statement 1: We have $e \in H \cap K$, which implies that it is not an empty set. For all $x, y \in H \cap K$, we have $xy \in H$ and $xy \in K$. So, $xy \in H \cap K$. For all $x \in H \cap K$, we have $x_l^{-1}, x_r^{-1} \in H \cap K$. Moreover, $\alpha(H \cap K) = \alpha(H) \cap \alpha(K) = H \cap K$. Similarly, we have $\beta(H \cap K) = H \cap K$.
Proof of statement 2: ($\Leftarrow$): Assume that $H \subseteq K$ or $K \subseteq H$, then $H \cup K = H$ or $H \cup K = K$. Therefore, $H \cup K$ is a BiHom-subgroup.
($\Rightarrow$): Suppose $H \nsubseteq K$. Then, there exists x, an element of H, such that $x \notin K$. For all $y \in K$ we have $xy \in H \cup K$. Since $H \cup K$ is a BiHom-subgroup, we have $xy \in H$ or $xy \in K$. However, xy may not be an element of K because if $xy = k \in K$, then by multiplying with $\beta(y_r^{-1})$, we obtain $(xy)\beta(y_r^{-1}) = k\beta(y_r^{-1}) \in K$. Using the BiHom-associativity condition (1) from the Definition 2.1 of BiHom-groups, we obtain $\alpha(x)(yy_r^{-1}) = \alpha(x)(e) \overset{D\ 2.1(3)}{=} \alpha^2(x)$, which implies that $\alpha^2(x) \in K$, that is, $x \in K$, which contradicts the assumed choice $x \notin K$. Therefore, $xy \in H$ for all $y \in K$. So, if we multiply by $\alpha(x_l^{-1})$, we get $\alpha(x_l^{-1})(xy) = h \in H$ (product of two elements of H), but $\alpha(x_l^{-1})(xy) \overset{D\ 2.1(1)}{=} (x_l^{-1}x)\beta(y) \overset{D\ 2.1(4)(3)}{=} \beta^2(y)$. Therefore, for all $y \in K$ we have $\beta^2(y) \in H$, which implies that $y \in H$. Then $K \subseteq H$. □

In the following, we provide the definition of the center (BiHom-center) of a BiHom-group. This notion trivially extends the ordinary notion of the center of a group, obtained when $\alpha = \beta = id$.

Definition 3.2. The center (BiHom-center) of a BiHom-group (G, α, β) is defined as

$$Z(G) = \{x \in G \mid \beta(x)\alpha(y) = \beta(y)\alpha(x), \forall y \in G\}. \tag{17}$$

Remark 3.2. If the element x is an element of the center of the BiHom group G. Then, x commutes with all elements a' of the BiHom-group G. Thus, $\beta(x)\alpha(a') = \beta(a')\alpha(x)$. In particular, taking $a' = \alpha^{-1}(a)$ yields $\beta(x)\alpha\big(\alpha^{-1}(a)\big) = \beta\big(\alpha^{-1}(a)\big)\alpha(x)$, and hence $\beta(x)a = \beta\big(\alpha^{-1}(a)\big)\alpha(x)$.

Proposition 3.6. *The center of a BiHom-group is a BiHom-subgroup.*

Proof. First, we have $e \in Z(G)$. As α and β commute by Corollary 2.7, for all $y \in G$,

$$\beta\alpha(y) \overset{C\ 2.7}{=} \alpha\beta(y) \quad \Rightarrow \quad e\alpha(y) \overset{D\ 2.1(3)}{=} \beta(y)e \quad \Rightarrow \quad \beta(e)\alpha(y) \overset{R\ 2.3}{=} \beta(y)\alpha(e).$$

Secondly, for all $x, y \in Z(G), a \in G$,

$$\beta(xy)\alpha(a) \overset{\text{D } 2.1(1)}{=} \alpha\beta(x)\big(\beta(y)\alpha\beta^{-1}(a)\big) \overset{\text{D } 3.2(17)}{=} \alpha\beta(x)\big(a\alpha(y)\big) \overset{\text{D } 2.1(1)}{=} \Big(\beta(x)a\Big)\beta\alpha(y)$$

$$\overset{\text{R } 3.2}{=} \Big(\beta(x)\alpha\alpha^{-1}(a)\Big)\beta\alpha(y) \overset{\text{D } 3.2(17)}{=} \Big(\beta\alpha^{-1}(a)\alpha(x)\Big)\beta\alpha(y) \overset{\text{D } 2.1(1)\text{C } 2.7}{=} \beta(a)\alpha(xy)$$

$$\Rightarrow xy \in Z(G).$$

Thus, $Z(G)$ is stable under the multiplication of G. Also, let $x \in Z(G)$ and x_{l}^{-1} and x_{r}^{-1} be respectively left and right inverse of x in G.

For all $y \in G$, we have

$$\beta(x)\alpha(y) \overset{\text{D } 3.2(17)}{=} \beta(y)\alpha(x),$$

$$\alpha\beta(x_{\mathrm{l}}^{-1})\Big(\beta(x)\alpha(y)\Big) = \alpha\beta(x_{\mathrm{l}}^{-1})\Big(\beta(y)\alpha(x)\Big), \quad \text{(left multiplication by } \alpha\beta(x_{\mathrm{l}}^{-1})\text{)},$$

$$\beta(x_{\mathrm{l}}^{-1}x)\beta\alpha(y) \overset{\text{D } 2.1(1)}{=} \beta(x_{\mathrm{l}}^{-1}y)\beta\alpha(x),$$

$$e\beta\alpha(y) \overset{\text{D } 2.1(4)\text{R } 2.3}{=} \beta(x_{\mathrm{l}}^{-1}y)\beta\alpha(x),$$

$$\beta^2\alpha(y) \overset{\text{D } 2.1(3)}{=} \beta(x_{\mathrm{l}}^{-1}y)\beta\alpha(x), \quad \text{(right multiplication by } \beta^2\alpha(x_{\mathrm{r}}^{-1})\text{)},$$

$$\beta^2\alpha(y)\beta^2\alpha(x_{\mathrm{r}}^{-1}) = \Big(\beta(x_{\mathrm{l}}^{-1}y)\beta\alpha(x)\Big)\beta^2\alpha(x_{\mathrm{r}}^{-1}),$$

$$\beta^2\alpha(yx_{\mathrm{r}}^{-1}) \overset{\text{D } 2.1(1)}{=} \alpha\beta\big((x_{\mathrm{l}}^{-1}y)(xx_{\mathrm{r}}^{-1})\big),$$

$$\beta^2\alpha(yx_{\mathrm{r}}^{-1}) \overset{\text{D } 2.1(4)(3)\text{R } 2.3}{=} \alpha^2\beta(x_{\mathrm{l}}^{-1}y),$$

$$\beta^2\alpha(y)\beta^2\alpha(x_{\mathrm{r}}^{-1}) = \alpha^2\beta(x_{\mathrm{l}}^{-1})\alpha^2\beta(y),$$

$$\beta\big(\beta\alpha(y)\big)\alpha\big(\beta^2(x_{\mathrm{r}}^{-1})\big) \overset{\text{C } 2.7}{=} \beta\big(\alpha^2(x_{\mathrm{l}}^{-1})\big)\alpha\big(\beta\alpha(y)\big).$$

Since $\alpha^2(x_{\mathrm{l}}^{-1}) = \beta^2(x_{\mathrm{r}}^{-1})$ (see Proposition 2.15), we deduce the following equalities

$$\beta\big(\beta\alpha(y)\big)\alpha\big(\alpha^2(x_{\mathrm{l}}^{-1})\big) = \beta\big(\alpha^2(x_{\mathrm{l}}^{-1})\big)\alpha\big(\beta\alpha(y)\big), \tag{18}$$

$$\beta\big(\beta\alpha(y)\big)\alpha\big(\beta^2(x_{\mathrm{r}}^{-1})\big) = \beta\big(\beta^2(x_{\mathrm{r}}^{-1})\big)\alpha\big(\beta\alpha(y)\big). \tag{19}$$

Since the maps α and β are bijective, $e \in Z(G)$, and the product is stable in $Z(G)$, $Z(G)$ is invariant under these maps, meaning $\alpha(Z(G)) = \beta(Z(G)) = Z(G)$. Therefore, from the equalities (18) and (19), we have for all $x \in Z(G) \implies \alpha^2(x_{\mathrm{r}}^{-1}), \beta^2(x_{\mathrm{l}}^{-1}) \in Z(G) \implies x_{\mathrm{r}}^{-1}, x_{\mathrm{l}}^{-1} \in \alpha^{-2}(Z(G)) = \beta^{-2}(Z(G)) = Z(G)$. Therefore, $Z(G)$ is stable under the inverses by the operation. This means $Z(G)$ is well-defined and is a BiHom-subgroup. □

Remark 3.3. For a BiHom-group G, the ordinary commutativity condition $xy = yx$ leads to the special case of a Hom-group. Indeed, if G is commutative, then for all $x, y \in G$, $xy = yx$, and hence, for $y = e$ and all $x \in G$, $\beta(x) \overset{\text{D } 2.1(3)}{=} ex = xe \overset{\text{D } 2.1(3)}{=} \alpha(x)$. Thus, G becomes a Hom-group. This is why we define the center of a BiHom-group as mentioned in Definition 3.2. This definition generalizes the notion of the center of a Hom-group or a group: if we consider $\alpha = \beta$, we obtain $xy = yx$, meaning the center of a Hom-group.

4. Lagrange theorem in BiHom-groups

In this section, we show that the analog of Lagrange's theorem is valid for BiHom subgroups of BiHom-groups, provided that the notions of cosets are suitably extended to BiHom cosets involving the twisting maps of the BiHom-group.

