Calibration of the Multiscale Stochastic Volatility Model via an Asymptotic Expansion Approach

by

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Abstract

We study a model of multiscale stochastic volatility for European option pricing. In this model there are two volatility factors. The first volatility factor is of fast scale of mean-reverting and the second one is of slow scale of mean-reverting. We review the useful calibration formula derived by Fouque et al.\cite{2} via an asymptotic expansion method. Our purpose of this study is two-fold. The first task is to implement this calibration formula to Swedish market data, more specifically to call option data of ABB stock during the time span of 4th Mar 2016 to 6th May 2016. We investigate how well the data fitting is assuming only the fast volatility factor and the full model with both fast and slow volatility factors. We study also the time stability of the fitted parameters over a time period of 10 weeks. The second task is to study the effect of the parameters included in the calibration formula on the volatility surface and smile.
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Chapter 1

Introduction

In the last 40 years options have become important in the world of finance, millions of people trade actively on a daily basis on different exchanges around the world. A call option gives the holder the right to buy the underlying asset at a certain date for a certain price. A put option gives the holder the right to sell the underlying asset at a certain date for a certain price. In the contract, there is a price decided for exercise which is called strike price or exercise price.

There exists different kind of options, two of the most traded ones are: American and European options. American options can be exercised anytime until maturity (expiration date) while European options can only be exercised at maturity.

Every option contract has two sides; on one side there is an investor who buys the option (long position), on the other side there is an investor who sells the option (short position). This leads to different payoffs for the buyer and seller of the option. The profit of the seller of the option means the buyer obtained a loss and vice-versa. There are four possible payoffs:

\( X_T \) is the final price of the underlying asset, \( K \) is the strike price and the payoff from a long position in a European call option is

\[ (X_T - K)^+ , \]

This means that the option will be exercised when \( X_T > K \).

The payoff from a short position in a European call option is

\[ -(X_T - K)^+ , \]

which is the opposite of the long position. In the same way the payoff from a long position in a European put option is

\[ -(X_T - K)^+ , \]
the payoff from a short position in a European put option is

\[-(K - X_T)^+,
\]

The complexity of computing option contracts push practitioners to base their work on mathematical models to price the derivatives. There are many ways to calculate the price of an option, the model introduced by Black and Scholes [1] considers the volatility to be constant for options with same maturity but different strike prices. This has been proved wrong by empirical studies, which show skewness in the implied volatility building a smiling shape. The lowest value of the implied volatility is seen at or near the money, whereas the implied volatility has bigger values for options in or out-of-the-money.

Researchers consider a stochastic volatility model to better price the options. In this thesis we consider a multiscale stochastic volatility model studied in Fouque et al. [2]. This model contains two correlated volatilities, one changing fast and one changing slowly.

The difficulty of computing the exact price of the option encourages mathematicians to consider its approximation derived using complex models. In Fouque et al. [2] an approximating option price formula is derived using asymptotic expansion method in the multiscale stochastic volatility model. Using this asymptotic expansion approach, a corresponding reduced calibration formula is also derived which serves as the starting point of the present thesis. Thanks to this calibration formula, only four so-called group market parameters are needed for calibration.

The contribution of the present thesis is to present the fitting results of the model via the calibration formula (Section 4.1, 4.2) and time stability of the parameters using Swedish market data (Section 4.3), more specifically all market call options data of ABB stock for 10 weeks. We would like to mention that such calibration study was carried out by Fouque, as described in [2], for S&P option data. However, it is desirable to investigate the calibration results to new market data. Another contribution here is to for the first time in the literature provide an analysis of the effect of the four group market parameters on implied volatility (Section 4.4, Section 4.5). Such study helps to understand the role played by each parameter and to predict better the behavior of the implied volatility in the future.
1.1 Literature Review

Black and Scholes made a huge contribution to the option pricing theory with their paper The Pricing of Options and Corporate Liabilities [1]. The Black-Scholes formula is one of the most popular formulas in finance in the pricing of a European option. The relation between the cross-sectional properties of option prices and the distribution of spot returns is the main advantage of the formula. The Black-Scholes formula explains well the stock options prices.

However the formula does have several limitations as explained by various authors: Rubinstein [10] explained the biasedness of the Black-Scholes formula; Melino, Turnbull [8] and Knoch [7] presented how the formula does not successfully price foreign currency option prices. The reason behind this limitation can be explained by the fact that the Black-Scholes formula considers the stock returns to be normally distributed with known mean and variance (Heston [5]). This Normal distribution does not capture the skewness and kurtosis observed in real market data.

Before the 1987 market crash, the graph of the implied volatility as function of strike price for fixed $t, x, T$ was often observed to be U-shaped with minimum at or near-the-money. Since then the volatility smile shows downward slope where the volatility decreases when the strike price increases. It means that the volatility used to price deep out-of-the-money put options is different to the volatility used to price deep in-the-money put options (it works likewise for call options).

Working on this problem, researchers found that since the Black-Scholes formula does not depend on the mean spot return, the doubts would be on the consideration of constant volatility. This led different authors such as Scott [11], Hull and White [6] and Wiggins [12] to start working on a stochastic volatility model.

The model proposed by Heston [5] considers a single factor stochastic volatility model where the level of the stock return in volatility are negatively correlated. However the model has the disadvantage of poorly reproducing the relationship between the volatility level and the slope of the smirk [2].

A method that reproduces better the relationship between the volatility level and the slope of the smirk was introduced by Christoffersen et al.[3] and it’s a two-factor stochastic volatility model; the first factor is fast mean-reverting with respect to the time horizon of a typical options contract, the second factor (slow factor) seems to not be mean reverting relative to this horizon even though it could be mean reverting over longer horizon.

In the thesis, we consider a variant of such multiscale stochastic volatility model introduced by Fouque et al. [2], the price of the underlying asset is
correlated with the volatility and the model gives a closed form solution of the European call option price. The model consists of the system of differential equations below; it contains a fast-scale and slow-scale volatilities under real-world measure:

\begin{align}
    dX_t &= \mu X_t dt + f(Y_t, Z_t) X_t dW_t^{(0)} \\
    dY_t &= \frac{1}{\epsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\epsilon}} \beta(Y_t) dW_t^{(1)} \\
    dZ_t &= \delta c(Z_t) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)}
\end{align}

(1.1)

Here \( X_t \) is the price of the underlying asset, \( \mu \) is the expected return, the function \( f(Y_t, Z_t) \) is the volatility function. We assume that \( f(Y_t, Z_t) \) is some positive and increasing function so that there is a solution to the system (1.1). The process \( W^{(0)} \) is a standard Brownian motion. \( X_t \) is a geometric Brownian motion in case the volatility is constant. \( Y_t \) and \( Z_t \) represent the fast and slow volatility processes. \( 1/\epsilon \) and \( \delta \) are the rates of mean reversion for \( Y_t \) and \( Z_t \) respectively. The coefficients \( \alpha(Y) \) and \( \beta(Y) \) (for the fast volatility factor) along with \( c(Z) \) and \( g(Z) \) (for the slow volatility factor) define the dynamics of the diffusion processes \( Y_t \) and \( Z_t \) under the physical measure \( P \) respectively.

The model (1.1) is transformed into risk neutral measure and the option price formula is approximated using an asymptotic expansion method. Calibration means that the gap between the market and mathematical models should be minimized. The concept is to elaborate a parametric model for the pricing of the underlying asset and try to find parameters that minimize the difference between the financial market data and the mathematical model.

In a complex model as in (1.1), calibration of all original model parameters is a very complicated task. Fortunately Fouque et al. [2] has derived, via the asymptotic expansion method, an approximating calibration formula (2.34) which involves only four group market parameters. This calibration formula, together with the first-order option pricing asymptotic expansion formula (2.21), serve as the main objects of study in this thesis.

1.2 Problem Formulation and the Aim of Thesis

We study the model fitting results and time stability of fitted parameters using the calibration formula (2.34). We address questions as: how well the model fits the market data using only the fast scale volatility and using both fast and slow scale volatility; How stable the fitted parameters are with
respect to time. All data have been collected by the authors directly from
the OMX Nordic exchange [9]. Approximately 120-180 market prices of call
options have been monitored on each Fridays for 10 weeks (4th march 2016
to 6 May 2016). Moreover we investigate the effect on volatility surface/smile
by varying the four group market parameters which are direct inputs to the
first-order asymptotic expansion pricing formula (2.21) and indirect input to
calibration formula (2.34). The aim is to gain deeper insights in the model
calibration results and explain the roles of market parameters in the volatility
surface/smile of model (1.1).

