Phasing Out a Polluting Input in a Growth Model with Directed Technological Change∗

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Abstract

This paper explores the potential conflict between economic growth and the environment, and the optimal long-run environmental policy. It formulates a growth model with directed technological change and focuses on the case with low elasticity of substitution between clean and dirty inputs in production. New technology is substituted for the polluting input, which results in a gradual decline in pollution along the optimal long-run growth path. In contrast to some recent work, the era of pollution and environmental policy is here not just a transitory phase in economic development. This result means that the government’s continuous efforts to reconcile economic growth and the environment will always be needed. The socially optimal policy includes a perpetual subsidy to ‘green’ research. The tax rate of pollution is monotonously increasing, while the pollution tax payments constitute a constant share of income. These policies result in a quite modest growth drag.

Keywords: Directed Technological Change, Pollution, Energy substitution, Growth drag.

JEL classification: O30, O31, O33, C65.

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1 Introduction

Economic growth is potentially harmful for the environment. This possible conflict can however be mitigated by technological change, for example in the process where clean production inputs are substituted for polluting inputs. Innovations with such capacities are not least important in the transformation of the energy system, where fossil fuels are gradually replaced by renewable sources of energy, such as solar and wind power. To analyze various issues related to these aspects of economic development, endogenous growth models have been presented where innovative efforts can be directed to make production cleaner. Such models are in particular useful for analyses of the optimal long-run environmental policy.

In a very influential article Acemoglu et al. (2012) (hereafter: AABH) use such a model to analyze the problem of climate change. Focusing on the case with a high elasticity of substitution between clean and polluting inputs in production, they show that a sustainable long-run growth can be obtained by merely a temporary policy that promotes ‘clean’ innovations. After this transitory period the clean input/technology will be superior, since it has become more competitive due to learning effects. This means that the era of pollution and environmental policy becomes just a limited episode in economic development. The policy recommendations from this analysis put a remarkably strong emphasis on subsidies to research in clean technologies.

\footnote{A higher material throughput tends to result in more emissions of wastes that are directly unhealthy and also damaging to ecosystems upon which we rely. The environmental history of the 20th century, boldly summarized in Table 12.1 of McNeill (2001), indeed shows many disturbing trends, for example in terms of lost species and increasing pollution. See also Rockström et al. (2009) for warnings that the trend starting with the Industrial Revolution may be leading to serious environmental instability.}

\footnote{See Yergin (2011) and Stern (2015) for broad expositions of recent developments in renewable-energy technologies.}

\footnote{This literature builds on Acemoglu (1998) and Acemoglu (2002). An early application to environmental issues is Grimaud and Rouge (2008). Another early article that considers both ordinary and environmentally-oriented research is Hart (2004). Smulders and de Nooij (2003) uses directed technological change to analyze the problem of energy conservation.}

\footnote{A subsequent paper, Acemoglu et al. (2016), even assumes an infinite elasticity of substitution in a model with more detailed descriptions of innovations, employment and production. An earlier paper in the same spirit is Tahvonen and Salo (2001), but their production structure is simpler and technological progress comes from a simple learning-by-doing effect.}
while the Pigouvian tax on pollution is quite moderate. The motivation for this is that the effects of the positive knowledge externalities in ‘clean’ research are so strong that it is not worth disturbing present production by a high pollution tax. These findings are closely linked to the assumption of a high elasticity of substitution.

This paper adds to the insights in AABH by exploring the case with low substitution possibilities in a somewhat different model. This generates a quite different growth path and very different policy implications. The case with low substitution possibilities is interesting in the light of the meta analysis in Stern (2012), which provides empirical support for the assumption of a low elasticity of substitution between different types of energy. The first important respect in which the model differs from AABH is that I here make a clear distinction between the ‘energy’ inputs and the technology components that are complementary to them.\(^5\) This explicit modeling of the saving of some of the polluting input by technological change is necessary to make feasible the phasing out of a polluting input in the case of complementarity. By contrast, AABH put all this together in a composite input, which makes it impossible to substitute technology for dirty energy.\(^6\) Second, the disutility of pollution is in AABH derived from the polluting input and the machines that are used together with it. Here I make the more natural assumption that the disutility stems solely from the dirty input.

A result of these modifications is that the efforts to reduce the use of the polluting input go on forever here. In AABH, the green technology comes out as the winner once and for all, after an intense initial effort to make it competitive. Thereby the conflict between growth and the environment is forever eliminated in the baseline case of their model.\(^7\) The assumptions

\(^5\)Although the inputs will be called (clean and dirty) energy the pollutants in this paper may be other substances than greenhouse gases, such as sulphur dioxide, tropospheric ozone and particles, which means that the environmental problem addressed here is not necessarily climate change. The model can thus be used to analyze the relationship between economic growth and pollution in general.

\(^6\)This choice between various ways of modeling the source of pollution in the model echoes a long-standing theme in the literature. For example, in Michel and Rotillon (1995) pollution is proportional to final output, while it is proportional to a polluting input in Schou (2000).

\(^7\)AABH have a middle case called ‘weak substitutes’, which means that the elasticity of substitution is higher than 1 but not sufficiently high. It has properties that are partially similar to what is found here, but even in this case clean innovations will become dominant in finite time so that subsidies to research are no longer needed. The tax on GHG emissions
chosen here imply that it will always be costly to keep pollution low and that the government’s continuous efforts to reconcile economic growth and the environment will always be needed. Behind this choice is the notion that it will always be possible to produce at lower costs by being more careless with the wastes of production.

An other notable difference in the results, compared to AABH, is that the optimal growth path has a considerably different character. Here the polluting input will be used all along the long-run growth path, although in a gradually decreasing quantity. The decline in the use of the dirty input is possible because of the growth in the technology factor that is complementary to it, which makes the use of the dirty input more effective. The intense research that makes this possible has an opportunity cost in terms of slower growth in a second technology factor, with the consequence that income growth is reduced. This is an instance of the tradeoff between growth and the environment that is so prevalent in models of this kind.

An expression is derived for the magnitude of the reduction in the growth rate that the optimal policy to phase out pollution causes. Some simple calculations indicate that this long-run growth drag may be quite modest or at least acceptable in the light of the improved environment: it is estimated that between 10% and 30% of the long-run growth rate is lost when the optimal environmental policy is implemented.

There are also considerable differences in the policy implications between this paper and AABH. First, the optimal pollution tax rate is more prominent here. It is monotonously increasing over time but the pollution tax payments will constitute a constant share of GDP on the long-run growth path, due to the downward trend in the quantity of pollution. Secondly, the optimal policy package includes a perpetual subsidy to clean research, in a range that is estimated to run from 15% to 30% of the cost. This contrasts with the result in AABH, where the corresponding subsidy declines to zero in finite time.

Economists have examined the relation between growth and pollution at least since the start of the limits-to-growth debate that followed the publication of Meadows et al. (1972). Two early articles that analyze the dampers that the optimal management of pollution puts on growth are not temporary. They do not analyze what time paths pollution and the tax take in this case.
Keeler, Spencer and Zeckhauser (1972) and Brock (1977). Since these models lack endogenous technological change it is not possible to analyze how different policy instruments can be used to stimulate research and development. For examples of early endogenous growth models with pollution, see Bovenberg and Smulders (1995) and Stokey (1998). An analysis of environmental problems in a growth model with creative destruction is found in Aghion and Howitt (1998, ch. 5). Grimaud (1999) describes a policy that makes the decentralized market economy follow the socially optimal growth path in this model. Reviews of this large literature are found in Brock and Taylor (2005), Xepapadeas (2005) and more recently in Smulders et al. (2014).

In addition to the papers cited above, the analysis in Golosov et al. (2014) of how an optimal tax on greenhouse gases should be designed is very interesting in this context. That paper follows the real-business-cycle literature more than the growth literature, thus excluding endogenous technological change. It includes, however, a quite detailed description of the carbon cycle. The model is used to simulate different possible development paths for the economy and for the climate. A similar analysis is performed by van der Ploeg and Withagen (2014), with more general assumptions about for instance the utility function and the costs of extracting oil.

Gerlagh, Kverndokk and Rosendahl (2014) study the optimal time path of an economic policy that is intended to support the development of clean energy. In addition to the policy instruments that have been mentioned above, they also use the patent life length. In a very interesting article Jones (2016) develops a growth model where some technologies are beneficial while others are harmful and even life-threatening. This leads to a tradeoff between safety and consumption growth, which may result in a consumption growth that is much lower than what is feasible.

Finally, the optimal solution to the model involves a kind of transition between two energy regimes. This puts an element of non-balanced growth into the model. Therefore the analysis is also based on the works by Kongsamut et al. (2001) and Acemoglu and Guerrieri (2008), where growth is balanced only asymptotically.

This paper proceeds as follows. The model is presented in Section 2.
Section 3 derives and analyzes the conditions for social optimum. In Section 4 the decentralized solution is derived and Section 5 develops the policies required to attain social optimum in a decentralized economy. Section 6 extends the model to stock pollution and Section 7 concludes the paper. Some derivations are put into Appendix A, while some longer (and standard) derivations are found online in Appendixes C and B.

2 The Model

The production structure of the model is very similar to that in Acemoglu (2002), and it will therefore be presented quite briefly. There are three types of firms, producing: (1) final output; (2) intermediate inputs; and (3) machines that are combined with ‘energy’ to produce the intermediate inputs. In addition, there are also innovating firms.

Final output is produced by a large number of firms in a competitive environment. They all use labor, \( L \), and two composite inputs, \( Y_D \) and \( Y_Z \). The representative production function is

\[
Y(t) = AL^{1-\alpha} \left[ \gamma_D Y_D(t)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma_Z Y_Z(t)^{\frac{\varepsilon-1}{\varepsilon}} \right]^\alpha. \tag{1}
\]

It is assumed that \( 0 < \alpha < 1 \), \( 0 < \gamma_D < 1 \), \( 0 < \gamma_Z < 1 \), \( \gamma_D + \gamma_Z = 1 \), and \( 0 < \varepsilon < \infty \). This is a constant-returns-to-scale function in \( L \), \( Y_D \) and \( Y_Z \). All technological change takes place in \( Y_D \) and \( Y_Z \), but since the overall function is a Cobb-Douglas, the improvements in technology spills over to labor as well.

The input \( Y_D \) is produced by a large number of competitive firms, using the factor \( D \), and a range of ‘machines’, \( x_D \), which are complementary to it. Similarly, producers of \( Y_Z \) use the factor \( Z \) and a number of \( Z \)-complementary machines, \( x_Z \). Assume the Cobb-Douglas form, such that

\[
Y_K = \frac{1}{1-\beta} \left[ \int_0^{N_K} x_K(j)^{1-\beta} dj \right] K^\beta, \quad K = D, Z. \tag{2}
\]

The variables \( N_D \) and \( N_Z \) denote the numbers of varieties of the machines. They grow endogenously, due to research efforts. This is the technological change of the model.

Now some differences from the previous literature can be pointed out. In contrast to Acemoglu (2002), the factors \( D \) and \( Z \) are here variable. They are both produced at constant extraction costs, in terms of the final output.
The unit cost of \( Z \) is \( c_Z \), whereas the unit cost of \( D \) is \( c_D \). Moreover, the input \( D \) generates a negative externality. A crucial difference between this model and AABH is that their entire \( Y_D \) is polluting, while only a part of it (\( D \)) is dirty here. The polluting part can here be saved by expansion of the technology factor \( N_D \), which does not happen in their model. This is essential for the possibility of phasing out the dirty input, by use of directed technological change, in the case with complementarity in the production function.

For concreteness, one can think of \( D \) as fossil energy and \( Z \) as renewable energy. The former clearly consists of non-renewable resources which run the risk of encountering problems of limited supply. This problem is however avoided by the result that \( D \) is declining in the socially optimal equilibrium, combined with the fact that non-renewable energy actually is available in huge quantities. Although petrol may be close to a production peak (or plateau), there remain large reserves of coal, fracking oil and natural gas; see Smil (2003) and Yergin (2011). In addition, environmental concerns will likely leave considerable amounts of the fossil fuels in the ground. For simplicity, the differences between the various sorts of fossil energy, with respect to polluting effects and production costs, will not be explicitly modeled here.\(^9\)

Firms in the third category, producing the machines \( x_K(j) \), all have a perpetual monopoly to produce their output. It costs \( \vartheta \) units of final output to produce one \( x_K(j) \), and for simplicity of notation it is assumed that \( \vartheta = 1 - \beta \). The \( x_K(j) \)'s are totally worn out directly after use, so there is a continuous reinvestment in machines.

There is an exogenous supply of scientists, \( S \). Denote the number of scientists that are involved in the development of blue-prints for the \( D \) sector by \( S_D \) and those who do research for the \( Z \) sector by \( S_Z \). The innovation

\(^9\)For examples on how a multitude of pollutants can be aggregated into one variable in practical CGE modelling, see Dellink et al. (2004).

\(^{10}\)Another simplification is that there exist no private property rights to any type of energy. The user costs are then simply \( c_D \) and \( c_Z \), respectively. In other words, these costs do not reflect any scarcity value and the price is not determined by the Hotelling rule. Based on results in earlier research, e.g. AABH, one can expect that an assumption of scarcity would increase the rate of research in the direction that makes production cleaner.
Possibilities frontier (IPF) is
\[
\dot{N}_D = \eta_D N_D f_D (S_D) \quad \text{and} \quad \dot{N}_Z = \eta_Z N_Z f_Z (S - S_D),
\]
where the constraint \( S = S_D + S_Z \) has been used and the parameters \( \eta_D \) and \( \eta_Z \) capture the levels of productivity in the two research sectors. It is assumed that \( f'_D > 0, f'_Z > 0, f''_D \leq 0 \) and \( f''_Z \leq 0 \). The presence of the knowledge stocks on the right-hand sides as usual means that innovators benefit from a knowledge base, built by earlier innovations.

Final output is used in consumption, \( C \), investment, \( X \), and for expenditures on energy, \( E \), which gives the equation \( Y = C + X + E \). The use of output for energy extraction is \( E = c_D D + c_Z Z \). Therefore the constraint can be rewritten as
\[
C = Y - (1 - \beta) \int_0^{N_D} x_D(j) dj - (1 - \beta) \int_0^{N_Z} x_Z(j) dj - c_D D - c_Z Z. \tag{4}
\]

The number of members in the representative household is \( L + S = M \), a constant (the population is thus stationary). Each member supplies one unit of labor. The utility function is
\[
U = \int_0^\infty e^{-\rho t} M \left[ \frac{(C/M)^{1-\theta} - 1}{1 - \theta} - \zeta \frac{D^{1+\kappa}}{1 + \kappa} \right] dt. \tag{5}
\]
It is assumed that the constants \( \rho, \kappa, \zeta \) and \( \theta \) are non-negative. The second term captures the disutility from pollution.\(^{11}\)

The economy embarks on a socially optimal growth path right from the beginning, starting from a state of the environment that may be problematic but not close to catastrophic. This implies a declining flow of pollution and, possibly with a delay, also a shrinking stock of pollution.\(^{12}\) As demonstrated in Section 6, the qualitative differences between a model with flow pollution and a model with stock pollution are then fairly small in the long run. For simplicity, pollution is therefore treated as a flow variable in this basic case.

\(^{11}\)Following Nordhaus (2008), Acemoglu et al. (2016) and Golosov et al. (2014) instead model the cost of pollution as lost production. Similarly, in a model with endogenous technological change, Bovenberg and Smulders (1995) let production output depend on natural capital which in turn is negatively affected by pollution.

\(^{12}\)For an illustration of such a path, in the context of global warming, see Figure 1.2 in Stern (2015).
3 Social Optimum

3.1 Static Allocation

For a considerable simplification of the exposition the solution can be analyzed in two steps. First, the output net of investment,

$$\tilde{Y} = Y - (1 - \beta) \int_0^{N_D} x_D(j) dj - (1 - \beta) \int_0^{N_Z} x_Z(j) dj,$$

is maximized with respect to \{x_D(j)\}_0^{N_D} and \{x_Z(j)\}_0^{N_Z}, subject to (1) and (2). This gives the static allocation of machines at given \(N_K\):s. According to Appendix A.1, all \(x_D(j)\) are equal in optimum and all \(x_Z(j)\) are equal in optimum. Using the obtained equations, the production function is reduced to

$$Y = A_0 \cdot \left[ \gamma_{D}^{\frac{\sigma-1}{\sigma}} (N_D) \right] + \gamma_{Z}^{\frac{\sigma-1}{\sigma}} (N_Z) - \frac{\alpha\beta}{1 - \alpha + \alpha\beta} \cdot (1 - \alpha + \alpha\beta),$$

where

$$A_0 = \left( \frac{AL^{1-\alpha} \alpha^\alpha (1-\beta)}{(1 - \beta) \alpha} \right)^{\frac{1}{1 - \alpha + \alpha\beta}}.$$

The constant \(\sigma = 1 + \beta(\varepsilon - 1)\) is an elasticity of substitution, which is made clear by equation (14) below. It also turns out that \(\tilde{Y} = Y - \alpha(1 - \beta)Y\), which means that (4) can be rewritten as

$$C = (1 - \alpha + \alpha\beta)Y - c_D D - c_Z Z.$$

The first term on the right-hand side of this equation is the output net of machine investments.

