Multi-Factor Extensions of the Capital Asset Pricing Model:  
An Empirical Study of the UK Market  

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Abstract

The point of this thesis is to compare classic asset pricing models using historic UK data. It looks at three of the most commonly used asset pricing models in Finance and tests the suitability of each for the UK market. The models considered are the Capital Asset Pricing Model (1964, 65 and 66) (CAPM), the Fama-French 3-Factor Model (1993) (FF3F) and the Carhart 4-Factor Model (1997) (C4F). The models are analysed using a 34 year sample period (1980-2014). The sample data follows the structure explained in Gregory et al (2013) and is compiled of stocks from the London Stock Exchange (LSE). The stocks are grouped into portfolios arranged by market capitalisation, book-to-market ratio, past 2-12 month stock return and past 12 month standard deviation of stock return. Statistical analysis is performed and the suitability of the models is tested using the methods of Black, Jensen & Scholes (1972), Fama & MacBeth (1973) and Gibbons, Ross & Shanken (1989). The results compare descriptive and test statistics across the range of risk factors and test portfolios for the each testing method on all three models. They show that although the UK market has some noticeable factor anomalies, none of the models clearly explains the 1980-2014 stock returns. However, of the three models, C4F shows the highest explanatory power in predicting stock returns.
Acknowledgements

I would firstly like to thank my supervisor, Lars Pettersson, whose knowledge, guidance and perseverance have been invaluable to me over the time I have been writing this paper. I recognise Lars as the one who introduced me to the subject of Modern Portfolio Theory and inspired me to take my knowledge into a career in Finance. For this I will be eternally grateful.

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I acknowledge the great work done by Professor Alan Gregory and his colleagues at Xfi - Centre for Finance and Investment at the University of Exeter. Their papers and data which they make available for research of the UK market are of major benefit to the field.

Finally I would like to pay tribute to my girlfriend Sonja, parents Malcolm and Aileen, and all of my family, friends, colleagues and lecturers who have helped and supported me over the years. This paper would not have been possible without you!

- Calum Johnson 2015
1 Motivation

2 Introduction

3 Background
   3.1 History of the UK Market
   3.2 Modern Portfolio Theory
   3.3 Risk
   3.4 The Investment Portfolio
      3.4.1 Portfolio Return
      3.4.2 Portfolio Variance
   3.5 The Efficient Frontier
   3.6 The Capital Market Line (CML)

4 Asset Pricing Models
   4.1 Single-Index (Market) Model
      4.1.1 Factor 1: Market (RMRF)
      4.1.2 The Capital Asset Pricing Model (CAPM)
   4.2 Multi-Factor Models
      4.2.1 Factor 2: Size (SMB)
      4.2.2 Factor 3: Value (HML)
      4.2.3 Fama-French 3-Factor Model (FF3F)
      4.2.4 Factor 4: Momentum (UMD)
      4.2.5 Carhart 4-Factor Model (C4F)

5 Formulation
   5.1 Data
   5.2 Construction of Factors
   5.3 Construction of Test Portfolios

6 Statistical Tests
   6.1 Descriptive Statistics
      6.1.1 Sample Mean
      6.1.2 Sample Variance
      6.1.3 Sample Standard Deviation
      6.1.4 Sample Skewness
      6.1.5 Sample Kurtosis
      6.1.6 Factor Correlation
      6.1.7 (Adjusted) $R^2$
   6.2 Hypothesis Testing
   6.3 Empirical Tests
      6.3.1 Black, Jensen & Scholes (1972)
      6.3.2 Fama-Macbeth (1973)
      6.3.3 Gibbons, Ross & Shanken (1989)
7 Results

