Sustainability for Portfolio Optimization

by

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Abstract

The 2007-2008 financial crash and the looming climate change and global warming have heightened interest in sustainable investment. But whether the shift is as a result of the financial crash or a desire to preserve the environment, a sustainable investment might be desirable. However, to maintain this interest and to motivate investors in indulging in sustainability, there is the need to show the possibility of yielding positive returns.

The main objective of the thesis is to investigate whether the sustainable investment can lead to higher returns.

The thesis focuses primarily on incorporating sustainability into Markowitz portfolio optimization. It looks into the essence of sustainability and its impact on companies by comparing different concepts.

The analysis is based on the 30 constituent stocks from the Dow Jones industrial average or simply the Dow. The constituents stocks of the Dow, from 2007-12-31 to 2018-12-31 are investigated. The thesis compares the cumulative return of the Dow with the sustainable stocks in the Dow based on their environmental, social and governance (ESG) rating. The results are then compared with the Dow Jones Industrial Average denoted by the symbol (^DJI) which is considered as the benchmark for my analysis.

The constituent stocks are then optimized based on the Markowitz mean-variance framework and a conclusion is drawn from the constituent stocks, ESG, environmental, governance and social asset results.

It was realized that the portfolio returns for stocks selected based on their environmental and governance ratings were the highest performers.

This could be due to the fact that most investors base their investment selection on the environmental and governance performance of companies and the demand for stocks in that category could have gone up over the period, contributing significantly to their performance.
Dedication

To my uncle, parents, beloved wife, daughter and entire family for all their support and prayers during my studies.
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Chapter 1

Introduction

In this chapter, I will present a background of the thesis and describe the topics that will be discussed in later chapters. I will also look into other research findings which will be considered and further investigated in the thesis. The chapter will also entail the motivation of the thesis and the outline of the report.

1.1 Background

Portfolio management is the systematic approach for attaining desired results while managing the associated risk. Markowitz who is credited with modern portfolio theory defined an efficient portfolio as the portfolio that has the highest possible potential return for a particular level of risk (see, [19]). An optimal portfolio considers the risk appetite of the investor. Portfolio optimization can, therefore, be defined as the maximization of the return of a particular risk or the minimization of the risk for a particular level of return.

Some investors are unconstrained whilst others have constraints that may include but not limited to, the degree of diversification, the coverage, minimum and maximum allocation of an asset class, type of asset class to invest in and other special needs. Assigning any of these constraints impacts the resulting returns and risks of the portfolio. In recent times, sustainability has become very critical in assessing the performance of companies. The sustainable investment will result in the optimal use of natural resources and preserve the global environment [16]. Companies are therefore seeking to add a constraint of socially desired investment to their portfolio in accordance with the principles of responsible investment of the United Nations (UN).

The attitude of investors is therefore shifting from just returns to paying more attention to environmental, social and governance issues when making investment decisions. As of 2014, there were more than 1200 signatories across the globe representing about US$ 35 trillion in assets under management who had bought into the principle of responsible investment (see, [21]). It is therefore not surprising that the socially responsible investment funds (SRI) have seen tremendous growth over the years. The increment in socially responsible investment funds has necessitated the creation of indicators to assess the performance of the funds and highlight the commitment of companies to sustainability.
Another reason for the heightened interest in sustainable investment is the global financial crisis (see, [23]). After the financial crisis, financial institutions have been called upon to be more responsible. The crisis made companies forward-looking and re-emphasized the need for sustainability. At the same time, investors might be genuinely thinking about impacting the environment and that may have informed their decision to invest in sustainability. But whether the shift is as a result of the financial crash or a desire to preserve the environment, a sustainable investment might be desirable. However, to maintain this interest and to motivate investors in indulging in sustainability, there is the need to show the possibility of yielding positive returns.

There have been a few studies on sustainable investment and returns. Some of the studies indicated that companies that were environmentally conscious yielded higher returns even during the financial crisis [11][16]. However, some of these studies have either focused on specific companies or used a short financial period in their analysis.

In order to increase knowledge about sustainable investment, one may have to consider longer periods to analyze and come to a reasonable conclusion. To my knowledge, previous studies have not contributed to relevant statistics on sustainable investment and return using extensive financial data [11][16].

### 1.2 Thesis objectives and report outline

The main objective of the thesis is to investigate whether sustainable investment can lead to higher returns.

The thesis will focus primarily on incorporating sustainability into Markowitz portfolio optimization. We will then look into the essence of sustainability and its impact on companies by comparing different concepts.

The thesis is structured this way. Firstly, I investigate recently published papers that expand the methodological spectrum of socially responsible investing by introducing mathematical models for portfolio choice. Secondly, I will elaborate on the theoretical framework and methodology for the thesis.

Readers who are unfamiliar with these concepts: Variance as a risk measure, returns, portfolio optimization, and sustainability should first read the chapters called Theoretical Framework and Sustainability in chapter 3, which explain these concepts that are necessary to follow the empirical analysis section.

Thirdly, chapter 4 which contains Data will give oversight and statistical representation of the data that will be used for the analysis. It will then compare an optimal portfolio with and without taking into account sustainability. The effects of sustainability on portfolio performance will then be examined.

The report ends with the final chapter called Conclusions.
1.3 Literature review

Investors are always at crossroads on the decision to maximize expected returns whilst minimizing the associated risk. Because investors demand a reward for higher risk, risky assets turn to have higher expected returns than a less risky asset. For an investment, the extra return obtained in excess of the risk-free rate of return is termed as the risk premium.

The core of modern portfolio theory is the Markowitz (1956) mean-variance (MV) optimization. Even though the theory is over 50 years, it forms the basis for modern-day finance and all new developments in asset allocation are based on some form of variation of the Markowitz theory [9]. Investors seek to distribute a fixed amount of capital among available assets with the motive of maximizing their investment. According to Markowitz portfolio selection, the portfolio risk can be said to be the variance of the portfolio return. It is therefore important to find a sustainable allocation that minimizes the risk of the expected return. The Markowitz problem is said to have a closed-form solution if the expected return vector and the covariance matrix of the returns of the underlying asset are known. However, in the real market, it is almost impossible to predetermine the expected return vector and the covariance matrix of the returns.

One of the major problems with the mean-variance optimization is that it is sensitive to uncertainty. Thus there is a possibility that the estimated expected return and the variance-covariance matrix of the returns can give an optimal portfolio which is unrealistic with a small change in the data set. Assigning equal weights helps to reduce this problem [9].

In recent times, it is increasingly becoming necessary for an investment decision to factor in sustainability [23]. This is because the supply function is irreversible as raw materials can only be used once [8]. There have therefore been many developed methods for evaluating the social and environmental performance of companies. The indexes associated with stock exchanges use methodologies that enable companies and aid stakeholders decision making. There has been a significant increase in the number of sustainability indexes over the period. In 2007, firms belonging to the S&P500 index from 1993 to 2008 were analyzed and it was realized that the market capitalization eliminated by selecting sustainable assets increases with time [13].

However, some of the motivation for companies to incorporate sustainability into their investment decisions is that it gives them access to knowledge (Corporate Sustainability Index (ISE, Índice de Sustentabilidade Empresarial) membership knowledge sharing), competitive advantage, resources availability over a long term and reputational value [21]. Moreover, the study of the panel data of the Financial Times Stock Exchange 350 Index (FTSE350) companies between 2006 and 2016 indicated that companies that factor in sustainability into their business decision-making processes engaged in business activities that enhanced their long-term efficiency and increased their shareholder wealth and corporate value [11]. The study also showed that corporate sustainable (CS) investment was incorporated into stock prices over time and investors that incorporated (CS) performance investment screens generated higher returns during peak periods and also reduced shareholders loses during the stock market crash.
1.4 Assumptions of the Markowitz theory

The Markowitz portfolio theory is said to be very robust and that explains why it forms the basis for modern finance. The Markowitz model for determining the optimal portfolio is based on returns, variances, and covariance of returns. The assumptions under the theory are [9]:

- All investors are rational and try to maximize their utility for a given level of income or money.
- Investors are risk-averse and try to minimize risk whilst maximizing return.
- Investors have free access to fair and correct information on risk and returns.
- The markets are efficient and absorb any information quickly and perfectly.
- Investors will always choose higher returns over lower returns for any level of risk.
- Investors base their decisions on expected returns and variance of these returns from the mean.

An efficient portfolio based on these assumptions is a portfolio of assets that gives a higher expected return for a chosen risk or a portfolio of assets that gives a lower risk for a chosen return. One way to achieve this is by diversification of securities. The unsystematic and company risk can be reduced by selecting securities and assets that are negatively correlated or has no correlation. Under portfolio diversification, Markowitz aims for the smallest possible attainable standard deviation, a negative (-1) coefficient of correlation and the covariance of assets within the portfolio to have a negative interactive effect. If all this can be achieved then the portfolio will have the smallest risk. In practice, the expected returns and covariance matrix are estimated from historical data.

Optimal portfolios are mean-variance efficient and the mean-variance efficiency (MVE) forms the basis for asset allocation and developing an optimal portfolio.

The main difference between a Markowitz efficient portfolio and an optimal portfolio is that a Markowitz efficient portfolio can be determined mathematically whilst an optimal portfolio is subjective to the risk appetite of the investor. The mathematical definition of risk or volatility in the field of portfolio selection are variance, semi-variance and the probability of an adverse outcome. Investment funds are allocated among competing classes of assets. The importance of diversification of an efficient portfolio of assets cannot be underestimated. This explains why investors manage their portfolio risk to an acceptable level based on the policies of the organization.
Chapter 2
Theoretical Framework

This section presents the complete set of theories and techniques which will be used as a foundation for my analysis based on which I will draw my conclusions. It will discuss the mathematical and financial concepts in portfolio management.

2.1 Modern portfolio theory

Modern portfolio theory is made up of several theories that are the foundation on which portfolio analysis and portfolio selection rest.

The Markowitz mean-variance portfolio selection forms the backbone of modern portfolio theory. The mean and variance discussed under the model are based on the portfolio returns.

2.1.1 Returns

Return can be considered as the money made or lost on an investment over time expressed as a fraction of the original investment. As expected, every prudent investor invests with the aim of making a profit. Returns are however situational and dependent on the financial data input used to measure it. In investment, expected returns mostly have a direct dependence on risk.

There are various types of returns and below are some of the variations of returns used in finance.

Net returns

Equation (2.1.1.1) defines a one period net return denoted by $R_t$. If the price of an asset at time, $t$ is denoted by $P_t$ and $P_{t-1}$ is the price of the preceding period to $P_t$, then the net return $R_t$ over the time interval $[t-1,t]$ without factoring in dividend is given by [17]:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$  \hspace{1cm} (2.1.1.1)

The Gross Return, $R_G$ is expressed as

$$R_G = R_t + 1 = \frac{P_t}{P_{t-1}}$$
Multiperiod gross returns

The multiperiod gross return, $R_G(n)$ is an $n$ period disjoint subintervals of a gross return. Thus if:

$$R_t(n) = \frac{P_t - P_{t-n}}{P_{t-n}} = \frac{P_t}{P_{t-n}} - 1,$$  \hspace{1cm} (2.1.1.2)

then following the equation (2.1.1.1)

$$R_G(n) = 1 + R_t(n) = \frac{P_t}{P_{t-n}} = \left(\frac{P_t}{P_{t-1}}\right)\left(\frac{P_{t-1}}{P_{t-2}}\right)\ldots\left(\frac{P_{t-n+1}}{P_{t-n}}\right)$$  \hspace{1cm} (2.1.1.3)

$$= (1 + R_t)(1 + R_{t-1})\ldots(1 + R_{t-n+1}),$$

where we use the notation $R_t = R_t(1)$ for simplicity.