With \(\sigma\) now defined, three important assumptions in this paper can be stated:

**Assumption 1:** (i) \(\sigma < 1\)  (ii) \(\theta > 1\)  (iii) \(1 - \alpha > \alpha\beta\).

The assumption that \(\sigma < 1\) is supported by empirical evidence in Stern (2012), where a meta analysis of 47 studies of interfuel substitution is presented. The unweighted average of the elasticities is below 1 and many of them are not statistically different from zero. Although the elasticity of substitution appears to be high in some situations, it may be more difficult to substitute between renewable and non-renewable energy in other parts of the economy. The differences could be explained by variations in the needs.
for complementary investments. Hence the elasticity of substitution can be lower in the aggregate than on the micro level and this is what transpires from Stern (2012).

There seems to be a fairly strong consensus that the intertemporal elasticity of substitution in consumption, $1/\theta$, is low, so that $\theta$ is larger than 1. For instance, Hall (1988), Attanasio and Weber (1993) and Hahm (1998) provide empirical evidence which supports this assumption. While the bulk of the evidence points in this direction, it should be noted that there are some studies that report lower values, for instance Vissing-Jørgensen and Attanasio (2003).

Finally, $1 - \alpha$ is the cost share of labor in production, which usually is around 2/3. Meanwhile $\alpha \beta$ is the cost share of energy, including the damage cost, as can be seen in Footnote 18. When the environmental damage cost is excluded this share is typically below 1/10. The addition of a substantial damage cost cannot reasonably raise the total of $\alpha \beta$ to anything close to 2/3 and therefore the third assumption is quite safe.\textsuperscript{13}

It is well known from for instance the literature on exhaustible resources, that the slowest growing (composite) input determines the growth rate of production when $\sigma < 1$ in a function like (6). This is because both inputs then are essential in production. See Appendix B.1 for a demonstration of this and Dasgupta and Heal (1979) for further discussions. For sustained growth, under our assumption that $\sigma < 1$, it is thus necessary that both $N_D D$ and $N_Z Z$ grow in the long run, and it would be wasteful if these products were growing at different rates.\textsuperscript{14}

\subsection{3.2 Conditions for Optimal Growth}

In the second step of the derivation of the optimal solution, the utility function in (5) is used together with the two constraints from (3), to form the current-value Hamiltonian

$$
\mathcal{H} = M \left[ \frac{(C/M)^{1-\theta} - 1}{1 - \theta} - \zeta D^{1+\kappa} \right] + \mu_D \eta_D N_D f_D (S_D) +
$$

\textsuperscript{13}The bracket in (6) exhibits decreasing returns to scale in the four variables $N_D$, $D$, $N_Z$ and $Z$ if $\frac{\alpha \beta}{1 - \alpha + \alpha \beta} < \frac{1}{2}$, i.e. if $1 - \alpha > \alpha \beta$.

\textsuperscript{14}Moreover, although $Z$ may appear to play a peripheral role in the analysis below, it is clear that we cannot disregard the clean variable (e.g. by putting $N_Z Z$ equal to a constant or even equal to zero) and just focus on technological progress that makes the use of $D$ more effective.

10
\[ + \mu_Z \eta_Z N_Z f_Z (S - S_D), \]

where \( \mu_D \) and \( \mu_Z \) are the co-state variables associated with \( N_D \) and \( N_Z \), respectively. The control variables in this problem are \( D \), \( Z \) and \( S_D \), while the state variables are \( N_D \) and \( N_Z \), with the initial values \( N_D(0) \) and \( N_Z(0) \). Finally, equations (6) and (7) must be taken into account when the Hamiltonian is maximized.

Assume that the functions \( f_D \) and \( f_Z \) are sufficiently steep near the origin to avoid a corner solution.\(^{15}\) The first-order condition for social optimum, with respect to \( S_D \), is then

\[ \mu_D \eta_D N_D f'_D (S_D) = \mu_Z \eta_Z N_Z f'_Z (S - S_D), \quad (8) \]

which is a requirement that the values of the marginal products of scientists should be equal in both research sectors.

Maximization of the Hamiltonian with respect to the energy inputs \( Z \) and \( D \) requires that

\[ \alpha \beta Y \gamma^\frac{\sigma - 1}{\sigma} (N_D D) = c_D + \zeta M^{1-\theta} C^{\theta} D^\alpha, \quad (9) \]

\[ \alpha \beta Y \gamma^\frac{\sigma - 1}{\sigma} (N_Z Z) = c_D + \zeta M^{1-\theta} C^{\theta} D^\alpha, \quad (10) \]

where

\[ [\bullet] \equiv \left[ \gamma^\frac{\sigma - 1}{\sigma} (N_D D) + \gamma^\frac{\sigma - 1}{\sigma} (N_Z Z) \right]. \]

In both cases the conditions say that the marginal product of the energy input, net of investment, should equal its cost. In equation (10) the cost includes, in addition to the extraction cost \( c_D \), the damage cost of pollution in terms of consumption \( \zeta M^{1-\theta} C^{\theta} D^\alpha = -\frac{\partial U}{\partial D}/(\partial U/\partial C) \).

The next pair of conditions are concerned with the necessary changes of the co-state variables.\(^{16}\) Using (3), (9) and (10), they are simplified to

\[ g_{\mu_D} = \rho - \frac{(C/M)^{-\theta}}{\mu_D N_D} \cdot \left( c_D + \zeta L^{1-\theta} C^{\theta} D^\alpha \right) D - g_{N_D} \quad (11) \]

\(^{15}\)An initial corner solution, with research in just one direction, cannot generally be ruled out. This would however only be for a transitory period because both \( N_D \) and \( N_Z \) must grow in the long run. To simplify the exposition we disregard this less interesting possibility by assuming (if needed) that \( \lim_{S_K \to 0} f_k (S_K) = \infty \) for \( K = D, Z \).

\(^{16}\)They read \( \dot{\mu}_D = \mu_D \rho - (C/M)^{-\theta} \alpha \beta Y \cdot [\bullet]^{-1} \gamma^\frac{\sigma - 1}{\sigma} (N_D D) \gamma^\frac{\sigma - 1}{\sigma} N_D^{-1} - \mu_D \eta_D f_D (S_D) \) and \( \dot{\mu}_Z = \mu_Z \rho - (C/M)^{-\theta} \alpha \beta Y \cdot [\bullet]^{-1} \gamma^\frac{\sigma - 1}{\sigma} (N_Z Z) \gamma^\frac{\sigma - 1}{\sigma} N_Z^{-1} - \mu_Z \eta_Z f_Z (S - S_D) \).
and

\[ g_{\mu Z} = \rho - \frac{(C/M)^{-\theta}}{\mu Z N_Z} \cdot c_Z Z - g_{N Z}. \]  

Throughout, the notation \( g_x \) is used to denote the growth rate of any variable, \( x \). Finally, the transversality conditions are

\[ \lim_{t \to \infty} e^{-\rho t} \mu_K N_K = 0, \quad K = D, Z. \] (13)

### 3.3 Balanced Growth

To simplify the subsequent expressions, the following composite variables are defined:

\[ \Delta \equiv \frac{N_Z Z}{N_D D}, \quad \omega \equiv \zeta M^{1-\theta} C^\theta D^\kappa \quad \text{and} \quad \xi \equiv N_Z^{\alpha \beta} Z^{\alpha - 1}. \]

Before stating the properties of the Balanced Growth Path (BGP) we develop a few expressions that provide some intuition for the results. By forming the ratio between (9) and (10) and defining \( \gamma \equiv \gamma_Z / \gamma_D \) the relative demand of \( Z \) and \( D \) is obtained:

\[ \frac{Z}{D} = \gamma^{\varepsilon} \left[ \frac{c_Z}{c_D + \omega} \right]^{-\sigma} \left( \frac{N_Z}{N_D} \right)^{\sigma - 1}. \] (14)

Since the expression within bracket is the relative cost of the two energy inputs, this shows how \( \sigma \) is an elasticity of substitution at given \( N_Z/N_D \). It follows from this equation that the relative demand for the dirty input, \( D \), declines as the pollution cost, \( \omega \), increases (\textit{ceteris paribus}).

Since the right-hand side of equation (9) is constant it follows that the marginal product of \( Z \) (net of investment) is unchanged over time. To eliminate \( Y \) from (9), the production function\(^{17} \) from (6) is used and this renders

\[ \alpha \beta A_1 \left( \gamma - \frac{\varepsilon}{\gamma} (\Delta)^{-\frac{\varepsilon - 1}{\sigma}} + 1 \right) \left[ \frac{\sigma^{\frac{\varepsilon - 1}{\sigma - 1}}}{\frac{\varepsilon}{\sigma - 1}} \frac{\sigma^{\frac{\varepsilon - 1}{\sigma - 1}}}{\frac{\varepsilon}{\sigma - 1}} \right]^{-1} \cdot \xi^{\frac{1}{1 - \alpha \beta \gamma}} = c_Z, \] (15)

where \( A_1 \equiv A_0 \cdot \gamma_Z^{\frac{\varepsilon}{\gamma - \frac{\varepsilon}{\gamma} (\Delta)^{-\frac{\varepsilon - 1}{\sigma}} + 1}} \left[ \frac{\sigma^{\frac{\varepsilon - 1}{\sigma - 1}}}{\frac{\varepsilon}{\sigma - 1}} \frac{\sigma^{\frac{\varepsilon - 1}{\sigma - 1}}}{\frac{\varepsilon}{\sigma - 1}} \right]^{-1} \cdot \xi^{\frac{1}{1 - \alpha \beta \gamma}}. \) This suggests that \( \Delta \) and \( \xi \) must be constant on the BGP.

By combining (9) and (10), the total cost of energy, including the externality, can be expressed as a share of output: \( c_D D + c_Z Z + \omega D = \alpha \beta Y.\(^{18} \)

\(^{17}\)Equation (6) can be written as \( \frac{Z}{D} = A_1 \cdot \left( \gamma - \frac{\varepsilon}{\gamma} (\Delta)^{-\frac{\varepsilon - 1}{\sigma}} + 1 \right) \left[ \frac{\sigma^{\frac{\varepsilon - 1}{\sigma - 1}}}{\frac{\varepsilon}{\sigma - 1}} \frac{\sigma^{\frac{\varepsilon - 1}{\sigma - 1}}}{\frac{\varepsilon}{\sigma - 1}} \right]^{-1} \cdot \xi^{\frac{1}{1 - \alpha \beta \gamma}}. \)

\(^{18}\)This means that \( \alpha \beta = (c_D D + c_Z Z + \omega D) / Y \), i.e. the cost share of energy including the damage cost.
Substitution of this into (7) gives
\[ \frac{C}{Z} = (1 - \alpha) \frac{Y}{Z} + \frac{\omega D}{Z}. \tag{16} \]

The constant marginal product of \( Z \) in (15) suggests that the average product \( Y/Z \) is constant as well, and this is easily confirmed; see footnote 17. A natural conjecture is then that all three ratios in (16) are constant in the long run.

The growth path of this model is however complicated by the fact that equation (14) is not entirely linear in the logs. To see the consequences of this, rewrite (14) as
\[ c Z \Delta^{1 - 1} = \gamma \epsilon / \sigma c D + \omega \frac{\omega D}{Z}. \]
Assume now that \( \Delta \) is constant (which appears necessary for a constant \( Y/Z \)-ratio according to (15)). If \( \omega \to \infty \) then \( \frac{cD + \omega}{\omega} \to 1 \) and the ratio \( \frac{\omega D}{Z} \) will also be constant asymptotically. This would give a ‘non-balanced’ element to the dynamics during the transition period, with constant growth rates in all variables only asymptotically, like in Acemoglu and Guerrieri (2008).

Appendix A.3 shows that these heuristically derived properties of the long-run growth path indeed hold. More precisely, it is there demonstrated that there exists a BGP with a unique long-run allocation of scientists, \( S_D^* \). On this BGP, the composite variables conjectured to be constant above actually are constant. This is summarized in the following Proposition.

**PROPOSITION 1**

(i) The socially optimal long-run allocation of scientists to the \( D \) research sector, \( S_D^* \), is implicitly given by
\[ g_{N_Z}(S_D^*) = \frac{(1 + \kappa)(1 - \alpha)}{(\theta + \kappa) \alpha \beta + (1 + \kappa)(1 - \alpha)} \cdot g_{N_D}(S_D^*), \tag{17} \]
where use has been made of (3) and the constraint \( S_Z = S - S_D \). Since \( g_{N_Z}(S_D) \) is decreasing in \( S_D \), while \( g_{N_D}(S_D) \) is increasing, the solution for \( S_D^* \) is unique.

(ii) On the Balanced Growth Path the composite variables \( \Delta, C/Z, \xi, Y/Z \) and \( \omega D/Z \) are constant.
(iii) Given part (ii), the definition of \( \omega \) and the solution for \( g_{Nz} = g_{Nz}(S^*_D) \) from (17), it follows that the model exhibits the following growth rates in the long run:

\[
\begin{align*}
g^*_Z &= g^*_Y = g^*_C = \frac{\alpha \beta}{1 - \alpha} g^*_N, \\
g^*_D &= -\frac{(\theta - 1)}{(1 + \kappa)} g^*_Z \quad \text{and} \quad g^*_\omega = \frac{(\theta + \kappa)}{(1 + \kappa)} g^*_Z. 
\end{align*}
\]

(18)

(Proof in Appendix A.3.)

It is clear from (17) that \( g_{Nd}(S^*_D) > g_{Nz}(S^*_D) \). The technological change is thus foremost directed to raise the effectiveness of the polluting input, \( D \), which facilitates the process of phasing it out. This relation between the growth rates holds even if the marginal utility of consumption is just moderately declining in consumption, i.e. \( \theta \in (0, 1] \). Only if utility is linear in consumption (\( \theta = 0 \)) and pollution (\( \kappa = 0 \)) will the two technology factors grow at the same rate.

A quite common result that emerges here is that pollution declines along the balanced growth path if and only if the marginal utility of consumption falls rapidly enough as consumption increases. In this case the requirement is that \( \theta > 1 \), which means that utility of consumption is bounded from above. With increasing limitations in the possibility to raise utility by more consumption, the decrease in pollution becomes essential for increased instantaneous utility over time. See for instance Stokey (1998) for a similar result.

The next notable finding is that the damage cost of pollution, \( \omega \), increases monotonously in the long run, despite of the downward trend in \( D \). This is explained by the growth in \( C \), which makes the marginal utility of consumption decline.\(^{19}\) In fact \( \omega \) grows even faster than the income, \( Y \), but \( D \) declines at a rate such that the ratio \( \omega D / Y \) remains constant on the BGP. The total cost of pollution, \( \omega D \), is therefore a constant share of output, which means that pollution does not imply an increasingly heavy burden on the growing economy. Thus, in contrast to AABH the economy will never escape the cost of pollution, but it is clearly something that can be handled.

\(^{19}\)The net effect is a higher MRS: \( \omega \equiv \zeta M^{1-\theta}C^\theta D^\omega = -\left(\partial U/\partial D\right)/\left(\partial U/\partial C\right) \) increases.
As argued after Assumption 1, it is reasonable to assume that $\alpha \beta < 1 - \alpha$, and probably by a wide margin. This means that the ratio on the right-hand side of the first equation in (18) is (much) smaller than unity. The common growth rate of consumption and production is therefore considerably smaller than the growth rate of the technology factor connected to the clean type of energy, let alone the even higher growth rate of $N_D$. This contrasts to many growth models, where (per capita) income grows at the same rate as some technology factor. One reason for this differing result is that the decline in the dirty input, $D$, must be compensated for by a high growth in its technology factor. For both types of energy, then, the growth is based more on the increase of the technology factors than on the growth of the energy inputs themselves. This similarity between the factors appears to depend on the complementarity between the energy inputs; given the decline in $D$, and the complementarity, it is not optimal to have $Z$ growing fast.