All simulation to perform the calibration and graphing the volatility smile
and implied volatility surface is done in MATLAB R2015b (on a desktop
computer running Microsoft Windows 7).

1.3 Disposition of the thesis

The thesis proceeds as follows. In Chapter 2 we review and explain the first
order approximation pricing formula and the calibration formula derived by
Fouque et al. [2] via the asymptotic expansion method. The main contri-
bution of this thesis is presented in chapters 3, 4. In Chapter 3 we present
the data, data processing and the observed market volatility surfaces/smiles.
Finally in Chapter 4 we collect all results of our experimental studies related
to the problems described above.
Chapter 2

The Model

2.1 Multiscale Stochastic Volatility Model

Given the price of European call option, practitioners used the Black and Scholes formula to define the implied volatility.

\[
C_{BS}(t, x) = xN(d_1) - Ke^{-r(T-t)}N(d_2),
\]

\[
d_1 = \frac{(\log(\frac{x}{K}) + (r + \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}},
\]

\[
d_2 = d_1 - \sigma\sqrt{T-t},
\]

where \(C_{BS}(t, x)\) is the European call option price, \(x\) is the current stock price, \(T - t\) is time to maturity, where \(T\) is the expiration date and \(t\) is time when the option is issued; \(K\) is strike price, \(\sigma\) is volatility, and \(r\) is the annualized risk-free interest rate, \(N(d)\) is the function of cumulative probability distribution for a standardized normal distribution.

The implied Volatility \((I)\) is the value that is needed to match the observed call option price \(C^{obs}\) to the Black Scholes call option price for a contract given the Strike Price \(K\) and maturity \(T\) as follows:

\[
C_{BS}(t, x; K, T; I) = C^{obs}
\]

Due to put-call parity the implied volatility will be the same for put and call options with same strike price and time to maturity.

The standard Black-Scholes theory considers the implied volatility to be constant but this is disproved by empirical studies, which show that the market prices implied volatilities are not constant given the same time to maturity but different strike prices.
The stochastic volatility model respects the following stochastic differential equation:

\[ dX_t = \mu_t X_t dt + f(Y_t) X_t dW_t^0 \]  

(2.3)

Starting with one-factor stochastic volatility model, the process \( Y \) is introduced, it is a one-dimensional \( \text{Itô's} \) process satisfying a stochastic differential equation driven by a second Brownian motion.

Considering \( f(Y_t) \) is a positive, increasing function; we have the following stochastic differential equation for \( Y_t \):

\[ dY_t = \alpha(Y_t) dt + \beta(Y_t) dW_t^1 \]  

(2.4)

\( \alpha : \mathbb{R} \to \mathbb{R}, \quad \beta : \mathbb{R} \to \mathbb{R}^+ \)

where \( W^1 \) is a standard Brownian motion, \( \alpha \) is the rate of mean reversion, \( \beta(Y_t) \) is a diffusion coefficient.

We have instantaneous correlation between \( W^0 \) and \( W^1 \) denoted by \( \rho(-1 \leq \rho \leq 1) \). We define it by

\[ d\langle W^0, W^1 \rangle_t = \rho dt \]  

(2.5)

This correlation can also be viewed as the correlation between stock price and volatility shocks.

\( \rho \leq 0 \) means that when the volatility increases, the stock price tends to decrease.

Another useful concept in the stochastic volatility model is the mean-reversion, it refers to a linear pull-back term in the drift of volatility driving factor \( Y \).

This brings us to the following new stochastic differential equation:

\[ dY_t = \alpha(m - Y_t) dt + \beta(Y_t) dW_t^1 \]  

(2.6)

where \( m \) represents the long-run mean of \( Y \).

This stochastic volatility model was introduced by Steven Heston[5] in 1993 and it is mainly based on three functions \( \alpha, \rho \) and \( \beta(Y_t) \). A high value of \( \alpha \) makes the variance process time dependent because a change in the variance from the mean is pulled back. \( \rho \) influences the skewness in the volatility smile. A negative correlation means that a higher volatility will be accompanied by lower strike price and vice versa. The diffusion function \( \beta(Y_t) \) has an impact on the kurtosis of the returns distribution and the steepness of the volatility smile. The kurtosis increases when \( \beta(Y_t) \) increases. One of the advantages of the stochastic volatility is its effect on the distribution of the stock price. Empirical studies show that the density function of stock
price has thicker tails when stochastic volatility is used than for constant volatility.

In this project, we will work with multifactor stochastic volatility model, in which we have a volatility that is driven by a 2-dimensional diffusion process \( Y \in \mathbb{R}^2 \). The difference between the two volatility processes is that one is driven by a fast rate of mean reversion whereas the other is driven by a slow rate of mean reversion. The volatility processes have Brownian motions (Wiener processes), which are correlated between each other and correlated with the Brownian motion driving the stock price.

\[
\begin{align*}
    dX_t &= \mu X_t dt + f(Y_t, Z_t)X_t dW_t^{(0)} \\
    dY_t &= \frac{1}{\epsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\epsilon}} \beta(Y_t) dW_t^{(1)} \\
    dZ_t &= \delta c(Z_t) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)}
\end{align*}
\]  

(2.7)

\[ \text{(2.8)} \]

### 2.2 Model Calibration

In the following sections, Section 2.2 and Section 2.3 we review and explain the asymptotic expansion formula (2.21) and the calibration formula (2.34) presented in Fouque et al[2].

Instead of deriving the exact pricing model, we consider its approximation, which is developed using the asymptotic expansion. The parameters of the new approximated option price are calibrated from the term structure of implied volatilities obtained from market data. The two factor stochastic volatility model considered under real world measure is transformed into risk-neutral measure \( P^* \) as Fouque et al [2] has derived:

\[
\begin{align*}
    dX_t &= r X_t dt + f(Y_t, Z_t)X_t dW_t^{(0)*} \\
    dY_t &= \left( \frac{1}{\epsilon} \alpha(Y_t) dt - \frac{1}{\sqrt{\epsilon}} \beta(Y_t) \Lambda_1(Y_t, Z_t) \right) dt + \frac{1}{\sqrt{\epsilon}} \beta(Y_t) dW_t^{(1)*} \\
    dZ_t &= \left( \delta c(Z_t) - \sqrt{\delta} g(Z_t) \Lambda_2(Y_t, Z_t) \right) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)*}
\end{align*}
\]  

(2.8)

The underlying asset \( X_t \) in (2.8) has:

- A constant instantaneous interest rate \( r \).
- A positive volatility function \( f(Y, Z) \), which is driven by two volatility processes \( Y, Z \) representing the fast and slow volatility factors respectively.
\( \Lambda_1(y, z) \) and \( \Lambda_2(y, z) \) represent the market prices of risk associated with the volatility factors \( Y \) and \( Z \) and they define the risk-neutral pricing measure \( p^* \). The market prices of risk are used as function of the volatility processes so that the \( X, Y, Z \) stay a Markov process under \( p^* \).

The Wiener processes \( W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*} \) are found using the Girsanov theorem with the following definition:

\[
dW_i = -\Lambda_i dt + dW_i^*
\]

with \( i=0,1,2 \). The Wiener processes \( W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*} \) considered under risk neutral probability, have the following correlation between them:

\[
d\langle W_t^{(0)*}, W_t^{(1)*} \rangle = \rho_1 dt, \quad (2.9)
\]
\[
d\langle W_t^{(0)*}, W_t^{(2)*} \rangle = \rho_2 dt, \quad (2.10)
\]
\[
d\langle W_t^{(1)*}, W_t^{(2)*} \rangle = \rho_{12} dt, \quad (2.11)
\]

where \( |\rho_1| < 1 \), \( |\rho_2| < 1 \), \( |\rho_{12}| < 1 \).