So by extension a multi-period return can be based on the gross return for $n$ periods and expressed as:

$$1 + R_t(n) = \frac{P_t}{P_{t-n}} = \prod_{i=0}^{n-1} (1 + R_{t-i})$$  \hspace{1cm} (2.1.1.4)

Log returns / Continuously compounding returns

In finance, we are mostly interested in the log returns because they are time additive (time consistent) and when the log returns for each period is normally distributed, then by adding the log returns we will get a result that is also normally distributed.

For a time interval of $[0, T]$ with the price of an asset at time 0 and $T$ being $P_0$ and $P_T$, respectively. The interval $[0, T]$ can be divided into $n$ equal distance intervals. Then based on the multi-period simple return, we assume that every $[t_{i-1}, t_i]$ sub-interval has a return $R$ which is the same, and represents an $n$th part of a one-period return over the interval and are represented by $R_{[0,T]}^*[1]$, then it implies that [22]:

$$R = \frac{R_{[0,T]}^*[1]}{n}$$

Thus following a similar argument from equation (2.1.1.4), the gross return over the time interval $[0, T]$ is:

$$R_{[0,T]}[n] = \prod_{i=1}^{n} (1 + R_{[t_{i-1}, t_i]}[1]) = (1 + R)^n = \left(1 + \frac{R_{[0,T]}^*[1]}{n}\right)^n$$  \hspace{1cm} (2.1.1.5)
where $t_0$ is the 0th term and $t_n$ is the last point, thus (2.1.1.5) can be expressed as:

$$R_{[0,T]}[n] = \frac{1}{P_0} \prod_{i=1}^{n-1} \frac{P_i}{P_{i-1}} = \frac{P_T}{P_0}$$

(2.1.1.6)

Based on equation (2.1.1.5) and (2.1.1.6)

$$\frac{P_T}{P_0} = \left(1 + \frac{R^*_T[1]}{n}\right)^n$$

(2.1.1.7)

As the subintervals $[t_{i-1}, t_i]$ becomes smaller the $n \to \infty$ hence:

$$\lim_{n \to \infty} \frac{P_T}{P_0} = \lim_{n \to \infty} \left(1 + \frac{R^*_T[1]}{n}\right)^n,$$

(2.1.1.8)

and based on the definition of exponential function,

$$\frac{P_T}{P_0} = e^{R^*_T[1]}$$

So a one-period Log return of an asset is therefore expressed as:

$$\ln\left(\frac{P_T}{P_0}\right) = R^*_T[1]$$

(2.1.1.9)

Representing a one-period log return as $R^L_{[0,T]}[1]$, it implies

$$R^L_{[0,T]}[1] = \ln\left(\frac{P_T}{P_0}\right) = \ln\left(1 + R_t[1]\right),$$

(2.1.1.10)

where $R_t$ is the simple return and by extension, an $n$ period log return is given by:

$$R^L_{[0,T]}[n] = \ln(1 + R_t(n))$$

$$= \ln\left(\frac{P_1}{P_0} \cdot \frac{P_2}{P_1} \cdots \frac{P_n}{P_{n-1}}\right)$$

$$= \sum_{i=1}^{n} \ln\left(\frac{P_i}{P_{i-1}}\right)$$

(2.1.1.11)

This reaffirms the reasons for using the log returns in finance given that it is easier to derive the time series properties of sums than of products [5].

**Factoring in dividend and interest payment on returns**

Shareholders are paid dividends when they invest in stocks and this must be accounted for in the calculation of returns. For bonds, the issuer owes the holder and is expected to pay an interest known as the coupon and that must also be accounted for. Thus if the interest or dividend is denoted by $D_{t+1}$ and is paid between time $t+1$ and $t$ then the net return at time $t$ is given by:

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1$$

(2.1.1.12)

where $P_{t+1}$ is known as the capital gain and $D_{t+1}$ is termed as income gain.
2.2 Portfolio construction

Investors are risk averse so provided two investments have equal returns, the investor will prefer the investment with less risk. This implies that if an investor wants higher returns then the investor must be willing to accept more risk. Investment decisions are therefore made based on the risk aversion of the investor.

The utility of an investor can be defined as the total satisfaction that one receives from consuming goods or services. According to Daniel Bernoulli who is credited with utility concept, for a rational person utility increases with wealth but at a decreasing rate [14]. Although it is very difficult to measure consumer’s utility, it can be determined indirectly from consumer behavior theories that indicate that consumers will strive to maximize their utility. In economics, therefore, the utility function is a mathematical function that ranks the alternatives when trying to maximize your choice in any situation. There are various forms of utility functions which include but not limited to the power utility function, exponential utility function and quadratic utility function. If an investor choice is based on the quadratic utility function, then it implies that it is a curve that has a decreasing gradient for larger risk when plotted. Assuming $U$ is a quadratic utility function and $w$ is wealth, then the utility of the wealth $w(U)$ can be expressed as

$$ U(w) = w - \Gamma w^2, $$

where $\Gamma$ is a risk-aversion coefficient.

From the Markowitz theory, risk aversion can be considered as the difference between the utility of expected wealth and the expected utility of wealth. This can, therefore, be said to be the risk premium and is expressed mathematically as:

$$ U[E(w)] - E[U(w)] $$

(2.2.0.1)

So following Amenc et al [2], based on equation (2.2.0.1), if :

- If $U[E(w)] \leq E[U(w)]$: then the individual is a risk lover and the utility function is convex
- If $U[E(w)] = E[U(w)]$: then the individual is risk-neutral and the utility function is linear
- If $U[E(w)] \geq E[U(w)]$: then the individual is risk averse and the utility function is concave

The plot in Figure 2.1 below shows the different risk preferences based on their respective utility curves.
The expected future value of a portfolio is unknown today because it depends on the random future prices of an asset. The stock prices are said to be stochastic and follow the random walk hypothesis (see, [15]) which states that, changes in stock prices are independent of each other and have the same distribution. This implies that the future movement of the stock price cannot be predicted by historical movement or trends.

According to Harry Markowitz (see, [19]), an optimal portfolio is constructed by maximizing the expected portfolio return for a given risk or by minimizing the risk for a given level of the expected return. Based on the theory, diversification helps to reduce the risk of a portfolio but due to the correlation between the returns on securities, the risk which is the variance cannot be eliminated entirely. The efficient portfolio is, therefore, the portfolio that has the highest expected return for a given level of risk.

Hence, any rational investor will pick the efficient portfolio in relation to his/her risk preference.

2.2.1 Risk and return

Based on [9] with little alteration to the parameters, the expected return of the \( i \) th asset return \( E(R_i) \), \( i = 1, \ldots, N \), is calculated as the probability adjusted mean.

\[
E(R_i) = p_{i1}r_{i1} + p_{i2}r_{i2} + \ldots + p_{iM}r_{iM}
\]

\[
= \sum_{j=1}^{M} p_{ij}r_{ij},
\]

where \( r_{ij} \) denotes the \( j \) th possible value of the \( i \) th asset return and \( p_{ij} \) stands for the probability of its realization for \( i = 1, \ldots, N \) and \( j = 1, \ldots, M \). If the probabilities of the outcomes are equally likely then the expected return of asset \( i \) can be expressed as the simple average as shown below:

\[
E(R_i) = \frac{\sum_{j=1}^{M} r_{ij}}{M}
\]
In portfolio analysis, a variance is of paramount interest because it shows how outcomes deviate from the average outcomes representing the portfolio risk. Assuming that the probabilities of the outcomes are equally likely, then the variance of the return on the \(i\) th asset is expressed as:

\[
\sigma_i^2 = \sum_{j=1}^{M} \frac{(r_{ij} - E(R_i))^2}{M}
\]  

(2.2.1.3)

However, in the event that the probabilities of the observations are different, then the variance of the return on the \(i\) th asset is expressed as:

\[
\sigma_i^2 = \sum_{j=1}^{M} p_{ij}(r_{ij} - E(R_i))^2
\]  

(2.2.1.4)

The standard deviation which is the square root of the variance and denoted by \(\sigma_i\) is expressed as a percentage and explains the average deviation of the observations from its expectation.

Generally, if an investor diversifies the portfolio, and \(w_i\) is the fraction or weight of the wealth invested in the \(i\) th assets, then the expected return of a portfolio is:

\[
E(R_p) = \mu_p = \sum_{i=1}^{N} w_i E(R_i)
\]  

(2.2.1.5)

For a two or more asset portfolio, the variance of a portfolio \(P\), represented by \(\sigma_p^2\) is not influenced by only the weight of the wealth \(w_i\) invested in the respective asset and the variance of each \(i\) th asset \(\sigma_i^2\) but it is also influenced by the covariance of the assets in the portfolio. This is demonstrated mathematically using a two-asset case where \(E(R_i)\) is the expected value of asset \(i\) and \(i = 1, 2\). The variance is given by:

\[
\sigma_p^2 = E(R_p - E(R_p))^2 = E \left[ w_1 R_1 + w_2 R_2 - (w_1 E(R_1) + w_2 E(R_2)) \right]^2
\]

\[
= E \left[ w_1 (R_1 - E(R_1)) + w_2 (R_2 - E(R_2)) \right]^2
\]

\[
= w_1^2 E \left[ (R_1 - E(R_1))^2 \right] + w_2^2 E \left[ (R_2 - E(R_2))^2 \right] + 2w_1w_2 E \left[ (R_1 - E(R_1))(R_2 - E(R_2)) \right]
\]

\[
= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \sigma_{12}
\]  

(2.2.1.6)

From the above equation, \(E \left[ (R_1 - E(R_1))(R_2 - E(R_2)) \right]\) is the covariance between assets 1 and 2 and it is denoted by \(\sigma_{12}\). Thus the variance of a portfolio of 2 assets is given by:

\[
w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \sigma_{12}
\]  

(2.2.1.7)
The covariance, $\sigma_{12}$ can be scaled to give us the correlation coefficient $\rho_{12}$ which lies between -1 and 1 and is given by the relationship:

$$ \rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \quad (2.2.1.8) $$

The case of a two-asset portfolio can be extended into the $N$ asset case by putting the variances and the covariances together as shown below:

$$ \sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} $$
$$ = \sum_{i=1}^{N} (w_i^2 \sigma_i^2) + \sum_{i=1}^{N} \sum_{j=1 \atop j \neq i}^{N} w_i w_j \sigma_{ij} \quad (2.2.1.9) $$

From equation (2.2.1.9) it is realized that the first part depends on the individual variances whilst the second part depends on the covariances.

Following the seminal paper of Harry Markowitz in 1952 (see, [19]), an optimal portfolio is constructed by maximizing the expected portfolio return for a given risk or by minimizing the risk for a given level of the expected return thus the mean and variance can be presented in a matrix form. We consider a portfolio with weights $w_i$. These weights can be put into a vector

$$ \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_N \end{pmatrix} $$

All portfolios are analyzed based on $\vec{w}$ and with the model constraints that:

- The investor has invested all his wealth thus the weights must sum up to one.
  $$ \sum_{i=1}^{N} w_i = 1 $$
- The investor cannot borrow an asset and sell it on the financial market (short selling is not allowed). So there are no negative weights.
  $$ 0 \leq w_i \leq 1 \text{ for } i = 1, \ldots, N $$

The expected returns of the assets, $\mu_i = E(R_i)$, are also collected into the mean vector

$$ \vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \cdots \\ \mu_N \end{pmatrix} $$

Thus comparing this to equation (2.2.1.5), the expected return of a portfolio $E(R_p)$ can be expressed as:

$$ E(R_p) = \mu_p = \sum_{j=1}^{N} w_i E(R_i) = \vec{\mu}^T \vec{w} = \vec{w}^T \vec{\mu} \quad (2.2.1.10) $$
The variances $\sigma_i^2 = \sigma_{ii} = \text{Cov}(R_i, R_i) = \text{Var}(R_i)$ and the covariance $\sigma_{ij} = \text{Cov}(R_i, R_j)$ are put into a matrix

$$\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \ldots & \sigma_{1N} \\
\sigma_{21} & \sigma_{22} & \ldots & \sigma_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{N1} & \sigma_{N2} & \ldots & \sigma_{NN}
\end{pmatrix}$$

The matrix $\Sigma$ is the so-called covariance matrix.