To sum up, the long-run growth path is characterized by a sustained increase in income along with a declining pollution. This could be called ‘sustainable development’ or ‘green growth’. It is obtained by allocating sufficient research efforts to the development of technologies that can be substituted for the polluting input. In other words, it is feasible and socially optimal with a balanced long-run growth where the growth rate in $N_D$ is much larger than the growth rate of $N_Z$, so that $D$ can decline and $Z$ increase, all in such a way that $\frac{N_Z Z}{N_D D}$ is constant.

The following rephrasing of the results may help to understand the difference between this model and AABH, and in fact show that they are more different than what appears at first sight. While in AABH protection of the environment requires that the major part of research is carried out in

---

20It is clear that there are limits to the supply of $Z$ in reality. In particular the supply of bio fuels is bounded by the ecosystem, and it competes with food production. For solar and wind power, on the other hand, the use is a minuscule share of the potential. There is room for a gigantic expansion of the number of solar panels and wind turbines, although not literally an exponential growth forever. It is therefore reassuring that $Z$ grows much slower than $N_Z$ on the BGP.

The present formulation of the model does not allow a constant $Z$, because the average product of $Z$ must be constant on the BGP (see footnote 17). One could probably get around this by making the extraction cost variable, but this would likely reduce the tractability of the model considerably.

21As always, it is crucial that the knowledge production functions in (3) are beneficial enough to allow steady growth rates of blueprints for new machines of both types; see Eriksson (2013) for a discussion about this.
the clean sector, it is the polluting sector that must employ most of the re-
searchers in the present paper. Somewhat paradoxically, it is therefore the
technological change in the polluting sector that makes production cleaner,
because it helps to reduce emissions by phasing out the polluting input
(e.g. oil, coal, natural gas). Research that develops the clean technology can
on the other hand be regarded as harmful to the environment, but of course
only in the narrow sense that it allows less research in the polluting sector.
All this might seem a bit odd, but it stems logically from the assumption
that the polluting energy input cannot be disposed of entirely (e.g. fossil
fuels in aeroplanes). The best one can do for the environment is to make
the use of it more effective, so that less of it is needed for a given energy
service.

* * *

For use in the comparison between the centralized and the decentralized
solutions the expressions for the costate variables, which are obtained by
integration of (11) and (12), are presented here. Assuming that the economy
has reached the BGP, so that there are constant growth rates, the costate
variables are (see Appendix A.2 for computations)

\[ \mu_D(t) = \frac{M^\theta C^{-\theta} \omega D}{\rho + (\theta - 1) g_Z} \cdot \left( \frac{c_D}{\omega} + 1 \right) \cdot \frac{1}{N_D} \]  \hspace{1cm} (19)

and

\[ \mu_Z(t) = \frac{M^\theta C^{-\theta} Z}{\rho + (\theta - 1) g_Z} \cdot c_Z \cdot \frac{1}{N_Z} \]  \hspace{1cm} (20)

Differentiation of these expressions with respect to time gives that \( g_{NK} + g_{\mu_K} = (1 - \theta) g_Z < 0 \). This means that the transversality conditions are
fulfilled.

4 Decentralized Model

This section analyzes the decentralized equilibrium of the model that is laid
out in Section 2. The following policy instruments are included:

1. A tax on the use of \( D \) that internalizes the negative pollution exter-
nality.
2. Subsidies to the use of \( \{x_D(j)\}_{0}^{N_D} \) and \( \{x_Z(j)\}_{0}^{N_Z} \). This is necessary because the monopolistic producers of these inputs charge prices that include a markup on the marginal cost.

3. Subsidies to 'clean' research that encourage innovations in the technology factor that is substituted for the dirty input, i.e. \( N_D \).

The inclusion of these instruments allows for a derivation of the economic policy that is necessary to make the decentralized economy follow the socially optimal balanced growth path in the long run. The next section is devoted to that.

4.1 Final Output

Assume that the prices of labor, \( L \), and the two composite inputs, \( Y_D \) and \( Y_Z \), are \( w \), \( p_D \) and \( p_Z \), respectively. The price of \( Y \) in (1) is normalized to unity by Walras’ law. We therefore have

\[
1 = w^{1-\alpha} \cdot \left[ \gamma^\varepsilon_p D P_D^{1-\varepsilon} + \gamma^\varepsilon_p Z P_Z^{1-\varepsilon} \right]^{\frac{\alpha}{1-\varepsilon}} ,
\] (21)

where the right-hand side is the unit cost function (see Appendix B.2 for computations). To simplify the subsequent expressions, the following assumption has been used here.

**Assumption 2:** \( A(1-\alpha) = ((1-\alpha)/\alpha)^\alpha \).

Cost minimization also implies the following factor demand functions, conditional on output and on input prices:

\[
L = (1-\alpha) \frac{Y}{w} , \quad Y_D = \alpha Y \frac{\gamma^\varepsilon_p D P_D^{1-\varepsilon}}{\gamma^\varepsilon_p D P_D^{1-\varepsilon} + \gamma^\varepsilon_p Z P_Z^{1-\varepsilon}} \quad \text{and} \quad Y_Z = \alpha Y \frac{\gamma^\varepsilon_p Z P_Z^{1-\varepsilon}}{\gamma^\varepsilon_p D P_D^{1-\varepsilon} + \gamma^\varepsilon_p Z P_Z^{1-\varepsilon}} .
\] (23)

A useful implication of (23) is

\[
p(t) \equiv \frac{p_Z(t)}{p_D(t)} = \gamma \frac{Y_D(t)^{-\frac{1}{\varepsilon}}}{Y_Z(t)^{-\frac{1}{\varepsilon}}} .
\] (24)

The left-hand side is the relative price of the two intermediate inputs and the right-hand side is the ratio of their marginal products.
4.2 Intermediate Production

Turning to the production of the inputs \( Y_D \) and \( Y_Z \), described in (2), the possibility of a pollution tax is now introduced. A tax adjustment of \( c_D \) is made, so that the unit cost of \( D \) perceived by the producers of \( Y_D \) is \( \bar{c}_D \). The analysis below will show what the necessary adjustment is.

There is also a need for a policy instrument that corrects for the monopoly behavior of the producers of machines. The price of \( x_K(j) \) is \( \chi_K(j) \), which will include a markup. To induce the producers of \( Y_K \) to buy \( x_K(j) \):s up to the point where the true marginal cost is equal to the marginal product, the price \( \chi_K(j) \) is multiplied by \( \varsigma \in (0, 1) \), so that \( 1 - \varsigma \) is the rate of subsidy.

The profit function of a firm that produces the intermediate input then is

\[
p_K Y_K - c_K X - \int_0^{N_K} \varsigma \chi_K(j) x_K(j) dj, \quad K = D, Z
\]

where \( c_K \), is \( \bar{c}_D \) and \( c_Z \) respectively. Taking (2) into account, the conditions for maximization of profits with respect to \( D, Z \) and \( x_K(j) \) are

\[
p_D \beta \frac{Y_D}{D} = \bar{c}_D, \quad p_Z \beta \frac{Y_Z}{Z} = c_Z
\]

and

\[
x_K(j) = \left( \frac{p_K}{\varsigma \chi_K(j)} \right)^{1/\beta} K, \quad j \in (0, N_K] \quad \text{and} \quad K = D, Z.
\]

Equation (25) simply says that the values of the marginal products of \( D \) and \( Z \) should equal their true marginal costs. In equation (26) it is worth noting that the demand for a machine is higher if a larger quantity of the complementary factor (\( K \)) is used. Moreover, if the price of the intermediate input \( (p_K) \) produced in this sector rises, it is optimal to use more of each machine. These are the well-known ‘market size effect’ and ‘price effect’, respectively, which influence profits of machine producers and thus the incentives to innovate. (See Acemoglu (2002) for a thorough discussion of this.)

4.3 Production of Machines

The producer of \( x_K(j) \) holds a perpetual monopoly to sell this machine variety. It was assumed above that the production cost of \( x_K(j) \) is \( \theta = 1 - \beta \).

The profit function of a firm with the right to produce variety \( j \) therefore is

\[
\pi_K(j) = (\chi_K(j) - \theta)x_K(j), \quad K = D, Z.
\]
Using (26) in (27), one finds that the optimal price is \( \chi_K(j) = 1 \). As in all models of this kind, this price is higher than the marginal cost. A correction of this market failure can be achieved by choosing \( \varsigma \) such that the price that buyers perceive is equal to the marginal cost, i.e. \( \varsigma \chi_K(j) = 1 - \beta \). For an optimal subsidy it is thus needed that \( \varsigma = 1 - \beta \).

### 4.4 Some Implications

At this point several expressions can be simplified by use of the privately optimal price \( \chi_K(j) = 1 \) in some of the above equations. Then some useful equations are developed. The computations are found in Appendix B.3.

A simplified expression for the profit of the producer of \( x_K(j) \) is

\[
\pi_K(j) = \frac{1 - \beta}{\varsigma} \frac{\varsigma K}{N_K}, \quad K = D, Z. \tag{28}
\]

The relative demand for the two energy inputs is

\[
\frac{Z}{D} = \gamma^\varsigma \left( \frac{N_Z}{N_D} \right)^{\sigma-1} \left( \frac{c_Z}{\bar{c}D} \right)^{-\sigma}. \tag{29}
\]

At given \( N_Z/N_D \), the demand is downward-sloping in the relative cost, \( c_Z/\bar{c}D \). Comparing (29) and (14) it is clear that they are identical if \( \bar{c}D = c_D + \omega \). The pollution externality can therefore be internalized by a tax equal to the social damage, \( \tau = \omega \), so that \( \bar{c}D = c_D + \tau \).

As demonstrated in Appendix B.4, the substitution of some of the expressions obtained here (and in Appendix B.3) into (1) gives the output in the decentralized economy:

\[
Y = A_2 \cdot \left[ \gamma_D^\varsigma \cdot (N_D D)^{\frac{\sigma-1}{\alpha}} + \gamma_Z^\varsigma \cdot (N_Z Z)^{\frac{\sigma-1}{\alpha}} \right]^{\frac{\alpha}{\sigma-1}} \frac{1}{1-\alpha + \alpha \beta}, \tag{30}
\]

where

\[
A_2 = \left( AL^{1-\alpha} \left[ \left( \frac{\varsigma + 1}{1-\beta} \right)^{\beta} \right]^{\frac{\alpha}{1-\alpha + \alpha \beta}} \right). \]

If \( \varsigma = 1 - \beta \) then \( \left( \frac{\varsigma + 1}{1-\beta} \right)^{\beta} = \frac{1}{(1-\beta)^{\beta}} \). This correcting subsidy would raise the use of ‘machines’ to the socially optimal level, and thus make this expression identical to the socially optimal output in (6).
4.5 Innovations

The value of a firm that owns the patent to an innovation is $V_K$. It equals the discounted stream of profits

$$V_K(t) = \int_t^\infty \exp \left[ - \int_t^v r(u')dv' \right] \pi_K(v)dv, \quad K = D, Z,$$

where $r$ is the market interest rate.

The market wage of scientists is $w_S$. A possibly needed subsidy to research in the technology that is substituted for the dirty input (i.e. $N_D$) is denoted by $\nu$. It is optimal for innovating firms to hire scientists up to the point where the net wage equals the value of the marginal product. This means that the conditions

$$(1 - \nu)w_S = V_D \eta_D N_D f'_D (S_D) \quad \text{and} \quad w_S = V_Z \eta_Z N_Z f'_Z (S_Z)$$

must be fulfilled, where research again is assumed in both sectors, because of high marginal products of research near the origin. These two equations can be combined to

$$V_D \eta_D N_D f'_D (S_D) = (1 - \nu)V_Z \eta_Z N_Z f'_Z (S_Z),$$

where the constraint $S = S_D + S_Z$ has been used.

On the BGP, i.e. when $S_D = S_D^*$, the conditions (8) and (32) can be written as

$$\frac{\eta_D N_D f'_D (S_D^*)}{\eta_Z N_Z f'_Z (S - S_D^*)} = \frac{\mu_Z}{\mu_D} \quad \text{and} \quad \frac{\eta_D N_D f'_D (S_D^*)}{\eta_Z N_Z f'_Z (S - S_D^*)} = \frac{(1 - \nu)V_Z}{V_D},$$

respectively. These two conditions imply the same $S_D^*$ if and only if

$$\nu = 1 - \frac{\mu_Z/\mu_D}{V_Z/V_D}. \quad (33)$$

A subsidy to research that expands $N_D$ is thus necessary if $\mu_Z/\mu_D < V_Z/V_D$. This would mean that $V_D$ is relatively low, indicating an under-valuation of $D$-saving research in the decentralized equilibrium, due to a failure to internalize the external knowledge effects of this research.\footnote{The pollution tax, $\tau$, of course has an effect on innovation but it can hardly correct both market failures. See Hattori (2017) for an analysis of how much a pollution tax encourages environmental innovation in a static model.}
4.6 Household

The representative household’s assets are $A = V_D N_D + V_Z N_Z$. All firms other than the ‘machine’ producers make zero profits. Therefore they have zero market value and provide no income flows. The assets change according to

$$\dot{A} = rA + wL + wS - C + \varrho,$$

where $\varrho$ is a lump-sum transfer, which may be negative.\footnote{Assuming budget balance at every point of time, the government’s budget constraint would be $\tau D = \nu w_S S_D + (1 - \varsigma) (x_D N_D + x_Z N_Z) + \varrho$. To the extent that the subsidies $\nu$ and $1 - \varsigma$ cannot be fully financed by the pollution tax revenues they are also financed through a lump-sum tax on the representative household (i.e. $\varrho < 0$).} Moreover, the initial value $A_0$ is a constraint in the household’s optimization problem.

The household has no influence on pollution and therefore simply chooses a time path for $C$ that maximizes (5) subject to (34) and the initial asset value. This gives the Euler equation

$$g_C = r - \rho \theta.$$  

(35)

There is also the transversality condition

$$\lim_{t \to \infty} \left( \exp \left( - \int_0^t r(s) ds \right) (V_L N_L + V_Z N_Z) \right) = 0,$$

(36)

which rules out excess accumulation of assets asymptotically.

5 Optimal environmental policies

This section examines what policy measures are necessary to make the development of the decentralized economy coincide with the growth path of the centrally planned economy in the long run.

5.1 Growth rates

As a first step Proposition 2 demonstrates that the long run growth rates of $D$, $N_D$, $Z$, $N_Z$ and $\omega$ are determined by the same conditions in the decentralized equilibrium as in the optimal solution, provided that the policy $\varsigma = 1 - \beta$ and $\tau = \omega$ is implemented.

**PROPOSITION 2** Assume that the subsidy to ‘machines’ is described...
The long run growth rates $g_D$, $g_{ND}$, $g_Z$, $g_{NZ}$ and $g_\omega$ of the decentralized equilibrium are ruled by the same conditions as in the optimal solution, the implications of which are described in Proposition 1.

The proof is provided in Appendix A.5.

5.2 Research Subsidy

It remains to derive the optimal subsidy to innovations that make $N_D$ expand (which in turn makes the use of $D$ more effective so that it can easier be phased out). The values of innovations are obviously important in this context. On a balanced growth path, where $\pi_K$ changes at the constant rate $g_{\pi_K}$, equation (31) can be integrated to

$$V_K(t) = \frac{\pi_K(t)}{r^* - g_{\pi_K}}, \quad K = D, Z,$$

where $r^*$ is the long-run market interest rate. Imposing again the optimal policy $\varsigma = 1 - \beta$ and $\bar{c}_D = c_D + c_D + \tau = \bar{c}_D + \omega$, the profits in equation (28) are simplified to

$$\pi_D(j) = \frac{(c_D + \omega) D}{N_D} \quad \text{and} \quad \pi_Z(j) = \frac{c_Z Z}{N_Z}. \quad (37)$$

This means that

$$V_D(t) = \frac{(c_D + \omega) D}{N_D} \cdot \frac{1}{r^* - g_{\pi_D}} \quad \text{and} \quad V_Z(t) = \frac{c_Z Z}{N_Z} \cdot \frac{1}{r^* - g_{\pi_Z}}. \quad (38)$$

To obtain an expression for the optimal research subsidy, equations (19), (20) and (38) are now used in (33) to get

$$\nu^* = \frac{g_{\pi_Z}^* - g_{\pi_D}^*}{\rho + \theta g_Z^* - g_{\pi_D}^*}.$$

This expression also uses $r^* = \rho + \theta g_C^*$, from (35), and $g_Z^* = g_C^*$. It directly follows that the subsidy should be equal to zero if $g_{\pi_D}^* = g_{\pi_Z}^*$. If, however, $g_{\pi_D}^* < g_{\pi_Z}^*$ then a positive subsidy is needed for social optimum.