The option price will be dependent on the current price \( X_t \) of the underlying asset, the present time \( t \) and the present values of the volatility processes \( Y_t \) and \( Z_t \).

Under risk neutral measure, the price of the European option \( P^{\epsilon,\delta} \), which depends on the parameters \( \epsilon, \delta \) is the expected discounted payoff:

\[
P^{\epsilon,\delta}(t, X_t, Y_t, Z_t) = E^*(e^{-r(T-t)}h(X_T)|X_t, Y_t, Z_t), \quad (2.12)
\]

where \( h(x) \) is the payoff function of the option.

From (2.8), the volatility factors \( Y \) and \( Z \), are not observable. In order to find the stock price, the functions \( f, \alpha, \beta, c, g, \Lambda_1, \Lambda_2 \), need to be estimated and this is not an easy task. Due to this complexity the option price is derived using the asymptotic expansion method:

\[
P^{\epsilon,\delta} \approx P_{BS} + P_{1,0}^{\epsilon} + P_{0,1}^{\delta} \quad (2.13)
\]

The option price is composed of the Black-Scholes price \( P_{BS} \), the first-order fast scale correction \( P_{1,0}^{\epsilon} \) and the first-order slow scale correction \( P_{0,1}^{\delta} \).

Here the terms related to the parameter \( \epsilon \) generate a singular perturbation problem and the terms related to the parameter \( \delta \) generate a regular
perturbation problem. A singular perturbation problem involves a small parameter, which cannot be approximated by equating the parameter to zero. The opposite to this problem is the regular perturbation problem.

The option price (2.13) is the solution of the following parabolic partial differential equation as written in Fouque et al. [2]:

\[
\left( \frac{1}{\epsilon} L_0 + \frac{1}{\sqrt{\epsilon}} L_1 + L_2 + \sqrt{\delta} M_1 + \delta M_2 + \frac{\delta}{\epsilon} M_3 \right) P^{\epsilon,\delta} = 0 \tag{2.14}
\]

with

\[
L_0 = \frac{1}{2} \beta^2(y) \frac{\partial^2}{\partial y^2} + \alpha(y) \frac{\partial}{\partial y}, \tag{2.15}
\]

\[
L_1 = \beta(y) (\rho_1 f(y,z) x \frac{\partial^2}{\partial x \partial y} - \Lambda_1(y,z) \frac{\partial}{\partial y}), \tag{2.16}
\]

\[
L_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y,z) x^2 \frac{\partial^2}{\partial x^2} + r(x \frac{\partial}{\partial x} - .), \tag{2.17}
\]

\[
M_1 = g(z) (\rho_2 f(y,z) x \frac{\partial^2}{\partial x \partial z} - \Lambda_2(y,z) \frac{\partial}{\partial z}), \tag{2.18}
\]

\[
M_2 = \frac{1}{2} g^2(z) \frac{\partial^2}{\partial z^2} + c(y) \frac{\partial}{\partial z}, \tag{2.19}
\]

\[
M_3 = \beta(y) \rho_{12} g(y) \frac{\partial^2}{\partial y \partial z}, \tag{2.20}
\]

- \(1/\epsilon L_0\) generates the fast factor \(Y\) under physical measure \(p\).

- \(L_1\) consists of derivatives to \(X\) and \(Y\) due to the covariance that exists between the price of the underlying asset and the fast volatility factor, it contains the derivative to \(Y\) in relation to the market price of risk \(\Lambda_1\).

- \(L_2\) consists of the change in time and the Black-Scholes operator at the volatility level \(f(y,z)\).

- \(M_1\) consists of derivatives to \(X\) and \(Z\) due to the covariance that exists between the price of the underlying asset and the slow volatility factor, it contains the derivative to \(Z\) in relation to the market price of risk \(\Lambda_2\).

- \(\delta M_2\) generates the slow factor \(Z\) under physical measure \(p\).
M₃ represents the covariance existing between the volatility processes Y and Z.

The asymptotic expansion of (2.13), derived by Fouque et al [2], leads to the first order approximation of the option price:

\[ P^* = P^*_{BS} + (T - t)(V^δ_0 \frac{\partial P^*_{BS}}{\partial \sigma} + V^δ_1 D_1 + V^ε_3 D_1 D_2 P^*_{BS}) \]  

(2.21)

This formula introduces the Greek Vega which is the rate of change of the option price with respect to the volatility of the underlying asset. \( V^δ_3 \) depends on parameter \( \epsilon \) and represents the effect of the fast volatility factor on the option price formula. The slow time scale effect on the option price formula is represented by the parameters \( V^δ_0 \) and \( V^δ_1 \) (depend on \( \delta \)). The influence of the correlation \( \rho_2 \) and the market price of risk \( \Lambda_2 \) are separated in the two parameters \( V^δ_0 \) and \( V^δ_1 \).

\[ D_1 = \frac{x}{\partial x} \]
\[ D_2 = x^2 \frac{\partial^2}{\partial x^2} \]

\( D_1 \) and \( D_2 \) are the logarithmic derivatives operators. (We refer to Fouque et al. [2] for the full explanation of \( D_1 \) and \( D_2 \)). Please observe that the group parameters \( \sigma^*, V^δ_0, V^δ_1, V^ε_3 \) (where \( \sigma^* \) is the corrected average volatility) are the ones to be calibrated in order to find the option price of the underlying asset comparing to the usual method of calibrating many original model parameters.

### 2.3 Call Prices and Implied Volatilities

Let consider a European call option with the following payoff function

\[ h(x) = (x - K)^+ \]

Starting with the Black-Scholes formula for European option price at the volatility \( \sigma^* \):

\[ P^*_{BS} = x N(d_1^*) - K e^{-\tau r} N(d_2^*), \]

with

\[ \tau = T - t \]
\[ d'_1 = \frac{\log(\frac{x}{K}) + (r + \frac{1}{2}\sigma^* \tau)}{\sigma^* \sqrt{\tau}} \]
\[ d'_2 = \frac{\log(\frac{x}{K}) - (r + \frac{1}{2}\sigma^* \tau)}{\sigma^* \sqrt{\tau}} \]

Formula (2.21) is written as follows:

\[ P^* = P^*_{BS} + \{ \tau V^\delta_0 + \tau V^\delta_1 D_1 + \frac{V^\prime_3}{\sigma^*} D_1 \} \frac{\partial P^*_{BS}}{\partial \sigma}, \quad (2.22) \]

We refer to Fouque et al. [2] thus we have:

\[ \frac{\partial P^*_{BS}}{\partial \sigma} = \tau \sigma^* D_2 P^*_{BS} \quad (2.23) \]

Vega is given by the following formula:

\[ \frac{\partial C^*_BS}{\partial \sigma} = \frac{x \sqrt{\tau} e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \quad (2.24) \]

Knowing that the implied volatility \( I \) is used in the Black-Scholes formula as mentioned in Section (2.1) we have the following equality:

\[ C^*_BS(I) = P^*, \quad (2.25) \]

where the left-hand side represents the call price computed using Black-Scholes formula with volatility \( I \) and the right-hand is the option price calculated using the stochastic volatility model.