So comparing this to equation (2.2.1.9), the variance of a portfolio ($\sigma_p^2$) is given by:

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = \vec{w}^T \Sigma \vec{w} \quad (2.2.1.11)$$

Diversification reduces the portfolio variance or risk to a certain level when the selected weights satisfy the two constraints under the Markowitz model. This is done by an investor spreading his wealth over an increasing number of assets. The part of the equation (2.2.1.9) which depends on individual variances reduces when this is done, and that translates into a decrease in the total variance of the portfolio. However, as an investor adds more assets the impact of diversification reduces. The portfolio is said to be fully diversified when the first part of equation (2.2.1.9) approaches zero. The second part of equation (2.2.1.9) which is dependant on the covariance can however not be eliminated by diversification because it contains the systematic or market risk [9].

This can be explained mathematically by looking at a portfolio of $N$ assets. If equal weight of wealth is invested in each asset then the weight invested is $1/N$. Substituting this into the equation (2.2.1.9), the variance $\sigma_p^2$ will be:

$$\sigma_p^2 = \sum_{i=1}^{N} \left( \frac{1}{N} \right)^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{1}{N} \frac{1}{N} \sigma_{ij} \right) \quad (2.2.1.12)$$

$$\sigma_p^2 = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{N} \right) + \frac{N-1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{\sigma_{ij}}{N(N-1)} \right) \quad (2.2.1.13)$$

Let

$$\overline{\text{Var}} = \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{N} \right) \quad \text{and} \quad \overline{\text{Cov}} = \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{\sigma_{ij}}{N(N-1)} \right) \quad (2.2.1.14)$$

be the average variance and the average covariance of the asset returns, respectively. Then

$$\sigma_p^2 = \frac{1}{N} \overline{\text{Var}} + \frac{N-1}{N} \overline{\text{Cov}}. \quad (2.2.1.15)$$
From equation (2.2.1.15), it is realized that as \( N \to \infty \) the first term goes to zero and the second term goes to the average covariance of the assets. Thus variance of the portfolio, \( \sigma_p^2 \) can be expressed as:

\[
\sigma_p^2 \approx \overline{Cov}
\]

This explains why the covariance that makes up the systematic risk of the portfolio cannot be eliminated by diversification.

### 2.2.2 Two asset portfolio as a function of expected return

Given a two asset portfolio, in order to maximize the expected portfolio return, one must invest only in the asset with the biggest expected return \( \mu_i \). Namely, if \( \mu_1 < \mu_2 \) and short selling is not allowed, then the expected portfolio return

\[
\mu_p = w_1 \mu_1 + w_2 \mu_2
\]

is maximized for \( w_1 = 0 \) and \( w_2 = 1 \). When short selling is allowed, then an investor will even sell the other asset to buy more of the asset with the biggest expected return.

For example, if \( \mu_1 = 0.25 \) and \( \mu_2 = 0.5 \), then a plot of \( \mu_p \) will be as as shown by Figure 2.2:

![Figure 2.2: A plot of \( \mu_p = \mu(w) = 0.25(1 - w) + 0.5w \)]

### 2.2.3 Computation of minimal variance

The variance function denoted by \( \sigma^2(w) \) is parabolic for \( \sigma_1, \sigma_2 > 0, \rho \in [-1, 1] \).

In order to determine the portfolio with the smallest variance we calculate the derivative of \( \sigma^2(w) \). Following equation (2.2.1.9)

\[
\sigma^2(w) = (1 - w)^2 \sigma_1^2 + w^2 \sigma_2^2 + 2(1 - w)w \rho_{12} \sigma_1 \sigma_2
\]

\[
\frac{\partial \sigma^2(w)}{\partial w} = -2(1 - w)\sigma_1^2 + 2w\sigma_2^2 + 2(1 - 2w) \rho_{12} \sigma_1 \sigma_2
\]

\[
= 2w(\sigma_1^2 + \sigma_2^2 - 2 \rho_{12} \sigma_1 \sigma_2) - 2\sigma_1^2 + 2\rho_{12} \sigma_1 \sigma_2
\]
From $\frac{\partial \sigma^2(w)}{\partial w} = 0$ we obtain

$$w = \frac{\sigma_1^2 - \rho_{12} \sigma_2 \sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12} \sigma_1 \sigma_2},$$

which minimizes $\sigma^2(w)$ since $(\sigma_1^2 + \sigma_2^2 - 2\rho_{12} \sigma_1 \sigma_2) > 0$.

### 2.2.4 Minimal variance when there is no short selling

When short selling is allowed (non-negative weights) and $|\rho_{12}| < 1$, then the portfolio with the smallest variance will have a weight $w_0$ in the second asset and $(1 - w_0)$ in the first asset with

$$w_0 = \frac{\sigma_1^2 - \rho_{12} \sigma_2 \sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12} \sigma_1 \sigma_2}.$$

If short selling is not allowed, then the minimal variance is attained when the weight of the second asset is

$$w_{\text{min}} = \begin{cases} 
0 & \text{if } w_0 < 0, \\
0 & \text{if } 0 \leq w_0 \leq 1, \\
1 & \text{if } w_0 > 1.
\end{cases}$$

This result follows from the observation that $\sigma^2(w)$ has a single global minimum at $w_0$ and thus it is an increasing function on $[0, 1]$ when $w_0 < 0$ and it is decreasing on $[0, 1]$ for $w_0 > 1$.

### 2.2.5 Impact of correlations on portfolio selection

Most investment choices involve a trade-off between risk and return which can be considered as a reward for taking a risk. This can be demonstrated by a two-asset case with expected returns $\mu_1$ and $\mu_2$. If the wealth is fully invested in these two assets with weights, $w$ in the first asset and the weight, $1 - w$ in the second asset, then following equation (2.2.1.5) the expected return of the portfolio can be expressed as:

$$\mu_p = w \mu_1 + (1 - w) \mu_2,$$

and its variance is:

$$\sigma_p^2 = w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w) \rho_{12} \sigma_1 \sigma_2 \quad (2.2.5.1)$$

It is realized that the relationship and impact on the portfolio between the two assets changes as the correlation value, $\rho_{12}$ keeps changing. If $\rho_{12}$ is +1, then the two assets move in the same direction and as a result, purchasing the two assets(diversification) does not reduce the risk. However if all other factors are held constant, then there is a higher payoff for diversification as $\rho_{12}$ gets closer to −1 since the assets will move in an opposite direction. This is shown in the plot below [9]:
This implies that for an \( n \) asset case, one can have various combinations of the assets based on their correlations and this can be extended to give us an idea of the portfolio possibilities curve along which all the possible combination will fall based on their mean return and standard deviation.

### 2.3 Efficient portfolios

The efficient frontier concept was introduced by Markowitz in his 1952 paper (see, [18]). As we already observed the paper assumed that the investor had fully invested and short sales were not allowed.

In theory, we could plot all risky assets and combinations of them based on their expected returns and standard deviations. However, we know that for an investment, the higher the risk the higher the return. As a result, if an investor wants to increase the expected return of a portfolio, then he/she must be willing to accept more risk.

**Minimal Variance Portfolio**

If \( P \) is a set of attainable portfolios, then the portfolio that has the smallest variance in \( P \) has weights denoted by:

\[
\overline{w}_{\text{min}}^T = \frac{\bar{1}^T \Sigma^{-1} \bar{1}}{\bar{1}^T \Sigma^{-1} \bar{1}},
\]

where the symbol \( \bar{1} \) stands for the vector of ones of the appropriate size. Since \( \Sigma \) is a covariance matrix, it is positive definite which implies that the denominator is positive.

The variance of this portfolio can be expressed as:

\[
\sigma^2_{\text{min}} = \frac{1}{\bar{1}^T \Sigma^{-1} \bar{1}}.
\]
Derivation of minimal variance portfolio

The classical derivation of minimal variance portfolio is based on the method of Lagrange multipliers, named after Lagrange (1736-1813). It is a strategy to find minimum or maximum of a function subject to constraints. We want to minimize:

\[ f(\vec{w}) = \vec{w}^T \Sigma \vec{w} \quad \text{under the constraint} \quad g(\vec{w}) = \vec{1}^T \vec{w} = 1. \]

Using the Lagrange multipliers \( \lambda \), we want to find the minimum of

\[ F(\vec{w}, \lambda) = \vec{w}^T \Sigma \vec{w} - \lambda (\vec{1}^T \vec{w} - 1) \]

We compute

\[
\frac{d}{dw_i} F(\vec{w}, \lambda) = \frac{d}{dw_i} \left( \sum_{i,j} w_i w_j \sigma_{i,j} - \lambda \sum_i w_i \right) \\
= 2 \sum_j w_j \sigma_{i,j} - \lambda = (2 \vec{w}^T \Sigma - \lambda \vec{1})_i \\
\frac{d}{d\lambda} F(\vec{w}, \lambda) = (\vec{1}^T \vec{w} - 1)
\]

where \((\vec{v})_i\) denotes the \(i\)th entry of the vector \(\vec{v}\). The first equations implies

\[ \vec{0}^T = 2\vec{w}^T \Sigma - \lambda \vec{1}^T \Rightarrow \frac{\lambda}{2} \vec{1}^T \Sigma^{-1} \vec{1} = \vec{w}^T \]

Further,

\[ 0 = (\vec{w}^T \vec{1} - 1) \Rightarrow 1 = \frac{\lambda}{2} \vec{1}^T \Sigma^{-1} \vec{1} \Rightarrow \lambda = 2 \frac{1}{\vec{1}^T \Sigma^{-1} \vec{1}} \]

As the solution of this optimization problem we obtain the analytical formula for the weights of the global minimal variance portfolio:

\[ \vec{w}_{\min}^T = \frac{\vec{1}^T \Sigma^{-1}}{\vec{1}^T \Sigma^{-1} \vec{1}}, \]

Portfolio variance, example I

If the covariance matrix is

\[ \Sigma = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix} \] with inverse \(\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & 0 \\ 0 & \frac{1}{\sigma_2^2} & 0 \\ 0 & 0 & \frac{1}{\sigma_3^2} \end{pmatrix}, \]
then for three uncorrelated assets we get

\[
\frac{1}{\sigma_{\text{min}}^2} = \vec{1}^T \Sigma^{-1} \vec{1}
\]

and

\[
\vec{1}^T \Sigma^{-1} = (1 \ 1 \ 1) \begin{pmatrix}
\frac{1}{\sigma_1^2} & 0 & 0 \\
0 & \frac{1}{\sigma_2^2} & 0 \\
0 & 0 & \frac{1}{\sigma_3^2}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sigma_1^2} & \frac{1}{\sigma_2^2} & \frac{1}{\sigma_3^2}
\end{pmatrix}
\]

So it implies that,

\[
\frac{1}{\sigma_{\text{min}}^2} = \vec{1}^T \Sigma^{-1} \vec{1} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}
\] (2.3.0.1)