7.1 Descriptive Statistics

7.1.1 Factors

7.1.2 Portfolio group 1 - Size and Value

7.1.3 Portfolio group 2 - Size and Momentum

7.1.4 Portfolio group 3 - Size, Value and Momentum

7.1.5 Portfolio group 4 - Standard Deviation

7.2 Black, Jensen & Scholes (1972) - Test Results

7.2.1 Portfolio group 1 - Size and Value

7.2.2 Portfolio group 2 - Size and Momentum

7.2.3 Portfolio group 3 - Size, Value and Momentum

7.2.4 Portfolio group 4 - Standard Deviation

7.3 Fama-Macbeth (1973) - Test Results

7.3.1 Portfolio group 1 - Size and Value

7.3.2 Portfolio group 2 - Size and Momentum

7.3.3 Portfolio group 3 - Size, Value and Momentum

7.3.4 Portfolio group 4 - Standard Deviation

7.4 Gibbons, Ross & Shanken (1989) - Test Results

7.4.1 Portfolio group 1 - Size and Value

7.4.2 Portfolio group 2 - Size and Momentum

7.4.3 Portfolio group 3 - Size, Value and Momentum

7.4.4 Portfolio group 4 - Standard Deviation

8 Conclusion

9 Appendix

I Appendix A - Mathematical Derivations

9.1 Portfolio Return and Variance

9.1.1 Portfolio Return

9.1.2 Portfolio Variance

9.2 Efficient Frontier: Mean-Variance Efficient Portfolios

9.3 Single-Index (Market) Model

9.3.1 Expected return on a security

9.3.2 Variance of return of a security

9.3.3 Co-variance between two securities

9.4 Multi-Factor Models

9.4.1 Expected return on a security

9.4.2 Variance of return of a security

9.4.3 Co-variance between two securities

9.5 Capital Asset Pricing Model (CAPM)

II Appendix B - Thesis Data

9.6 Market Data
List of Tables

1. Descriptive statistics of Factors .................................................. 40
2. Correlation of Factors ................................................................. 41
3. Descriptive statistics of portfolios sorted by Size and Value ............. 41
4. Descriptive statistics of portfolios sorted by Size and Momentum ...... 42
5. Descriptive statistics of portfolios sorted by Size, Value and Momentum 43
6. Descriptive statistics of portfolios sorted by Standard Deviation ...... 44
7. BJS t-test for 25 portfolios arranged by the Size and Value ............. 45
8. BJS t-test for 25 portfolios arranged by Size and Momentum .......... 45
9. BJS t-test for 27 portfolios arranged by Size, Value and Momentum - CAPM model ............................................................... 46
10. BJS t-test for 27 portfolios arranged by Size, Value and Momentum - FF3F model .............................................................. 46
11. BJS t-test for 27 portfolios arranged by Size, Value and Momentum - C4F model .............................................................. 47
12. BJS t-test for 25 portfolios arranged by Standard Deviation ........... 49
13. FM test for 25 portfolios arranged by Size and Value ..................... 50
14. FM test for 25 portfolios arranged by Size and Momentum ............. 50
15. FM test for 27 portfolios arranged by Size, Value and Momentum ..... 51
16. FM test for 25 portfolios arranged by Standard Deviation ............. 51
17. GRS $\chi^2$ and F tests for 25 portfolios arranged by Size and Value . 52
18. GRS $\chi^2$ and F tests for 25 portfolios arranged by Size and Momentum 52
19. GRS $\chi^2$ and F tests for 27 portfolios arranged by Size, Value and Momentum 53
20. GRS $\chi^2$ and F tests for 25 portfolios arranged by Standard Deviation . 53
21. Extract of data on monthly returns for Size by Book-to-Market sorted Portfolios ................................................................. 72
22. Extract of data on monthly returns for Size by Momentum sorted Portfolios 72
23. Extract of data on monthly returns for Size by Book-to-Market by Momentum sorted Portfolios ..................................................... 73
24. Extract of data on monthly returns for Standard Deviation sorted Portfolios 73
25. Extract of data on monthly returns for the Factors .......................... 73
List of Figures

<table>
<thead>
<tr>
<th></th>
<th>The Efficient Frontier with Capital Market Line</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>The Security Market Line</td>
<td>25</td>
</tr>
</tbody>
</table>
1 Motivation

My interest in the understanding of financial markets and optimal investing has developed over many years. I completed a BSc in Mathematics at the University of Glasgow in 2010 but as I didn’t really specialise in one particular area I was unsure what career to pursue. This reason, combined with a job market still struggling to recover from The Great Recession, resulted in me working for a variety of employers who did not require my mathematical and analytical skills.