The next step is to calculate the difference between the implied volatility \( I \) and the corrected average volatility \( \sigma^* \), obtaining results in powers of \( \sqrt{\epsilon} \) and \( \sqrt{\delta} \):

\[ I - \sigma^* = \sqrt{\epsilon} I_{1,0} + \sqrt{\delta} I_{0,1} + ..., \quad (2.26) \]

The (2.25) is expanded using (2.24) and (2.26) in order to obtain the formula for the implied volatility with the calibrated parameters. (The reader may refer to Fouque et al.[2] for further explanations)

\[ C^*_BS(\sigma^*) + (\sqrt{\epsilon} I_{1,0} + \sqrt{\delta} I_{0,1}) \frac{\partial C^*_BS(\sigma^*)}{\partial \sigma} + ... = P^*_{BS} + \left\{ \tau V^\delta_0 + \tau V^\delta_1 D_1 + \frac{V^\prime_3}{\sigma^*} D_1 \right\} \frac{\partial P^*_{BS}}{\partial \sigma} + ..., \quad (2.27) \]
The fast scale terms ($\sqrt{\epsilon}$) and slow scale terms ($\sqrt{\delta}$) are matched separately due to the following equality $P^*_BS = C_{BS}(\sigma^*)$:

$$\sqrt{\epsilon} I_{1,0} \frac{\partial C_{BS}(\sigma^*)}{\partial \sigma} = \frac{V^*_3}{\sigma^*} D_1 \frac{\partial P^*_BS}{\partial \sigma}, \quad (2.28)$$

$$\sqrt{\delta} I_{0,1} \frac{\partial C_{BS}(\sigma^*)}{\partial \sigma} = \left\{ \tau V^*_0 + \tau V^*_1 D_1 \right\} \frac{\partial P^*_BS}{\partial \sigma}, \quad (2.29)$$

the expression for Vega is differentiated with respect to $x$ refer to Fouque et al [2]:

$$D_1 \frac{\partial C_{BS}}{\partial \sigma} = (1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}}) \frac{\partial C_{BS}}{\partial \sigma} \quad (2.30)$$

For $\sigma = \sigma^*$, we obtain:

$$\sqrt{\epsilon} I_{1,0} = \frac{V^*_3}{\sigma^*} (1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}}) = \frac{V^*_3}{2\sigma^*} (1 - \frac{2r}{\sigma^{*2}}) + \left( \frac{V^*_3}{\sigma^{*3}} \right) \log(\frac{K}{x}) \quad (2.31)$$

$$\sqrt{\delta} I_{0,1} = \tau V^*_0 + \tau V^*_1 (1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}}) = \tau \left\{ \frac{V^*_0}{2} (1 - \frac{2r}{\sigma^{*2}}) \right\} + \left( \frac{V^*_1}{\sigma^{*2}} \right) \log(\frac{K}{x}). \quad (2.32)$$

Log-moneyness to maturity ratio is defined by

$$LMMR = \frac{\log(\frac{K}{x})}{\tau} \quad (2.33)$$

And the first order approximation for the implied volatility is

$$I \approx \sigma^* + \sqrt{\epsilon} I_{1,0} + \sqrt{\delta} I_{0,1}$$

The implied volatility takes the simple form of

$$I \approx b^* + \tau b^\delta + (a^\epsilon + \tau a^\delta) LMMR \quad (2.34)$$

$b^*, b^\delta, a^\epsilon, a^\delta$ are the parameters defined in terms of functions $(\sigma^*, V^*_0, V^*_1, V^*_3)$ as follows:

$$b^* = \sigma^* + \frac{V^*_3}{2\sigma^*} (1 - \frac{2r}{\sigma^{*2}}), \quad a^\epsilon = \frac{V^*_3}{\sigma^{*3}} \quad (2.35)$$

$$b^\delta = V^*_0 + \frac{V^*_1}{2} (1 - \frac{2r}{\sigma^{*2}}), \quad a^\delta = \frac{V^*_1}{\sigma^{*2}} \quad (2.36)$$
The calibration formula (2.34) is very important for the experimental studies in this thesis. In one of our tasks we investigate the influences of these useful group market parameters \((\sigma^*, V_0^\delta, V_1^\delta, V_2^\delta)\) in option pricing formula (2.21) on the volatility surface/smile of the multi-scale model (1.1). It is therefore convenient to derive closed-form expressions of these group market parameters in terms of the coefficients \((a^\epsilon, b^*, a^\delta, b^\delta)\) in (2.34). We determine them by inverting the parameters \((a^\epsilon, b^*, a^\delta, b^\delta)\) keeping only the terms of power \((\sqrt{\epsilon})\) and \((\sqrt{\delta})\).

We transform (2.35):

\[
V_3^\epsilon = a^\epsilon \sigma^{*3} \\
b^* = \sigma^* + \frac{a^\epsilon \sigma^{*2}}{2} (1 - \frac{2r}{\sigma^{*2}}) \\
b^* = \sigma^* + \frac{a^\epsilon \sigma^{*2}}{2} - ra^\epsilon \\
\frac{a^\epsilon \sigma^{*2}}{2} + \sigma^* - (b^* + ra^\epsilon) = 0.
\]

thus we have

\[
\sigma^* = b^* + a^\epsilon \left(r - \frac{b^*}{2}\right).
\]

we can write \(V_3^\epsilon\) in terms of order \(\sqrt{\epsilon}\) as

\[
V_3^\epsilon = a^\epsilon \sigma^{*3} = a^\epsilon b^{*3} + ...,
\]

The calibrated formulas are:

\[
\sigma^* = b^* + a^\epsilon \left(r - \frac{b^*}{2}\right), \quad V_3^\epsilon = a^\epsilon b^{*3} \tag{2.37}
\]

\[
V_0^\delta = b^\delta + a^\epsilon \left(r - \frac{b^*}{2}\right), \quad V_1^\delta = a^\delta b^{*2} \tag{2.38}
\]

Here we use the two step linear regression process suggested in Fouque et al[2]. The implied volatility is a function of time to maturity \(T\) and strike price \(K\) \(\{I(T_i, K_{i,j})\}\) and it’s clear that it has different strike prices for each maturity. To find \(a^\epsilon, b^*, a^\delta, b^\delta\) we need to proceed to the fitting of the approximated implied volatility \(I\) to the large data of strike prices and time to maturity. Given that the exercising days are less compared to the strike prices, it is best to start the fitting maturity wise. We perform a linear regression of the implied volatilities on the corresponding LMMRs

\[
(LMMR)_{i,j} = \frac{\log\left(\frac{K_{i,j}}{x}\right)}{\tau_i}
\]
With $x$ representing the spot price of the underlying asset, $\tau_i$ is the exercising day for the corresponding option. We use the least-square method to estimate the coefficients $\hat{a}_i$ and $\hat{b}_i$ which minimize:

$$\min_{(a_i,b_i)} \sum_j (I(T_i, K_{i,j}) - (a_i(LMMR)_{i,j} + b_i))^2$$

(2.39)

A second linear regression of the estimates $\hat{a}_i$ and $\hat{b}_i$ on time to maturity $\tau_i$ separately in order to obtain estimates of the intercept $\hat{a}^\epsilon$ and slope $\hat{a}^\delta$

$$\min_{(a_0, a_1)} \sum_i \{\hat{a}_i - (a_0 + a_1 \tau_i)\}^2$$

(2.40)

And the intercept $\hat{b}^\epsilon$ and slope $\hat{b}^\delta$

$$\min_{(b_0, b_1)} \sum_i \{\hat{b}_i - (b_0 + b_1 \tau_i)\}^2$$

(2.41)
Chapter 3

Calibration to real market data

3.1 The data

In this section we give a brief information of the data we used. In the Subsection 3.1.1, we describe the stock options, the sources for the option data and details concerning the chosen stock options.

Section 3.1.2 gives definitions of the data used for computing the implied volatility such as the price of the option, the exercise price, time to maturity, etc. In Section 3.1.3, we explain the method of cleaning data in order to keep consistent data. In the subsections 3.1.4 and 3.1.5 we introduce the parameters used in the calibration process and we plot the volatility smile and volatility surface obtained from calibrated parameters.

3.1.1 Option and market properties

In this project we use ABB stock options traded on Nasdaq OMX Nordic. ABB, a Swedish-Swiss corporation operating in almost 100 countries, is one of the leading companies in providing power and automation technologies for utility and industrial customers worldwide. The choice of ABB stock option is because its one of the most active stocks on Nasdaq OMX Nordic. We consider the call option in our thesis with exercise at maturity.

3.1.2 Option data definition

We extract the current option prices, strike prices, and time to maturity for each option. We regroup the options according to their expiring date (time to maturity).
Price ($u$): Here we take the average between ask (minimum price that a seller is willing to accept for a security) and bid for each option (maximum price that the buyer is willing to pay for a security) for a given time and exercise price.

Strike price ($K$): Price that have been decided for the owner of the option to buy (call) or sell (put) the underlying security. There are different strike prices for a given time to maturity. Strike prices tend to increase for options far from expiration.

Time to maturity ($\tau$): We consider the European option, in which options can only be exercised at maturity. We measure the time to maturity in years, taking into account the trading days only.

Annualized risk-free interest rate ($r$): rate of interest that can be earned taking into account that there is no risk. We use the following formula.

$$\tau = \frac{T - t}{252}$$

Where $T$ is the expiration date, $t$ is time when the option is issued. The value 252 is the conventional trading days.