From equation (2.3.0.1)

\[
\sigma_{\text{min}}^2 = \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}}
\]

The portfolio, with smallest variance has weights:

\[
\vec{w}_{\text{min}}^T = \sigma_{\text{min}}^2 \vec{1}^T \Sigma^{-1}
\] (2.3.0.2)

So substituting $\vec{1}^T \Sigma^{-1}$ into equation (2.3.0.2)

\[
\vec{w}_{\text{min}}^T = \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}} \begin{pmatrix}
\frac{1}{\sigma_1^2} & \frac{1}{\sigma_2^2} & \frac{1}{\sigma_3^2}
\end{pmatrix}
\]

Portfolio variance, example II

If we have a covariance matrix

\[
\Sigma = \begin{pmatrix}
0.4 & 0.3 & 0.3 \\
0.3 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.4
\end{pmatrix}
\]

and the inverse $\Sigma^{-1} = \begin{pmatrix}
7 & -3 & -3 \\
-3 & 7 & -3 \\
-3 & -3 & 7
\end{pmatrix}$,

then

\[
\frac{1}{\sigma_{\text{min}}^2} = (1 \ 1 \ 1) \begin{pmatrix}
7 & -3 & -3 \\
-3 & 7 & -3 \\
-3 & -3 & 7
\end{pmatrix} \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} = 3
\]
and

\[ \bar{1}T \Sigma^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 7 & -3 \\ -3 & -3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \]

Therefore, the smallest variance portfolio has weights:

\[ \bar{w}_{\min}^T = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \]

So the weights of the smallest variance is:

\[ \bar{w}_{\min}^T = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \]

**Minimal variance line**

Following [27] with a little alteration to the parameters, if

\[ c_{1n} = \bar{1}T \Sigma^{-1} \bar{1} = \bar{\mu}^T \Sigma^{-1} \bar{1} \quad c_{nn} = \bar{\mu}^T \Sigma^{-1} \bar{\mu} \quad c_{11} = \bar{1}T \Sigma^{-1} \bar{1} . \]

Then the portfolio with the smallest variance among attainable portfolios with expected return \( \bar{\mu}_R \) has weights

\[ \bar{w}^T = \frac{c_{nn} - \mu_R c_{1n} \bar{1}T \Sigma^{-1} \bar{1}}{c_{11} c_{nn} - c_{1n}^2} + \frac{\mu_V c_{11} - c_{1n} \bar{\mu}^T \Sigma^{-1} \bar{\mu}}{c_{11} c_{nn} - c_{1n}^2} \]

**Derivation of the minimum variance line**

The Lagrange multipliers are used once again and minimize \( \bar{w}^T \Sigma \bar{w} \), with the constraints \( \bar{w}^T \bar{1} = 1 \) and \( \bar{w}^T \bar{\mu} = \mu_R \). In other words, we want to minimize

\[ G(\bar{w}, \lambda, \vartheta) = \bar{w}^T \Sigma \bar{w} - \lambda (\bar{w}^T \bar{1} - 1) - \vartheta (\bar{w}^T \bar{\mu} - \mu_V) \]

Just as in the earlier computations: From \( \frac{d}{d w_i} = 0 \) we obtain

\[ \bar{w}^T = \frac{\lambda}{2} \bar{1}^T \Sigma^{-1} + \frac{\vartheta}{2} \bar{\mu}^T \Sigma^{-1} \]

The additional conditions are that \( \frac{d}{d \lambda} = 0 \) and \( \frac{d}{d \vartheta} = 0 \) so we obtain

\[ 1 = \bar{w}^T \bar{1} = \frac{\lambda}{2} \bar{1}^T \Sigma^{-1} \bar{1} + \frac{\vartheta}{2} \bar{\mu}^T \Sigma^{-1} \bar{\mu}, \]

\[ \mu_R = \bar{w}^T \bar{\mu} = \frac{\lambda}{2} \bar{1}^T \Sigma^{-1} \bar{\mu} + \frac{\vartheta}{2} \bar{\mu}^T \Sigma^{-1} \bar{\mu} . \]

Solving for \( \lambda \) and \( \vartheta \) gives the stated result.
Example III

Let

\[
\begin{align*}
\vec{\mu}^T &= (0.30 \ 0.21 \ 0.24) \\
\sigma_1 &= 0.23 \quad \sigma_2 = 0.26 \quad \sigma_3 = 0.21 \\
\rho_{12} &= 0.4 \quad \rho_{13} = 0.2 \quad \rho_{23} = 0.3
\end{align*}
\]

To compute the minimal variance portfolio and the associated returns, we create a covariance matrix from the data as shown below:

\[
\Sigma = \begin{pmatrix}
\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\
\rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\
\rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2
\end{pmatrix} = \begin{pmatrix}
0.0529 & 0.02392 & 0.00966 \\
0.02392 & 0.0676 & 0.01638 \\
0.00966 & 0.01638 & 0.09
\end{pmatrix}
\]

So for the three assets in the covariance matrix, the minimal variance is

\[
\frac{1}{\sigma_{min}^2} = \vec{1}^T\Sigma^{-1}\vec{1}
\]

Therefore by computation,

\[
\frac{1}{\sigma_{min}^2} = \vec{1}^T\Sigma^{-1}\vec{1} = 22.1861
\]

But the portfolio, with smallest variance has weights:

\[
\vec{w}_{min}^T = \sigma_{min}^2 \vec{1}^T\Sigma^{-1} = (0.0530 \ 0.5524 \ 0.3946)
\]

Therefore,

\[
\vec{w}_{min}^T\vec{\mu} = (0.0530 \ 0.5524 \ 0.3946) \begin{pmatrix} 0.30 \\ 0.21 \\ 0.24 \end{pmatrix} = 0.2266
\]

Portfolio to a given return

Following [27], to compute the portfolio with \(\mu_R = 0.2\) and the smallest variance, we first calculate

\[
\begin{align*}
c_{1n} &= \vec{1}^T\Sigma^{-1}\vec{\mu} = 6.7586 \\
c_{nn} &= \vec{\mu}^T\Sigma^{-1}\vec{\mu} = 1.2087 \\
c_{11} &= \vec{1}^T\Sigma^{-1}\vec{1} = 38.9695
\end{align*}
\]

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Then, we compute

\[

c_{11}c_{nn} - c_{1n}^2 = 1.4224 \\
c_{nn} - \mu Rc_{1n} = -0.1431 \\
\mu Rc_{11} - c_{1n} = 1.0353
\]

\[
\tilde{w}^T = \frac{(c_{nn} - \mu Rc_{1n})\tilde{1}^T\Sigma^{-1} + (\mu Rc_{11} - c_{1n})\tilde{\mu}^T\Sigma^{-1}}{c_{11}c_{nn} - c_{1n}^2}
\]

\[
= \begin{pmatrix} 0.7445 & -0.2555 & 0.511 \end{pmatrix}
\]

\[
\tilde{w}^T\tilde{\mu} = 0.2
\]

**Variance of the minimum variance portfolio with return, \(\mu_p\)**

\[
\sigma_p^2 = \tilde{w}^T\Sigma\tilde{w}
\]

\[
= \left( \frac{c_{nn} - \mu p c_{1n}}{c_{11}c_{nn} - c_{1n}^2} \tilde{1}^T\Sigma^{-1} + \frac{\mu p c_{11} - c_{1n}}{c_{11}c_{nn} - c_{1n}^2} \tilde{\mu}^T\Sigma^{-1} \right)\Sigma\tilde{w}
\]

\[
= \left( \frac{c_{nn} - \mu p c_{1n}}{c_{11}c_{nn} - c_{1n}^2} \tilde{w}^T + \frac{\mu p c_{11} - c_{1n}}{c_{11}c_{nn} - c_{1n}^2} \tilde{\mu}^T\tilde{w} \right)
\]

\[
= \frac{1}{c_{11}} + \frac{(\mu_p - c_{1n}/c_{11})^2}{c_{nn} - c_{1n}^2/c_{11}}
\]

The **Efficient frontier** in the mean-standard deviation space is the upper part of the hyperbola:

\[
(\mu - \mu_{min})^2 = s(\sigma_p^2 - \sigma_{min}^2),
\]

where \(\mu_{min} = \frac{c_{1m}}{c_{11}}\) and \(\sigma_{min}^2 = \frac{1}{c_{11}}\) represent the expected return and the variance of the global minimum variance (GMV) portfolio, and \(s = \frac{c_{mm} - c_{1n}^2}{c_{11}}\) is the slope parameter of the efficient frontier.

### 2.3.1 Maximum Sharpe ratio portfolio

Investors invest in risky assets with an expectation of higher returns. Risk-adjusted returns can be computed with the Sharpe ratio. The Sharpe ratio is defined as the expected excess return per unit risk. Thus given \(E(R_i)\) to be expected return on \(i\) th asset, \(R_f\) as the risk-free rate and \(\sigma_i\) to be the standard deviation of the \(i\) th asset, then Sharpe ratio can be computed as:

\[
\frac{E(R_i) - R_f}{\sigma_i}
\]

The excess return obtained by investing in a risky asset \((E(R_i) - R_f)\) is known as the risk premium. If the Sharpe ratio of an asset being analyzed is negative then an investor is better-off investing at the risk-free rate. Therefore for a portfolio, the **Sharpe ratio** can be expressed
as:
\[
\frac{\mu_p}{\sigma_p} = \frac{\bar{w}^T \bar{\mu}}{\sqrt{\bar{w}^T \Sigma \bar{w}}}
\]  
(2.3.1.1)
when \( R_f = 0 \).

The risk-adjusted performance of a portfolio is positively correlated with the value of the Sharpe ratio. Using the techniques demonstrated earlier, we get the weights of the maximum Sharpe ratio portfolio as:

\[
\bar{w}_{max}^T = \frac{\bar{\mu}^T \Sigma^{-1}}{\mu^T \Sigma^{-1} \bar{1}}.
\]

It is also known as the **maximum drift portfolio**.

**Maximum Sharpe ratio portfolio for independent assets**

Let us assume that we have three independent assets. How does \( \bar{w}_{max} \) look like? If the assets are independent, then \( \Sigma \) are of the form [27]:

\[
\Sigma = \begin{pmatrix}
\sigma_1^2 & 0 & 0 \\
0 & \sigma_2^2 & 0 \\
0 & 0 & \sigma_3^2
\end{pmatrix}
\]

with inverse \( \Sigma^{-1} = \begin{pmatrix}
\frac{1}{\sigma_1^2} & 0 & 0 \\
0 & \frac{1}{\sigma_2^2} & 0 \\
0 & 0 & \frac{1}{\sigma_3^2}
\end{pmatrix} \).

So that

\[
\bar{w}_{max}^T = \frac{1}{\mu^T \Sigma^{-1} \bar{1}} \begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3
\end{pmatrix}.
\]

At this example we also see that we cannot consider a bond, as in this formula all \( \sigma \) should be non-zero.