To develop the above equation, use (18) together with the expression for $\pi_D$ and $\pi_Z$ in (37). This gives the changes

$$g_{\pi_Z}^* = \frac{1 - \alpha - \alpha \beta}{\alpha \beta} \cdot g_Z^* \quad \text{and} \quad g_{\pi_D}^* = \frac{1 - \alpha - \alpha \beta}{\alpha \beta} \cdot g_Z^* - \frac{(\theta + \kappa)}{(1 + \kappa)} \cdot g_Z^*.$$
Since $1 - \alpha > \alpha \beta$ the profits in both sectors have declining trends. Looking back at the profit expressions in (37) it is clear that they are both positively related to the quantities of the energy inputs, $D$ and $Z$ respectively. These are the market size effects. The price effects appear through the $N_K$’s, which lower the profits as they grow. The falling profits in both sectors are thus explained by strongly negative price effects. For $\pi_D$ this trend is reinforced by the decline of $D$ over time, thus making $g_{\pi D}^* < g_{\pi Z}^*$ and hence the subsidy is necessary. Due to the complementarity between the inputs, the fall in $D$ holds back the increase of $Z$.

By use of the above expressions for the growth rates of $\pi_D$ and $\pi_Z$ the optimal subsidy can be expressed in terms of parameters and one growth rate:

$$\nu^* = \frac{(\theta + \kappa)}{(1 + \kappa)} \frac{\rho}{\frac{g_Z}{g_Z} + (\theta - 1) + \frac{(1 - \alpha)}{\alpha \beta} + \frac{(\theta + \kappa)}{(1 + \kappa)}}. $$

This subsidy is zero only in the quite uninteresting case where $\theta = \kappa = 0$, which would mean that $g_{\pi D}^* = g_{\pi Z}^*$. In the general case there should be a subsidy to research in the $D$ sector. As expected, the direct tax on pollution ($\tau$) is thus not sufficient to achieve the socially optimal direction of technological change; it is also necessary to impose a subsidy that internalizes the positive external knowledge effects of innovation that facilitate the future development of blue-prints for machines that substitute for the polluting input.

One can however expect that the subsidy is moderate in size, not least since the expression $(1 - \alpha)/(\alpha \beta)$ in the denominator is a quite large number. For a simple estimation of the magnitude of the subsidy, assume first that $\alpha = 0.4$, $\beta = 0.25$, $\kappa = 1$, $\theta = 2$, $\rho = 0.015$, and $g_Z = 0.02$. Then $\nu \approx 0.16$. As a ‘robustness’ test, the parameters are considerably stretched in the directions that make $\nu$ higher: assume that $\alpha = 0.4$, $\beta = 0.5$, $\kappa = 0.5$, $\theta = 2.5$, $\rho = 0.001$, and $g_Z = 0.03$. Then $\nu \approx 0.31$. The optimal subsidy to clean research is thus, according to these calculations, somewhere between 16 and 31 percent of the costs on the long-run growth path.

\[\text{AABH call } \rho = 0.001 \text{ the ‘Stern case’ and } \rho = 0.015 \text{ the ‘Nordhaus case’. As expected, the lower discount rate calls for a higher subsidy. (Recall that } g_Z \text{ is equal to the income growth rate.)}\]
5.3 Growth drag

This sub-section looks into the growth effect of environmental policy. More precisely it estimates how much slower the economy will grow when the optimal economic policy is implemented, compared to when it is not. The details are found in Appendix A.4.

If there is no environmental policy then $\tau = 0$ and $\bar{c}$ is constant in (29). Differentiation of this equation under the assumption that $\Delta$ is constant, now renders $g_{ND} = g_{NZ}$ and consequently $g_D = g_Z$. In comparison with the finding in the previous sub-section, where $g_{NZ}^* < g_{ND}^*$, there is now a reallocation of scientists in the direction of $N_Z$-enhancing research. Since $N_Z$ now grows faster, so does $Z$ and therefore income; see equation (18). The environmental policy thus means imposing a drag on income growth. (Conversely, a laissez-faire equilibrium implies a steadily growing pollution.)

The magnitude of this drag depends on the innovation possibilities frontier. For a simple estimation we suppose that a linear approximation of (3) is appropriate for the range of $S_D$ that is examined here. This would imply that $f_D(S_D) = S_D$ and $f_Z(S_Z) = S_Z$.

In the case where policy measures are taken to reduce pollution to the optimal growth rate of income is denoted by $g_Y^l$, where the ‘$l$’ signifies the linear case. To compare this with the linear case where pollution is disregarded, we define $g_Y^{ln}$ where the ‘$ln$’ signifies the ‘linear case with no policy’. Appendix A.4 derives the following relation between the two growth rates:

$$g_Y^{ln} = \left(1 + \frac{\theta + \kappa}{1 + \alpha \beta \frac{\eta_D}{\eta_D + \eta_Z}}\right) g_Y^l.$$  \hfill (39)

The parenthesis shows how much higher the growth rate would be if pollution were ignored.

For a benchmark estimation of the difference between the growth rates, assume that productivity is equal in both types of research, so that $\frac{\eta_Z}{\eta_D + \eta_Z} = \frac{1}{2}$. The share of income that goes to labor, $1 - \alpha$, is assumed to be 0.6, while the energy share of income is $\alpha \beta = 0.1$. (The latter is admittedly at the lower end, considering that the damage cost is included, but higher values will be used below.) It then follows that $\frac{\alpha \beta}{1 - \alpha} = \frac{1}{6}$. Finally, assume that $\theta = 2$ and $\kappa = 1$, so that $\frac{\theta + \kappa}{1 + \kappa} = \frac{3}{2}$.\footnote{Choices of $\theta$ around 2 are not unusual; see for instance AABH and Jones (2016).} Substitution of this into the above
equation renders
\[ g_Y^{ln} = \left(1 + \frac{1}{8}\right) g_Y^{lY} \quad \text{or} \quad g_Y^{lY} = \left(1 - \frac{1}{9}\right) g_Y^{ln}. \]

By implementing the optimal policy the economy loses a mere 11 percent of the growth rate in this case. While going from a 2% percent growth rate to a rate of 1.78% will have significant effects on income in the long run, it does not seem unbearable, considering the benefit of the declining pollution.

To examine the sensitivity of the above estimate, values of the parameters are now chosen such that they appear to slightly exaggerate the growth drag. First assume that \( \eta_Z = 2\eta_D \), i.e. the productivity of research in the \( Z \) is double as high as the corresponding productivity in the \( D \) sector, so that \( \frac{\eta_Z}{\eta_D + \eta_Z} = \frac{2}{3} \). It is also assumed that \( \alpha \beta = 0.12 \) and \( \theta = 2 + \kappa \), from which it follows that \( \frac{\alpha \beta}{1 - \alpha} = \frac{1}{5} \) and \( \frac{\theta + \kappa}{1 + \kappa} = 2 \). Then
\[ g_Y^{ln} = \left(1 + \frac{4}{15}\right) g_Y^{lY} \quad \text{or} \quad g_Y^{lY} = \left(1 - \frac{4}{19}\right) g_Y^{ln}. \]

In this case the economy would lose about 21% of the growth rate if it takes measures to reduce pollution to optimal levels. This is a considerable cost, but far from a doomsday scenario.\(^{27}\)

### 6 Stock Pollution

Many harmful emissions stay in nature for long times, notably greenhouse gases. It is therefore reasonable to model pollution as a stock variable in many applications. As this section shows, however, this would not change the qualitative long-run results much if the economy follows an optimal path with declining flows of pollution, because the stock will then eventually follow a similar path.

Assume that the polluting emissions, \( D \), are accumulated into the stock \( Q \). The change in this pollution stock is ruled by the equation
\[ \dot{Q} = D - \tau(Q), \quad (40) \]

\(^{26}\)Maintaining \( \theta = 2 \), the assumption that \( \theta = 2 + \kappa \) would mean pressing \( \kappa \) down to zero. This would have the quite implausible implication that the marginal damage of pollution does not become more severe when \( D \) increases.

\(^{27}\)For a final example the assumptions that \( \frac{\alpha \beta}{1 - \alpha} = 2 \) and \( \frac{\eta_Z}{\eta_D + \eta_Z} = \frac{2}{3} \) are maintained, while the environmental damage cost is raised so that \( \alpha \beta = 0.2 \). This means that \( \frac{\alpha \beta}{1 - \alpha} = \frac{1}{3} \) and \( g_Y^{ln} = \left(1 + \frac{12}{27}\right) g_Y^{lY} \quad \text{or} \quad g_Y^{lY} = \left(1 - \frac{12}{39}\right) g_Y^{ln}. \) Even in this stretched case the economy would not lose more than 31% percent of the growth rate.
where $\varphi(Q) > 0$ represents the natural decay of $Q$ and $\varphi'(Q) > 0$. Assume also that nature’s capacity to transform $Q$ into less harmful substances is weakened when the pollution stock exceeds some critical level, $\bar{Q}$. This amounts to the assumption that $\varphi''(Q) < 0$ for $Q \geq \bar{Q}$. At lower stress levels, however, the relationship is simply assumed to be linear: $\varphi(Q) = \varphi Q$ for $Q < \bar{Q}$, where $\varphi$ is a constant.

Utility is now a function of the pollution stock. The Hamiltonian therefore becomes

$$H = M \left[ \frac{(C/M)^{1-\theta} - 1}{1-\theta} - \zeta \frac{Q^{1+\kappa}}{1+\kappa} \right] + \mu_D \eta D f_D (S_D) + \mu_Z \eta Z f_Z (S - S_D) + \mu_Q (D - \varphi(Q)),$$

where $\mu_Q$ is the costate variable of $Q$. This variable is negative and captures the future stream of disutilities from an additional unit of $Q$.

Working through the usual procedure, five equations emerge that rule the growth rates of the central variables. They are identical to the equations for the case with flow pollution, with the exception that $\omega$ is replaced by $\varpi \equiv -(C/M)^\theta \mu_Q > 0$ as the damage cost of pollution, and thus $g_\omega$ is substituted for $g_\omega$. The solutions for the growth rates will therefore be the same as in Proposition 1, with $g_\omega$ replaced by $g_\varpi$. Thus, modeling pollution as a flow variable instead of a stock variable does not make a large qualitative difference if the economy follows an optimal path with declining pollution. Appendix A.6 provides the details.

7 Conclusions

This paper has analyzed the long-run optimal growth path in a model with a polluting input. Due to endogenous and directed technological change, the economy can afford to improve the living conditions with respect to income and environmental quality over time.

The specific constraint imposed on the model is a low substitutability between the clean and the polluting input in production. In spite of this non-beneficial assumption the model exhibits a balanced growth path with

\[ \text{An example that can be used to motivate this slowing decay of the pollution stock is the enforcing effect of methane releases, caused by the thawing of permafrost in e.g. Siberia. They are results of global warming but they also accelerate that same process. (There may even be a tipping point beyond which $\varphi(Q) < 0.$)} \]
increasing consumption and decreasing pollution. This requires that most of the research efforts are directed in a way such that the polluting input can be phased out without a severe loss of output. A cost of this research policy is that it holds back income growth, but this growth drag is estimated to be fairly modest.

The first policy measure required for social optimum is an environmental tax equal to the marginal damage of pollution. This ensures the usual internalization of the externality. Secondly, it is necessary to subsidize the research that facilitates the process of phasing out the polluting input; the tax on pollution is not a sufficient regulation of pollution. Finally, there is a need to correct the mark-up pricing on ‘machines’ by a subsidy.

The policy analysis has been carried out under the assumption that there are no political failures. It is clear that decisions to hand out subsidies to research in reality are influenced by lobbying. In addition, there is a well-known difficulty for policy makers to pick the winners. In the context of the present model this would amount to deciding which research projects are the most promising to help saving on the polluting input. It would be interesting for future research to include these complications in the analysis.
References


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A Appendix 1

A.1 Computations for Section 3.1

The condition for the maximization of output net of investment, with respect to $x_K(j)$, is

$$
\frac{\alpha}{1-\beta} \cdot Y \left[ \gamma_D Y_D(t) \frac{\epsilon-1}{\alpha} + \gamma_Z Y_Z(t) \frac{\epsilon-1}{\alpha} \right]^{-1} \cdot \gamma_K Y_K(t) \frac{1}{\epsilon} \cdot K^\beta = x_K(j)^\beta.
$$

Since the LHS is independent of $j$, all varieties of $x_K$ are used in equal quantities and the $j$ index of $x_K$ can be left out. Using (1) to eliminate the bracket:

$$
x_K = \left( \frac{\alpha}{1-\beta} \cdot Y \frac{\alpha+1-\epsilon}{\alpha} \cdot (AL^{1-\alpha}) \frac{\epsilon-1}{\alpha} \cdot \gamma_K Y_K(t) \frac{1}{\epsilon} \right)^{1/\beta} \cdot K, \quad K = D, Z.
$$

Substituting these expressions into (2) (using the definition $\sigma = 1+\beta(\epsilon-1)$):

$$
Y_K = \left( \frac{\alpha^{1-\beta}}{1-\beta} \right)^{\frac{\epsilon}{\sigma}} \cdot Y \frac{\alpha+1-\epsilon(1-\beta)}{\alpha} \cdot (AL^{1-\alpha}) \frac{(\epsilon-1)(1-\beta)}{\alpha} \cdot \gamma_K \frac{\epsilon}{\sigma} \cdot (N_K K)^{\frac{\epsilon}{\sigma}}, \quad K = D, Z.
$$

Substitution of these into (1) gives

$$
Y = \left( \frac{AL^{1-\alpha} \alpha^{(1-\beta)}}{(1-\beta)^{\alpha}} \right)^{\frac{\epsilon}{\sigma}} \cdot \gamma_D \left( N_D D \right)^{\frac{\epsilon-1}{\sigma}} \gamma_D + \gamma_Z \left( N_Z Z \right)^{\frac{\epsilon-1}{\sigma}} \cdot \gamma_K \left( N_K K \right)^{\frac{\epsilon}{\sigma}} \cdot \left( N_K K \right)^{\frac{\epsilon}{\sigma}}.
$$

This is equivalent to (6).

To express machine investments as a fraction of output, $Y_K$ is substituted back into $x_K$:

$$
x_K = \left( \frac{\alpha}{1-\beta} \right)^{\frac{1}{\beta}} \left( \frac{\alpha^{1-\beta}}{1-\beta} \right)^{-\frac{\epsilon}{\sigma}} \cdot Y \frac{\alpha+1-\epsilon}{\alpha} \cdot (AL^{1-\alpha}) \frac{(\epsilon-1)}{\alpha} \cdot \gamma_K \frac{\epsilon}{\sigma} \cdot (N_K K)^{-\frac{1}{\sigma}}.
$$

Now multiply both sides by $N_K$ and define $X_K \equiv N_K x_K$:

$$
X_K = \left( \frac{\alpha}{1-\beta} \right)^{\frac{1}{\beta}} \left( \frac{\alpha^{1-\beta}}{1-\beta} \right)^{-\frac{1}{\sigma}} \cdot Y \frac{\alpha+1-\epsilon}{\alpha} \cdot (AL^{1-\alpha}) \frac{(\epsilon-1)}{\alpha} \cdot \gamma_K \frac{\epsilon}{\sigma} \cdot (N_K K)^{1-\frac{1}{\sigma}}.
$$

Thus the total use of machines is

$$
X_D + X_Z = \left( \frac{\alpha}{1-\beta} \right)^{\frac{1}{\beta}} \left( \frac{\alpha^{1-\beta}}{1-\beta} \right)^{-\frac{1}{\sigma}} \cdot Y \frac{\alpha+1-\epsilon}{\alpha} \cdot (AL^{1-\alpha}) \frac{(\epsilon-1)}{\alpha} \cdot \left[ \gamma_D \left( N_D D \right)^{1-\frac{1}{\sigma}} + \gamma_Z \left( N_Z Z \right)^{1-\frac{1}{\sigma}} \right].
$$
Using $\sigma = 1 + \beta(\varepsilon - 1)$ again, we can rewrite this as

$$X_D + X_Z = \alpha \cdot Y.$$ 

The final line is equal to 1, by (6), so we have $X_D + X_Z = \alpha \cdot Y$. The expenditures on investments are therefore $(1 - \beta)(X_D + X_Z) = \alpha (1 - \beta) \cdot Y$, which has been used in (7).