Definition 4.1. Let G be a BiHom-group and $H \leq G$ be a BiHom-subgroup. Let $x \in G$. Denote $\beta(x)H = \{\beta(x)h \mid h \in H\}$ and $H\alpha(x) = \{h\alpha(x) \mid h \in H\}$. The sets $\beta(x)H$ and $H\alpha(x)$ are called the BiHom-left coset and the BiHom-right coset of H in G, respectively. We define $(G/H)_l = \{\beta(x)H \mid x \in G\}$ and $(G/H)_r = \{H\alpha(x) \mid x \in G\}$ as the set of all the BiHom–left cosets and the set of all the BiHom-right cosets, respectively.

The set of BiHom-left cosets defines an equivalence relation on G, as demonstrated in the following proposition.

Proposition 4.1. *Let H be a BiHom-subgroup of the BiHom-group G. For all $x, y \in G$, the following statements are equivalent:*

1) $\beta(x)H \cap \beta(y)H \neq \emptyset$;
2) $\beta(x)H = \beta(y)H$;
3) $\beta(y_l^{-1}x)H = H$, *that is,* $y_l^{-1}x \in H$.

Proof. 1 $\Rightarrow$ 2: Statement 1 implies that there exists $h_x, h_y \in H$ such that $\beta(x)h_x = \beta(y)h_y$. For all $h \in H$, $\alpha^{-1}(h_x) \times \big(\alpha^{-2}(h_x^{-1})\beta^{-2}(h)\big) = \alpha^{-2}(h_x h_x^{-1})\beta^{-1}(h) = e\beta^{-1}(h) = h$. Thus, for every $h \in H$, there is $h' = \alpha^{-2}(h_x^{-1})\beta^{-2}(h) \in H$, such that $h = \alpha^{-1}(h_x)h'$. So,

$$\begin{aligned}\beta(x)h &= \beta(x)\Big(\alpha^{-1}(h_x)h'\Big) = \Big(\alpha^{-1}\Big(\beta(x)h_x\Big)\Big)\beta(h')\\ &= \Big(\alpha^{-1}\Big(\beta(y)h_y\Big)\Big)\beta(h'), \quad \text{(Replace } \beta(x)h_x \text{ by } \beta(y)h_y\text{)}\\ \beta(x)h &= \beta(y)(h_y h') \in \beta(y)H.\end{aligned}$$

Thus, $\beta(x)H \subseteq \beta(y)H$. Similarly, one can show that $\beta(y)H \subseteq \beta(x)H$.
2 $\Rightarrow$ 3: If 2 holds, then there exists $h, h' \in H$, such that $\beta(x)h = \beta(y)h'$, then

$$\begin{aligned}\alpha\beta(y_l^{-1})\Big(\beta(x)h\Big) &= \alpha\beta(y_l^{-1})\Big(\beta(y)h'\Big), \quad \text{(left multiplication by } \alpha\beta(y_l^{-1})\text{)}\\ \beta(y_l^{-1}x)\beta(h) &= \beta(y_l^{-1}y)\beta(h'),\\ \beta\big((y_l^{-1}x)h\big) &= e\beta(h'),\\ (y_l^{-1}x)h &= \beta(h'),\\ \Big((y_l^{-1}x)h\Big)\beta(h_r^{-1}) &= \beta(h')\beta(h_r^{-1}), \qquad \text{(right multiplication by } \beta(h_r^{-1})\text{)}\\ \alpha(y_l^{-1}x)(hh_r^{-1}) &= \beta(h'h_r^{-1}),\\ \alpha^2(y_l^{-1}x) &= \beta(h'h_r^{-1}) \in H.\end{aligned}$$

Thus, $\alpha^2(y_l^{-1}x) \in H$, so $y_l^{-1}x \in H$, then $\beta(y_l^{-1}x)H = H$.
3 $\Rightarrow$ 1: Suppose $\beta(y_l^{-1}x)h = h'$. Then $\beta(y_l^{-1}x)h = eh'' = \beta(e)h'' = \beta(y_l^{-1}y)h''$. Therefore, there exists $h, h'' \in H$ such that $\beta(y_l^{-1}x)h = \beta(y_l^{-1}y)h''$. Using BiHom-associativity (1) in Definition 2.1 yields

$$\begin{aligned}\alpha^{-1}\beta(y_l^{-1})\big(\beta(x)\beta^{-1}(h)\big) &\overset{\text{D } 2.1(1)}{=} \alpha^{-1}\beta(y_l^{-1})\big(\beta(y)\beta^{-1}(h'')\big)\\ &\overset{\text{P } 2.6:1}{\Rightarrow} \beta(x)\beta^{-1}(h) = \beta(y)\beta^{-1}(h'').\end{aligned}$$

On the left side, $\beta(x)\beta^{-1}(h) \in \beta(x)H$ since $\beta^{-1}(h) \in H$. On the right side, $\beta(y)\beta^{-1}(h'') \in \beta(y)H$ because $\beta^{-1}(h'') \in H$. Then, $\beta(x)H \cap \beta(y)H \neq \emptyset$. □

Remark 4.1. Similarly, if we consider the set of BiHom-right cosets, these properties become

1) $H\alpha(x) \cap H\alpha(y) \neq \emptyset$.
2) $H\alpha(x) = H\alpha(y)$.
3) $H\alpha(xy_r^{-1}) = H$, that is, $xy_r^{-1} \in H$.

Next, we suppose that the BiHom-group G is finite. To extend a fundamental result of finite group theory to the theory of BiHom-groups (Lagrange's theorem), we start with the following lemma.

Lemma 4.2. *Let H be a BiHom-subgroup of a finite BiHom-group (G, α, β). Then, for all elements $x \in G$, the following equalities of cardinalities hold: $|\beta(x)H| = |H\alpha(x)| = |H|$.*

Proof. Consider the mapping defined by $f : H \to \beta(x)H$ where $f(h) = \beta(x)h$. It is injective because $\beta(x)h = \beta(x)h'$ implies $h = h'$. Because H is finite, f is also bijective, establishing a one-to-one correspondence between the elements of the two sets. Thus, $|\beta(x)H| = |H|$. Similarly, consider the map $g : H \to H\alpha(x)$, we obtain $|H\alpha(x)| = |H|$. □

Theorem 4.3. *(Lagrange theorem in BiHom-group) Let G be a finite BiHom-group and H be a BiHom-subgroup of G. Then, the cardinality $|H|$ divides the cardinality $|G|$.*

Proof. According to Proposition 4.1, the set of BiHom-left cosets defines a partition of G. Let $A = \{x_1, \dots, x_n\}$ be a family of representatives of this partition. Then, $G = \bigcup_{x\in G} \beta(x)G = \bigcup_{x_i\in A} \beta(x_i)G$. Thus, $|G| = \sum_{i=1}^{n} |\beta(x_i)H| = \sum_{i=1}^{n} |H| = n|H|$ from Lemma 4.2. Then, $|H|$ divides $|G|$. □

5. Normal BiHom-subgroup and BiHom-quotient groups

Proposition 5.1. *Let $H \leq_{\alpha,\beta} G$, and x is an element of G. The following statements are equivalent.*

1) $H\alpha(x) = \beta(x)H$.
2) $\forall h \in H, \Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) \in H$.
3) $H = \Big(\beta(x)H\Big)\beta\alpha(x_r^{-1})$.
4) $\forall h \in H, \alpha\beta(x_l^{-1})\Big(\beta(x)h\Big) \in H$.
5) $\alpha\beta(x_l^{-1})\Big(\beta(x)H\Big) = H$.

Proof. 1 $\Rightarrow$ 2: Assume that for all $h \in H$, there exists $h' \in H$ such that $\beta(x)h = h'\alpha(x)$. Then, for all $h \in H$,

$$\Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) = \Big(h'\alpha(x)\Big)\beta\alpha(x_r^{-1}) \overset{D\ 2.1(1)}{=} \alpha(h')\alpha(x.x_r^{-1}) \overset{D\ 2.1(4)R\ 2.3}{=} \alpha(h')e \overset{D\ 2.1(3)}{=} \alpha^2(h') \in H.$$

In other words, $\Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) \in H$ for all $h \in H$.

2 $\Rightarrow$ 3: We only need to show that $H \subseteq \Big(\beta(x)H\Big)\beta\alpha(x_r^{-1})$. For all $h \in H$, we have

$$h = \alpha^2(h') \overset{D\ 2.1(3)}{=} \alpha(h')e \overset{R\ 2.3}{=} \alpha(h')\alpha(e) \overset{D\ 2.1(4)}{=} \alpha(h')\alpha(x.x_r^{-1}) \overset{D\ 2.1(2)(1)}{=} \Big(h'\alpha(x)\Big)\beta\alpha(x_r^{-1}).$$

Note that

$$\begin{aligned} h'\alpha(x) &= \beta\beta^{-1}(h')\alpha(x) \overset{D\ 2.1(3)}{=} \Big(e\beta^{-1}(h')\Big)\alpha(x) \overset{D\ 2.1(4)R\ 2.3}{=} \Big(\alpha^{-2}\beta(xx_r^{-1})\beta^{-1}(h')\Big)\alpha(x) \\ &\overset{P\ 2.9:1}{=} \Big(\Big(\alpha^{-2}\beta(x)\alpha^{-2}\beta(x_r^{-1})\Big)\beta^{-1}(h')\Big)\alpha(x) \overset{D\ 2.1(1)}{=} \Big(\alpha^{-1}\beta(x)\Big(\alpha^{-2}\beta(x_r^{-1})\beta^{-2}(h')\Big)\Big)\alpha(x) \\ &\overset{D\ 2.1(1)}{=} \beta(x)\Big(\Big(\alpha^{-2}\beta(x_r^{-1})\beta^{-2}(h')\Big)\beta^{-1}\alpha(x)\Big) = \beta(x)a, \end{aligned} \tag{20}$$

where $a = \Big(\big(\alpha^{-2}\beta(x_r^{-1})\beta^{-2}(h')\Big)\beta^{-1}\alpha(x)\Big)$. Due to

$$\begin{aligned} \beta^2(a) &= \beta^2\Big(\Big(\alpha^{-2}\beta(x_r^{-1})\beta^{-2}(h')\Big)\beta^{-1}\alpha(x)\Big) \overset{D\ 2.1(2)}{=} \Big(\alpha^{-2}\beta^2\beta(x_r^{-1})h'\Big)\beta\alpha(x) \\ &\overset{P\ 2.15}{=} \Big(\beta(x_l^{-1})h'\Big)\beta\alpha(x) = \Big(\beta(x_l^{-1})h'\Big)\beta\alpha(x) \in H, \quad \text{(by hypothesis 5.1:2)} \end{aligned}$$

$a = \Big(\Big(\alpha^{-2}\beta(x_r^{-1})\beta^{-2}(h')\Big)\beta^{-1}\alpha(x)\Big) \in H$. So, using equality (20), we infer

$$h = \Big(h'\alpha(x)\Big)\beta\alpha(x_r^{-1}) \overset{(20)}{=} \Big(\beta(x)a\Big)\beta\alpha(x_r^{-1}) \in \Big(\beta(x)H\Big)\beta\alpha(x_r^{-1}), \quad \forall h \in H.$$

Therefore, $H \subseteq \Big(\beta(x)H\Big)\beta(\alpha(x_r^{-1})$.