**Characterisation of efficient portfolios**

We can conclude that all efficient portfolios in the efficient frontier \( \bar{w} \neq \bar{w}_{min} \) satisfy the equation below in (2.3.1.2) for some real numbers \( \mu', \gamma \)

\[
\gamma \bar{w}^T = \bar{\mu}^T \Sigma^{-1} - \mu'^T \Sigma^{-1}
\]  
(2.3.1.2)

Thus, all efficient portfolios are of the form

\[
\bar{w}^T = a\bar{w}_{max} + (1 - a)\bar{w}_{min},
\]

where \( \bar{w}_{min} \) is the minimal variance portfolio and \( \bar{w}_{max} \) is the maximum Sharpe ratio portfolio.
Explanation

Take any efficient portfolio \( \mathbf{w} \neq \mathbf{w}_{\text{min}} \) and draw the tangent line to the efficient frontier that passes through this portfolio. Then the slope of this tangent equals \( \frac{\mathbf{w}^T (\mathbf{\mu} - \mathbf{\mu'})}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}} \). It is the maximal slope that still hit the efficient frontier (it was a tangent). So this is the maximum of the function

\[
F(\mathbf{w}, \lambda) = \frac{\mathbf{w}^T (\mathbf{\mu} - \mathbf{\mu'})}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}} - \lambda (\mathbf{1}^T \mathbf{w} - 1)
\]

Derivate with respect to \( \mathbf{w} \)

\[
0 = \frac{\mathbf{\mu}^T}{\sigma_p} - \frac{\mathbf{w}^T (\mathbf{\mu} - \mathbf{\mu'})}{\sigma_p^3} \mathbf{w}^T \Sigma - \lambda \mathbf{1}^T
\]

Multiplying by \( \mathbf{w} \) gives

\[
0 = \frac{\mu_p}{\sigma_p} - \frac{\mu_p - \mu'}{\sigma_p} - \lambda
\]

As a solution of this equation we get \( \lambda = \frac{\mu'}{\sigma_p} \).

Tangent portfolio

When short sales are allowed, then the weights can take negative values but they must still sum up to one. Thus the constraint that \( \sum_{i=1}^{N} w_i = 1 \) is still applicable.

However, by introducing a riskless asset \( R_f \), and an investor having an unlimited lending and borrowing option at a risk-free rate, the constraint can be removed. This is because one can borrow at the risk-free rate and invest in asset \( i \). Thus if one invest \( w \) fraction of his initial funds in asset \( i \), it is possible that \( w > 1 \). Assuming the investor puts \( w \) fraction of funds in asset \( i \) he/she will put \( (1 - \mathbf{1}^T \mathbf{w}) \) fraction of funds in the riskless asset. The combined expected return can, therefore, be expressed as [9]

\[
E(R_c) = (1 - w)R_f + wE(R_i)
\]  

(2.3.1.3)

The combined risk is

\[
\sigma_c = \left[ (1 - w)^2 \sigma_f^2 + w^2 \sigma_i^2 + 2w(1 - w) \sigma_i \sigma_f \rho_{fi} \right]^{1/2}
\]

And since there is no risk for risk-free, \( \sigma_f = 0 \) and \( \sigma_c = w \sigma_i \)

\[
w = \frac{\sigma_c}{\sigma_i}
\]  

(2.3.1.4)

Substituting equation (2.3.1.4) into equation (2.3.1.3) and rearranging the terms:

\[
E(R_c) = R_f + \left( \frac{E(R_i) - R_f}{\sigma_i} \right) \sigma_c
\]  

(2.3.1.5)

22
The expression in the big bracket is the Sharpe ratio and represents the slope of the equation. The Sharpe ratio of an efficient portfolio is termed as the market portfolio.

A capital market line (CML) of an efficient frontier is a graph of the expected return of a portfolio consisting of all possible proportions between the market portfolio and a risk-free asset.

Drawing from the risk-free rate in the mean-variance space, the tangency portfolio is tangent to the efficient frontier and has the maximum Sharpe ratio.

**Attainable portfolios in the investor universe**

If all possible portfolios in the investor universe are plotted it will be realized that for the same value of risk (variance/standard deviation), there will always be portfolios that will give a higher return than others until we reach some point. The portfolio which will correspond to this point will be dominating other portfolios with the same fixed level of risk. This portfolio will be one of the point of the efficient frontier. All portfolios that dominate other portfolios for the given level of risk will determine the efficient frontier [12]. The efficient frontier can, therefore, be considered as a set of all efficient portfolios.

The combination of assets that gives the lowest variance or risk of all efficient portfolios is known as the minimum variance portfolio (MVP) and it lies on the extreme left boundary. The figure below shows the efficient frontier, which is the upper boundary of the plot:

![Figure 2.4: Efficient frontier for the investor universe](image)

The yellow star is the global minimum variance portfolio whilst the red star is the maximum Sharpe ratio.
Chapter 3

Sustainability

This section presents the mathematical and financial concept of sustainability and how it can be incorporated into the classical portfolio management and optimization.

3.1 Sustainable investment

Sustainable investment is basically an investment of funds with a sustainable perspective. This type of investment concentrates on returns and sustainability with much focus on climate change, risk and ESG factors.

Generally, it is believed that on average, sustainable companies outperform non-sustainable companies [1], however, there is a likelihood that at least one non-sustainable company could outperform a sustainable company. Thus sustainable investment is not based entirely on returns but also on the ethical point of view.

There are many ways for investors to include sustainability into their portfolio optimization. The simplest way, however, is to invest in only sustainable assets. Currently, many investment firms decide as per company policy to invest in stocks that are sustainable. For instance, a company may by policy take a decision to invest in only fossil free assets or get rid of investment funds that are unethical (Divestment).

Following S.Herzel et al (see, [13]) screening is the most straight forward way to introduce socially responsible (ESG) constraints into a portfolio optimization with focus on sustainability. If $L$ is the minimum acceptable level of return, then from equation (2.2.1.13) we get

$$E(R_p) = \bar{w}^T \mu \geq L$$  \hspace{1cm} (3.1.0.1)

Let $\tau(L)$ denote the ratio between the Sharpe ratios of the optimal screened portfolio with the optimal unscreened portfolio for a particular level of expected return $L$. Then the sustainability price for the expected level of return can be given by:

$$p(L) = 1 - \tau(L)$$  \hspace{1cm} (3.1.0.2)

where $\tau(L) \leq 1$. 

24
Derivation

Let $\Omega$ denote the set of all portfolios which satisfy 3.1.0.1 and let $\Omega_s$ be a subset of $\Omega$ which is restricted to the portfolios that include the stocks of sustainable companies only. Then, from the definition of $\tau(L)$ we get

$$
\tau(L) = \frac{\text{Sharpe ratio of the optimal screened portfolio}}{\text{Sharpe ratio of the optimal unscreened portfolio}}
$$

\[
\max_{\bar{w} \in \Omega_s} \frac{\bar{w}^T \bar{\mu}}{\sqrt{\bar{w}^T \Sigma \bar{w}}} \leq 1,
\]

since $\Omega_s \subseteq \Omega$.

3.2 Sustainability rating

As global warming continues to become endemic, international bodies continue to adopt policies to enhance sustainability. Paramount among some of these actions are the United Nations 2030 agenda for sustainable development which the EU is fully committed to as the front-runner.

The growing interest in sustainable investment has resulted in the need for international committees to pass standards for sustainability reporting. The index called AccountAbility’s AA1000 series standard is used by many organizations to demonstrate their sustainability performance [3]. The agencies responsible for sustainability ratings of companies develop a rank for the companies based on their sustainability report together with other information that will be available to them. The agencies score a number of factors and aggregate them into a score that is coherent in a particular investment.

3.2.1 Corporate sustainability indexes

There are several indexes for comparing the commitment of companies to sustainability. The first sustainability index was launched by Dow Jones in collaboration with RobecoSAM in 1999 by the New York Stock exchange [6].

But with the increasing interest in sustainability, there are now many other sustainability indexes [6]. An investor who focuses on sustainable investment will decide on the sustainability index of his/her choice.

Then based on the sustainability index, one can give a constraint on the total rank of stocks in the optimal portfolio. Thus the stocks that will make up the optimal portfolio must not go beyond the preferred rating based on the sustainability appetite of the investor.
3.3 Modeling sustainability value and return

Following the G. Dorfleitner, S. Utz (see, [7]), one can determine the sustainability return of every single investment return with respect to a set $f$ of factors based on an existing sustainability rating. This can then be incorporated based on the sustainability target of the investor.

The targeted sustainability return can be expressed as $TSR_i^{[s,t]}(F, \omega)$, where $F$ is the factor, $\omega$ is the state and $[s,t]$ is the investment period for the $i$th investment which is given by a sustainability rating. The targeted sustainability returns are random variables. The targeted sustainability value can be obtained from the targeted sustainability return if we know the initial wealth $V_i^s$ invested in the $i$th investment.

The targeted sustainability value denoted by $TSV_i^{[s,t]}$ can be expressed as:

$$TSV_i^{[s,t]}(F, \omega, V_i^s) = V_i^s \times TSR_i^{[s,t]}(F, \omega)$$

(3.3.0.1)

The preference of the investor can however be shaped by making $\delta \in \mathbb{R}$ be a real number and $F \in f$ be a factor of a sustainable interest. Let $\tau$ be the ratio of the sharpe ratio as defined in section 3.1, then the strength of a sustainable impact on the investor can then be denoted by $\delta(F, \tau)$. Then for investor $\tau$ underlisted holds:

1. $\delta(F, \tau) > 0$: Factor $F$ has a positive impact on investment decision.
2. $\delta(F, \tau) = 0$: The investor is indifferent with respect to factor $F$.
3. $\delta(F, \tau) < 0$: The investor rejects the interpretation of target sustainability of factor $F$.

3.3.1 Sustainability value

Following the notations for the target sustainability value, if $\Pi = \{\text{investor preferences}\}$ then the sustainability value $SV_i^{[s,t]} : \Omega \times \Pi \times \mathbb{R} \to \mathbb{R}$ of a $i$th investment is a real random number with sample space $\Omega$ representing the non-monetary value an investor $\tau$ receives at maturity $t$. This can therefore be expressed as:

$$SV_i^{[s,t]}(\omega, \tau, V_i^s) = \sum_{F \in f} \delta(F, \tau)TSV_i^{[s,t]}(F, \omega, V_i^s)$$

(3.3.1.1)

Thus it depends on the state $\omega$, the initial wealth $V^s$ and the preferred $\tau$.

3.3.2 Sustainability return

Following a similar argument for sustainability value a sustainability return $SR_i^{[s,t]} : \Omega \times \Pi \to \mathbb{R}$ of a $i$th investment is a real random number with sample space $\Omega$, in period $[s,t]$ and an
investor $\tau$ in preference space $\Pi$. This can therefore be expressed as:

$$SR_i^{[s,t]}(\omega, \tau) = \frac{SV_i^{[s,t]}(\omega, \tau, V_i^s)}{V_i^s}$$

$$= \sum_{F \in f} \delta(F, \tau)TSV_i^{[s,t]}(F, \omega, V_i^s)$$

(3.3.2.1)

### 3.3.3 Sustainable portfolio return

By extension, if the weights of $N$ assets are $w_1, w_2, \ldots, w_N$ and their corresponding sustainability returns are $SR_1^{[s,t]}, SR_2^{[s,t]}, \ldots, SR_N^{[s,t]}$ then we can combine the classical portfolio returns with the concept of sustainability. So from equation (2.2.1.5) the sustainable portfolio return can be therefore be expressed as:

$$SR_p^{[s,t]} = \sum_{i=1}^{N} w_i SR_i^{[s,t]}$$

(3.3.3.1)

### 3.4 Sustainable investment in a long term

Investors are starting to consider how to make more money by finding situations where companies are smart about the environment and save cost thus make more money. With an increasing interest in sustainability, it is realized that there is starting to be a change in consumer preferences. Companies that adapt to these changes are likely to thrive whilst those that do not may fail.

In December 2018, Johnson & Johnson saw the worst two-day slide in their stock price in more than 16 years. Their shares dropped by 14% wiping out more than 50 billion in market value based on claims that its baby powder contained asbestos and their metal hip replacements were defective.