In 2012 I started working within the wealth management wing of the UK’s second largest bank. There I was first introduced to the concept of investment portfolios. Though the role was only operational, it sparked a desire to learn more about how these portfolios were constructed and how these strategies were developed.

In autumn 2012 I made a bold decision to move to Sweden and return to education. This MSc in Financial Engineering at Mälardalen University has included many courses which have developed my financial and mathematical knowledge.

However what struck a personal chord were the courses of Portfolio Theory taught by Lars Pettersson. I took the first Portfolio Theory course in the first semester of my Financial Engineering programme. Lars approached these courses with a focus on understanding. Studying from the 8th Edition of Modern Portfolio Theory and Investment Analysis by Elton, Gruber, Brown & Goetzmann, I learned of mean-variance efficiency, the importance of security covariance and the efficient frontier. This knowledge was then utilised in an optimisation project using historic data from the Swedish market. Co-operating with team members from many different countries I thoroughly enjoyed researching and presenting our results.

After gaining this knowledge I was particularly keen to build my understanding of investment strategies and asset pricing. I was pleased to get the opportunity to take the second Portfolio Theory course offered at the University in spring 2014. Lars continued the course and started with the Capital Asset Pricing Model. He connected it to the previously learned Single-Index model then introduced the idea of multiple factors contributing to stock price movements.

Then, unlike any course I had previously taken in Scotland or Sweden, the class were assigned to teach the remaining lectures. This really encouraged me to understand Modern Portfolio Theory at a much deeper level. I was given the task of talking about Market Efficiency, basing it around Chapter 17 of Elton, Gruber, Brown & Goetzmann’s book.

This led me to the work of Eugene Fama and his pioneering papers with Kenneth French on Multi-Factor asset pricing models. Though these theories are not new and have been tested extensively, there is still relevance in comparing a number of models with alternative and more recent data.
Up to this point my research had been primarily based around the US and Swedish markets. This is down to the papers, books and lectures to which I had previously been exposed. However being from the UK, a country with its own extensive financial history, I really wanted to test the theories that were developed using NYSE data on the LSE.
2 Introduction

Since the development and implementation of asset pricing models in finance there have been criticisms and subsequent searches for new and improved models. It is obvious that the 'one model fits all' concept is not possible when simply considering all the differences that exist between investors. The sheer complexity of the world’s financial system suggests that even if all investors were indeed rational, the idea that there is only one optimal investment portfolio to hold and one way to calculate it is fanciful. However, until a model is developed that can sufficiently explain stock price movements and consistently deliver a positive return for investors, the search for new and improved models will continue.

Stock markets have existed for centuries, however it wasn’t until after the Great Depression of the 1920’s and 30’s that a shift towards more 'intelligent investing' started. Benjamin Graham was one of the pioneers of this approach. His work focussed on the due diligence required to make sound investment decisions.

However the real starting point in what can be classified as Modern Portfolio Theory was the publication of the 1950’s articles by Harry Markowitz and his subsequent book - Portfolio Selection: Efficient Diversification of Investments (1959).

Markowitz proposed a mean-variance approach to investing which inspired the work of Sharpe (1964), Lintner (1965) and Mossin (1966) who independently put forward a theory that would become the Capital Asset Pricing Model (CAPM). The CAPM model values an asset by adding a scaled market risk premium to the current risk-free return. The market risk-premium is scaled by a market sensitivity factor calculated using covariances between assets and the total market returns.

The intuitively simple theories of Markowitz and the CAPM have been around for over half a century. However, the real breakthrough of successful implementation into common industry practice came later. This was primarily down to the digital revolution of technology and the efficiencies that have come with it. When Markowitz’s theories were first published, there was no access to the computers and programmes we use today. The lengthy procedure of simplifying the input data by hand required for portfolio analysis was removed with the introduction of computers. This massively increased the efficiency of optimal portfolio calculations. The introduction of computers at affordable prices to the public has also made these computational procedures available to almost everyone.