3.1.3 Data cleaning

Data need some treatment in order to be consistent, it means that we need to remove some unreliable data entries. This procedure consists of removing:

- Options with daily closing bid and ask quotes, which are very low and obviously out-of-the-money options. in our case we removed the options prices ($u$) less than 0.50SEK. Options with a high strike price tend have lower option price ($u$).
- Options missing either the bid or ask quote.
- Options for which Black-Scholes did not return implied volatility. This procedure is performed in MATLAB to compute the implied volatility using the Black-Scholes model for European options.
3.1.4 Parameters

The calibration process consists of adjusting the model parameters so that they fit well the market prices. We obtain our calibrated parameters through the first order approximation of the implied volatility. Here we need to compute the market implied volatility using the Black-Scholes model with parameters $K, x_0$ (stock price at time 0), $\tau, u, r$. (defined in Section 3.1.3). The strike price $K$ ranges from 125 to 210, time to maturity belongs to the set $\{6, 11, 16, 21, 31, 51, 96, 161, 226\}$ counted according to the working days on Nasdaq. To obtain $\tau$ we divide working days by total trading days on Nasdaq for 2016(252). ABB is a non-dividend paying stock and the annualized interest rate is equal to 5 percent.

For simplicity, we assign the parameters:

$$a_0 = b^\delta; a_1 = b^*; a_2 = a^\delta; a_3 = a^\epsilon;$$

3.1.5 Volatility smile and implied volatility surface

The values for the calibrated parameters are shown in the Table (3.1) below and we use them as the starting points for the experiments performed in sections 4.4-4.5. We plot volatility smiles and implied volatility surfaces for nine different maturities without making any changes into the parameters and check for the option behaviour.

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>-0.1608</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0.2675</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.2626</td>
</tr>
<tr>
<td>$a_3$</td>
<td>-0.0582</td>
</tr>
</tbody>
</table>

Table 3.1: Values of calibrated parameters ($a_0, a_1, a_2, a_3$)

<table>
<thead>
<tr>
<th>$\sigma^*$</th>
<th>0.2667</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_0^\delta$</td>
<td>-0.1571</td>
</tr>
<tr>
<td>$V_1^\delta$</td>
<td>0.0188</td>
</tr>
<tr>
<td>$V_3^\delta$</td>
<td>-0.0011</td>
</tr>
</tbody>
</table>

Table 3.2: Values of group parameters ($\sigma^*, V_0^\delta, V_1^\delta, V_3^\delta$)
In this first experiment we considered a stock price driven by two volatility factors (slow and fast factor). In figures (3.1) and (3.2) we observe the skewness in the volatility smile. For time to maturity 1 to time to maturity 6, we observe a negative correlation between $I$ and $K$, where the implied volatility decreases as strike price increases. The correlation between $I$ and $K$ becomes positive from time to maturity 7 to time to maturity 9. We also observe a high fluctuation in implied volatility for $\tau$ closer to maturity and the steepness of the curve decreases as time to maturity increases.

The change in correlation can be explained by the parameters of the slope in the first-order approximation of the implied volatility:
\[ I \approx b^* + \tau b^\delta + (a^\epsilon + \tau a^\delta)LMMR \] (3.1)

Notice that the parameters \( a^\epsilon \) and \( a^\delta \) have different signs; for small values of \( \tau \), the slope of the volatility smile is negative and positive as \( \tau \) increases (from time to maturity 7, \( \tau = 0.3810 \)).
Chapter 4

Experiments and Results

In this part of the thesis we compare the fitting of the volatility smile ($I$ against $LMMR$) using only the fast volatility factor model and using both the fast and slow volatility factors model. Here we analyze how well these two models capture the range of maturities. Then we test the parameters stability over a time period of ten weeks. If parameter time stability holds, it is an indication that the model is reliable and can be used for further studies on option pricing for example. Next we study the behavior of volatility smile by changing some of the parameters of the option price formula and compare them with the original volatility smile. We change one of the market group parameters ($\sigma^*, V_0^\delta, V_1^\delta, V_3^\epsilon$) while keeping the other parameters constant and look at the change in the approximated implied volatility. Here we have 3-D plot in which the implied volatility is a function of strike price and time to maturity (implied volatility surface). We look at the skewness and the rate of change of implied volatility.

4.1 Experiment 1: Fast Volatility Factor

In this section, we compare the data fitting of implied volatility against $LMMR$ produced by a one single volatility factor (fast factor) with the same fitting using two factors. When we ignore the slow factor, the approximation of the implied volatility is as follow:

$$I \approx b^* + a^\epsilon(LMMR)$$

In figure (4.1) the circles are obtained from ABB stock options data on 4th March 2016, there are nine maturities and they increase anti-clockwise with
the shortest maturities at the bottom-right. The solid straight line is plotted using the calibrated parameters \((a^* , b^*)\). The fast factor model struggles to capture the implied volatility for the range of maturities.

![Diagram](image)

Figure 4.1: implied volatilities as a function of LMMR for ABB stock options on 4th March 2016. The circles are obtained from ABB stock options data and the line \(b^* + a^*(LMMR)\) generates the results using estimated parameters from only fast scale factor

### 4.2 Experiment 2: Fast and Slow Volatility Factors

The model using the fast and slow volatility factors follows the same procedure as explained in Section (2.3), where we perform the two-step regression to obtain the estimated parameters \((a^* , b^* , a^\delta , b^\delta )\). Figure (4.2) shows the plot of the slope coefficients \(\hat{a}_i\) of LMMR, the straight line \((a = a^* + a^\delta \tau)\) is increasing due to the fact that strike prices increases for higher maturities. Figure (4.3) shows the intercept \(\hat{b}_i\) with a decreasing straight line \((b = b^* + b^\delta \tau)\). In figure (4.4), we plot the implied volatilities against LMMR (represented by the circles) using data obtained from ABB stock options on 4th March 2016, the solid lines are obtained from the approximated implied volatility:

\[
I \approx b^* + \tau b^\delta + (a^* + \tau a^\delta)LMMR
\]

The fast and slow factors model shows the ability to capture better the range of maturities. The maturities increase anti-clockwise with the shortest ma-
turities at the bottom-right.

Figure 4.2: Term structure fit on 4th March 2016. The circles on the top figure represent the slope coefficients $\hat{a}_i$ of LMMR (first step fitting) and the solid line is fitted in the second regression $(a^\epsilon + \tau a^\delta)$.

Figure 4.3: Term structure fit on 4th March 2016. The circles on the top figure represent the intercept $\hat{b}_i$ of LMMR (first step fitting) and the solid line is fitted in the second regression $(b^\epsilon + \tau b^\delta)$. 
**4.3 Experiment 3: Parameter Time Stability**

In this section, we study how stable are the calibrated parameters during a period of 10 weeks. We extract data on ABB call options on each Friday at 13:00 from Nasdaq Nordic website (http://www.nasdaqomxnordic.com/) starting 4th March 2016. When the time stability of the parameters is respected, it means that the model captures well the aspects of the dynamics of the underlying in order to better price and hedge path-dependent options. Figure (4.5) shows the time evolution of the estimated parameters $(a^*, b^*, a^\delta, b^\delta)$ over a period of 10 weeks. One sees that there is not much difference between the values of each parameter $(a_0, a_1, a_2, a_3)$ over time. This shows that the estimated parameters are relatively stable over this period of 10 weeks. Figure (4.6) shows the time stability of the market group parameters $(\sigma^*, V_0^\delta, V_1^\delta, V_3^\epsilon)$ computed over the same period of ten weeks. The mean values of the group parameters over time period of 10 weeks are:

\[
\sigma^* = 0.2014, \quad V_0^\delta = -0.1306, \quad V_1^\delta = 0.0118, \quad V_3^\epsilon = -0.0003
\]

$\sigma^*$ indicates that the corrected average volatility for ABB stock option. The other parameters ($V^\epsilon$s) are small as predicted by the asymptotic theory.
Figure 4.5: Time stability of the fitted parameters (a0, a1, a2, a3)

Figure 4.6: Time stability of the group parameters ($\sigma^*, V_0^\delta, V_1^\delta, V_3^\epsilon$)

One sees that in Figure 4.6 the parameters $V_1^\delta$ and $V_3^\epsilon$ are stable over the time period of ten weeks with no significant jumps whereas $\sigma^*$ and $V_0^\delta$ shows small and insignificant jumps and this makes the model more reliable.
4.4 Experiment 4: Change in the market group parameters

In this experiment we test how the change in group parameters \((\sigma^*, V_0^\delta, V_1^\delta, V_3^\epsilon)\) affects the approximated implied volatility. The change in implied volatility affects directly the option price of the underlying asset. The asymptotic theory [2] predicts that \((\sigma^*)\) has the significant effect on the approximated implied volatility given that the other parameters \((V's)\) are small. We check if this theory holds for our calibrated parameters.