For instance, with the EU laws intended to reduce carbon dioxide (CO2) emissions, car manufacturers who invest a lot in technology to improve their emissions are likely to thrive in the long run. A typical example is Cummins and Navistar which compete in the heavy-duty truck engine market. When there was a new pollution regulation that required a brand new emission technology, Navistar felt their old engine platform would meet the regulation. Cummins on the other hand invested in new emission technology. Around 2010 Navistar had to pay a fine on all their engines that were noncompliant and by 2012 they had to abandon their entire engine platform and eventually started buying engines from Cummins. It then made loses until 2017 when it posted its first profit since 2011. This shows how costly sustainability can be to a company and the value of its stock. Below is a plot of the stocks of Navistar and Cummings over the period for emphasis:
Figure 3.1: Comparison of stocks of Navister and Cummins
Chapter 4

Implementation

This section contains the problem statement, the goals of the thesis, the data and data source that would be used for the analysis. I started off by computing and comparing the cumulative return of sustainable and unsustainable stocks. I then optimized sustainable stocks and analyzed the returns in relation to the benchmark and unsustainable stocks.

4.1 Problem statement

Sustainability for Portfolio Optimization has been a main consideration in the financial sector considering the increasing interest of investors towards a sustainable future. The main objective of the thesis is to access the impact of factoring sustainability into portfolio optimization from both the financial and ethical point of view.

Goals of chapter

1. Model the stock index which is our benchmark for the optimization based on all the stocks under consideration.

2. Compare it with the optimization after screening the stocks based on ESG ratings.

3. Establish a relationship between the maximization of returns and the minimization of volatility.

4. Examine if sustainability has a positive impact on portfolio optimization.

4.1.1 Data and data source

This section discusses the source of the data and how the risk-free rate was calculated.
4.1.2 Data source and description

The analysis will be based on Dow Jones industrial average or simply the Dow, which is a stock market index that consists of 30 of America’s largest companies from a wide range of industries. Table 4.1 shows the list of stocks that make up the Dow.

<table>
<thead>
<tr>
<th>Stocks</th>
<th>Ticker Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apple Inc</td>
<td>AAPL</td>
</tr>
<tr>
<td>American Express Company</td>
<td>AXP</td>
</tr>
<tr>
<td>The Boeing company</td>
<td>BA</td>
</tr>
<tr>
<td>Caterpillar Inc.</td>
<td>CAT</td>
</tr>
<tr>
<td>Cisco System</td>
<td>CSCO</td>
</tr>
<tr>
<td>Chevron Corp</td>
<td>CVX</td>
</tr>
<tr>
<td>The Walt Disney Company</td>
<td>DIS</td>
</tr>
<tr>
<td>DowDuPont Inc.</td>
<td>DWDP</td>
</tr>
<tr>
<td>The Goldman Sachs Group, Inc.</td>
<td>GS</td>
</tr>
<tr>
<td>The Home Depot, Inc.</td>
<td>HD</td>
</tr>
<tr>
<td>International Business Machines Corporation</td>
<td>IBM</td>
</tr>
<tr>
<td>Intel Corporation</td>
<td>INTC</td>
</tr>
<tr>
<td>Johnson &amp; Johnson.</td>
<td>JNJ</td>
</tr>
<tr>
<td>JPMorgan Chase &amp; Co.</td>
<td>JPM</td>
</tr>
<tr>
<td>The Coca-Cola Company</td>
<td>KO</td>
</tr>
<tr>
<td>McDonald’s Corporation</td>
<td>MCD</td>
</tr>
<tr>
<td>3M Company</td>
<td>MMM</td>
</tr>
<tr>
<td>Merck &amp; Co., Inc.</td>
<td>MRK</td>
</tr>
<tr>
<td>Microsoft Corporation.</td>
<td>MSFT</td>
</tr>
<tr>
<td>NIKE, Inc.</td>
<td>NKE</td>
</tr>
<tr>
<td>Pfizer Inc.</td>
<td>PFE</td>
</tr>
<tr>
<td>The Procter &amp; Gamble Company</td>
<td>PG</td>
</tr>
<tr>
<td>The Travelers Companies, Inc.</td>
<td>TRV</td>
</tr>
<tr>
<td>UnitedHealth Group Inc.</td>
<td>UNH</td>
</tr>
<tr>
<td>United Technologies Corporation</td>
<td>UTX</td>
</tr>
<tr>
<td>Verizon Communications Inc.</td>
<td>VZ</td>
</tr>
<tr>
<td>Visa Inc.</td>
<td>V</td>
</tr>
<tr>
<td>Walgreens Boots Alliance, Inc.</td>
<td>WBA</td>
</tr>
<tr>
<td>Walmart Inc.</td>
<td>WMT</td>
</tr>
<tr>
<td>Exxon Mobil Corporation</td>
<td>XOM</td>
</tr>
</tbody>
</table>

Table 4.1: Dow Jones Industrial Average stocks

The data under consideration is an 11-year time period from 31st December 2007 to 31st December 2018. The historical stock prices and the ESG ratings of the 30 stocks were downloaded from Yahoo finance via (https://finance.yahoo.com/). The ESG ratings are from 0 – 100 with the best performers receiving 100.
A 10-year treasury rate by month was downloaded from (http://www.multpl.com/10-year-treasury-rate/table/by-month) with an additional 1 year treasury. The risk-free rate of 2.65% was then calculated by averaging the treasury rate for the 11 year period under consideration.

In order to compare the performance of my optimization, I also download the Dow Jones Industrial Average data for the same period, to be applied in the comparison section. The Dow Jones Industrial Average which is the benchmark for my analysis is denoted by the symbol (^DJI) and is based on the 30 constituent stocks under consideration.

### 4.1.3 Price performance plot

The price performance plot is used to monitor the performance of a portfolio based on the price. In this section, I compare the average price of the constituent stocks of Dow and Jones with the price of the Dow Jones Industrial Average and the result is presented in Figure 4.1.

![Figure 4.1: Portfolio price performance of all the 30 stocks](image)

It is realized that the price performance of the portfolio of 30 stocks is very close to the performance of the Dow Jones Industrial Average due to the fact that the Dow Jones Industrial Average is based on the 30 constituent stocks.

This forms the premise for using the Dow Jones Industrial Average as the benchmark for my analysis.

### 4.1.4 Research method

The returns of the stocks can be estimated from the historical data by using the continuously compounded approach or the discrete approach. The thesis uses the discrete approach to find
the log returns of the stocks between two time periods, \( t \) and \( t - 1 \) is calculated using the formula from equation (2.1.1.11) and following [25]

\[
R_t = \ln \frac{P_t}{P_{t-1}}
\]

After calculating the return, the variance and covariance matrix can be calculated using the numpy package built-in function in python [20].

Denote \( \vec{w} \) as the weight vector of the portfolio, \( \sigma_p^2 \) as the variance of the portfolio. \( \Sigma \) as the variance and covariance matrix of the log-return, \( \vec{\mu} \) is the mean return vector of the individual stocks.

Then the optimization problem that maximizes the Sharpe ratio is given by [10]:

\[
\frac{\vec{w}^T \vec{\mu} - R_f}{\sqrt{\vec{w}^T \Sigma \vec{w}}} \leq 1 \quad \text{subject to}\quad w_i \leq 1, \quad \vec{w}^T \vec{1} = 1, \quad \vec{1}^T = [1, 1, \cdots 1]
\]

I assume a no-short salling scenario so the weights of the stocks are between 0 and 1.

To demonstrate the impact of sustainability, I optimized the portfolio value of my stocks based on the 30 stocks under consideration and compare it to the benchmark. Under this, I optimized for both the maximum Sharpe ratio and minimum volatility.

Then based on the stock screening concept for adding a sustainability constraint to portfolio optimization, I decided to generate a sustainable portfolio by including only stocks with an ESG rating greater or equal to 67 in my portfolio (Screening).

A similar screening technique was used based on the environmental, social and governance sustainability rating. The screened stocks were modelled to generate a cumulative return, Maximum Sharpe ratio and Minimum Volatility and the results were compared to ascertain the differences between these portfolios.

### 4.2 Results and analysis

In this section, I will first go through the sectorial analysis.

I will then present the log returns of the constituent stocks of the Dow. Afterward, I will demonstrate the effect of incorporating sustainability technique into portfolio optimization by simulating a portfolio value for all stocks under consideration. The result will then be compared with the portfolio value of sustainable stocks that are selected based on their ESG rating.

The results generated by each of the techniques are compared to the benchmark to ascertain their performance.

#### 4.2.1 Sectorial analysis

The Dow and Jones is made up of several industries in the USA economy. Below is a presentation of the various sectors and industries that make up the constituent stocks of the Dow.
<table>
<thead>
<tr>
<th>Ticker Symbols</th>
<th>Sector</th>
<th>Industry</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL</td>
<td>Information Technology</td>
<td>Technological Hardware Storage Peripherals</td>
</tr>
<tr>
<td>AXP</td>
<td>Financials</td>
<td>Consumer Finance</td>
</tr>
<tr>
<td>B A</td>
<td>Industrials</td>
<td>Aerospace Defense</td>
</tr>
<tr>
<td>CAT</td>
<td>Industrials</td>
<td>Machinery</td>
</tr>
<tr>
<td>CSCO</td>
<td>Information Technology</td>
<td>Communications Equipment</td>
</tr>
<tr>
<td>CVX</td>
<td>Energy</td>
<td>Oil Gas Consumable fuels</td>
</tr>
<tr>
<td>DIS</td>
<td>Communication Services</td>
<td>Entertainment</td>
</tr>
<tr>
<td>DWDP</td>
<td>Materials</td>
<td>Chemicals</td>
</tr>
<tr>
<td>GS</td>
<td>Financials</td>
<td>Capital Market</td>
</tr>
<tr>
<td>HD</td>
<td>Consumer Discretionary</td>
<td>Speciality Retail</td>
</tr>
<tr>
<td>IBM</td>
<td>Information Technology</td>
<td>IT Services</td>
</tr>
<tr>
<td>INTC</td>
<td>Information Technology</td>
<td>Semiconductors and Semiconductor Equipment</td>
</tr>
<tr>
<td>JNJ</td>
<td>Health Care</td>
<td>Pharmaceuticals</td>
</tr>
<tr>
<td>JPM</td>
<td>Financials</td>
<td>Banks</td>
</tr>
<tr>
<td>KO</td>
<td>Consumer Staples</td>
<td>Beverages</td>
</tr>
<tr>
<td>MCD</td>
<td>Consumer Discretionary</td>
<td>Hotels Restaurants Leisure</td>
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<td>MMM</td>
<td>Industrials</td>
<td>Industrial Conglomerates</td>
</tr>
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<td>MRK</td>
<td>Health Care</td>
<td>Pharmaceuticals</td>
</tr>
<tr>
<td>MSFT</td>
<td>Information Technology</td>
<td>Software</td>
</tr>
<tr>
<td>NKE</td>
<td>Consumer Discretionary</td>
<td>Textiles, Apparels, Luxury Goods</td>
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<tr>
<td>PFE</td>
<td>Health Care</td>
<td>Pharmaceuticals</td>
</tr>
<tr>
<td>PG</td>
<td>Consumer Staples</td>
<td>Household Products</td>
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<td>Health Care</td>
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<td>Diversified Telecommunication Services</td>
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<td>Food Staples Retailing</td>
</tr>
<tr>
<td>WMT</td>
<td>Consumer Staples</td>
<td>Food Staples Retailing</td>
</tr>
<tr>
<td>XOM</td>
<td>Energy</td>
<td>Oil Gas Consumable fuels</td>
</tr>
</tbody>
</table>