3 $\Rightarrow$ 1: For all $h \in H$, we have $\alpha^2(h) \in H$, and it follows that

$$\alpha^2(h) \overset{\text{D } 2.1(3)}{=} \alpha(h)e, \overset{\text{D } 2.1(4)\text{R } 2.3}{=} \alpha(h)\alpha(xx_r^{-1}), \overset{\text{D } 2.1(1)}{=} \Big(h\alpha(x)\Big)\beta\alpha(x_r^{-1}).$$

Since $\alpha^2(h) \in H$ and $H \subseteq \big(\beta(x)H\big)\beta(\alpha(x_r^{-1})$, there is $h' \in H$ such that $\alpha^2(h) = \Big(\beta(x)h'\Big)\beta\alpha(x_r^{-1})$. Then, $\Big(h\alpha(x)\Big)\beta\alpha(x_r^{-1}) = \alpha^2(h) = \Big(\beta(x)h'\Big)\beta\alpha(x_r^{-1})$. Therefore, by Proposition 2.6:2 (right cancellation property), $h\alpha(x) = \beta(x)h'$, that is, $H\alpha(x) \subseteq \beta(x)H$. Similarly, $\big(\beta(x)H\big)\beta(\alpha(x_r^{-1}) \subseteq H$, for any $h \in H$, there exists $h' \in H$ such that $\Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) = h'$, and since $h' \in H$, there exists $h'' \in H$ such that $h' = \alpha^2(h'') \in H$. Thus, we obtain $\Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) = h' = \alpha^2(h'') = \Big(h''\alpha(x)\Big)\beta\alpha(x_r^{-1})$. Using the right cancellation property 2.6:2 (right cancellation property) in the last equality yields $\beta(x)h = h''\alpha(x)$. Therefore, $\beta(x)H \subseteq H\alpha(x)$. Consequently, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Similarly, by applying the same reasoning but switching the terms on the right with those on the left, we obtain (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). □

Definition 5.1. In a BiHom-group G, we say that a BiHom-subgroup H is a BiHom-Normal subgroup if $\beta(x)H = H\alpha(x)$ for all $x \in G$ (or one of the equivalent properties from Proposition 5.1 holds for all $x \in G$).

Example 5.2. The center of a BiHom-group $Z(G)$ is BiHom-normal in G. Indeed, we proved in Example 2.11 that $Z(G)$ is a subgroup of G. It remains to verify BiHom-normality. We must show that $Z(G)$ satisfies statement 2 of Proposition 5.1, that for all $x \in G$ and $h \in Z(G)$, we have $\Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) \in Z(G)$. Let $x \in G$ and $h \in Z(G)$. Then,

$$\Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) \overset{\text{D } 2.1(1)}{=} \alpha\beta(x)\Big(h\alpha(x_r^{-1})\Big) \overset{\text{D } 3.2}{=} \alpha\beta(x)\Big(\beta(x_r^{-1})\alpha\beta^{-1}(h)\Big)$$
$$\overset{\text{D } 2.1(1)}{=} \Big(\beta(x)\beta(x_r^{-1})\Big)\alpha(h) \overset{\text{D } 2.1(2)(4)\text{R } 2.3}{=} e\alpha(h) \overset{\text{D } 2.1(3)}{=} \alpha^2(h).$$

Since $\alpha\big(Z(G)\big) = Z(G)$ because it is a BiHom-subgroup, it follows that $\alpha^2(h) \in Z(G)$ for all $h \in Z(G)$. Therefore, $\Big(\beta(x)h\Big)\beta\alpha(x_r^{-1}) \in Z(G), \quad \forall h \in Z(G), \quad \forall x \in G$. Thus, $Z(G)$ is normal in G.

If a BiHom-subgroup H of a BiHom-group G is BiHom-normal, then $(G/H)_r = (G/H)_l$, and we denote the quotient as G/H. The BiHom-operation of G can be extended to the quotient set, yielding a BiHom-structure compatible with the operation of G.

The BiHom-quotient structure is given by the following proposition.

Proposition 5.3. *Let H be a BiHom-normal subgroup of G, and G/H be the set of all BiHom-left cosets of H. Then G/H forms a BiHom-group under the operation defined by*

$$\beta(x)H \dot{-} \beta(y)H = \beta(xy)H, \quad \overline{\alpha}\big(\beta(x)H\big) = \beta\alpha(x)H,$$
$$\overline{\beta}\big(\beta(x)H\big) = \beta^2(x)H,$$

that is $(G/H, \dot{-}, \overline{\alpha}, \overline{\beta})$ is a BiHom-group. If there is no confusion, we denote $\beta(x)H$ as $\overline{x}$.

Proof. To show that the operation $\dot{-} : G/H \times G/H \to G/H$ is well-defined, assume that $\overline{x} = \overline{a}$ and $\overline{y} = \overline{b}$, meaning that $a_l^{-1}x \in H$ and $b_l^{-1}y \in H$. Our goal is to prove that $\overline{x} \dot{-} \overline{y} = \overline{a} \dot{-} \overline{b}$. Since $\overline{x} \dot{-} \overline{y} = \overline{xy}$ and $\overline{a} \dot{-} \overline{b} = \overline{ab}$, we must show that $\overline{xy} = \overline{ab}$. Note that

$$(ab)_l^{-1}(xy) \overset{\text{P } 2.14:1}{=} \Big(\alpha^{-1}\beta(b_l^{-1})\beta^{-1}\alpha(a_l^{-1})\Big)(xy) \overset{\text{D } 2.1(1)}{=} \beta(b_l^{-1})\Big(\beta^{-1}\alpha(a_l^{-1})\beta^{-1}(xy)\Big)$$
$$\overset{\text{D } 2.1(1)}{=} \beta(b_l^{-1})\Big(\beta^{-1}(a_l^{-1}x)y\Big).$$

Since $\overline{x}=\overline{a}$, that is, $\beta(x)H=\beta(a)H$, there exists $h\in H$, such that $a_l^{-1}x=h$. Then, we have $\beta(b_l^{-1})\Big(\beta^{-1}(a_l^{-1}x)y\Big)=\beta(b_l^{-1})\big(\beta^{-1}(h)y\big)=\beta(b_l^{-1})\big(h'y\big)$ for some $h'\in H$.

By the normality of H, we have $H\alpha(\alpha^{-1}(y))=\beta(\alpha^{-1}(y))H$. Then, there exists $h''\in H$ such that $h'y=\beta(\alpha^{-1}(y))h''$. Therefore, we can express $(ab)_l^{-1}(xy)$ as

$$(ab)_l^{-1}(xy)=\beta(b_l^{-1})\big(h'y\big)=\beta(b_l^{-1})\Big(\beta\alpha^{-1}(y)h''\Big)\overset{\text{D }2.1(1)}{=}\beta\alpha^{-1}(b_l^{-1}y)\beta(h'').$$

Since $\overline{y}=\overline{b}$, that is, $b_l^{-1}y=h_3\in H$, we have $\beta\alpha^{-1}(b_l^{-1}y)\beta(h'')=\beta\alpha^{-1}(h_3)\beta(h'')$ belongs to H (product of two elements of H). Thus, $(ab)_l^{-1}(xy)$ is in H, that is, $\overline{ab}=\overline{xy}$. Therefore, we have demonstrated that operation $\dot{-}$ is well-defined. Let us verify that $\overline{\alpha}:G/H\to G/H$ is well-defined. For this, assume that $\overline{x}=\overline{a}$, then $a_l^{-1}x\in H$ and so $\alpha(a_l^{-1}x)\in H$. Therefore, $\alpha(a_l^{-1})\alpha(x)\in H$, that is, $\overline{\alpha(x)}=\overline{\alpha(a)}$. Thus, $\overline{\alpha}(\overline{x})=\overline{\alpha}(\overline{a})$. We verify that $\overline{\beta}:G/H\to G/H$ is well-defined. Assume that $\overline{x}=\overline{a}$, then $a_l^{-1}x\in H$, and so $\beta(a_l^{-1}x)\in H$. Therefore, $\beta(a_l^{-1})\beta(x)\in H$, that is, $\overline{\beta(x)}=\overline{\beta(a)}$. Thus, $\overline{\beta}(\overline{x})=\overline{\beta}(\overline{a})$. Then, the mappings $\overline{\alpha}$ and $\overline{\beta}$ are well-defined.