In the figures below we have increased each of the parameters from 20,40 and 60 percent and observed the changes in implied volatility surface. We analyze the shape of the surface, the skew and smile and the effect on the correlation between implied volatility and the strike price. The calibration is performed on 4th March 2016 on ABB stock options and we have the exercising period ranging from the day the data is extracted to the end of the year 2016.

Figure 4.7: Implied volatility surface with changes of \(\sigma^*\) from 20-60 percent

Figure (4.7) shows that changing \(\sigma^*\) while keeping the other market group parameters constant:

- Significant change in the approximated implied volatility for shorter maturities. Implied volatility decreases as strike price increases.
• Increase in the values of the estimated slope coefficients ($\hat{a}_i$) and the estimated intercept ($\hat{b}_i$).

• Increase in the long-run mean level of the implied volatility from approximately 15 to approximately 27 percent.

• Decrease in the rate of convergence to the long-run mean of implied volatility. This means that $\sigma^*$ affects the fast mean-reverting rate $\epsilon$.

• Change in the volatility skew, we observe an increase in implied volatility for lower strike prices. A change in the skew means that changing $\sigma^*$ affects the correlations $\rho_1$ and $\rho_2$. We can conclude that we observe more fear from investors that the market will crash because they expect the prices to fall, reason why the volatility increases (crashophobia).

![Implied volatility surface with changes of $V_0^\delta$ from 20-60 percent](image)

Figure 4.8: Implied volatility surface with changes of $V_0^\delta$ from 20-60 percent

Figure (4.8) shows that changing $V_0^\delta$ while keeping the other market group parameters constant:

• Do not affect significantly the shape of the implied volatility surface for shorter maturities.
The implied volatility converges fast to its long run mean for longer maturities. This shows that $V_0^\delta$ is related to the slow mean reverting rate.

Affect the skew for longer maturities with higher strike prices. We observe that the implied volatility for shorter maturities remain relatively unchanged, while it decreases significantly for longer maturities.

Since $V_0^\delta$ has a small value therefore it does not affect the shape of the implied volatility surface significantly for a change up to 60 percent.

Figure 4.9: Implied volatility surface with changes of $V_1^\delta$ from 20-60 percent

Figure (4.9) shows that changing $V_1^\delta$ while keeping the other market group parameters constant:

- Affect the rate of mean reversion and the volatility reaches very fast its long run mean.
- We observe higher values of implied volatilities for shorter maturities with same strike prices, whereas for longer maturities we obtain lower
values of implied volatility and the long run mean level has decreased. Changing $V^\delta_1$ affects the volatility skew for longer maturities. This is explained by the fact that $V^\delta_1$ is proportional to $\rho_2$.\[2\]

**Figure 4.10:** Implied volatility surface with changes of $V^\varepsilon_3$ from 20-60 percent

Figure (4.10) shows that changing $V^\varepsilon_3$ while keeping the other market group parameters constant:

- We observed lower implied volatility values for same strikes and maturities, thus the changes in $V^\varepsilon_3$ affects the implied volatility skew for shorter maturities, therefore the correlation $\rho_1$ for shorter maturities is affected, given that $V^\varepsilon_3$ is of order $\sqrt{\varepsilon}$ which is the rate for fast mean reversion.

- The long run mean level for implied volatility is not affected.

Our findings agree with the asymptotic theory because $\sigma^*$ has the highest value of all the other parameters and changing it from 20 to 60 percent has a significant impact on the shape of the volatility surface, as it affects the long-run mean level and the diffusion of the volatility.
4.5 Experiment 5: Effect of $V_3^ε$ on Correlation

Fouque et al [2] proposes that for a positive diffusion $\beta(y)$ and for a positive function $f(y, z)$ and increasing in $y$, the parameter $V_3^ε$ has the sign of $\rho_1$ where $\rho_1$ is the correlation between the fast volatility factor and the stock price of the underlying asset. Here we investigate through experiment that this proposition is true. We compare the change in volatility smile and implied volatility surface obtained from our first calibration (data obtained from the market) with the volatility smile and implied volatility surface in which we change the sign of $V_3^ε$ keeping the other parameters unchanged.
Figures (3.1) (3.2) (negative $V_3^\epsilon$) show that the correlation between the strike price and the implied volatility is negative for maturities 1 to 6 and positive for maturities 7 to 9. For maturity 1 to 6, the implied volatility is high for lower strike prices attaining values of 60 %. As time to maturity increases, the implied volatility decreases. Comparing with figures (4.11) and (4.12), we see that the correlation between the strike price and implied volatility has changed for maturities 1 to 6 while it remained the same for maturities 7 to 9.
Conclusion

Fouque et al\cite{2} made a valuable contribution to the study of option pricing in a multiscale stochastic volatility model with his approximated option price formula derived using an asymptotic expansion method. In the classical model, practitioners had to estimate seven parameters, using complicated methods. With his approximated option price formula, Fouque reduced those parameters to only four market parameters. In this thesis we gave a brief review and explanation of the model, the derivation of the approximated option price formula and the calibration of the parameters needed in the formula done by Fouque.

We implemented the multiscale stochastic volatility model to Swedish market data for ABB stock options and performed five experiments:

- In experiments 1 and 2, studying the fitting of the range of maturities considering first the one-single scale volatility model (only fast volatility factor) then the multiscale stochastic volatility model (fast and slow volatility factors), proved that the latter captures well the implied volatility for the range of maturities. This is explained by the fact that the multiscale stochastic volatility model, to the opposite of the fast volatility factor model, separates the values of the volatility for each strike price maturity wise, thanks to the approximated implied volatility formula (2.34) derived by Fouque et al. \cite{2}. However the model struggles to fit the shortest maturities; the solutions to this problem proposed by Fouque et al. \cite{2} is to use a periodic modulation of the fast time scale with period corresponding to the monthly expiration cycles of traded options and the second-order asymptotic theory.

- In Experiment 3, the study of the calibrated parameters stability over a period of 10 weeks showed that there were no significant jumps in the parameters values. Parameter time stability is an important feature for Fouque’s model because it proves its reliability for future option pricing and hedging purposes.
• The investigation of the effect of the market group parameters ($V$’s) on the volatility surface in Experiment 4 helped to understand the impact of each parameter on implied volatility.

  - Increasing $\sigma^*$ up to 60% affects the overall shape of the volatility surface. The long run mean level of the volatility and the slope of the volatility skew are increased. This proves that $\sigma^*$ is proportional to the correlations between the stock price of the underlying asset and the fast and slow volatility factors respectively.
  - $\sigma_0^*$ affects both the volatility skew and the rate of convergence of the volatility to its long run mean level for longer maturities, however we do not see a significant change in the surface for shorter maturities.
  - $\sigma_1^*$ affects the rate of convergence of the volatility to its long run mean level and the volatility skew for longer maturities. $\sigma_1^*$ is related to the correlation between the slow volatility factor and the stock price of the underlying asset as proposed by Fouque et al. [2].
  - A change in $\sigma_3^*$ decreased the slope of the volatility skew for shorter maturities. This proved that $\sigma_3^*$ is related to $\rho_1$ (correlation between the stock price of the underlying asset and the fast volatility factor).