### A.2 Co-state variables

We now integrate (11) and (12), to obtain expressions for the costate variables. Define

$$R_D = M^\theta \cdot \left( \frac{C_D}{\omega} \cdot \frac{\alpha}{1 - \alpha + \alpha \beta} \right) \cdot \left( \frac{A L 1^{\alpha \beta \alpha (1 - \beta)}}{(1 - \beta)^{\alpha \beta}} \cdot \left[ \frac{\gamma_D}{\alpha \beta} \cdot ((N_D D)^{1 - \frac{1}{\alpha \beta}} + \gamma_Z (N_Z Z)^{1 - \frac{1}{\alpha \beta}}) \right] \right).$$

Then (11) and (12) can be written as:

$$\dot{\mu}_D(t) + \mu_D(t)(g_{N_D}(t) - \rho) = -R_D(t)$$

and

$$\dot{\mu}_Z(t) + \mu_Z(t)(g_{N_Z}(t) - \rho) = -R_Z(t).$$

Using the integrating factors $\exp\left[ \int (g_{N_K}(t) - \rho) \, dt \right]$, we have

$$\mu_D(t) = e^{-\int (g_{N_D}(t) - \rho) \, dt} \left( -\int e^{\int (g_{N_D}(t) - \rho) \, dt} R_D(t) \, dt + C_D \right)$$

and

$$\mu_Z(t) = e^{-\int (g_{N_Z}(t) - \rho) \, dt} \left( -\int e^{\int (g_{N_Z}(t) - \rho) \, dt} R_Z(t) \, dt + C_Z \right),$$

where $C_D$ and $C_Z$ are constants of integration.

Now assume that the economy is on a BGP, at least since some time $t_0 < t$. That is, all growth rates are constant from time $t_0$, including the growth rate of $R_K$, $g_{R_K}$. The values of the co-states at time $t$ thus are

$$\mu_D(t) = e^{-(g_{N_D} - \rho)(t - t_0)} \left( -R_D(t_0) \int_{t_0}^{t} e^{(g_{N_D} + g_{R_D} - \rho)(t - s)} \, ds + C_D \right)$$

and

$$\mu_Z(t) = e^{-(g_{N_Z} - \rho)(t - t_0)} \left( -R_Z(t_0) \int_{t_0}^{t} e^{(g_{N_Z} + g_{R_Z} - \rho)(t - s)} \, ds + C_Z \right).$$
Integrating:

\[ \mu_D(t) = e^{-(g_{ND} - \rho)(t-t_0)} \left[ \frac{R_D(t_0)}{g_{ND} + g_{RD} - \rho} + C_D \right] - \frac{R_D(t_0)e^{g_{RD}(t-t_0)}}{g_{ND} + g_{RD} - \rho} \]

and

\[ \mu_Z(t) = e^{-(g_{NZ} - \rho)(t-t_0)} \left[ \frac{R_Z(t_0)}{g_{NZ} + g_{RZ} - \rho} + C_Z \right] - \frac{R_Z(t_0)e^{g_{RZ}(t-t_0)}}{g_{NZ} + g_{RZ} - \rho} \]

For constant growth rates it is required that

\[ \frac{R_K(t_0)}{g_{NK} + g_{RK} - \rho} + C_K = 0, \quad K = D, Z. \]

Choosing the \( C_K \):s such that this is fulfilled, we have

\[ \mu_D(t) = \frac{R_D(t)}{\rho - (g_{ND} + g_{RD})} \]

and

\[ \mu_Z(t) = \frac{R_Z(t)}{\rho - (g_{NZ} + g_{RZ})}, \]

where \( R_K(t) = R_K(t_0)e^{g_{RK}(t-t_0)} \). Inspection of the definitions of the \( R_K \):s shows that \( g_{RK} = (1 - \theta)g_Z - g_{NK} \). (In the case of \( R_D \) this holds asymptotically, when \( \omega \) gets large.) Recalling also the definitions of \( R_K \) we have (19) and (20).

### A.3 Dynamics

We now define the additional composite variables

\[ \chi \equiv \frac{\mu Z N Z}{\mu_D N_D}, \quad \hat{\omega} \equiv \frac{c_D + \omega}{\omega}, \quad \Omega \equiv \frac{\omega D}{\Omega}, \quad \Lambda \equiv \frac{Z^{1-\theta}}{\mu_D N_D}, \quad \hat{\omega} \equiv \frac{C}{\hat{Z}} \quad \text{and} \quad \hat{d} \equiv \frac{D^{1+\kappa}}{\hat{Z}^{1-\gamma}}. \]

We also define

\[ \bar{\Delta} \equiv 1 + \hat{\gamma} \hat{\omega} (\Delta) \frac{e^{\Delta}}{\sigma}, \quad \underline{\beta} \equiv \frac{(1 - \alpha + \alpha \beta)}{(1 - \alpha)} - \sigma, \]

\[ \Gamma(S_D) \equiv \frac{(\theta + \kappa)\alpha \beta}{1 - \alpha} g_{NZ} (S_D) + (1 + \kappa) (g_{NZ} (S_D) - g_{ND} (S_D)), \]

\[ g_N (S_D) \equiv g_{NZ} (S_D) - g_{ND} (S_D) \]

and

\[ \Psi \equiv \hat{\gamma} (\theta + \kappa) \underline{\beta} + \bar{\Delta} \left[ \theta \Omega \left( \sigma - \hat{\omega} + (\sigma - 1) \hat{\omega} \frac{(1 - \alpha)}{\alpha \beta} \right) + \hat{\omega} (\sigma + \sigma \kappa) \right]. \]
Appendix C shows that the dynamics of the model can be described by the following system\(^{29}\) of equations:

\[
g_{\Delta} = \frac{\sigma \Delta}{\Psi} \cdot [\hat{c} \cdot \Gamma(S_D) + (\hat{c} - \theta \Omega)(\hat{\omega} - 1)g_N(S_D)] \tag{41}
\]

\[
g_{\bar{\omega}} = (1 - \hat{\omega}) \left[ \sigma^{-1} g_{\Delta} - g_N(S_D) \right] \tag{42}
\]

\[
g_{S_D} = \frac{cz \Lambda}{\mathcal{E}(S_D)\hat{c}^2} \left( \gamma - \sigma \Delta \frac{1 - \alpha}{\sigma} - \eta \frac{f_Z'(S - S_D)}{f_D'(S_D)} \right) \tag{43}
\]

\[
g_{\hat{c}} = -\frac{1}{\sigma} \left[ \Gamma(S_D) + (\hat{\omega} - 1)g_N(S_D) \right] + \frac{\Delta(\sigma \kappa + \hat{\omega}) + (\theta + \kappa)\hat{\omega}}{\theta \sigma \Delta} \cdot g_{\Delta} \tag{44}
\]

\[
g_{\hat{d}} = \Gamma(S_D) - \frac{(\theta + \kappa)\hat{\omega} + \Delta \sigma(1 + \kappa)}{\Delta \sigma} \cdot g_{\Delta} \quad \text{and} \tag{45}
\]

\[
g_{\lambda} = \frac{(\theta - 1)\hat{\omega}}{\sigma \Delta} \cdot g_{\Delta} + (1 - \theta) \frac{\alpha \beta}{1 - \alpha} g_N(S_D) (S_D) + \zeta M \cdot \Lambda \cdot \hat{c} \cdot \hat{d} - \rho. \tag{46}
\]

We can now characterize the BGP, where all changes are equal to zero. First, when \(g_{\hat{d}} = 0\) and \(g_{\Delta} = 0\) equation (45) immediately implies that \(\Gamma(S_D) = 0\). This means that \(S_D^*\) is given by

\[
\frac{(\theta + \kappa)\alpha \beta + (1 + \kappa)(1 - \alpha)}{1 - \alpha} g_N(S_D) = g_N(Z_D^*), \tag{47}
\]

which is equivalent to (17). Turning to equation (42) we see that \(g_{\bar{\omega}} = 0\) and \(g_{\Delta} = 0\) imply that \((1 - \hat{\omega})g_N(S_D) = 0\). Since, by (47), \(g_N(S_D) \neq 0\) it must be that

\[
\hat{\omega}^* = 1. \tag{48}
\]

When \(g_{S_D} = 0\) (43) implies

\[
(\Delta^*)^{1 - \sigma} = \frac{\gamma}{\sigma} \eta \frac{f_Z'(S - S_D^*)}{f_D'(S_D^*)}. \tag{49}
\]

Finally, (46) is reduced to

\[
\zeta M \cdot \Lambda^* \cdot \hat{\omega}^* \hat{c}^* \hat{d}^* = \rho + (\theta - 1) \frac{\alpha \beta}{1 - \alpha} g_N(S_D^*). \tag{50}
\]

**Characterizing the BGP:**

Among other things, it follows that \(\Delta\) and \(C/Z\) are constant on the BGP.

\(^{29}\)Note that \(\Omega = \zeta M^{1 - \theta} \cdot \hat{c}^0 \cdot \hat{d}^0\).
Then by (15) \( \xi \) must also be constant and by Footnote 17 \( Y/Z \) must be constant. A final implication, by (16), is that \( \omega D/Z \) is constant. We therefore have

\[
g_N D + g_D = g_N z + g_z, \tag{51}
\]

\[
g_\omega + g_D - g_Z = 0 \tag{52}
\]

and

\[
\alpha \beta g_{NZ} = (1 - \alpha) g_z. \tag{53}
\]

Moreover, by the definition of \( \omega, g_C = g_\omega/\theta - \kappa g_D/\theta \). Using \( g_C = g_Z \) (since \( C/Z \) does not change on the long-run growth path) this can be rewritten as

\[
g_\omega = \theta g_Z + \kappa g_D. \tag{54}
\]

Solving equations (51)—(54) gives

\[
g_D = \frac{(1 - \theta)}{(1 + \kappa)} g_z, \hspace{1cm} g_z = \frac{\alpha \beta}{(1 - \alpha)} g_{NZ} \hspace{1cm} \text{and}
\]

\[
g_N D = \frac{(1 + \kappa)(1 - \alpha) + (\theta + \kappa) \alpha \beta}{(1 + \kappa)(1 - \alpha)} g_N z. \tag{55}
\]

These expressions are used to obtain equation (18) in Proposition 1.

**A.4 Growth drag**

Absent environmental policy, \( \tau = 0 \) and thus \( \bar{c} \) is constant in (29). Differentiation of this equation under the assumption that \( \Delta \) is constant gives \( g_{ND} = g_N Z \). Equation (51) then implies that \( g_D = g_Z \). Let the super-index \( n \) signify ‘no policy’. In comparison to the socially optimal solution, where \( g^*_N Z < g^* N D \), the fact that \( g^n_{N Z} = g^n_{ND} \) here implies a reallocation of scientists in the direction of \( N_Z \)-enhancing research, such that \( g^n_{NZ} > g^n_{N Z} \) and \( g^n_{ND} < g^n_{ND} \). Since \( N_Z \) grows faster in this regime, so does \( Z \) and therefore income. That is,

\[
g^*_Y = g^*_Z = \frac{\alpha \beta}{(1 - \alpha)} g^*_N z \hspace{1cm} < g^n_Y = g^n_Z = \frac{\alpha \beta}{(1 - \alpha)} g^n_{NZ}.
\]

For a useful version of the Innovation Possibilities Frontier, the first equation in (3) is solved for \( S_D = f_D^{-1}(g_{ND}/\eta_D) \) and substituted into the second equation of (3). The result is

\[
g_{NZ} = \eta_Z f_Z (S - f_D^{-1}(g_{ND}/\eta_D)). \tag{56}
\]
This implies a negative relation between \( g_{NZ} \) and \( g_{ND} \), i.e. a tradeoff. Assume now a linear approximation of (3), such that \( f_D(S_D) = S_D \) and \( f_Z(S_Z) = S_Z \). Then by (3) \( S_D = g_{ND}/\eta_D \) and (56) is modified to

\[
g_{NZ} = \tilde{S} - \frac{\eta_Z}{\eta_D} g_{ND},
\]

where \( \tilde{S} \equiv \eta_Z S \).

In the case where policy measures are taken to reduce pollution to the optimal level, equations (55) still apply (because the centralized and the decentralized solutions coincide). Combining the last one of them with (57):

\[
g^l_{NZ} = \frac{(1 + \kappa)(1 - \alpha)\eta_D}{(1 + \kappa)(1 - \alpha)(\eta_D + \eta_Z) + (\theta + \kappa)\alpha\beta\eta_Z} \tilde{S},
\]

where the ‘l’ signifies the linear case. Since \( g_Y = g_Z = \frac{\alpha^2}{(1 - \alpha)} g_{NZ} \), this implies that

\[
g^l_Y = \frac{\alpha\beta(1 + \kappa)}{(1 + \kappa)(1 - \alpha)(\eta_D + \eta_Z) + (\theta + \kappa)\alpha\beta\eta_Z} \eta_D \tilde{S}.
\]

To compare this with the linear case where pollution is disregarded, note that \( g_{NZ} = g_{ND} \) implies that (57) can be solved for \( g_{NZ} = \frac{\eta_D}{\eta_D + \eta_Z} \tilde{S} \). Since \( g_Y = g_Z = \frac{\alpha\beta}{(1 - \alpha)} g_{NZ} \) here as well, the income growth rate now is

\[
g^b_Y = \frac{\alpha\beta}{(1 - \alpha)} \frac{\eta_D}{\eta_D + \eta_Z} \tilde{S},
\]

where the ‘b’ signifies the ‘linear case with no policy’. The relation between the two growth rates is obtained by elimination of \( \eta_D \tilde{S} \) between (58) and (59). The result is (39).

### A.5 Growth Rates in Equilibrium

On the BGP \( \Delta \) is constant. Therefore (51) applies here as well. Next, the pollution externality is internalized by the inclusion of the tax \( \tau = \omega \) in (29), such that \( \bar{c}_D = c_D + \tau = c_D + \omega \). Differentiation of equation (29) then gives that

\[
g_{ND} - g_{NZ} = \frac{\omega}{c_D + \omega} g_\omega.
\]

Asymptotically, as \( \omega \to \infty \), it holds that \( g_{ND} - g_{NZ} = g_\omega \). Using this in (51) gives (52).

Appendix B.5 shows how some expressions from Section 4 can be used to derive the marginal product of \( Z \). Assuming that \( \varsigma = 1 - \beta \) this equation
is equivalent to (15). The time derivative of this gives equation (53) if \( \Delta \) is constant.

Since it is the possibility to have a socially optimal growth in a policy-adjusted decentralized equilibrium that is investigated here, \( C/Z \) is again taken to be constant in the long run. The definition of \( \omega \) still applies and therefore (54) can be used here as well.

It thus emerges that the growth rates \( g_D, g_{ND}, g_Z, g_{NZ} \) and \( g_\omega \) in the decentralized equilibrium are determined by equations that are identical to those in the social optimum, (51)—(54) and (3), if the optimal policy \( \zeta = 1 - \beta \) and \( \tau = \omega \) is implemented. The decentralized growth rates are then socially optimal, thus coinciding with those in Proposition 1.

### A.6 Stock Pollution

Maximizing this Hamiltonian in Section 6 with respect to \( S_D \) and \( Z \) gives (8) and (9) respectively. Similarly, the conditions (11)–(13) are unchanged by this modification of the model. Three new conditions emerge, however. First, instead of (10) it must now be that

\[
\frac{\alpha \beta Y}{\bullet} \cdot \frac{\gamma D^2 (N_D D)^{\frac{\sigma - 1}{\sigma}}}{D} = c_D + \varpi,
\]

where \( \varpi \equiv -(C/M)^\theta \mu_Q > 0 \) is the damage cost of pollution. The necessary change in \( \mu_Q \) is ruled by

\[
\dot{\mu}_Q = \mu_Q \rho + \mu_Q \varphi'(Q) + M \zeta Q^\kappa
\]

and, finally, an additional transversality condition is

\[
\lim_{t \to \infty} e^{-\rho t} \mu_Q Q = 0.
\]

It thus appears that \( \varpi \) here takes the place of \( \omega \) and that the long-run growth rates are unchanged, except that \( g_\varpi \) replaces \( g_\omega \). The following paragraphs demonstrate that this is true.