Now, we can proceed to verify the BiHom-group structure. The mapping $\overline{\alpha}$ is multiplicative, since $\overline{\alpha}\big(\overline{xy}\big)=\overline{\alpha}\big(\beta(x)H\beta(y)H\big)=\overline{\alpha}\big(\beta(xy)H\big)=\beta\alpha(xy)H=\beta\alpha(x)\beta(\alpha(y)H=\big(\beta\alpha(x)H\big)\big(\beta\alpha(y)H\big)=\overline{\alpha}(\overline{x})\overline{\alpha}(\overline{y})$. Furthermore, it is bijective since $\overline{\alpha}(\overline{x})=\overline{\alpha}(\overline{y})$ implies $\beta\alpha(x)H=\beta\alpha(y)H$, so $\alpha(x)_l^{-1}\alpha(y)\in H$, then $\alpha(x_l^{-1}y)\in H$, so $x_l^{-1}y\in H$, then $\overline{x}=\overline{y}$. Similarly, we have $\overline{\beta}(\overline{x}\ \overline{y})=\overline{\beta}(\overline{x})\overline{\beta}(\overline{y})$, and if $\overline{\beta}(\overline{x})=\overline{\beta}(\overline{y})$, then $\overline{x}=\overline{y}$; that is, it is bijective.

Let us verify the BiHom-associativity. If $x,y,z\in H$, then $\overline{x},\overline{y},\overline{z}\in G/H$ and

$$\begin{aligned}\overline{\alpha}(\overline{x})\dot{-}\big(\overline{y}\dot{-}\overline{z}\big)=\overline{\alpha(x)}\big(\overline{yz}\big)&=\big(\beta\alpha(x)H\big)\big(\beta(yz)H\big)=\Big(\beta\alpha(x)\beta(yz)\Big)H\\&\overset{\text{C }2.7\text{D }2.1(1)}{=}\Big(\beta(xy)\beta^2(z)\Big)H=\big(\beta(xy)H\big)\big(\beta^2(z)H\big)\\&=\Big(\beta(x)H\beta(y)H)\Big)\beta^2(z)H=(\overline{x}\dot{-}\overline{y})\dot{-}\overline{\beta}(\overline{z}).\end{aligned}$$

BiHom-unitality holds since $\overline{e}=\beta(e)H=H\in G/H$ is the BiHom-unity. We verify

$$\overline{x}\dot{-}\overline{e}=\big(\beta(x)H\big)\big(\beta(e)H\big)\overset{\text{D }2.1(2)}{=}\beta(xe)H\overset{\text{D }2.1(3)}{=}\overline{xe}=\overset{\text{D }2.1(3)}{=}\overline{\alpha(x)}=\overline{\alpha}(\overline{x}).$$
$$\overline{e}\dot{-}\overline{x}=\big(\beta(e)H\big)\big(\beta(x)H\big)\overset{\text{D }2.1(2)}{=}\beta(ex)H\overset{\text{D }2.1(2)}{=}\overline{ex}\overset{\text{D }2.1(3)}{=}\overline{\beta(x)}=\overline{\beta}(\overline{x}).$$

The BiHom-invertibility holds since for all $x\in G$, there exist $x_r^{-1},x_l^{-1}\in G$ such that

$$\overline{x}\dot{-}\overline{x_r^{-1}}=\beta(x)H\beta(x_r^{-1})H=\beta(xx_r^{-1})H=\beta(e)H=H=\overline{e},$$
$$\overline{x_l^{-1}}\dot{-}\overline{x}=\beta(x_l^{-1})H\beta(x)H=\beta(x_l^{-1}x)H=\beta(e)H=H=\overline{e}.$$

Thus, we obtain a new BiHom-group $(G/H,\dot{-},\overline{\alpha},\overline{\beta})$, which is referred to as the BiHom-quotient group. □

6. Homomorphisms of BiHom-groups

We now define the homomorphisms of BiHom-groups as mappings that preserve the BiHom-structure.

Definition 6.1. A homomorphism of BiHom-groups $(G,\mu_G,\alpha_G,\beta_G)$ and $(K,\mu_K,\alpha_K,\beta_K)$ is a mapping $f:(G,\mu_G,\alpha_G,\beta_G)\to(K,\mu_K,\alpha_K,\beta_K)$, such that

1) $f\circ\mu_G=\mu_K\circ f$;
2) $f\circ\alpha_G=\alpha_K\circ f$;
3) $f\circ\beta_G=\beta_K\circ f$.

Moreover, if f is bijective, we say that f is an isomorphism.

Proposition 6.1. *If $f:(G,\mu_G,\alpha_G,\beta_G)\to(K,\mu_K,\alpha_K,\beta_K)$ is a homomorphism of BiHom-groups, then*

1) $f(e_G)=e_K$.
2) *For all $x\in G$, $f(x_l^{-1})=(f(x))_l^{-1}$.*

3) *For all* $x \in G$, $f(x_r^{-1}) = (f(x))_r^{-1}$.

Proof. 1: For all $x \in G$,

$$\mu_K(f(x), f(e_G)) = f(\mu_G(x, e_G)) = f\alpha_G(x) = \alpha_K f(x) = \mu_K(f(x), e_K).$$

Then, by the law of cancellation, we get $f(e_G) = e_K$.
2: For all $x \in G$, $\mu_K(f(x_l^{-1}), f(x)) = f(\mu_G(x_l^{-1}, x)) = f(e_G) = e_K$. Therefore, the left inverse of $f(x)$ is $f(x_l^{-1})$.
3: For all $x \in G$, $\mu_K\big(f(x), f(x_r^{-1})\big) = f\big(\mu_G(x, x_r^{-1})\big) = f(e_G) = e_K$. Thus, the right inverse of $f(x)$ is $f(x_r^{-1})$. □

Definition 6.2. Let $f : (G, \mu_G, \alpha_G, \beta_G) \to (K, \mu_K, \alpha_K, \beta_K)$ be a homomorphism of BiHom-groups. The kernel and image of f are defined respectively by
$\ker f = \{x \in G \ : \ f(x) = e_K\}$ and $\operatorname{Im} f = \{f(x) \ : \ x \in G\}$.

Proposition 6.2. *If* $f : (G, \mu_G, \alpha_G, \beta_G) \to (K, \mu_K, \alpha_K, \beta_K)$ *is a homomorphism of BiHom-groups, then*

1. *The kernel* $\ker f$ *is a BiHom-normal subgroup of* G.
2. *The image* $\operatorname{Im} f$ *is a BiHom-subgroup of* K.

Proof. 1: For all $x, y \in \ker f$, $f(\mu_G(x, y)) \overset{\text{D } 6.1:1}{=} \mu_K(f(x), f(y)) = \mu_K(e_K, e_K) = e_K$. Thus, $\ker f$ is closed under the product. For all $x \in \ker f$, we have

$$\mu_K(f(x_l^{-1}), f(x)) \overset{\text{D } 6.1:1}{=} f(\mu_G(x_l^{-1}, x)) = f(e_G) \overset{\text{Prop.}6.1:1}{=} \alpha_K(e_K) = \alpha_K f(x_l^{-1}).$$

This implies $f(x_l^{-1}) = e_K$, from α_K being bijective. Thus, for all $x \in \ker f$, $x_l^{-1} \in \ker f$. From Proposition 6.1:1, $e_K \overset{\text{R } 2.3}{=} \overset{\text{D } 2.1(4)}{=} f\big(\mu_G(x, x_r^{-1})\big) \overset{\text{D } 6.1:1}{=} \mu_K\big(f(x), f(x_r^{-1})\big)$ implying that $\beta_K f(x_r^{-1}) = e_K$, that is, $x_r^{-1} \in \ker f$, by Definition 2.1(3), Remark 2.3 and the bijectivity of β_K. Also, $f\alpha_G(x) \overset{\text{D } 6.1:2}{=} \alpha_K f(x) = \alpha_K(e_K) \overset{\text{R } 2.3}{=} e_K$, and $f\beta_G(x) \overset{\text{D } 6.1:3}{=} \beta_K f(x) = \beta_K(e_K) \overset{\text{R } 2.3}{=} e_K$. Hence, for all $x \in \ker f$, $\alpha_G(x), \beta_G(x) \in \ker f$. Therefore, $\ker f$ is a BiHom-subgroup of G. To prove BiHom-normality, for all $x \in G$ and all $h \in \ker f$, we have

$$\begin{aligned} f\Big(\big(\beta_G(x)h\big)\beta_G\alpha_G(x_r^{-1})\Big) &\overset{\text{D } 6.1:1}{=} \Big(f\beta_G(x)f(h)\Big)f\beta_G\alpha_G(x_r^{-1}) \\ &\overset{\text{D } 6.1:3}{=} \Big(\beta_K f(x)f(h)\Big)\beta_K\alpha_K f(x_r^{-1}) = \Big(\beta_K f(x)e_K\Big)\beta_K\alpha_K f(x_r^{-1}) \\ &\overset{\text{D } 2.1(3)}{=} \beta_K\alpha_K f(x)\beta_K\alpha_K f(x_r^{-1}) \overset{\text{D } (6.1):1}{=} \beta_K\alpha_K\big(f(x)f(x_r^{-1})\big) \\ &\overset{\text{D } 2.1(4)\ \text{P } 6.1:1}{=} \beta_K\alpha_K\big(f(xx_r^{-1})\big) \overset{\text{R } 2.3}{=} \beta_K\alpha_K(e_K) = e_K, \end{aligned}$$

that is, $\forall x \in G, \forall h \in \ker f$: $\big(\beta_G(x)h\big)\beta_G\alpha_G(x_r^{-1}) \in \ker f$. Therefore, $\ker f$ is a BiHom-normal subgroup.
2: $\operatorname{Im} f$ is not empty of course, $f(e_G) \overset{\text{P } 6.1:1}{=} e_K \in \operatorname{Im} f$. For all $f(x), f(y) \in \operatorname{Im} f$, we have

$$f(\mu_G(x, y)) \overset{\text{D } 6.1:1}{=} \mu_K(f(x), f(y)) \in \operatorname{Im} f.$$

Hence, $\operatorname{Im} f$ is closed under the product in K. For all $f(x) \in \operatorname{Im} f$, we have

$$(f(x))_l^{-1} \overset{\text{P } 6.1:2}{=} f(x_l^{-1}) \in \operatorname{Im} f, \quad (f(x))_r^{-1} \overset{\text{P } 6.1:3}{=} f(x_r^{-1}) \in \operatorname{Im} f.$$

Moreover, for all $f(x) \in \operatorname{Im} f$, we have

$$\alpha_K f(x) \overset{\text{D } 6.1:2}{=} f\alpha_G(x) \in \operatorname{Im} f, \quad \beta_K f(x) \overset{\text{D } 6.1:3}{=} f\beta_G(x) \in \operatorname{Im} f.$$

This implies that $\alpha_K(\operatorname{Im} f) \subseteq \operatorname{Im} f$ and $\beta_K(\operatorname{Im} f) \subseteq \operatorname{Im} f$. □

Proposition 6.3. *If* $f : (G, \mu_G, \alpha_G, \beta_G) \to (K, \mu_K, \alpha_K, \beta_K)$ *is a homomorphism of BiHom-groups, then* f *is injective if and only if* $\ker f = \{e_G\}$.