• In Experiment 5 we studied the proposition by Fouque et al. [2] that $\sigma_3^*$ has the sign of $\rho_1$ in case $\beta(y)$ is positive and $f(y, z)$ is increasing in $y$. The proposition proved to be true for the volatility skew with shorter maturities. Changing the sign of $\sigma_3^*$ from negative to positive changed the slope of the volatility skew for maturity 1 to 6 while maturity 7 to 9 remained unchanged. Our own interpretation is that $\sigma_3^*$, which is of order $\sqrt{\epsilon}$ has a significant effect on the $\rho_1$ for shorter maturities while there may be a change in $\rho_1$ for longer maturities depending on the value of $\sigma_3^*$.

Suggestion for further studies on this topic:

• The time stability of the parameters was studied using a weekly calibration for a period of 10 weeks. A daily calibration of the parameters for at least six months would be helpful to prove whether the parameter values remain relatively stable over a long period.

• Fouque et al. [2] propose that $\sigma_3^*$ has the sign of $\rho_1$ and this was investigated in this thesis. Fouque proposes that $\sigma_1^*$ has the sign of $\rho_2$.
and this was not investigated in this thesis. A further study would be to analyse using real market data if this proposition is true.
Fulfillment of Thesis Objectives

This section is a requirement of the Swedish National Agency for Higher Education to Bachelor theses in Mathematics, mathematical statistics financial mathematics and actuarial science. The section contains five objectives obtained from "How to Write a Thesis?" written by Sergei Silvestrov, Anatoliy Malyarenko and Dmitrii Silvestrov. We will discuss on every objective to explain how the objectives were fulfilled.

Objective 1:
"For Bachelor degree, student should demonstrate knowledge and understanding in the major field of study, including knowledge of the field’s scientific basis, knowledge of applicable methods in the field, specialization in some part of the field and orientation in current research questions."

The thesis is about the pricing of options in a multiscale stochastic volatility model. It uses theory from various areas in mathematics such as probability theory, differential calculus, vector algebra and statistics. The authors discuss the background of the thesis topic in the introduction and literature review sections. From the breakthrough of the Black-Scholes formula [1] to the relatively new multiscale stochastic volatility model introduced by Christoffersen et al. [3], the history of the study on option pricing is given. In mathematics and in other areas of science, models are supposed to reproduce in the best way possible the observations made in the real world. The authors explain the limitations of the Black-Scholes model (skew in the volatility smile observed in real market is not predicted by the model), the introduction and the limitations of a stochastic volatility model by Heston [5](disadvantage of poorly reproducing the relationship between the volatility level and the slope of the smirk)and the multiscale stochastic volatility model, which reproduces better the reality observed in the market. The method used in the thesis, which is similar to Christoffersen’s model, is introduced by Fouque et al [2] and its option price formula is derived with help of an asymptotic expansion technique.

Objective 2:
For Bachelor degree, the student should demonstrate the ability to search,
To complete this project, the authors mainly referred to the book "Multiscale stochastic volatility for equity, interest rate and credit derivatives" by Fouque et al.[2] (chapters 1-5) and different articles obtained online thanks to the license from the Malardalen University’s library. These articles helped to understand the history of the option pricing study up to Fouque’s model.

Empirical studies performed in this project used data from NASDAQ OMX Nordic, particularly the ABB stock options. Data of approximately 120 to 180 market prices of call options were extracted every Friday for a period of 10 weeks. Microsoft Excel was used to process and clean data. Simulation to perform calibration, analysis and interpretation of the results were performed in MATLAB R2015b on a desktop computer running Microsoft Windows 7.

The model used considers an underlying asset driven by two volatility factors, a fast ($Y$) and a slow ($Z$) mean-reverting factor. Given the complexity of estimating the parameters of the model in order to price an option, an asymptotic expansion method is used to derive a first-order approximation option price formula and the implied volatility. The new approximated option price formula uses a reduced number of four calibrated parameters.

**Objective 3:**

*For Bachelor degree, the student should demonstrate the ability to independently identify, formulate and solve problems and to perform tasks within specified time frames.*

The task of the project was to briefly review and explain the multiscale stochastic volatility model and the derivation of the approximated option price formula $P^*$ by Fouque et al [2] using an asymptotic expansion method. We reviewed the calibration formula by Fouque to obtain a group parameters ($\sigma^*, V^0, V^1, V^2$) used in $P^*$. The next task in this project was to evaluate the method that fit better market data using only the fast volatility factor and both the fast and slow volatility factors. Here results showed that the two-factor volatility model captures better the range of maturity. After that the authors analyzed the calibrated parameters time stability obtained from a weekly calibration that was performed on ABB stock options for a period of 10 weeks. The results showed that there were no significant jumps between the values of each parameter; we also give the mean values of the parameters over this period. This stability of the model parameters is a proof that the model is reliable for pricing and hedging purposes. The next task was to investigate the effect of the calibrated parameters on the volatility smile and surface. The procedure was to increase the value of one of the parameters while keeping the other parameters constant, this study helped to observe
that some parameters had an impact on the volatility surface for shorter maturities while other parameters had an impact for longer maturities. This impact consists of a change in volatility skew, a change in correlation between stock price and volatility and a change in the long-run mean of the volatility. In the last task of the project, the authors investigated the effect of the parameter $V_3'$ on the correlation between the stock price and the fast volatility factor ($\rho_1$). Here the goal was to test a proposition by Fouque et al[2] that $V_3'$ has the sign of $\rho_1$. Our results after taking different signs of $V_3'$ proved that the sign of the slope of the volatility smile changes for shorter maturities while it stayed unchanged for longer maturities.

**Objective 4:**

"For Bachelor degree, the student should demonstrate the ability to present orally and in writing and discuss information, problems and solutions in dialogue with different groups."

In this thesis the authors have dedicated two chapters in order to draw conclusions for the main objective of the project. In Chapter 3 it is presented how the data was collected and used in the model, moreover the method of calibration is presented in a way that the reader can easily follow and graphs of data were labeled and captioned with the help of "Latex" and for the graphical parts of the thesis MATLAB software program was used to ensure a professional look to the thesis. A separate appendix is attached to the thesis for the MATLAB codes.

An oral presentation will take place in June 10th 2016, where the authors will explain their solutions and research results to the audience, there will be facility for the audience to ask question regarding the research outcomes.

**Objective 5:**

"For bachelor degree, the student should demonstrate ability in the major field of study make judgments with respect to scientific, societal and ethical aspects."

In Chapter 1 and 2 of the thesis the authors have introduced the major theory part of the project, In Chapter 3 the data and the method of calibration and some results from calibration are described with the use of real market data, In the theory part of the thesis the citation is clearly mentioned whenever it was needed. In Chapter 4 different experiments were applied on the parameters including the model and the results were demonstrated and explained in a manner where the reader can easily follow and understand the topic. Eventually the authors have presented their ability of using the field of mathematics in finance. In the acknowledgments section the authors show their gratitude to the people who contributed in the achievement of this project.
References


[9]. NASDAQ OMX Nordic (http://www.nasdaqomxnordic.com)

[10]. Rubinstein, M. "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23,1976 through August 31,1978" Journal-


Appendix A

Black-Scholes Implied Volatility and Data Cleaning

The following scripts and data were used in MATLAB in order to compute Black-Scholes implied volatility and log-moneyness to maturity ratio, to facilitate the calibration procedure of the parameters. For the Black-Scholes implied volatility, we used the built-in MATLAB function blsimpv(s,k,r,tau,u), which returns the implied volatility given the spot price, strike price, risk-free interest rate, time to maturity and option price.