A combination of (9) and (60) gives \( c_Z Z + c_D D = \alpha \beta Y - \varpi D \). Substitution of this into (7) renders

\[
\frac{C}{Z} = (1 - \alpha) \frac{Y}{Z} + \frac{\varpi D}{Z}
\]

instead of (16). Along the BGP, where \( \Delta \) is constant, equations (51) and (53) are still valid. With \( \frac{\varpi D}{Z} \) constant on the BGP, equation (52) is now
replaced by
\[ g_\varpi + g_D - g_Z = 0. \]  
(64)

The definition of \( \varpi \) implies that \( g_C = g_\varpi / \theta - g_{\mu Q} / \theta \). On a BGP where \( g_C = g_Z \) it must therefore be that
\[ g_\varpi = \theta g_Z + g_{\mu Q}. \]  
(65)

To make this equation more like (54), note first that the growth rate of \( Q \) is constant along the BGP. On an optimal growth path with declining pollution the economy eventually reaches the region where \( Q < \bar{Q} \). Equation (40) can then be rewritten as \( g_Q = \frac{D}{\bar{Q}} - \varphi \) and this implies that
\[ g_D = gQ. \]  
(66)

Moreover, on the BGP (61) can be written as \( g_{\mu Q} = \rho + \varphi + M \zeta Q^* / \mu Q \) when \( Q < \bar{Q} \). The final term is constant only if
\[ \kappa g_Q = g_{\mu Q}. \]  
(67)

Equations (66) and (67) together imply that \( g_{\mu Q} = \kappa g_D \), which means that (65) can be written as
\[ g_\varpi = \theta g_Z + \kappa g_D. \]  
(68)

This is what emerges instead of (54). Finally, equation (56) is unaffected by the introduction of stock pollution.

To sum up, equations (51), (53), (64), (68) and (56) are five equations in the variables \( g_D, g_{N_D}, g_Z, g_{N_Z} \) and \( g_\varpi \). This is identical to the system (51)—(56), with the exception that \( g_\varpi \) is substituted for \( g_\omega \). The solutions for the growth rates will therefore be the same as in Proposition 1, with \( g_\omega \) replaced by \( g_\varpi \).30 Thus, modeling pollution as a flow variable instead of a stock variable does not make a large qualitative difference if the economy follows a path with declining pollution.

30Given these solutions, expressions for \( g_Q \) and \( g_{\mu Q} \) on the BGP can be derived by use of (66) and (67). From equation (67) it follows that \( g_{\mu Q} + g_Q = (1 + \kappa)g_Q < 0 \), where the inequality rests on the assumption that \( \theta > 1 \). This means that the transversality condition in (62) will be satisfied.
B Appendix (Not for Publication)

B.1 Feasible growth rates

Differentiation of (6) with respect to time gives

\[ g_Y = \alpha \beta \frac{\gamma_D^{\frac{1}{\sigma}} (g_N^D + g_D) + \gamma_Z^{\frac{1}{\sigma}} (\Delta^{\frac{1}{\sigma}} - \gamma_D^{\frac{1}{\sigma}} (g_N^Z + g_Z))}{\gamma_D^{\frac{1}{\sigma}} + \gamma_Z^{\frac{1}{\sigma}} (\Delta^{\frac{1}{\sigma}} - \gamma_D^{\frac{1}{\sigma}})} \tag{69} \]

where we have defined \( \Delta \equiv \frac{N^Z}{D^N} \). When \( g_N^D + g_D = g_N^Z + g_Z \), and thus \( \Delta \) is constant, (69) is reduced to

\[ g_Y = \frac{\alpha \beta}{1 - \alpha + \alpha \beta} \cdot (g_N^D + g_D) \tag{70} \]

Similar results apply when the growth is not ‘balanced’ so that \( \Delta \) is not constant over time. We have the following results. If \( g_N^D + g_D < g_N^Z + g_Z \) then \( \lim_{t \to \infty} \Delta = \infty \). It then follows that

\[ \lim_{t \to \infty} g_Y = \frac{\alpha \beta}{1 - \alpha + \alpha \beta} \cdot (g_N^D + g_D) \quad \text{if} \quad 0 < \sigma < 1 \]

and

\[ \lim_{t \to \infty} g_Y = \frac{\alpha \beta}{1 - \alpha + \alpha \beta} \cdot (g_N^Z + g_Z) \quad \text{if} \quad \sigma > 1. \]

In the reverse case, when \( g_N^D + g_D > g_N^Z + g_Z \) we have \( \lim_{t \to \infty} \Delta = 0 \). It then follows that

\[ \lim_{t \to \infty} g_Y = \frac{\alpha \beta}{1 - \alpha + \alpha \beta} \cdot (g_N^Z + g_Z) \quad \text{if} \quad 0 < \sigma < 1 \]

and

\[ \lim_{t \to \infty} g_Y = \frac{\alpha \beta}{1 - \alpha + \alpha \beta} \cdot (g_N^D + g_D) \quad \text{if} \quad \sigma > 1. \]

These results can be summarized as follows. The slowest growing (composite) input determines the growth rate when \( 0 < \sigma < 1 \), because both inputs are needed (‘essential’) in this case. When the elasticity of substitution is higher, one of the inputs can dominate production, while the other is much less important, as for the dirty input in AABH.

B.2 Computations for Section 4.1

B.2.1 Lagrangean and foc

The Lagrangean for the cost-minimization problem in the final output sector is:

\[ C^F = wL + p_D Y_D + p_Z Y_Z - \lambda \left( AL^{1-\alpha} \left[ \gamma_D Y_D(t)^{\frac{1}{\sigma}} + \gamma_Z Y_Z(t)^{\frac{1}{\sigma}} \right]^{\alpha\gamma} - Y \right). \]
The first-order conditions are:

\[ w = \frac{1 - \alpha}{L} Y, \quad (71) \]

\[ p_D = \lambda AL^{1-\alpha} \left[ \gamma_D Y_D(t) \frac{\alpha-1}{\alpha} + \gamma_Z Y_Z(t) \frac{\alpha-1}{\alpha} \right] \cdot \gamma_D Y_D(t)^{-\frac{1}{\alpha}} \cdot \gamma_Z Y_Z(t)^{-\frac{1}{\alpha}}. \quad (72) \]

and

\[ p_Z = \lambda AL^{1-\alpha} \left[ \gamma_D Y_D(t) \frac{\alpha-1}{\alpha} + \gamma_Z Y_Z(t) \frac{\alpha-1}{\alpha} \right] \cdot \gamma_D Y_D(t)^{-\frac{1}{\alpha}} \cdot \gamma_Z Y_Z(t)^{-\frac{1}{\alpha}}. \quad (73) \]

We simplify by taking the ratio of (73) and (72):

\[ p(t) \equiv \frac{p_Z(t)}{p_D(t)} = \gamma Y_Z(t)^{-\frac{1}{\alpha}} / Y_D(t)^{-\frac{1}{\alpha}}, \quad (74) \]

where \( \gamma \equiv \gamma_Z / \gamma_D \). Similarly, the ratio of (71) and (72) is

\[ \frac{w}{p_D} = \frac{(1 - \alpha)}{\alpha \gamma_D} \cdot \left[ \gamma_D Y_D(t) \frac{\alpha-1}{\alpha} + \gamma_Z Y_Z(t) \frac{\alpha-1}{\alpha} \right] \cdot Y_D(t)^{1/\alpha}. \quad (75) \]

**B.2.2 Conditional demands of** \( Y_D \) **and** \( Y_Z \), **step I**

To obtain the first versions of the conditional demand functions of \( Y_D \) and \( Y_Z \) (still functions of \( L \)), we start by rewriting (74):

\[ Y_D = p^\frac{\alpha}{\alpha-1} Y_Z. \quad (76) \]

Substituting this into the constraint of the cost minimization problem and solving for the conditional demand:

\[ Y_Z = Y^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} L^{\frac{\alpha-1}{\alpha}} p_Z^{-\frac{\alpha-1}{\alpha}} \gamma_Z^{\frac{\alpha}{\alpha-1}} \left[ \gamma_D p_D^{1-\alpha} + \gamma_Z p_Z^{1-\alpha} \right]^{1/\alpha}. \quad (77) \]

Combining this with (76):

\[ Y_D = Y^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} L^{\frac{\alpha-1}{\alpha}} p_D^{-\frac{\alpha-1}{\alpha}} \gamma_D^{\frac{\alpha}{\alpha-1}} \left[ \gamma_D p_D^{1-\alpha} + \gamma_Z p_Z^{1-\alpha} \right]^{1/\alpha}. \quad (78) \]

These demands are still functions of the third input, \( L \).
B.2.3 Conditional demand of $L$

To eliminate $L$, we must first obtain the conditional demand for this variable. Therefore, we solve (75) for $L$:

$$L = \frac{p_D (1 - \alpha)}{\omega} \cdot \left[ \gamma_D Y_D(t) \left( \frac{t^{\alpha}}{\gamma} \right) + \gamma_Z Y_Z(t) \left( \frac{t^{\alpha}}{\gamma} \right) \right] \cdot Y_D(t)^{\frac{1}{\alpha}}. \quad (79)$$

Using now (77) and (78):

$$L = \frac{p_D (1 - \alpha)}{\omega} \cdot \left[ \gamma_D \left( \frac{p_D^{-\varepsilon} p_D^{\varepsilon}}{p_D^{-\varepsilon} p_D^{\varepsilon}} \right) \left( \frac{t^{\alpha}}{\gamma} \right) \right] \cdot \left( \frac{Y^\frac{1}{\alpha} A^{-\frac{1}{\alpha}} \alpha L^{-\frac{\alpha}{\alpha}} \left[ \gamma_D p_D^{1-\varepsilon} + \gamma_Z p_Z^{1-\varepsilon} \right]^{\frac{1}{1+\varepsilon}} \right)^{\frac{1}{\alpha}}. \quad (80)$$

Simplifying:

$$L = \frac{1}{\omega} \frac{(1 - \alpha)}{\alpha} \cdot Y^\frac{1}{\alpha} A^{-\frac{1}{\alpha}} \alpha L^{-\frac{\alpha}{\alpha}} \left[ \gamma_D p_D^{1-\varepsilon} + \gamma_Z p_Z^{1-\varepsilon} \right]^{\frac{1}{1+\varepsilon}}. \quad (81)$$

Solving for $L$:

$$L = \left( \frac{1 - \alpha}{\omega} \right)^{\frac{\alpha}{\alpha}} \cdot Y^\frac{1}{\alpha} A^{-\frac{1}{\alpha}} \alpha L^{-\frac{\alpha}{\alpha}} \left[ \gamma_D p_D^{1-\varepsilon} + \gamma_Z p_Z^{1-\varepsilon} \right]^{\frac{1}{1+\varepsilon}}. \quad (82)$$

B.2.4 Conditional demands of $Y_D$ and $Y_Z$, step II

Equation (80) can now be used to obtain the ‘real’ conditional demands of $Y_D$ and $Y_Z$. First, substituting (80) into (77) to eliminate $L$:

$$Y_Z = Y^\frac{1}{\alpha} A^{-\frac{1}{\alpha}} p_Z^{-\varepsilon} \gamma_Z \left[ \gamma_D p_D^{1-\varepsilon} + \gamma_Z p_Z^{1-\varepsilon} \right]^{\frac{1}{1+\varepsilon}} \cdot \left( \left( \frac{1 - \alpha}{\omega} \right)^{\frac{\alpha}{\alpha}} \cdot \left( Y A^{-1} \right)^{\frac{\alpha}{\alpha}} \left[ \gamma_D p_D^{1-\varepsilon} + \gamma_Z p_Z^{1-\varepsilon} \right]^{\frac{1}{1+\varepsilon}} \right)^{\frac{\alpha}{\alpha}}. \quad (81)$$

Similarly,

$$Y_D = \left( \frac{w\alpha}{1 - \alpha} \right)^{1-\alpha} \cdot Y^\frac{1}{\alpha} A^{-\frac{1}{\alpha}} \gamma_D \cdot \left[ \gamma_D p_D^{1-\varepsilon} + \gamma_Z p_Z^{1-\varepsilon} \right]^{\frac{1}{1+\varepsilon}}. \quad (82)$$

Equations (80), (81) and (82) thus give the conditional demands for $L$, $Y_Z$ and $Y_D$.

\(^{31}\)That is, functions only of $Y, w, p_D$ and $p_Z$. 

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B.2.5 Cost Function

Using (80), (81) and (82) in the cost expression \( C^Y = wL + p_D Y_D + p_Z Y_Z \):

\[
C^Y = w^{1-\alpha} \left(\frac{1-\alpha}{\alpha}\right)^\alpha \cdot \frac{Y}{A} \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha}{1-\varepsilon}}
\]

\[
+ w^{1-\alpha} \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \cdot \frac{Y}{A} \cdot p_{\hat{D}}^{1-\varepsilon} \gamma_{\hat{D}} \cdot \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha\alpha+1}{1-\varepsilon}}
\]

\[
+ w^{1-\alpha} \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \cdot \frac{Y}{A} \cdot p_{\hat{Z}}^{1-\varepsilon} \gamma_{\hat{Z}} \cdot \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha\alpha+1}{1-\varepsilon}}.
\]

Simplifying:

\[
C^Y = \frac{Y}{A} \cdot w^{1-\alpha} \cdot \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha}{1-\varepsilon}} \cdot \left(\frac{1-\alpha}{\alpha}\right)^\alpha \cdot \left\lbrack \frac{1}{1-\alpha} \right\rbrack. \quad (83)
\]

This is the cost function. Invoking Assumption 2, i.e. that \( A = \left(\frac{1-\alpha}{\alpha}\right)^\alpha \cdot \frac{1}{1-\alpha} \):

\[
C^Y = Y w^{1-\alpha} \cdot \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha}{1-\varepsilon}}. \quad (84)
\]

This becomes the unit cost function in (21) of the main text, if we put \( Y = 1 \).

B.2.6 The Wage

An expression for the wage is obtained by first solving (80) for \( w \):

\[
w = \left(\frac{1-\alpha}{\alpha}\right) \cdot \left(\frac{Y}{AL}\right) \cdot \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}. \quad (85)
\]

Now solve (21) for \( \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} = w^{1-\frac{1}{\alpha}} \), and use this in (85):

\[
w = \left(\frac{1-\alpha}{\alpha}\right)^\alpha \cdot \frac{Y}{AL}.
\]

Since we have assumed that \( \left(\frac{1-\alpha}{\alpha}\right)^\alpha \cdot \frac{1}{\alpha} = 1 - \alpha \), this means that \( w = (1 - \alpha) \cdot Y / L \), which is equivalent to (22) in the main text.

B.2.7 Simplified Conditional Demands for \( Y_D \) and \( Y_Z \)

Finally, we simplify equations (81) and (82). Solving (21) for \( w = \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha}{1-\varepsilon}} \frac{1}{1-\alpha} \)
and substituting into (81):

\[
Y_Z = \left(\frac{\alpha}{1-\alpha} \cdot \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha}{1-\varepsilon}} \cdot \frac{1}{1-\alpha} \right)^{1-\alpha} \cdot \frac{Y}{A} p_{\hat{Z}}^{1-\varepsilon} \gamma_{\hat{Z}} \cdot \left[ \gamma_{\hat{D}} p_{\hat{D}}^{1-\varepsilon} + \gamma_{\hat{Z}} p_{\hat{Z}}^{1-\varepsilon} \right]^{\frac{\alpha\alpha+1}{1-\varepsilon}}.
\]

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or
\[ Y_Z = \left( \frac{\alpha}{1 - \alpha} \right)^{1 - \alpha} \cdot \frac{1}{A} \cdot Y \cdot \frac{\gamma_Z p_Z^{-\varepsilon}}{\gamma_D p_D^{-\varepsilon} + \gamma_Z p_Z^{-\varepsilon}}. \]

Rewriting the earlier assumption [Assumption 2] \((\frac{1 - \alpha}{\alpha})^{\frac{1}{\frac{1}{\alpha} - \alpha}} \cdot \frac{1}{A} = 1 - \alpha\) as \((\frac{1 - \alpha}{\alpha})^{1 - \alpha} \cdot \frac{1}{A} = \alpha\), we have
\[ Y_Z = \alpha \cdot Y \cdot \frac{\gamma_Z p_Z^{-\varepsilon}}{\gamma_D p_D^{-\varepsilon} + \gamma_Z p_Z^{-\varepsilon}}. \] (86)

Similarly, the simplified conditional demand for \(Y_D\) is
\[ Y_D = \alpha \cdot Y \cdot \frac{\gamma_D p_D^{-\varepsilon}}{\gamma_D p_D^{-\varepsilon} + \gamma_Z p_Z^{-\varepsilon}}. \] (87)

### B.3 Computations for Section 4.4

Substituting the price \(\chi_K(j) = 1\) into (26):
\[ x_K(j) = \left( \frac{p_K}{\varsigma} \right)^{1/\beta} K, \quad K = D, Z. \] (88)

The profit function (27) is reduced to
\[ \pi_K(j) = \beta \left( \frac{p_K}{\varsigma} \right)^{1/\beta} K, \quad K = D, Z. \] (89)

The uniform demand of machines, implied by (88), also means that the intermediate inputs in (2) are reduced to
\[ Y_K = \frac{1}{1 - \beta} \left( \frac{p_K}{\varsigma} \right)^{\frac{1 - \beta}{\alpha}} \varsigma K \quad K = D, Z. \] (90)

Equation (25) can here be used to eliminate \(K\) and \(Y_K\) from (90):
\[ \frac{\beta}{1 - \beta} \left( \frac{p_K}{\varsigma} \right)^{\frac{1 - \beta}{\alpha}} \varsigma p_K \varsigma K = c_K, \quad K = D, Z. \] (91)

Equation (28) is obtained by combining (89) and (91).