Proof. ($\Rightarrow$) For all $x \in \ker f$, $f(x) = e_K \overset{\text{P } 6.1:1}{=} f(e_G)$. Then $x = e_G$, since f is injective, that is, $\ker f = \{e_G\}$.
($\Leftarrow$) If $f(x) = f(y)$, then

$$f\big(\mu_G(x, y_r^{-1})\big) = \mu_K\big(f(x), f(y_r^{-1})\big) \overset{\text{P } 6.1:3}{=} \mu_K\big(f(x), f(y)_r^{-1}\big) = e_K \overset{\text{P } 2.8:3}{\Longrightarrow} x = y. \quad \square$$

Example 6.4. Consider a normal BiHom-subgroup $H \lhd G$. Then, there is a canonical surjective BiHom-group homomorphism from (G, α, β) to the quotient BiHom-group as defined in Proposition 5.3, $(G/H, \overline{\alpha}, \overline{\beta})$. It sends an element x to the BiHom-coset determined by x. Note that $\pi : G \to G/H$ for all $x \in G$ such that $\pi(x) = \beta(x)H$. For all $x, y \in G$, $\pi(xy) = \beta(xy)H = \beta(x)H\beta(y)H = \pi(x)\pi(y)$. For all $x \in G$, $\pi\alpha(x) = \beta\alpha(x)H = \overline{\alpha}\big(\beta(x)H\big) = \overline{\alpha}\big(\pi(x)\big)$, and $\pi\beta(x) = \beta^2(x)H = \overline{\beta}\big(\beta(x)H\big) = \overline{\beta}\big(\pi(x)\big)$. By definition, every element m of the quotient G/H can be expressed as $m = \beta(x)H$ for some $x \in G$. Therefore, $\pi(x) = \beta(x)H = m$. Thus, π is well-defined and surjective. Moreover, we have $\pi(x) = e_{G/H} \Leftrightarrow \beta(x)H = H \Leftrightarrow x \in H$, and then $\ker\pi = H$.

Proposition 6.5. *If $f : (G, \mu_G, \alpha_G, \beta_G) \to (K, \mu_K, \alpha_K, \beta_K)$ is a homomorphism of BiHom-groups, H is a BiHom-subgroup of G, and M is a BiHom-subgroup of K, then*

1. *The image $f(H)$ of a BiHom-subgroup is a BiHom-subgroup of K.*
2. *The inverse image of a BiHom-subgroup $f^{-1}(M) = \{x \in G \ : \ f(x) \in M\}$ is a BiHom-subgroup of G.*

Proof. 1: Since $f(e_G) \overset{\text{P } 6.1:1}{=} e_K \in f(H)$, it follows that $f(H)$ is not empty. For all $f(x), f(y) \in f(H)$, we have $f(\mu_G(x, y)) \overset{\text{D } 6.1:1}{=} \mu_K(f(x), f(y)) \in f(H)$. Thus, $f(H)$ is closed under the product of K.

Moreover, for all $f(x) \in f(H)$,

$$(f(x))_l^{-1} \overset{\text{P } 6.1:2}{=} f(x_l^{-1}) \in f(H) \quad \text{and} \quad (f(x))_r^{-1} \overset{\text{P } 6.1:3}{=} f(x_r^{-1}) \in f(H), \tag{21}$$

$$\alpha_K f(x) \overset{\text{D } 6.1:2}{=} f\alpha_G(x) \in f(H) \quad \text{and} \quad \beta_K f(x) \overset{\text{D } 6.1:3}{=} f\beta_G(x) \in f(H), \tag{22}$$

that is, $\alpha_K f(H) \subseteq f(H)$ and $\beta_K f(H) \subseteq f(H)$. Then, $f(H)$ is a BiHom-subgroup of K.

2: Let M be a BiHom-subgroup of K. We must prove that $f^{-1}(M)$ is a BiHom-group.

(a): For all $x, y \in f^{-1}(M)$, $f\big(\mu_G(x, y)\big) \overset{\text{P } 6.1:1}{=} \mu_K\big(f(x), f(y)\big) = \mu_K(m, m') \in M$. Thus, $f^{-1}(M)$ is closed under the product.

(b): For all $x \in f^{-1}(M)$ (that is, $f(x) = m \in M$), we have

$$\begin{aligned} e_K &\overset{\text{P } 6.1:1}{=} f(e_G) \overset{\text{D } 2.1(4)}{=} f\big(\mu_G(x_l^{-1}, x)\big) \overset{\text{D } 6.1:1,2}{=} \mu_K\big(f(x_l^{-1}),\ f(x)\big) \\ &= \mu_K\big(f(x_l^{-1}),\ m\big) \overset{\text{P } 6.1:2}{=} \mu_K\big(f(x)_l^{-1},\ m\big). \end{aligned}$$

If we multiply the members of the above equality by $\beta_K(m_r^{-1})$, then we obtain

$$\begin{aligned} \mu_K\Big(\mu_K\big(f(x)_l^{-1}, m\big),\ \beta_K(m_r^{-1})\Big) &\overset{\text{D } 2.1(1)}{=} \Big(\mu_K\big(\alpha_K f(x)_l^{-1}, \mu_K(m,\ m_r^{-1})\Big) \\ &\overset{\text{D } 2.1(4)(3)}{=} \alpha_K^2(f(x)_l^{-1}), \end{aligned}$$

$$\mu_K\big(e_K,\ \beta_K(m_r^{-1})\big) = \beta_K^2(m_r^{-1}).$$

Thus, $\alpha_K^2 f(x_l^{-1}) = \beta_K^2(m_r^{-1}) \in M$, because M is a BiHom-subgroup. Also, $\alpha_K^2 f(x_l^{-1}) \in M$ implies that $f(x_l^{-1}) \in M$. So, $x_l^{-1} \in f^{-1}(M)$. Similarly, for all $x \in f^{-1}(M)$, we have

$$e_K \overset{\text{P } 6.1:1}{=} f\mu_G\big(x, x_r^{-1}\big) \overset{\text{D } 6.1:1}{=} \mu_K\big(m, f(x_r^{-1})\big).$$

If we multiply the members of above equality by the term $\alpha_K(m_r^{-1})$, then we obtain

$$\begin{aligned} \mu_K\Big(\alpha_K(m_r^{-1}),\ \mu_K\big(m,\ f(x)_r^{-1}\big)\Big) &\overset{\text{D } 2.1(1)}{=} \mu_K\Big(\mu_K\big(m_r^{-1},\ m\big),\ \beta_K f(x)_r^{-1}\Big) \\ &\overset{\text{D } 2.1(4)(3)}{=} \beta_K^2(f(x)_r^{-1}). \end{aligned}$$

Also, $\mu_K\big(\alpha_K(m_r^{-1}),\ e_K\big) \overset{\text{D } 2.1(3)}{=} \alpha_K^2(m_r^{-1})$. Therefore, $\alpha_K^2 f(x_r^{-1}) = \beta_K^2(m_r^{-1}) \in M$, because M is a BiHom-subgroup. Therefore, $\alpha_K^2 f(x_r^{-1}) \in M$ implies $f(x_r^{-1}) \in M$. So, $x_r^{-1} \in f^{-1}(M)$.

(c): Additionally, we have $f\alpha_G(x) \overset{\text{D } 6.1:2}{=} \alpha_K f(x) = \alpha_K(m) \in M$. So, $\alpha_G(x) \in f^{-1}(M)$. Also, $f\beta_K(x) \overset{\text{D } 6.1:3}{=} \beta_K f(x) = \beta_K(m) \in M$. So, $\beta(x) \in f^{-1}(M)$. Hence, $\alpha(x), \beta(x) \in f^{-1}(M)$ for all $x \in f^{-1}(M)$.