Data cleaning is needed to remove the implied volatilities that do not have a number (NaN) and their corresponding strike prices and time to maturities.

```matlab
r = 0.05;
s = 149.3;
k = [125 130 135 140 145 150 155 127.5 132.5 137.5 142.5 147.5
    125 130 ... 135 140 150 155 127.5 132.5 137.5 142.5 147.5 125
    130 135 140 ... 145 150 155 127.5 132.5 137.5 142.5 147.5 135 140
    145 150 155 160 ... 170 137.5 142.5 147.5 100 110 120 125 130 135
    140 145 150 155 160 ... 165 127.5 132.5 137.5 142.5 147.5 125 130
    135 140 145 150 155 160 ... 165 170 127.5 132.5 137.5 142.5 147.5
    90 100 110 115 120 125 130 135 ... 140 145 150 160 170 180 90 100
    110 120 130 140 150 170 190 90 100 ... 110 120 130 140 150 170 190
    210];
tau_days = [6111621315196161226];
dayConvention = 252;
time = tau_days/dayConvention;

tau = [0.0238 0.0238 0.0238 0.0238 0.0238 0.0238 0.0238 0.0238
    0.0238 0.0238 ... 0.0238 0.0238 0.0437 0.0437 0.0437 0.0437 0.0437
    0.0437 0.0437 0.0437 ... 0.0437 0.0437 0.0437 0.0437 0.0635 0.0635
```
u = [24.75 19.75 14.875 10 5.75 2.175 0.6 22.25 17.5 12.625 7.75 3.85 24.75 ... 9.125 7.25 4.5 25 ... 20 15.375 10.875 6.75 3.65 1.65 22.5 17.75 13.125 8.75 5.125 15.625 ... 11.125 7.25 4.175 2.1 0.9 0.165 13.375 9.25 5.5 49.625 39.875 29.875 ... 25.25 20.25 15.75 11.75 8.125 5 2.925 ... 1.5 0.725 22.75 18.125 13.625 ... 9.875 6.375 25.5 20.875 16.5 12.875 9.375 6.375 4.25 2.625 1.5 0.9 ... 23.125 18.75 14.75 11.125 7.875 ... 59.875 50.125 40.375 35.625 31.125 ... 40.375 31.25 23.125 15.625 10 3.4 1.1 59.75 49.875 40.375 31.125 16.125 11.125 4.375 1.725 0.65];

lmmr = log(k/s) ./ tau;

impliedvol = blsimpv(s, k, r, tau, u);

isnan(impliedvol);
d = find(isnan(impliedvol) == 0);
impliedvol(d) = [];
lmmr(d) = [];
tau(d) = [];
k(d) = [];
u(d) = [];

x1 = find(tau == 0.0238);
tau1 = tau(x1);
lmmr1 = lmmr(x1);
impliedvol1 = impliedvol(x1);
k1 = k(x1);

x2 = find(tau == 0.0437);
tau2 = tau(x2);
lmmr2 = lmmr(x2);
impliedvol2 = impliedvol(x2);
k2 = k(x2);

x3 = find(tau == 0.0635);
tau3 = tau(x3);
lmmr3 = lmmr(x3);
impliedvol3 = impliedvol(x3);
k3 = k(x3);

x4 = find(tau == 0.0833);
tau4 = tau(x4);
lmmr4 = lmmr(x4);
impliedvol4 = impliedvol(x4);
k4 = k(x4);

x5 = find(tau == 0.123);
tau5 = tau(x5);
lmmr5 = lmmr(x5);
impliedvol5 = impliedvol(x5);
k5 = k(x5);

x6 = find(tau == 0.2024);
tau6 = tau(x6);
lmmr6 = lmmr(x6);
impliedvol6 = impliedvol(x6);
k6 = k(x6);

x7 = find(tau == 0.381);
tau7 = tau(x7);
lmmr7 = lmmr(x7);
impliedvol7 = impliedvol(x7);
k7 = k(x7);

x8 = find(tau == 0.6389);
tau8 = tau(x8);
lmmr8 = lmmr(x8);
impliedvol8 = impliedvol(x8);
k8 = k(x8);

x9 = find(tau == 0.8968);
tau9 = tau(x9);
lmmr9 = lmmr(x9);
\text{impliedvol9}=\text{impliedvol}(x9);
k9=k(x9);

\text{LMMR} = \text{lmmr1}; \text{lmmr2}; \text{lmmr3}; \text{lmmr4}; \text{lmmr5}; \text{lmmr6}; \text{lmmr7}; \text{lmmr8}; \text{lmmr9};
K= k1; k2; k3; k4; k5; k6; k7; k8; k9;
I= \text{impliedvol1}; \text{impliedvol2}; \text{impliedvol3}; \text{impliedvol4}; \text{impliedvol5}; \text{impliedvol6}; \text{impliedvol7}; \text{impliedvol8}; \text{impliedvol9};
Appendix B

Calibrated Parameters of the First-Order Approximated Implied Volatility

The function was used to compute the parameters using the given input data.
This function calibration(LMMR,I,time) follows the previous script (A) and it computes the parameters of the implied volatilities. It includes the function lsqcurvefit(fun,pguess,x1,x2), which uses least-square method to solve nonlinear curve-fitting problems.

function [a0,a1,a2,a3]=calibration(LMMR,I,time)

    for i=1:length(time);
        x1= LMMRi; x2=Ii;

        fun=@(u,x1) u(1)*x1+u(2);
        uguess = [x1(1),x2(1)];
        u = lsqcurvefit(fun,uguess,x1,x2);
        a(i)=u(1); b(i)=u(2); end

        fun1=@(v,time) v(1)*time+v(2);
        vguess = [time(1),a(1)];
        v = lsqcurvefit(fun1,vguess,time,a);
        m=v(1);
        n=v(2);
fun2=@(w,time) w(1)*time+w(2);
wguess = [time(1),b(1)];
= lsqcurvefit(fun2,wguess,time,b);
o=w(1);
p=w(2);

\[
a0= m;\]
\[
a1=n;\]
\[
a3 = o;\]
\[
a2= p;\]

\[
\text{coeff} = [a1 \, a0 \, a3 \, a2];\]
end
Appendix C

Implied Volatility using Calibrated Parameters

The function below was used to approximate the implied volatility using the equation $I \approx b^* + \tau b^* + (a^* + \tau a^*)LMMR$ and plot the volatility smile and apply polynomial fitting to the volatility smile.

```matlab
function [IV]=ImpVol(a0,a1,a2,a3,s,time)
    this function will use equation (I=a1+a0tau+(a3+a2tau)LMMR) to compute approximated implied volatility. and plot the volatility smile using polynomial fittings.

    Inputs:
a0,a1,a2,a3,s,time
    Outputs
    IV

    K=[125:(210-125)/29:210];
    lmmr=log(K./s)./time(1);
    B=a3;
    C=time(1)*a2;
    D=B+C;
    E=D';
    W=E.*lmmr;
    a=ones(length(K),1).*a1;
    a=a';
    b=time(1).*a0;
    BB=ones(length(K),1).*b;
    BB=BB';
```
IV=(a+BB+W);

    K=K';
IV=IV';
load census;
fitpoly2=fit(K,IV,'poly2')
Plot the fit with the plot method.
plot(fitpoly2,K,IV)
Move the legend to the top left corner.
xlabel('Strike Price','FontSize',14,'FontWeight','bold','Color','b')
ylabel('Implied Volatility','FontSize',14,'FontWeight','bold','Color','b')
title('Time to maturity')
end
Appendix D

Implied Volatility Surface

The following script will generate the implied volatility surfaces found in experiments 1, 2 and 3.

The script will plot the volatility surface using equation \( I = a_1 + a_0 \tau + (a_3 + a_2 \tau) \text{LMMR} \) to compute approximated implied volatility and plot the volatility surface.

\[
K = [125: (210 - 125)/29:210];
\]

\[
\text{Tau} = [0.896825396825397: (0.0238095238095238 - 0.896825396825397) / (\text{length(kkk)} - 1): 0.0238095238095238];
\]

\[
\text{lmmr} = \log(K./s)./\text{Tau};
\]

\[
B = \text{ones(size(Tau))}.*a_3;
\]

\[
C = \text{Tau}.*a_2;
\]

\[
D = B + C;
\]

\[
E = D';
\]

\[
w = E*\text{lmmr};
\]

\[
a = \text{ones(length(K))}.*a_1;
\]

\[
b = \text{Tau}.*a_0;
\]

\[
BB = \text{ones(length(K),1)}*b;
\]

\[
\text{IV} = (aaa + BB + www);
\]

\[
surf(\text{Tau},K,\text{IV})
\]

title('Implied Volatility Surface')

xlabel('Time to maturity', 'FontSize', 14, 'FontWeight', 'bold', 'Color', 'b')
ylabel('Strike price', 'FontSize', 14, 'FontWeight', 'bold', 'Color', 'b')
zlabel('Implied volatility', 'FontSize', 14, 'FontWeight', 'bold', 'Color', 'b')