Next, we can form some useful ratios. First, we divide the two expressions in (91) by each other:
\[ p = \left( \frac{N_D c_Z}{N_Z c_D} \right)^{\beta}. \] (92)
This gives the optimal $p$ from the supply side (i.e. in the views of firms producing $Y_K$ and its components). To include the demand side (firms buying $Y_K$ to produce $Y$), we use (90) in (24):

$$p = \gamma^\beta \sigma^\gamma \left( \frac{N_Z}{N_D D} \right)^{-\frac{\beta}{\gamma}} \left( = \gamma^\beta \sigma^\gamma \cdot \Delta^{-\frac{\beta}{\gamma}} \right).$$

(93)

[Recall that $\sigma \equiv \beta \varepsilon + 1 - \beta = \varepsilon - (\varepsilon - 1)(1 - \beta).$] Finally, we combine the supply and demand sides in (92) and (93), to eliminate $p$ and get (29).

B.4 Decentralized Output

To derive (30), we first substitute (90) into (1):

$$Y = AL^{1-\alpha} \left( \frac{\delta^\beta}{\gamma^\beta \sigma^\gamma} \right)^\alpha \left[ \gamma_D \left( p_D^{\frac{\beta}{\gamma}} \right)^{\frac{\alpha}{\beta}} (N_D D)^{\frac{\alpha}{\gamma}} - 1 \right] \cdot \left( 1 - \sigma^\alpha \cdot \frac{N_D D}{AL} \cdot \frac{\gamma_D}{\Delta} \cdot \frac{\delta^\beta}{\gamma^\beta \sigma^\gamma} \cdot \frac{\alpha}{\beta} \right).$$

To eliminate $p_K$, we first solve (21) for

$$p_D = w^{\frac{\alpha - 1}{\alpha}} \cdot (\gamma_D + \gamma_Z p^{1-\varepsilon})^{\frac{1}{1-\varepsilon}} \quad \text{and} \quad p_Z = w^{\frac{\alpha - 1}{\alpha}} \cdot (\gamma_D p^{\varepsilon-1} + \gamma_Z)^{\frac{1}{1-\varepsilon}}.$$

Substitute these into the previous equation:

$$Y = AL^{1-\alpha} \left( \frac{\delta^\beta}{\gamma^\beta \sigma^\gamma} \right)^\alpha \cdot \left( (1 - \sigma^\alpha) \cdot \frac{N_D D}{AL} \cdot \frac{\gamma_D}{\Delta} \cdot \frac{\delta^\beta}{\gamma^\beta \sigma^\gamma} \cdot \frac{\alpha}{\beta} \right).$$

Now use (93) to eliminate $p$ and (22) to eliminate $w$:

$$Y = AL^{1-\alpha} \left( \frac{\delta^\beta}{\gamma^\beta \sigma^\gamma} \right)^\alpha \cdot \left( (1 - \sigma^\alpha) \cdot \frac{N_D D}{AL} \cdot \frac{\gamma_D}{\Delta} \cdot \frac{\delta^\beta}{\gamma^\beta \sigma^\gamma} \cdot \frac{\alpha}{\beta} \right).$$

Collecting terms:

$$Y = \left( A^\beta L^{1-\alpha} \left( \frac{\delta^\beta}{\gamma^\beta \sigma^\gamma} \right)^\alpha \cdot (1 - \sigma^\alpha)^{\alpha + \beta - 1 - \alpha \beta} \right)^{\frac{1}{1 - \alpha + \alpha \beta}}.$$

By Assumption 2 $(1 - \sigma)^{\alpha + \beta - 1 - \alpha \beta} = A^{1-\beta} \alpha^{\alpha(1-\beta)}$, so we have (30).
B.5 Marginal product of \( Z \)

To compute the marginal product of \( Z \) we first combine (25) and (90):

\[
\frac{\beta}{1-\beta} \left( \frac{1}{\zeta} \right)^{1-\beta} \left( p_Z \right)^{\frac{\beta}{1-\beta}} N_Z = c_Z.
\]

Now substitute \( p_Z = w^{\frac{\alpha-1}{\alpha}} \cdot \left[ \gamma_D p^{\sigma-1} + \gamma_Z \right]^{\frac{1}{\sigma}} \) from (21):

\[
\frac{\beta}{1-\beta} \left( \frac{1}{\zeta} \right)^{1-\beta} \left( w^{\frac{\alpha-1}{\alpha}} \right)^{\frac{\alpha-1}{\alpha\sigma}} \left[ \gamma_D (p)\varepsilon^{-1} + \gamma_Z \right]^{\frac{1}{\sigma(\varepsilon-1)}} N_Z = c_Z.
\]

Now substitute (22) and (93):

\[
\frac{\beta}{1-\beta} \left( \frac{1}{\zeta} \right)^{1-\beta} \left( \frac{1-\alpha}{\alpha} \right)^{\frac{\alpha-1}{\alpha\sigma}} \cdot (Y)^{\frac{\alpha-1}{\alpha\sigma}} \left[ \gamma^{-\frac{\epsilon}{\sigma}} \cdot \Delta^{-\frac{(\sigma-1)}{\sigma}} + 1 \right]^{\frac{1}{\sigma-1}} \gamma_Z N_Z = c_Z,
\]

where we have used \( \sigma - 1 = \beta(\varepsilon - 1) \) and \( \frac{\epsilon \beta(\sigma-1)}{\sigma} - \varepsilon = \frac{\epsilon(\sigma-1-\sigma)}{\sigma} = -\frac{\varepsilon}{\sigma} \).

Equation (30) can be written as

\[
Y = A_2 \gamma_Z^{\frac{\sigma}{\sigma-1}} \cdot \left[ \gamma^{-\frac{\epsilon}{\sigma}} \cdot \left( \Delta \right)^{-\frac{(\sigma-1)}{\sigma}} + 1 \right]^{\frac{1}{\sigma-1}} \gamma_Z \cdot (N_Z Z)^{\frac{\alpha\beta}{1-\alpha-\sigma}}.
\]

Substitute this into the previous equation

\[
\left( \frac{\beta}{1-\beta} \left( \frac{1}{\zeta} \right)^{1-\beta} \left( \frac{1-\alpha}{\alpha} \right)^{\frac{\alpha-1}{\alpha\sigma}} \right) \cdot A_2 \gamma_Z^{\frac{\sigma}{\sigma-1}} \cdot \left[ \gamma^{-\frac{\epsilon}{\sigma}} \cdot \left( \Delta \right)^{-\frac{(\sigma-1)}{\sigma}} + 1 \right]^{\frac{1}{\sigma-1}} \gamma_Z \cdot (N_Z Z)^{\frac{\alpha\beta}{1-\alpha-\sigma}} = c_Z.
\]

To simplify the constant, assume that \( \zeta = 1 - \beta \). Recall also the definition of \( A_2 \). The constant (except the \( \gamma_Z \) factor) then is

\[
\alpha\beta \cdot \left( \left( \frac{1}{1-\beta} \right)^{\alpha} L^{1-\alpha} A \cdot \alpha^{\alpha(1-\beta)} \right)^{\frac{1}{1-\alpha+\sigma}}.
\]

Using this in (94):

\[
\alpha\beta \cdot \left( \left( \frac{1}{1-\beta} \right)^{\alpha} L^{1-\alpha} A \cdot \alpha^{\alpha(1-\beta)} \right)^{\frac{1}{1-\alpha+\sigma}} \gamma_Z^{\frac{\sigma}{\sigma-1}} \cdot \left[ \gamma^{-\frac{\epsilon}{\sigma}} \cdot \left( \Delta \right)^{-\frac{(\sigma-1)}{\sigma}} + 1 \right]^{\frac{1}{\sigma-1}} \gamma_Z \cdot (N_Z Z)^{\frac{\alpha\beta}{1-\alpha-\sigma}} = c_Z.
\]

This is equation (15).
C Balanced Growth Path (Not for publication)

Define the additional composite variables

\[ \chi \equiv \frac{\mu Z_N}{\mu D N_D}, \quad \hat{\omega} \equiv \frac{c_D + \omega}{\omega}, \quad \Omega \equiv \frac{\omega D}{Z}, \quad \Lambda \equiv \frac{Z^{1-\theta}}{\mu D N_D}, \quad \hat{\gamma} \equiv \frac{C}{Z} \quad \text{and} \quad \hat{d} \equiv \frac{D^{1+\kappa}}{Z^{1-\theta}}. \]

We will here derive six equations in six variables and their growth rates, in order to characterize the BGP. The variables are \( \Delta, \hat{\omega}, S_D, \hat{c}, \Lambda \) and \( \hat{d} \).

C.1 Deriving the six equations

C.1.1 Delta

The definition of \( \Delta \) gives

\[ g_\Delta = g_{N_Z} + g_Z - g_{N_D} - g_D. \]  

\((96)\)

Differentiating (14) with respect to time:

\[ g_{Z} - g_{D} = \sigma \frac{\omega}{c_D + \omega} g_{\omega} + (\sigma - 1) \left( g_{N_Z} - g_{N_D} \right). \]

Substitute this into (96):

\[ \frac{1}{\sigma} g_\Delta = \frac{\omega}{c_D + \omega} g_{\omega} + g_{N_Z} (S_D) - g_{N_D} (S_D), \]

\((97)\)

where \( g_{N_Z} (S_D) \) and \( g_{N_D} (S_D) \) are given by (3).

Since \( \omega \) will grow on the BGP it is useful to replace it by something that is constant in the long run. We therefore define

\[ \hat{\omega} \equiv \frac{c_D + \omega}{\omega} \quad \Rightarrow \quad g_{\hat{\omega}} = -\frac{c_D}{c_D + \omega} g_{\omega}. \]

As \( \omega \) grows, \( \hat{\omega} \to 1 \) and \( g_{\hat{\omega}} \to 0 \). This definition means that

\[ \omega = \frac{c_D}{\hat{\omega} - 1} \quad \Rightarrow \quad g_{\omega} = -\frac{\hat{\omega}}{\hat{\omega} - 1} g_{\hat{\omega}}. \]

We note that \( \frac{\omega}{c_D + \omega} g_{\omega} = \frac{1}{1-\hat{\omega}} \cdot g_{\hat{\omega}} \). Substitute this into (97):

\[ \frac{1}{\sigma} g_\Delta = \frac{1}{1-\hat{\omega}} \cdot g_{\hat{\omega}} + g_{N_Z} (S_D) - g_{N_D} (S_D). \]  

\((98)\)
C.1.2 Omega-hat

Next, rewrite (14) and multiply by $D/Z$:

$$c_Z \gamma^{-\frac{\epsilon}{2} - \frac{\sigma}{2}} \Delta^{\frac{1-\sigma}{2}} = \hat{\omega} \Omega.$$  \hfill (99)

Note that $\Omega = \zeta M^{1-\theta} \cdot \hat{c}^\theta \cdot \hat{d}$. Thus $g_\Omega = \theta g_e + g_d$ and differentiation of (99) gives

$$g_\Omega = \frac{1 - \sigma}{\sigma} g_\Delta - \theta g_e - g_d.$$  \hfill (100)

C.1.3 Scientists

Using the definition $\chi \equiv \frac{\mu_Z N_Z}{\mu_D N_D}$, to get $g_\chi = g_{\mu_D} + g_N - g_{\mu_D} - g_N$, and invoking (11) and (12):

$$g_\chi = M^\theta \Lambda \cdot (\hat{c})^{-\theta} \left( \left( \frac{c_D + \omega}{Z} \right) - c_Z \chi^{-1} \right).$$  \hfill (101)

Recall that (8) implies that

$$\chi^{-1} = \frac{\mu_D N_D}{\mu_Z N_Z} = \frac{\eta_f f_Z (S - S_D)}{\eta_D f_D (S_D)}.$$

Using this and (99) in (101), we have

$$g_\chi = M^\theta \Lambda \cdot (\hat{c})^{-\theta} c_Z \left( \gamma^{-\frac{\epsilon}{2} - \frac{\sigma}{2}} \Delta^{\frac{1-\sigma}{2}} - \eta f_Z f_D (S_D) \right).$$  \hfill (102)

To get the change in $S_D$, we differentiate (8) with respect to time

$$\left( \frac{f_D'' S_D}{f_D} + \frac{f_Z'' S_D}{f_Z} \right) g_{S_D} = g_\chi.$$

Combining this with (102):

$$g_{S_D} = \frac{M^\theta \Lambda \cdot (\hat{c})^{-\theta} c_Z}{\mathcal{E}(S_D)} \left( \gamma^{-\frac{\epsilon}{2} - \frac{\sigma}{2}} \Delta^{\frac{1-\sigma}{2}} - \eta f_Z f_D (S_D) \right),$$  \hfill (103)

where we have defined

$$\mathcal{E}(S_D) \equiv \left( \frac{f_D'' S_D}{f_D} + \frac{f_Z'' S_D}{f_Z} \right).$$
C.1.4 Differentiated consumption constraint

Equation (9) implies that \( \frac{Y}{Z} = \frac{c_Z}{\alpha \beta} \cdot \left[ \gamma^{-\frac{1}{\sigma}} (\Delta)^{-\frac{\sigma+1}{\gamma}} + 1 \right] \). The use of this in (16) gives

\[
\hat{c} = \frac{(1 - \alpha)c_Z}{\alpha \beta} \left( \gamma^{-\frac{1}{\sigma}} \Delta^{-\frac{1}{\sigma}} + 1 \right) + \Omega.\tag{104}
\]

Taking logs and differentiating with respect to time (recall that \( \Omega = \zeta M^{1-\theta} \cdot \hat{c} \cdot \hat{d} \) and thus \( \Omega g_{\Omega} = \zeta M^{1-\theta} \cdot \hat{c} \cdot \hat{d} (\theta g_{\hat{c}} + g_{\hat{d}}) \)):

\[
g_{\hat{c}} = (\hat{c})^{-1} \cdot \left( \frac{(1 - \alpha)c_Z}{\alpha \beta \gamma^{\frac{1}{\sigma}}} \cdot \frac{1 - \sigma}{\sigma} \cdot \Delta^{1-\frac{\sigma}{\gamma}} g_{\Delta} + \zeta M^{1-\theta} \cdot \hat{c} \cdot \hat{d} (\theta g_{\hat{c}} + g_{\hat{d}}) \right).
\]

Solve for \( g_{\hat{c}} \) (using \( \Omega = \zeta M^{1-\theta} \cdot \hat{c} \cdot \hat{d} \)):

\[
g_{\hat{c}} = \frac{(1 - \alpha)c_Z (1 - \sigma) \Delta^{1-\frac{\sigma}{\gamma}}}{(\hat{c} - \theta \Omega) \alpha \beta \gamma^{\frac{1}{\sigma}}} g_{\Delta} + \frac{\Omega}{(\hat{c} - \theta \Omega)} g_{\hat{d}}.\tag{105}
\]