Thus, $f^{-1}(M)$ is a BiHom-subgroup of G. □

Theorem 6.6 *(First Isomorphism Theorem). If $f : (G, \mu_G, \alpha_G, \beta_G) \to (K, \mu_K, \alpha_K, \beta_K)$ is a surjective homomorphism of BiHom-groups, then $G/\ker f \cong K$.*

Proof. $G/\ker f$ is a BiHom-group by Proposition 6.2. Then we can define a homomorphism $\tilde{f} : G/\ker f \to \operatorname{Im} f$ by $\tilde{f}(\beta(x)\ker f) = f(x)$. First, let us check that $\tilde{f}$ is well-defined:

$$\begin{aligned}\beta(x)\ker f = \beta(y)\ker f &\Rightarrow \mu_G(x_l^{-1}, y) \in \ker f,\\ &\Rightarrow \mu_K\big(f(x_l^{-1}), f(y)\big) = e_K, \text{ and } \mu_K\big(f(x_l^{-1}), f(x)\big) = e_K\\ &\overset{\text{P } 2.6:2}{\Rightarrow} f(x) = f(y) \text{ (by right cancellation).}\end{aligned}$$

Thus, $\tilde{f}(\overline{x}) = \tilde{f}(\overline{y})$ implies that $\tilde{f}$ is well defined. The mapping $\tilde{f}$ is a homomorphism as

$$\begin{aligned}\tilde{f}\big(\overline{\mu_G}(\overline{x},\ \overline{y})\big) &= \tilde{f}\big(\overline{\mu_G(x,\ y)}\big) = f\big(\mu_G(x,\ y)\big)\\ &= \mu_K(f(x),\ f(y)) = \overline{\mu_K}\big(\tilde{f}(\overline{x}),\ \tilde{f}(\overline{y})\big).\end{aligned}$$

Since f is surjective, for all $y \in K$, $f(x) = y$ for some $x \in G$. Thus, $\tilde{f}(\overline{x}) = \overline{y}$.

To show that $\tilde{f}$ is injective, suppose that $\tilde{f}(\overline{x}) = \tilde{f}(\overline{y})$. This implies $f(x) = f(y)$, and thus, $\mu_K\big(f(x)_l^{-1},\ f(y)\big) = f\big(\mu_G(x_l^{-1}),\ y\big) = e_K$, which means $x_l^{-1}y \in \ker(f)$. Hence, $\beta(x)\ker(f) = \beta(y)\ker(f)$, which implies $\overline{x} = \overline{y}$. Thus, $\tilde{f}$ is an isomorphism. □

Corollary 6.7. *Let $f : (G, \mu_G, \alpha_G, \beta_G) \to (K, \mu_K, \alpha_K, \beta_K)$ be a homomorphism of BiHom-groups. Then, $G/\ker f \cong \operatorname{Im} f$.*

Proof. If we consider only the image of f ($\operatorname{Im} f$) in place of K, we obtain a surjection. Therefore, we can apply the previous theorem. □

Proposition 6.8. *Let H, N be two BiHom-subgroups of G, and H be normal in G. Then $NH = HN$ is a BiHom-subgroup of G.*

Proof. First, we show that $NH = HN$. For all $n \in N$, we have $n = \beta(n')$ for some $n' \in N$. Thus, $nH = \beta(n')H \overset{\text{P } 5.1:1}{=} H\alpha(n') = Hn'' \subset HN$, that is, $NH \subset HN$. Similarly, we find that, for all $n \in N$, $Hn = H\alpha(n') \overset{\text{P } 5.1:1}{=} \beta(n')H = n''H \subset NH$. Thus, $HN \subset NH$. Thus, $NH = HN$.

Secondly, we show that NH is a BiHom-subgroup. For all $n, n' \in N$ and all $h, h' \in H$,

$$\begin{aligned}(nh)(n'h') &= \alpha(n)\Big(h\big(\beta^{-1}(n'h')\big)\Big)\\ &\overset{\text{D } 2.1(1)}{=} \alpha(n)\Big(\Big(\alpha^{-1}(h)\beta^{-1}(n')\Big)h'\Big), \quad by \quad HN = NH\\ &= \alpha(n)\Big((n''h'')h'\Big) \overset{\text{D } 2.1(1)}{=} \alpha(n)\Big(\alpha(n'')\big(h''\beta^{-1}(h')\big)\Big)\\ &\overset{\text{D } 2.1(1)}{=} \big(n\alpha(n'')\big)\Big(\beta(h'')h'\Big) \in NH.\end{aligned}$$

For any $nh \in NH$, we have

$$(nh)_l^{-1} \overset{\text{P } 2.14:1}{=} \alpha^{-1}(\beta(h_l^{-1}))\beta^{-1}(\alpha(n_l^{-1})) \in HN = NH, \tag{23}$$

$$(nh)_r^{-1} \overset{\text{P } 2.14:2}{=} \alpha^{-1}(\beta(h_r^{-1}))\beta^{-1}(\alpha(n_r^{-1})) \in HN = NH. \tag{24}$$

Also, $\alpha(NH) \overset{\text{D } 2.1(2)}{=} \alpha(N)\alpha(H) = NH$ and, similarly, $\beta(NH) \overset{\text{D } 2.1(2)}{=} \beta(N)\beta(H) = NH$. Thus, NH is a BiHom-subgroup of G. □

Theorem 6.9 *(Second Isomorphism Theorem). Let H, N be two BiHom-subgroups of G with H normal in G. Then,*

1) *$H \cap N$ is a normal BiHom-subgroup of N.*
2) *H is a normal BiHom-subgroup of NH.*
3) *$N/(H \cap N) \cong NH/H$.*

Proof. 1: By the Proposition 6.8, $H \cap N$ is a BiHom-subgroup, let's prove that it is normal. For all $n \in N$ and all $x \in H \cap N$, we have

$$\big(\beta(n)x\big)\beta(\alpha(n_r^{-1})) \in N, \quad \text{(Product of elements of } N\text{)}$$
$$\big(\beta(n)x\big)\beta\alpha(n_r^{-1}) \in H. \quad \text{(Normality of } H \text{ Prop.5.1:3)}$$

Thus, $\big(\beta(n)x\big)\beta\alpha(n_r^{-1}) \in H \cap N$, which shows that $H \cap N$ is normal in N.
2: NH is a BiHom-subgroup of G containing H. For all $nh \in NH$ and all $x \in H$, we have

$$\big(\beta(nh)x\big)\beta\alpha\big((nh)_r^{-1}\big) \in H \quad \text{(Normality of } H \text{ in } G \text{ Prop.5.1:3)}.$$

3: We consider the homomorphism f as the restriction of the canonical surjection π (defined in Example 6.4; $\pi_{|_N} : N \to G/H$) given by $\pi_{|_N}(x) = \beta(x)H$, composed with the restriction of α ($\alpha_{|_N} : N \to N$) given by $\alpha(x) = xe$. We get $f = \pi_{|_N} \circ \alpha_{|_N} : N \to G/H$. For all $x \in N$, $f(x) = \pi(xe) = \beta(\alpha(x))H$ is a homomorphism of BiHom-groups. We show that

a) $\mathrm{Im}(f) = NH/H$: For all $y \in NH/H$, there exists $n \in N$ and $h \in H$, such that $y = \beta(nh)H$. We have $f(n) = \beta(\alpha(n))H$, and

$$\alpha(n)_l^{-1}\big(nh\big) \overset{\text{D } 2.1(1)}{=} \big(n_l^{-1}n\big)\beta(h) \overset{\text{D } 2.1(3)}{=} \beta^2(h) \in H.$$

Thus, $f(n) = \beta\alpha(n)H = \beta(nh)H = y$. Therefore, $\mathrm{Im}(f) = NH/H$.

b) $\ker(f) = H \cap N$:

$$\begin{aligned}\ker(f) &= \{n \in N \mid f(n) = e_{G/H}\}\\ &= \{n \in N \mid \beta\alpha(n)H = e_{G/H}\} = \{n \in N \mid n \in H\} = H \cap N.\end{aligned}$$

Therefore, by Corollary 6.7, $N/(H \cap N) \cong NH/H$. □

Theorem 6.10 *(The Third Isomorphism Theorem). Let M and N be two normal BiHom-subgroups of (G, α, β), and suppose that N is also a BiHom-subgroup of (M, α, β). Then the BiHom-quotient group $(M/N, \overline{\alpha}, \overline{\beta})$ is a normal BiHom-subgroup of $(G/N, \overline{\alpha}, \overline{\beta})$, and $(G/N)/(M/N) \cong G/M$.*

Proof. For all $\beta(m)N \in M/N$ and all $\beta(x)N \in G/N$, we have

$$\overline{\beta}\big(\beta(x)N\big)\big(\beta(m)N\big)\overline{\alpha}\left(\overline{\beta}\big(\beta(x_r^{-1})N\big)\right) = \big(\beta^2(x)N(\beta(m)N)\big)\alpha(\beta^2(x_r^{-1})N)$$
$$= \left(\beta\big(\beta(x)m\big)\alpha(\beta^2(x_r^{-1})\right)N = \overline{\beta}\big((\beta(x)m)\alpha(\beta(x_r^{-1}))N.$$

Since M is normal in G, we have $(\beta(x)m)\alpha(\beta(x_r^{-1})) \in M$, so $\overline{\beta}\big((\beta(x)m)\alpha(\beta(x_r^{-1})\big)N \in M/N$. Therefore, M/N is normal in G/N, and $(G/N)/(M/N)$ is a BiHom-group.
Next, we define a map $f : G/N \to G/M$ by $f(\beta(x)N) = \beta(x)M$.

a) f is well-defined. Since $\beta(x)N = \beta(y)N$ implies $x_l^{-1}y \in N \subseteq M$, it follows that $\beta(x)M = \beta(y)M$.

b) f is a homomorphism. For all $x, y \in G$,

$$f(\beta(x)N\beta(y)N) = f(\beta(xy)N) = \beta(xy)M = \beta(x)M\beta(y)M = f(x)f(y).$$

Thus, f is a homomorphism. Furthermore, for all $x \in G$, $\beta(x)M \in G/M$, so $f(\beta(x)N) = \beta(x)M$, which proves that f is surjective.

c) $\ker(f) = M/N$. Suppose that $\beta(x)N \in \ker(f)$, so $f(\beta(x)N) = e_{G/M} = M$. This implies $\beta(x)M = M$, hence $x \in M$. Therefore, $\ker(f) = \{\beta(x)N \mid x \in M\} = M/N$.