Table 4.2: Industrial and sectorial presentation of Dow and Jones constituents stocks

4.2.2 Mean log returns of the constituent stocks

The annual mean log returns of the constituent stocks are presented in Figure 4.2 and Table 4.3.
Table 4.3: Mean log return of constituent stocks

<table>
<thead>
<tr>
<th>Ticker</th>
<th>Mean Log Returns</th>
<th>Ticker</th>
<th>Mean Log Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL</td>
<td>0.236</td>
<td>MCD</td>
<td>0.143</td>
</tr>
<tr>
<td>AXP</td>
<td>0.093</td>
<td>MMM</td>
<td>0.107</td>
</tr>
<tr>
<td>B A</td>
<td>0.164</td>
<td>MRK</td>
<td>0.091</td>
</tr>
<tr>
<td>CAT</td>
<td>0.081</td>
<td>MSFT</td>
<td>0.142</td>
</tr>
<tr>
<td>CSCO</td>
<td>0.074</td>
<td>NKE</td>
<td>0.185</td>
</tr>
<tr>
<td>CVX</td>
<td>0.063</td>
<td>PFE</td>
<td>0.110</td>
</tr>
<tr>
<td>DIS</td>
<td>0.131</td>
<td>PG</td>
<td>0.059</td>
</tr>
<tr>
<td>DWDP</td>
<td>0.087</td>
<td>TRV</td>
<td>0.113</td>
</tr>
<tr>
<td>GS</td>
<td>0.013</td>
<td>UNH</td>
<td>0.194</td>
</tr>
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<td>0.198</td>
<td>UTX</td>
<td>0.064</td>
</tr>
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<td>IBM</td>
<td>0.025</td>
<td>V</td>
<td>0.228</td>
</tr>
<tr>
<td>INTC</td>
<td>0.106</td>
<td>VZ</td>
<td>0.101</td>
</tr>
<tr>
<td>JNJ</td>
<td>0.094</td>
<td>WBA</td>
<td>0.079</td>
</tr>
<tr>
<td>JPM</td>
<td>0.101</td>
<td>WMT</td>
<td>0.081</td>
</tr>
<tr>
<td>KO</td>
<td>0.086</td>
<td>XOM</td>
<td>0.009</td>
</tr>
</tbody>
</table>

4.2.3 Analysis of all 30 stocks

In order to calculate the maximum Sharpe ratio and minimum volatility for the 30 stocks I generated the optimal weights that will be allocated to each of the 30 stocks and the results are presented in Figure 4.3.
The plot on the left is the weights of the maximum Sharpe ratio portfolio whilst the one on the right represents the minimum volatility weights. It is realized that 6 stocks make up the maximum Sharpe ratios portfolio whereas 7 stocks make up the minimum volatility of the 30 constituent stocks. No weight (zero weight) is assigned to the remaining stocks. The maximum Sharpe ratio portfolio assigns a considerable weight to McDonald’s (MCD) followed by Apple (AAPL). The minimum volatility portfolio, however, assigns the highest weight to Johnson Johnson (JNJ) followed by McDonald’s. From the statistics based on the daily log returns presented below, the mean return for MCD, AAPL and JNJ are 0.00057, 0.00094, 0.00037 respectively and their corresponding standard deviations are 0.0116, 0.01923, 0.0107.

So even though the returns on AAPL is very high, it has a relatively high volatility. Perhaps that could be the reason the minimum volatility portfolio did not assign any weight to AAPL but assigned the highest weight to JNJ which has the a relatively low volatility.

Even though the return on MCD is lower than AAPL it is quite stable and high as compared to many others in the set. That could be a contributing factor for the allocation of highest weight to MCD followed by AAPL in the maximum Sharpe ratio portfolio.

### 4.2.4 Statistics for the maximum Sharpe ratio and minimum volatility stocks

This section presents the statistics of the daily log returns of assets in the maximum Sharpe ratio and minimum volatility portfolio.
Figure 4.4: Statistics of stocks that make up the maximum Sharpe ratio portfolio

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Figure 4.5: Statistics of stocks that make up the minimum volatility portfolio

37
From the statistics, one realizes that the maximum Sharpe ratio concentrates extensively on higher returns. This is because, in the Maximum Sharpe ratio scenario, the investor cares more about the return (see, [9]).

The stocks that make up the minimum variance portfolio, however, have a relatively lower standard deviation or volatility.

### 4.2.5 Efficient frontier

The efficient frontier of the maximum Sharpe portfolio return and minimum volatility based on the 30 constituent stocks is presented in Figure 4.6.

From the plot and based on the result in appendix (A.4) the maximum Sharpe ratio of the constituent stocks is 0.19733 and the minimum volatility is 0.090659. The plot shows the expected return in relation to the volatility. As we move from the blue towards the red, the volatility increases with the expected return.

### 4.2.6 Cumulative returns

The cumulative value based on all the 30 constituent stocks in the Dow is presented in Figure 4.7 (see, [24]).
From the plot, the cumulative return for the 30 stock portfolio is 2.64 and that of the benchmark is 1.45. The cumulative return, therefore, outperforms the benchmark.

### 4.3 Screening

Screening is used to incorporate a sustainability constraint by selecting only assets with a rating greater or equal to 67. The results for the selection based on ESG, environmental, social and governance rating is presented in Table 4.4.
4.4 Analysis of ESG stocks

The stocks with an ESG rating greater or equal to 67 were chosen and analyzed. Out of the 30 constituent stocks, only 10 met the sustainability constraint that was added to the classical portfolio optimization. As a result, these 10 stocks were analyzed and the results are presented below.

4.4.1 Asset correlation matrix

The asset correlation matrix for the ESG stocks is presented in Figure 4.8.

![Asset Correlation for ESG Rating Portfolio](image)

Figure 4.8: Asset correlation matrix of ESG stocks
4.4.2 ESG optimal weight

I once again calculated the optimal weight for the sustainable stocks based on the maximum Sharpe ratio and minimum volatility concept. The results are presented in Figure 4.9.

![Weights Max_Sharpe vs Min_Var Portfolio ESG Rating](image)

Figure 4.9: Optimal weight for ESG stocks

From the plot, 3 stocks were selected from the pool of sustainable stocks and weights were assigned to them for the maximum Sharpe ratio. The minimum variance portfolio, however, has 4 stocks. Based on the statistics it is realized that the stocks of the maximum Sharpe ratio have high returns whilst that of the minimum variance portfolio has lower volatility.

4.4.3 Efficient frontier

The efficient frontier of the maximum Sharpe ratio and minimum volatility based on the ESG stocks is presented in Figure 4.10.

![Efficient Frontier ESG](image)

Figure 4.10: Efficient frontier of ESG stocks
From the plot the maximum Sharpe portfolio return is 0.168268 whilst the minimum volatility portfolio return is 0.082318 (see, Appendix B.3). It is realized that both underperformed the portfolio of 30 constituent stocks.

### 4.4.4 Cumulative returns

The cumulative value generated from the 10 stocks that make up the ESG sustainable stocks based on their ESG rating is presented in Figure 4.11.

![Figure 4.11: Cumulative returns of ESG stocks](image)

From the plot, the cumulative return for the ESG stock portfolio is 2.45 and that of the benchmark is 1.45. Thus the cumulative return clearly outperforms the benchmark.

It is however realized that the cumulative return of the sustainable portfolio is lower than that of the unsustainable or unscreened portfolio.

### 4.5 Analysis of environmental stocks

In this section, I will select stocks based on their environmental ratings and the results are presented.

#### 4.5.1 Asset correlation matrix

The correlation matrix for the environmental stocks are presented in Figure 4.12.
It is realized that 16 stocks have an environmental rating greater or equal to 67. This indicates the commitment of more companies in ensuring that they engage in practices that are environmental friendly.

From the matrix, it is realized that the strongest correlation is between JPM and GS stocks. This could be due to the fact that both stocks are in the asset management industry.

### 4.5.2 Mean log returns of the environmental stocks

The mean log returns of the environmental friendly stocks are also presented in Figure 4.13.

From the plot, it is realized that Apple has the highest return and Visa is close in second. Goldman Sachs however has the least return in the set.
4.5.3 Environmental optimal weight

I then calculated the optimal weight for the environmental sustainable stocks based on the maximum Sharpe ratio and minimum volatility. The attained results are presented in figure 4.14.

![Weights Max_Sharpe vs Min_Var Portfolio ESG Rating](image)

Figure 4.14: Optimal weight for environmental stocks

As seen from the plot, 6 stocks make up the maximum Sharpe ratio. The highest weight allocation of 27.9% is to AAPL whilst the least weight of 2.44% is to JNJ. The minimum variance portfolio however has 5 stocks with JNJ being assigned the highest weight of 45.6%.

4.5.4 Efficient frontier

The efficient frontier of the maximum Sharpe ratio and minimum volatility based on the environmental sustainable stocks is presented in Figure 4.15.

![Efficient Frontier Environmental](image)

Figure 4.15: Efficient Frontier of Environmental stocks
From the plot, the maximum Sharpe portfolio return and the minimum volatility portfolio return are 0.2118418 and 0.088589 respectively (see, Appendix C.3). It is realized that both results outperformed the ESG stocks. The maximum Sharpe portfolio return also outperforms the portfolio of 30 stocks. The minimum volatility portfolio is slightly lower than the portfolio of 30 stocks, however, the expected returns are approximately the same.

4.5.5 Cumulative returns

The cumulative value generated from the 16 stocks that make up the environmental friendly stocks is presented in Figure 4.16.

![Cumulative Returns for the Environmental Rating Portfolio vs Benchmark](image)

Figure 4.16: Cumulative returns of environmental stocks

From the plot, the cumulative return for the environmental stock portfolio is 3.00. This is over 100% increment on the benchmark. The result indicates that the cumulative returns for the environmental sustainable stocks outperform both the benchmark and the ESG stocks.

4.6 Analysis of governance stocks

In this section, the stocks were selected based on their governance ratings. Thus stocks with a governance rating greater or equal to 67 were selected and analyzed and the results are presented.

4.6.1 Asset correlation matrix

The asset correlation matrix for governance friendly stocks are presented in Figure 4.17.
It is realized that 13 stocks satisfy the governance screening constraint. This indicates that, there is also a relatively good commitment of companies towards good governance.

From the matrix, it is realized that the strongest correlation is between MSFT and INTC stocks. They are both in the information technology industry so that could be the reason for the strong correlation.

### 4.6.2 Mean log returns of the governance stocks

The mean log returns of the governance stocks are also presented in Figure 4.18.

From the plot, it is realized that Visa has the highest return and the home depot is close in second. Cisco has the least return in the set.
4.6.3 Governance optimal weight

The optimal weight for the governance stocks based on the maximum Sharpe ratio and minimum volatility are computed and the results are presented in Figure 4.19.

![Weights Max_Sharpe and Minimum_Volatility for Governance Rating Portfolio](image)

Figure 4.19: Optimal weight for governance stocks

As seen from the plot, 4 stocks make up the maximum Sharpe ratio with Visa assigned the highest weight of 39% whilst the least weight of 8.6% is assigned to JNJ. The minimum variance portfolio however has seven stocks with JNJ being assigned the highest weight of 57.1%. It is interesting how the weight allocation of JNJ is so high. As discussed earlier it may be due to its relatively small standard deviation and a reasonable expected return.

4.6.4 Efficient frontier

The efficient frontier of the maximum Sharpe portfolio return and minimum volatility based on the sustainable governance portfolio is presented in Figure 4.20.
From the plot the maximum Sharpe portfolio return is 0.199601 whilst the minimum volatility portfolio return is 0.101051 (see, Appendix D.2). From the results, the return value based on the maximum Sharpe ratio out-performed both the ESG and constituent stock portfolio but was lower than the environmental sustainable stocks. The minimum volatility portfolio return however out-performed all the classifications under consideration.