C.1.5 Lambda

The definition of \( \Lambda \equiv \frac{Z^{1-\theta}}{\mu \rho N_D} \) implies that \( g_{\Lambda} = (1 - \theta)g_z - g_{\mu_D} - g_{N_D} \). Rewrite equation (11): \( g_{\mu_D} + g_{N_D} = \rho - M^\theta \cdot \Lambda \cdot (\hat{c})^{-\theta} \cdot \hat{\omega} \Omega \). Combining these two equations:

\[
g_{\Lambda} = (1 - \theta)g_z + M^\theta \cdot \Lambda \cdot (\hat{c})^{-\theta} \cdot \hat{\omega} \Omega - \rho.
\]

To eliminate \( g_z \) we differentiate (15) with respect to time (and use the definition of \( \xi \)):

\[
g_z = \frac{\alpha \beta}{1 - \alpha} g_{N_z} - \frac{((1 - \alpha + \alpha \beta) / \sigma) - (1 - \alpha)}{(1 - \alpha) \left( 1 + \gamma^{\frac{1}{\sigma}} (\Delta)^{-\frac{\sigma+1}{\gamma}} \right)} g_{\Delta}.\tag{106}
\]

Now use (106) in the previous equation (recall that \( \Omega = \zeta M^{1-\theta} \cdot \hat{c} \cdot \hat{d} \)):

\[
g_{\Lambda} = (\theta - 1) \left( \frac{(1 - \alpha + \alpha \beta)}{(1 - \alpha) \sigma} \right) - 1 \left( 1 + \gamma^{\frac{1}{\sigma}} (\Delta)^{-\frac{\sigma+1}{\gamma}} \right) g_{\Delta} - (\theta - 1) \frac{\alpha \beta}{1 - \alpha} g_{N_z} (S_D) + \zeta M \cdot \Lambda \cdot \hat{\omega} \hat{d} - \rho.\tag{107}
\]

C.1.6 \( d \)-hat

The time derivative of \( \hat{d} = \frac{D^{1+s}}{Z^{1-\sigma}} \) is \( g_{\hat{d}} = (1 + \kappa)g_D - (1 - \theta)g_z \). We first develop two expressions to substitute into this expression. First, equation (106) gives

\[
(1 - \theta)g_z = (1 - \theta) \left( \frac{\alpha \beta}{1 - \alpha} g_{N_z} - \frac{((1 - \alpha + \alpha \beta) / \sigma) - (1 - \alpha)}{(1 - \alpha) \left( 1 + \gamma^{\frac{1}{\sigma}} (\Delta)^{-\frac{\sigma+1}{\gamma}} \right)} g_{\Delta} \right).
\]

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Next, equation (96) gives \((1 + \kappa)g_D = (1 + \kappa) (g_{N_Z} + g_Z - g_{N_D} - g_\Delta)\). Using (106):

\[
(1 + \kappa)g_D = (1 + \kappa) \left( g_{N_Z} + \frac{\alpha \beta}{1 - \alpha} g_{N_Z} - \frac{(1 - \alpha + \alpha \beta)/\sigma - (1 - \alpha)}{(1 - \alpha) \left(1 + \frac{\gamma}{\sigma} (\Delta) \frac{\sigma - 1}{\sigma}\right)} g_\Delta - g_{N_D} - g_\Delta \right). 
\]

Substituting these two equations into the time derivative of \(\hat{g}_D\):

\[
g_{\hat{d}} = \left( \frac{\theta + \kappa}{1 - \alpha} \right) g_{N_Z} (S_D) + (1 + \kappa) (g_{N_Z} (S_D) - g_{N_D} (S_D)) \\
\quad - \left( \frac{\theta + \kappa}{1 - \alpha} \right) + (1 - \theta) + (1 + \kappa) \frac{\gamma}{\sigma} (\Delta) \frac{\sigma - 1}{\sigma} g_\Delta. 
\]

To sum up, the six equations in the variables \(\Delta, \hat{\omega}, S_D, \hat{c}, \Lambda\) and \(\hat{d}\) are (98), (100), (103), (105), (107) and (108).

**C.2 Solving the system**

Equation (103) does not contain any other change than the \(g_{S_D}\) on the right hand side. Equation (107) can be dealt with later, when an expression for \(\Delta\) is obtained. We therefore focus on the other four equations initially.

**C.2.1 Deriving an equation for \(g_\Delta\)**

Substitute (108) into (105) (to eliminate \(g_{\hat{d}}\)):

\[
g_{\hat{c}} = \frac{\Omega}{(\hat{c} - \theta \Omega)} \left( \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{N_Z} (S_D) + (1 + \kappa) (g_{N_Z} (S_D) - g_{N_D} (S_D)) \right) + \\
\quad + \left( \frac{\Delta \frac{1 - \sigma}{\sigma} + \gamma \frac{\hat{c}}{\sigma} (1 - \sigma)}{1 + \gamma \frac{\hat{c}}{\sigma} (\Delta) \frac{\sigma - 1}{\sigma}} \right) (\hat{c} - \theta \Omega) \alpha \beta \gamma \frac{\hat{c}}{\sigma} g_\Delta \\
\quad - \frac{\Omega \alpha \beta \gamma \frac{\hat{c}}{\sigma} \left( (\theta + \kappa) \left( \frac{1 - \alpha + \alpha \beta}{1 - \alpha} \right) + (1 - \theta) + (1 + \kappa) \frac{\gamma}{\sigma} (\Delta) \frac{\sigma - 1}{\sigma} \right)}{(\hat{c} - \theta \Omega) \alpha \beta \gamma \frac{\hat{c}}{\sigma} \left( 1 + \gamma \frac{\hat{c}}{\sigma} (\Delta) \frac{\sigma - 1}{\sigma} \right)} g_\Delta. 
\]

Thus \(g_{\hat{c}}\) is here a function of just one change: \(g_\Delta\).

The next step is to make the equation for \(g_{\hat{z}}\), (100), an equation in just one change, \(g_\Delta\), by eliminating \(g_{\hat{c}}\) and \(g_{\hat{d}}\). First, substitute (108) into (100) (to eliminate \(g_{\hat{d}}\)):

\[
g_{\hat{z}} = (1 + \kappa) (g_{N_D} (S_D) - g_{N_Z} (S_D)) - \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{N_Z} (S_D) - \theta g_{\hat{c}}
\]
In this equation

\[ g = \frac{(\theta + \kappa) \left( \frac{(1-\alpha+\alpha\beta)}{(1-\alpha)} \right) + 1 - \sigma \theta + (1 + \sigma \kappa) \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}}}{\sigma \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)} g \Delta. \]  

(110)

Second, we substitute (109) into (110) (to eliminate \( g \hat{c} \)):

\[ g = \frac{\hat{c}}{\hat{c} - \theta \Omega} \left( (1 + \kappa) (g_{N_D}(S_D) - g_{N_Z}(S_D)) - \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{N_Z}(S_D) \right) \]

\[ + \frac{\hat{c} \alpha \beta \gamma \hat{\xi} \left( (\theta + \kappa) \left( \frac{(1-\alpha+\alpha\beta)}{(1-\alpha)} \right) + 1 - \sigma \theta + (1 + \sigma \kappa) \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)}{(\hat{c} - \theta \Omega) \alpha \beta \gamma \hat{\xi} \sigma \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)} g \Delta \]

\[ + \frac{\theta (\sigma - 1) \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right) \left( \Omega \alpha \beta \gamma \hat{\xi} + \Delta^{\frac{1}{\alpha}} (1 - \alpha) c_Z \right)}{(\hat{c} - \theta \Omega) \alpha \beta \gamma \hat{\xi} \sigma \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)} g \Delta. \]

Equation (14) can be written as \( c_Z \Delta^{\frac{1}{\alpha}} = \gamma \hat{\xi} \hat{\omega} \). Using this in the above equation:

\[ g = \frac{\hat{c}}{\hat{c} - \theta \Omega} \left( (1 + \kappa) (g_{N_D}(S_D) - g_{N_Z}(S_D)) - \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{N_Z}(S_D) \right) \]

\[ + \frac{\hat{c} \left( (\theta + \kappa) \left( \frac{(1-\alpha+\alpha\beta)}{(1-\alpha)} \right) + 1 - \sigma \theta + (1 + \sigma \kappa) \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)}{(\hat{c} - \theta \Omega) \sigma \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)} g \Delta \]

\[ + \frac{\theta \Omega (\sigma - 1) \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right) \left( 1 + \hat{c} \frac{(1-\alpha)}{\alpha \beta} \right)}{(\hat{c} - \theta \Omega) \sigma \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)} g \Delta. \]

To prepare for the substitution into (98) it is useful to separate out the expression \( g \Delta / \sigma \):

\[ g = \frac{\hat{c}}{\hat{c} - \theta \Omega} \left( (1 + \kappa) (g_{N_D}(S_D) - g_{N_Z}(S_D)) - \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{N_Z}(S_D) \right) \]

\[ + \frac{\hat{c} \left( (\theta + \kappa) \left( \frac{(1-\alpha+\alpha\beta)}{(1-\alpha)} \right) - \sigma \theta + \sigma \kappa \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)}{(\hat{c} - \theta \Omega) \sigma \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)} g \Delta \]

\[ + \frac{\theta \Omega \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right) \left( \sigma + (\sigma - 1) \hat{\omega} \frac{(1-\alpha)}{\alpha \beta} \right)}{(\hat{c} - \theta \Omega) \sigma \left( 1 + \gamma \hat{\xi} (\Delta)^{\frac{x-1}{\alpha}} \right)} g \Delta + \frac{1}{\sigma} g \Delta. \]  

(111)

In this equation \( g \) is a function of one change only, namely \( g \Delta \).
Now, finally, we use (111) to eliminate $g_\varnothing$ in (98) and thus obtain the equation where $g_\Delta$ is not a function of any other change. We get:

$$g_\Delta = \frac{\sigma \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right)}{\Psi}.$$  

\[
\cdot \left( \hat{c} \cdot (\theta + \kappa) \frac{\alpha \beta}{1 - \alpha} \cdot g_{Nz}(S_D) - \left[ \hat{c} \cdot (1 + \kappa) + (\hat{c} - \theta \Omega) \cdot (\hat{\omega} - 1) \right] \cdot (g_{N_D}(S_D) - g_{N}(S_D)) \right),
\]

where

$$\Psi \equiv \hat{c} \cdot (\theta + \kappa) \left( \frac{(1 - \alpha + \alpha \beta)}{1 - \alpha} - \sigma \right) + \left[ \hat{c} \cdot (\theta - \Omega) \cdot (1 - \hat{\omega}) \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right) \right] \cdot \left[ \theta \Omega \left( \sigma - \hat{\omega} + (\sigma - 1) \hat{\omega} \left( \frac{1 - \alpha}{\alpha \beta} \right) \right) + \hat{c} \cdot (\hat{\omega} + \sigma \kappa) \right].$$

C.2.2 Substituting $g_\Delta$ back

In the light of (112) (using the definition of $\Psi$) equation (111) can be simplified to

$$g_\varnothing = \frac{\hat{c} \cdot \left( 1 + \kappa \right) \left( g_{N_D}(S_D) - g_{N}(S_D) \right) - \left( \theta + \kappa \right) \frac{\alpha \beta}{1 - \alpha} \cdot g_{Nz}(S_D) \right) +$$

$$+ \frac{\Psi \cdot (\hat{c} - \theta \Omega) \cdot (1 - \hat{\omega}) \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right)}{(\hat{c} - \theta \Omega) \cdot \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right)} \cdot g_\Delta.$$  

Now substitute (112) into this

$$g_\varnothing = \left( \Psi \cdot (\hat{c} - \theta \Omega) \cdot (1 - \hat{\omega}) \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right) \right) \cdot \left( (\hat{\omega} - 1) \cdot (g_{Nz}(S_D) - g_{ND}(S_D)) \right) +$$

$$+ \frac{\hat{c} \cdot (1 - \hat{\omega}) \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right) \cdot \left( \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} \cdot g_{Nz}(S_D) \right)}{\Psi} \cdot \left( 1 + \kappa \right) \cdot \left( g_{N_D}(S_D) - g_{N}(S_D) \right) \right) \cdot \left( g_{N_D}(S_D) - g_{N}(S_D) \right).$$

This is an equation where $g_\varnothing$ is not a function of any other change.

Next we turn to the expression for $g_\psi$. Recall first that, by (14) and the definition of $\hat{\omega}$, $c_Z \hat{\omega} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} = \gamma \hat{z} \hat{\omega} \Omega$. Equation (109) can therefore be simplified to

$$g_\psi = \frac{\Omega}{(\hat{c} - \theta \Omega)} \left( \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} \cdot g_{Nz}(S_D) + (1 + \kappa) \cdot \left( g_{N_D}(S_D) - g_{ND}(S_D) \right) \right) +$$

$$+ \frac{\Omega \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right) \cdot (1 - \sigma) \cdot \left( 1 + \kappa \right)}{(\hat{c} - \theta \Omega) \cdot \left( 1 + \gamma \hat{z} \left( \Delta \right)^{\frac{\sigma - 1}{\alpha \beta}} \right)} \cdot g_\Delta.$$  

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Substituting (112) into this (and simplifying by use of the definition of $\Psi$):

$$g^c = \frac{\Omega \left( 1 + \gamma \hat{\omega} \left( \Delta \right)^{\frac{\sigma - 1}{\sigma}} \right) \left[ \hat{\omega} \left( \frac{1 - \alpha}{\alpha \beta} \right) \left( 1 - \sigma \right) + \hat{\omega} - \sigma \right]}{\Psi}.$$

$$\cdot \left( \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{NZ}(SD) + (1 + \kappa) (g_{NZ}(SD) - g_{ND}(SD)) \right) +$$

$$\frac{\Omega \left( 1 + \gamma \hat{\omega} \left( \Delta \right)^{\frac{\sigma - 1}{\sigma}} \right) \left[ \frac{1 - \alpha}{\alpha \beta} \hat{\omega} \left( 1 - \sigma \right) - \sigma (1 + \kappa) \left( 1 + \gamma \hat{\omega} \left( \Delta \right)^{\frac{\sigma - 1}{\sigma}} \right) \right]}{\Psi} \cdot (\hat{\omega} - 1) (g_{NZ}(SD) - g_{ND}(SD)) . \tag{114}$$

This is an equation where $g^c$ is not a function of any other change.

Next, we go for an equation for $g^d$, by substituting (112) into (108):

$$g^d = \frac{\left( 1 + \gamma \hat{\omega} \left( \Delta \right)^{\frac{\sigma - 1}{\sigma}} \right) \left[ \theta \Omega \left( \sigma - \hat{\omega} + (\sigma - 1) \hat{\omega} \left( \frac{1 - \alpha}{\alpha \beta} \right) + \hat{c} (\hat{\omega} - \sigma) \right) \right]}{\Psi} .$$

$$\cdot \left( \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{NZ}(SD) + (1 + \kappa) (g_{NZ}(SD) - g_{ND}(SD)) \right)$$

$$+ \frac{(\theta + \kappa) \left( \frac{1 - \alpha + \alpha \beta}{1 - \alpha} - \sigma \right) + \sigma (1 + \kappa) \left( 1 + \gamma \hat{\omega} \left( \Delta \right)^{\frac{\sigma - 1}{\sigma}} \right)}{\Psi} \cdot ((\hat{c} - \theta \Omega) (\hat{\omega} - 1) (g_{NZ}(SD) - g_{ND}(SD))) . \tag{115}$$

This is an equation where $g^d$ is not a function of any other change.

Finally, we substitute (112) into (107):

$$g^L = \frac{(\theta - 1) \left( \frac{1 - \alpha + \alpha \beta}{1 - \alpha} - \sigma \right)}{\Psi} \hat{c} \left( \frac{(\theta + \kappa) \alpha \beta}{1 - \alpha} g_{NZ}(SD) + (1 + \kappa) (g_{NZ}(SD) - g_{ND}(SD)) \right) +$$

$$\frac{(\theta - 1) \left( \frac{1 - \alpha + \alpha \beta}{1 - \alpha} - \sigma \right)}{\Psi} \cdot (\hat{c} - \theta \Omega) (\hat{\omega} - 1) (g_{NZ}(SD) - g_{ND}(SD))$$

$$+(1 - \theta) \frac{\alpha \beta}{1 - \alpha} g_{NZ}(SD) + \zeta L \cdot \Lambda \cdot \hat{\omega} d - \rho . \tag{116}$$

This is an equation where $g^L$ is not a function of any other change.

Equations (103), (112)–(116) are simplified to (41)–(46) in Appendix A.3.