Applying the First Isomorphism Theorem (Theorem 6.6) yields $(G/N)/(M/N) \cong G/M$. □

6.1. Direct product of BiHom-groups

Definition 6.3. Let (G, α_G, β_G) and (K, α_K, β_K) be two BiHom-groups. Consider the set $G \times K$ equipped with the operation $(x, y).(a, b) = (xa, yb)$ for all $x, a \in G$ and $y, b \in K$. Define the maps $\alpha = \alpha_G \times \alpha_K : G \times K \to G \times K$ and $\beta = \beta_G \times \beta_K : G \times K \to G \times K$ by $\alpha(x, y) = (\alpha_G(x), \alpha_K(y))$, $\beta(x, y) = (\beta_G(x), \beta_K(y))$. Then, $G \times K$ with such operations and maps is called the direct product of G and K.

Proposition 6.11. *Under the notations and constructions proposed in Definition 6.3. We claim that* $(G \times K, \alpha_G \times \alpha_K, \beta_G \times \beta_K)$ *is a BiHom-group.*

Proof. The BiHom-associativity for all $x_1, x_2, x_3 \in G$ and $y_1, y_2, y_3 \in K$ is shown by

$$\begin{aligned}\alpha(x_1, y_1)\big((x_2, y_2)(x_3, y_3)\big) &= (\alpha_G(x_1), \alpha_K(y_1))\big(x_2x_3, y_2y_3\big)\\ &= \big(\alpha_G(x_1)(x_2x_3), \alpha_K(y_1)(y_2y_3)\big)\\ &= \big((x_1x_2)\beta_G(x_3)), (y_1y_2)\beta_K(y_3)\big)\\ &= \big(x_1x_2, y_1y_2\big)(\beta_G(x_3), \beta_K(y_3))\\ &= \big((x_1, y_1)(x_2, y_2)\big)\beta(x_3, y_3).\end{aligned}$$

BiHom-unitality is satisfied with the BiHom-unity element $(e_G, e_K) \in G \times K$:

$$(x, y)(e_G, e_K) = (xe_G, ye_K) = (\alpha_G(x), \alpha_K(y)) = \alpha(x, y),$$
$$(e_G, e_K)(x, y) = (e_Gx, e_Ky) = (\beta_G(x), \beta_K(y)) = \beta(x, y).$$

The BiHom-invertibility is demonstrated by

$$(x, y)(x_l^{-1}, y_l^{-1}) = (x_l^{-1}x, y_l^{-1}y) = (e_G, e_K), \quad (x, y)_l^{-1} = (x_l^{-1}, y_l^{-1}),$$
$$(x, y)(x_r^{-1}, y_r^{-1}) = (xx_r^{-1}, yy_r^{-1}) = (e_G, e_K), \quad (x, y)_r^{-1} = (x_r^{-1}, y_r^{-1}).$$

The map α is multiplicative since

$$\begin{aligned}\alpha\big((x_1, y_1)(x_2, y_2)\big) &= \alpha(x_1x_2, y_1y_2) = \big(\alpha_G(x_1x_2), \alpha_K(y_1y_2)\big)\\ &= \big(\alpha_G(x_1)\alpha_G(x_2), \alpha_K(y_1)\alpha_K(y_2)\big)\\ &= \big(\alpha_G(x_1), \alpha_K(y_1)\big)\big(\alpha_G(x_2), \alpha_K(y_2)\big) = \alpha(x_1, y_1)\alpha(x_2, y_2).\end{aligned}$$

Similarly, $\beta\big((x_1, y_1)(x_2, y_2)\big) = \beta(x_1, y_1)\beta(x_2, y_2)$. Thus, the conditions for a BiHom-group are satisfied. Then, $(G \times K, \alpha_G \times \alpha_K, \beta_G \times \beta_K)$ is a well-defined BiHom-group. □

Theorem 6.12. *Let* H *and* K *be BiHom-subgroups of* G *that satisfy*

1) $G = HK$;
2) $H \cap K = \{e\}$;
3) $\forall h \in H, k \in K : \beta(h)\alpha(k) = \beta(k)\alpha(h)$.

Then, $G \cong H \times K$.

Proof. Define $f : H \times K \to G$ as $(h, k) \mapsto hk$. First, we show that f is an isomorphism. By applying a mapping $\alpha^{-1}\beta^{-1}$ in the equality of statement 3, we get

$$\alpha^{-1}(h)\beta^{-1}(k) = \alpha^{-1}(k)\beta^{-1}(h)\ \forall (h, k) \in H \times K.$$

Then, for all $(h, k), (h', k') \in H \times K$, we have

$$\begin{aligned}&f\big((h, k)(h', k')\big) = f(hh', kk') = (hh')(kk') = \big(\alpha^{-1}(hh')k\big)\beta(k')\\ &= \Big(h\big(\alpha^{-1}(h')\beta^{-1}(k)\big)\Big)\beta(k') = \Big(h\big(\alpha^{-1}(k)\beta^{-1}(h')\big)\Big)\beta(k') = \Big(\alpha^{-1}(hk)h'\Big)\beta(k')\\ &= (hk)(h'k') = f(h, k)f(h', k').\end{aligned}$$

For all $(h, k) \in H \times K$, we observe that

$$f(\alpha \times \alpha(h, k)) = f(\alpha(h), \alpha(k)) = \alpha(h)\alpha(k) = \alpha(hk) = \alpha(f(h, k)).$$

Similarly, $f(\beta \times \beta(h, k)) = f(\beta(h), \beta(k)) = \beta(h)\beta(k) = \beta(hk) = \beta(f(h, k))$. Then, f is a homomorphism of the BiHom-groups. It is evident that f is a surjection because $G = HK$. For all $(h, k) \in \mathrm{Ker}(f)$, we have

$$f(h, k) = hk = e_G, h = k_r^{-1} \in H \cap K = \{e_G\},$$
$$h = e_G = k, \text{ that is, } \mathrm{Ker}\, f = \{(e_G, e_G)\}.$$

Therefore, f is an injection. Thus, $G \cong H \times K$. □

Remark 6.1. We can replace the equality in statement 3 in the theorem with the normality of H and K, and in the general case, we obtain the following statement.

Corollary 6.13. *Let H, K be normal BiHom-subgroups of G, such that $H \cap K = \{e\}$. Then $H \times K \cong HK$.*

Proof. First, since H is normal, we have $HK = KH$. Consequently, HK is a BiHom-subgroup of G, which we denote by M. This allows us to apply the previous theorem to M. To do so, we need to prove that for all $h \in H$ and $k \in K$, the following condition of Theorem 6.12 (3) holds $\beta(h)\alpha(k) = \beta(k)\alpha(h)$. To achieve this, we proceeded as follows. For all $h \in H$ and $k \in K$, we have

$$\Big(\beta(h)\alpha(k)\Big)\Big(\beta(h_r^{-1})\alpha(k_r^{-1})\Big) \overset{\text{D } 2.1(1)}{=} \alpha\beta(h)\Big(\alpha(k)\Big(h_r^{-1}\beta^{-1}\alpha(k_r^{-1})\Big)\Big)$$
$$\overset{\text{D } 2.1(1)}{=} \alpha\beta(h)\Big(\big(kh_r^{-1}\big)\alpha(k_r^{-1})\Big) = \alpha\beta(h)a,$$

where $a = \Big((kh_r^{-1})(\alpha(k_r^{-1})\Big)$. Then,

$$\beta(a) = \beta\Big((kh_r^{-1})(\alpha(k_r^{-1})\Big) \overset{\text{D } 2.1(2)}{=} \big(\beta(k)h'\big)\beta\alpha(k_r^{-1}) \in H. \tag{25}$$

(by Prop.5.1:2 normality of H)

Therefore, $\Big(\beta(h)\alpha(k)\Big)\Big(\beta(h_r^{-1})\alpha(k_r^{-1})\Big) = \alpha\beta(h)a \in H$. Also,

$$\Big(\beta(h)\alpha(k)\Big)\Big(\beta(h_r^{-1})\alpha(k_r^{-1})\Big) \overset{\text{D } 2.1(1)}{=} \Big(\Big(\alpha^{-1}\beta(h)k\Big)\beta(h_r^{-1})\Big)\beta\alpha(k_r^{-1}) = b\beta\alpha(k_r^{-1}),$$

where, $b = \Big(\big(\alpha^{-1}\beta(h)k\big)\beta(h_r^{-1})\Big)$. Then,

$$\alpha(b) = \alpha\Big(\big(\alpha^{-1}\beta(h)k\big)\beta(h_r^{-1})\Big) \overset{\text{D } 2.1(2)}{=} \Big(\beta(h)\alpha(k)\Big)\alpha\beta(h_r^{-1})) \in K, \tag{26}$$

(by Prop.5.1:2 normality of K)

Therefore, $\Big(\beta(h)\alpha(k)\Big)\Big(\beta(h_r^{-1})\alpha(k_r^{-1})\Big) = b\beta\alpha(k_r^{-1}) \in K$. Hence,

$$\Big(\beta(h)\alpha(k)\Big)\Big(\beta(h_r^{-1})\alpha(k_r^{-1})\Big) \in H \cap K = \{e\}.$$

Thus, $\Big(\beta(h)\alpha(k)\Big)\Big(\beta(h_r^{-1})\alpha(k_r^{-1})\Big) = e$, which implies

$$\Big(\beta(h)\alpha(k)\Big) \overset{\text{R } 2.4}{=} \Big((\beta(h_r^{-1})\alpha(k_r^{-1})\Big)_l^{-1} \overset{\text{P } 2.14:1\text{R } 2.4}{=} \alpha^{-1}\beta\alpha(k)\beta^{-1}\alpha\beta(h) \overset{\text{C } 2.7}{=} \beta(k)\alpha(h). \tag{27}$$

Finally, since $M = HK$, $H \cap K = \{e\}$, and the last equality in (27) holds in M, we can apply Theorem 6.12 restricted to $M = HK$. As a result, we obtain $H \times K \cong HK$. □

7. Some differences between groups and BiHom-groups

This section shows that the order of an element in BiHom-groups differs fundamentally from that in the classical case. This example also highlights the failure of Sylow's theorems in BiHom-groups.