4.6.5 Cumulative returns

The cumulative portfolio value generated from the stocks with a good governance rating is presented in Figure 4.21.

From the plot, the cumulative return for the governance portfolio is 3.21. This is also over 100% increment on the benchmark and the best return in all the models.
4.6.6 Comparison of results

This section shows the summary results for the cumulative return, maximum Sharpe and minimum volatility portfolio for all the models.

![Cumulative Returns for all Portfolios vs Benchmark](image)

Figure 4.22: Cumulative returns for all portfolios vs benchmark

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Table 4.5: Maximum Sharpe and minimum volatility portfolio results
Chapter 5

Conclusions

The report uses the theoretical framework of the Mean-variance portfolio optimization and introduces an additional constraint based on the sustainability rating to investigate the Dow and Jones constituent stocks. The Dow Jones Industrial average was used as the benchmark.

I started off by generating the maximum Sharpe portfolio and minimum volatility portfolio of the 30 constituent stocks of the Dow. Then, I generated its cumulative return and compared it with the benchmark. From the results, the portfolio outperformed the benchmark.

After that, I factored in a constraint based on a sustainability rating greater or equal to 67 and calculated the maximum Sharpe portfolio, minimum volatility portfolio and cumulative return of the new portfolio (ESG) of 10 stocks. The result was then compared with the benchmark and it also outperformed the benchmark. However, it was realized that it underperformed the unscreened portfolio. If that happens then the difference between the screened and unscreened portfolio is what is termed as the cost of sustainability.

I then decided to investigate further by screening the constituent stocks based on their environmental sustainability ratings. After the screening, the portfolio had 16 stocks meeting the constraint. The maximum Sharpe portfolio, minimum volatility portfolio, and cumulative return were then computed based on the 16 environmental sustainable stocks. The cumulative return and maximum Sharpe portfolio outperformed the benchmark, ESG portfolio and the portfolio of 30 stocks (unscreened). The minimum volatility portfolio, if rounded up to 2 decimal places will however have approximately the same expected return as the unscreened or unsustainable portfolio.

When the same concept was applied to stocks based on their governance sustainability rating, the results also outperformed the unscreened portfolio. In fact, the portfolio based on governance rating had the maximum return for the minimum volatility portfolio in the classified group. However, the maximum Sharpe return was a little lower than that of the environmental portfolio.

The results for social ratings presented in Appendix E and the comparison table shows that the performance of the portfolio based on social ratings was the lowest in the classification.

Thus one could infer that with an increasing interest in sustainability, the focus of shareholders is beginning to shift towards the consideration of non-financial criteria such as governance and environmental criteria in their investment decision making [26]. From the results, it was realized that the ESG stocks do not always outperform non-sustainable stocks. This motivates the assertion that if the motivation of investors to invest in good companies is because they yield higher returns, then those investors are not socially responsible but are only pursuing a management strategy.

However, the good news is that the stocks selected based on both environmental and governance ratings out-performed the unscreened portfolio. Thus it can be concluded that assets that implement

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environmentally friendly and have a sustainable governance structure strive over the long term. One of the main reasons why the returns on assets with a good environmental and governance rating assets outperformed the unscreened could be as a result of the shifting of consumer interest towards assets and goods that are environmentally friendly and have a sustainable governance structure.

With this change in a paradigm shift and the sensitization on sustainability, investors who do not incorporate sustainability into their investment decision will eventually lose their goodwill and that may affect their bottom line.

The results from the social stocks, however, underperformed the unsustainable stocks. This may be likely as a result of less attention being given to social sustainability factors. So perhaps if we start to conscientize people on the social responsibilities of organizations, I believe companies with a good social rating will also eventually outperform the unscreened portfolio and that will have a significant impact on the performance of the ESG stocks. Thus the need for sustainability in modern portfolio optimization cannot be underestimated.

5.1 Future work

The main parameters of the Markowitz model which are the mean, standard deviation and correlation matrix of the asset returns were estimated using historical data. The use of historical data, however, introduces two main problems. These are the estimation error and the stationarity of the model parameters (see,[4]). When only modern data are used to estimate the parameters, it may lead to an estimation error and return data from about 50 years back may not have a lot of impact on current returns.

I decided to use an 11-year data in my analysis but future work can extend the time horizon and investigate into the estimation error and how they can be improved.

Future work can also conduct the analysis based on different indices to confirm the findings and conclusions drawn from the Dow index.
Fulfilment of thesis Objective

This part will discuss how the thesis requirement of the Swedish National Agency for Higher education for a 2 years Masters has been met. Every masters thesis can be awarded after a prospective student has been evaluated and satisfied the 6 objectives. I have demonstrated how my thesis satisfies the objectives for a master’s degree in mathematics with specialization in financial engineering by stating and discussing how each objective is fulfilled.

Objective 1
For Master degree, student should demonstrate knowledge and understanding in the major field of study, including both broad knowledge in the field and substantially deeper knowledge of certain parts of the area as well as insight into current research and development

The thesis starts off with an introduction to modern portfolio theory and centers on the Markowitz model which is the backbone of modern portfolio theory. It demonstrates the significance of the Markowitz model in the generation of the efficient frontier and its usefulness in making decisive investment decisions. The author investigates sustainability and tries to identify the reasons for the upsurge interest. Literature from books, journals, and genuine internet websites was reviewed extensively. The first chapter discusses studies that have been conducted into this area and the contributing factors for the upsurge. The theoretical framework of modern portfolio theory is discussed in chapter 2 whilst the 3rd chapter tries to narrow down to the mathematical approach in factoring sustainability into the mean-variance framework. The author demonstrates the impact of sustainability by considering a real-world scenario based on historical data and drawing conclusions from my findings.

Objective 2
For Master Degree, student should demonstrate deeper methodological knowledge in the major field of study

The first part of the thesis discusses the objectives and motivation that informed the decision of the author to study the sustainability for portfolio optimization. I demonstrated how a financial problem can be formulated into a mathematical problem and using python and statistics, I presented a coherent framework for addressing the problem. The author presented a comprehensive introduction and theoretical framework to serve as the grounds to bring a reader with a little or no knowledge in modern portfolio theory to a level where he can appreciate and interpret the objectives of the thesis. It goes further to discuss the more technical aspect of trying to incorporate sustainability into the Markowitz mean-variance framework. Current research based on many articles and journals were reviewed for the development of the thesis. The author further discusses the sustainability rating and how it is generated. The reader can now use the knowledge in the correlation matrix, mean-variance optimization, sustainability rating, statistics and the investor preference to try and solve the problem of sustainability for portfolio optimization. The author has further demonstrated how to generate and compare the asset performance of sustainable stocks and non-sustainable stocks based on their cumulative returns. If a reader decides to extend this study, it will be great to analyze the sustainability for portfolio optimization by considering the estimation error in using historical data. A look at the same topic based on the value at risk will also be quite interesting.

Objective 3
For Master degree, student should demonstrate the ability to critically and systematically integ-
rate knowledge and to analyze, assess and deal with complex phenomena, issues and situations even with limited information

The author puts together complex theories from different sources into a coherent paper. The paper was impacted by the financial and mathematical knowledge and skillset attained by the author during his studies at Märlardalen University. The author demonstrated the desire to push the boundaries of available knowledge by digging deeper into the mean-variance optimization framework in relation to sustainability. He also went beyond the usual comparison of sustainable performance by using the ESG and analyzed the impact of the environmental, social and governance ratings on their own merit. The author considered results from different research work and journals to decide on the direction of the thesis and real-world data was analyzed to draw the conclusions. The process for translating the mathematical problem into a programming code is not trivial.

Objective 4
For Master degree, student should demonstrate the ability to critically, independently and creatively identify and formulate issues and to plan and carry out advanced tasks within specified time frames, thereby contributing to the development of knowledge and to evaluate this work.

With the continuous increase in interest in sustainability, the author decided to look into the motivation for investors and how it can be sustained with by considering not just the mathematical solution but also from the ethical point of view. The author decided on the direction of the thesis with well-defined timelines. The author used python to demonstrate various tests to ascertain the performance of sustainable assets and non-sustainable assets. The demonstration was based on the Dow and Jones stocks in the US market.

Objective 5
For Master degree, student should demonstrate ability in both national and international contexts, orally and in writing to present and discuss their conclusions and the knowledge and arguments behind them, in dialogue with different groups.

The author gathered and cited knowledge from journals and authors from all over the world and presented them in a universally accepted language and concept. He also tried to be expansive to bring readers with little or no knowledge in modern portfolio theory to a level where they can appreciate and interpret the thesis. My findings will be demonstrated by a verbal presentation of my thesis that will be supported by a pictorial presentation of findings.

Objective 6
For Master degree, student should demonstrate ability in the major field of study make judgments taking into account relevant scientific, social and ethical aspects, and demonstrate an awareness of ethical issues in research and development

The author tries to address a trending issue using financial engineering. He recognizes that some institutions engage in sustainable investment from the ethical point of view whilst others do to maximize returns. He tries to demonstrate this from a mathematical point of view. Both schools of thought were demonstrated in the paper. The author demonstrated that it is possible to yield higher returns by engaging in sustainable investment but went further to demonstrate there are times when a sustainable
investment will underperform the non-sustainable investment. When this happens, then investment deci-
sions will be made from the ethical point of view. He, however, concluded that with the growing
interest sustainability, sustainable investment is likely to outperform the non-sustainable investment
over the long term. I acknowledge that I had to fall on other peoples work as the basis for my paper and
I have tried to duly recognize their inputs to the best of my ability. The materials used as references
in this thesis are all published. The Data for my analysis was called from yahoo finance. The author
will like to end by stating that incorporating these theories and findings into any investment decision
making comes with an associated risk.
Appendix A

Constituent stocks

Figure A.1: Distribution plot for all 30 stocks
Figure A.2: Weight plot for 30 stocks
### Results for symbol AAPL

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Figure A.4: Maximum Sharpe ratio and minimum volatility value

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\begin{align*}
\text{return} & = \text{volatility} \\
\text{Max Sharpe portfolio} & = 0.191733 \quad 0.190015 \\
\text{Min Volatility portfolio} & = 0.090659 \quad 0.139912
\end{align*}
\]
Appendix B

ESG stocks

Figure B.1: Distribution plot for ESG stocks
Figure B.2: Weight plot for ESG stocks

Figure B.3: Maximum Sharpe ratio and minimum volatility value

Figure B.4: Price performance plot for ESG Stocks
Appendix C

Environmental stocks

Figure C.1: Distribution plot for environmental stocks
Figure C.2: Weight plot for environmental stocks

![Weight plot for environmental stocks](image)

Figure C.3: Maximum Sharpe ratio and minimum volatility value

```
returns  volatility
Max Sharpe portfolio  0.211841  0.218157
returns  volatility
Min_Volatility portfolio  0.68588  0.149776
```
Appendix D

Governance stocks

Figure D.1: Distribution plot for governance stocks

Figure D.2: Maximum Sharpe ratio and minimum volatility value
Figure D.3: Price performance plot for governance stocks
Appendix E

Social stocks

Figure E.1: Asset correlation matrix of social stocks
Figure E.2: Distribution plot for social stocks

Figure E.3: Weights plot for social stocks
Figure E.4: Efficient frontier of social stocks

Figure E.5: Maximum Sharpe ratio and minimum volatility value

Figure E.6: Cumulative returns for social stocks
Bibliography