Example 7.1. Let $\mathbf{S}_3$ be the symmetric group of degree 3, which includes all permutations of the set $\{1, 2, 3\}$ under composition. The elements of $\mathbf{S}_3$ are defined in the sequel

$$\mathbf{S}_3 = \{\mathrm{id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (2\ 1\ 3)\}.$$

We will twist $\mathbf{S}_3$ using Proposition 2.9 to obtain a BiHom-group, using the two morphisms α and β defined as follows:

$$\alpha(\sigma) = \sigma \quad \text{and} \quad \beta(\sigma) = (1\,2\,3)\sigma(2\,1\,3), \quad \text{for all } \sigma \in \mathbf{S}_3$$

We denote this BiHom-group by $(\mathbf{S}_3, \otimes, \alpha, \beta)$, where the binary operation $\otimes$ is defined by

$$\sigma \otimes \gamma = \alpha(\sigma)\beta(\gamma) = \sigma\ (1\,2\,3)\gamma(2\,1\,3), \quad \forall \sigma, \gamma \in \mathbf{S}_3.$$

The mappings α and β act as follows:

σ	$\alpha(\sigma)$
id	id
(1 2)	(1 2)
(1 3)	(1 3)
(2 3)	(2 3)
(1 2 3)	(1 2 3)
(2 1 3)	(2 1 3)

σ	$\beta(\sigma)$
id	id
(1 2)	(1 3)
(1 3)	(2 3)
(2 3)	(1 2)
(1 2 3)	(1 2 3)
(2 1 3)	(2 1 3)

Using these definitions, the multiplication table for the BiHom-group $(\mathbf{S}_3, \otimes, \alpha, \beta)$ is derived, illustrating how the elements of the group combine under the operation $\otimes$.

$\otimes$	id	(1 2)	(1 3)	(2 3)	(1 2 3)	(2 1 3)
id	id	(1 3)	(2 3)	(1 2)	(1 2 3)	(2 1 3)
(1 2)	(1 2)	(1 2 3)	(2 1 3)	id	(1 3)	(2 3)
(1 3)	(1 3)	id	(1 2 3)	(2 1 3)	(2 3)	(1 2)
(2 3)	(2 3)	(2 1 3)	id	(1 2 3)	(1 2)	(1 3)
(1 2 3)	(1 2 3)	(1 2)	(1 3)	(2 3)	(2 1 3)	id
(2 1 3)	(2 1 3)	(2 3)	(1 2)	(1 3)	id	(1 2 3)

From this one can conclude the following properties:

1) The cardinality $|(\mathbf{S}_3, \otimes, \alpha, \beta)|$ of the BiHom-group $(\mathbf{S}_3, \otimes, \alpha, \beta)$ is the cardinality $|\mathbf{S}_3| = 6$ of $\mathbf{S}_3$.
2) The BiHom-group $(\mathbf{S}_3, \otimes, \alpha, \beta)$ is not commutative, as $(1\,2) \otimes (1\,3) = (2\,1\,3) \neq \mathrm{id} = (1\,3) \otimes (1\,2)$.
3) The BiHom-group $(\mathbf{S}_3, \otimes, \alpha, \beta)$ is monogenic, meaning it is generated by a single element. Thus, $(\mathbf{S}_3, \otimes, \alpha, \beta) = \langle(1\,2)\rangle = \langle(1\,3)\rangle = \langle(2\,3)\rangle$. For example, using the definition that $\langle S\rangle$ is the smallest BiHom-subgroup containing S, we get $(1\,2) \in \langle(1\,2)\rangle$, and since a BiHom-subgroup is closed under multiplication, we have

$$\begin{aligned}
(1\,2) \otimes (1\,2) &= (1\,2\,3) \in \langle(1\,2)\rangle,\\
(1\,2) \otimes (1\,2\,3) &= (1\,3) \in \langle(1\,2)\rangle,\\
(1\,2) \otimes (1\,3) &= (2\,1\,3) \in \langle(1\,2)\rangle,\\
(1\,3) \otimes (1\,2) &= \mathrm{id} \in \langle(1\,2)\rangle,\\
\mathrm{id} \otimes (1\,3) &= (2\,3) \in \langle(1\,2)\rangle.
\end{aligned}$$

Therefore, the BiHom-subgroup generated by $(1\,2)$ is indeed the entire BiHom-group $\mathbf{S}_3$. Hence, $\langle(1\,2)\rangle = \mathbf{S}_3$. Similarly, we obtain $\langle(1\,3)\rangle = \langle(2\,3)\rangle = \mathbf{S}_3$. Therefore, the theory of BiHom-groups is different from traditional group theory; being monogenic (generated by a single element) does not imply commutativity.

4) The power of an element is not clear in a BiHom-group, because x^n has no meaning in a BiHom-structure. We need parentheses, and if we define a left power by $x_l^n = x \otimes x_l^{n-1}$ and similarly a right power by $x_r^n = x_r^{n-1} \otimes x$, then we obtain different results. For example,

$$\begin{aligned}
(1\,2)_l^3 &= (1\,2) \otimes \big((1\,2) \otimes (1\,2)\big) = (1\,2) \otimes (1\,2\,3) = (1\,3),\\
(1\,2)_r^3 &= \big((1\,2) \otimes (1\,2)\big) \otimes (1\,2) = (1\,2\,3) \otimes (1\,2) = (1\,2).
\end{aligned}$$

Therefore, $(1\,2)_l^3 \neq (1\,2)_r^3$.

Consequently, some fundamental theorems of group theory regarding the order of elements, such as Cauchy's theorem, remain conjectures in the BiHom-structure.

5) The first Sylow theorem confirms the existence of a subgroup of order p^k in a group of order mp^k such that $p \nmid m$. For example, in group $\mathbf{S}_3$ of order $6 = 2 \times 3$, the theorem confirms the existence of subgroups of order 2 and subgroups of order 3. However, despite the order of the BiHom-group $(\mathbf{S}_3, \otimes, \alpha, \beta)$ also being $6 = 2 \times 3$, there is no BiHom-subgroup of order 2 in $(\mathbf{S}_3, \otimes, \alpha, \beta)$.

Indeed, if H is a BiHom-subgroup of $(\mathbf{S}_3, \otimes, \alpha, \beta)$, then
If H contains (1 2) or (1 3) or (2 3), then $H = \mathbf{S}_3$ is the entire BiHom-group, meaning that the order of H is 6. If H does not contain such elements, then H will contain (1 2 3) or (2 1 3), and in both cases, $H = \{\mathrm{id}, (1\ 2\ 3), (2\ 1\ 3)\}$. Suppose $(1\ 2\ 3) \in H$, then

$$(1\ 2\ 3) \otimes (1\ 2\ 3) = (2\ 1\ 3) \in H,$$

$$(1\ 2\ 3) \otimes (2\ 1\ 3) = \mathrm{id} \in H.$$

If $A = \{\mathrm{id}, (1\ 2\ 3), (2\ 1\ 3)\}$, A is a subgroup of the group $\mathbf{S}_3$. We have $\beta(1\ 2\ 3) = (2\ 1\ 3)$ and $\beta(2\ 1\ 3) = (1\ 2\ 3)$, then $\beta(A) = A$ and $\alpha(A) = A$. Thus, by Proposition 3.2, A is a BiHom-subgroup of the BiHom-group $(\mathbf{S}_3, \otimes, \alpha, \beta)$, that is, $H = \{\mathrm{id}, (1\ 2\ 3), (2\ 1\ 3)\}$.

Similarly, if H contains (2 1 3), we get $H = \{\mathrm{id}, (1\ 2\ 3), (2\ 1\ 3)\}$.

Therefore, the BiHom-subgroups of the BiHom-group $(\mathbf{S}_3, \otimes, \alpha, \beta)$ are as follows

1) The trivial BiHom-subgroup {id}, of order 1.
2) One proper BiHom-subgroup $H = \{\mathrm{id}, (1\ 2\ 3), (2\ 1\ 3)\}$, of order 3.
3) The entire BiHom-group $\mathbf{S}_3$, of order 6.

Therefore, there exists no BiHom-subgroup of order 2, despite the order of $|\mathbf{S}_3| = 6$. Thus, the Sylow theorems fail in BiHom-groups, meaning that the confirmation of the existence of a BiHom-subgroup of order p^k in a BiHom-group of order mp^k is not true.

Remark 7.1. Studies indicate that the category of BiHom-groups is larger than that of Hom-groups and traditional groups. However, this broader scope can present challenges in obtaining results that are well-established for classical groups, primarily due to the loss of the associativity condition, which is replaced in BiHom-groups by BiHom-associativity.

Conclusion

In this study, we introduced and studied the structure of BiHom-groups, highlighting their fundamental properties and open questions. Key challenges remain, such as defining the order of elements, extending classical results such as Cauchy's theorem and the class equation, and developing a systematic approach for linearizing BiHom-groups into BiHom-algebras. Addressing these questions could bridge the gap between BiHom-groups and non-associative algebraic structures, opening new research directions in algebra and mathematical physics. We hope that this study inspires further exploration of BiHom-groups and their applications.

Acknowledgements

The work described in this paper was supported by a research grant from University Mohamed El Bachir El Ibrahimi of Bordj Bou Arreridj 34030, Algeria, and from the Ministry of Higher Education and Scientific Research, Algeria. The project is titled "Algèbre et Analyse Fonctionnelle Appliquées" (Project (PRFU) code C00L03UN340120230004). Finally, the authors would like to express their gratitude to the referees for carefully reading the paper and providing valuable suggestions and comments.

Data availability

No data was used for the research described in the article.

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