Although CAPM has become a landmark model in asset pricing there are many who doubt its credibility. Richard Roll (1977) produced a critique where he suggested it is impossible to create or observe a truly diversified market portfolio. This is due to the

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1In the mean-variance sense, explained in Section 3.2.
2Explained in Sections 4.1.1 and 4.1.2.
fact that any market index used as a proxy for the true market portfolio cannot capture every available investment.

Other researchers have also found the CAPM to be inefficient when valuing assets. Influenced by Benjamin Graham’s 1930’s theories of value investing, Stattman (1980) and Rosenberg et al (1985) found superior asset returns on US firms with higher Book-to-Market ratios. This effect was also found on the Japanese market by Chan et al (1991). Banz (1981) found evidence that small firms attain significantly higher returns.

These anomalies in the CAPM theory led Eugene Fama and Kenneth French to their 1992 paper. They too found evidence supporting both size and value effects. This resulted in the pioneering Fama-French 3-factor asset pricing model (1993).

Jagadesh & Titman (1993) performed tests on assets measured and grouped by previous performance. They tested a range of different different strategies performed over varying periods of investment. Their results showed that assets attaining higher returns for the previous 2-12 months yielded higher returns for a following short holding period of 3-12 months. Carhart (1997) combined this theory of a Momentum factor in asset returns with the Fama-French model to propose a 4-factor asset pricing model.

A number of studies on asset pricing models have been done recently on the UK market. Gregory et al (2013) performed an empirical study and have created a UK database to enable the wider research community to carry out analysis like that of Fama & French for the UK market. Michou, Mouselli & Stark (2014) analyse the differences between previous UK studies.

According to Elton et al (2011) many investors are currently hiring active managers to deal with their finances. These investors hold a belief that their manager is the one who can ‘beat the market’. The reality however is that index funds set up and utilised by passive managers to mimic the market and factors like size, value and momentum, consistently out-perform the majority of active managers around the world. Also little evidence exists of viable strategies to predict the superior managers.

This overconfidence is naturally inherent in people and Elton et al (2011) mention a study done at a top US university where they asked a class of students if they expected they would finish in the top 10% of their class. 87.5% indicated they did.

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3This is the value on the firms balance sheet divided by the firms market capitalisation. More in Section 4.2.1.
4See Section 4.2.1
5See Section 4.2.2
6Active Manager - Acts on their own forecasts.
7Passive Manager - Trades on a mechanical rule using past data.
Consequently sound passive investment strategies are vital options for investors and choosing the best fitting model for each market is essential.

The aim of this thesis is to assess the suitability of asset pricing models in the UK. The models being considered are the Capital Asset Pricing Model (CAPM), the Fama-French 3-Factor Model (FF3F) and the Carhart 4-Factor Model (C4F). These are three of the most popular models in the world of business and investments today.

This thesis applies three tests common to the literature on asset pricing. The first test is based on Black, Jensen & Scholes (1973). This is a one-step time-series test that determines whether the models fit the market by assessing the $\alpha$ intercepts on the sample test portfolios. A t-statistic is then calculated to see if these are statistically significantly close to zero.

The second test is similar to Fama-Macbeth (1973). This is a two-step cross-sectional test that first performs a time-series regression to obtain $\beta$ values for the test portfolios. It then performs a second regression across the test portfolios to obtain $\gamma$ and $\alpha$ values. The significance of these is measured with t-tests.

The third test is the multivariate test of Gibbons, Ross & Shanken (1989). This is performed across the time-series and cross-section of test portfolios simultaneously to obtain an F-value. The value of this test shows whether $\alpha$ is jointly equal to zero and if the models fit the sample data. Finally a $\chi^2$ test measures the models for goodness of fit.

The thesis begins by looking at the history and background of asset pricing. It then introduces the key concepts of Modern Portfolio Theory. It goes on to develop the theory behind asset pricing models, considering a single-index model then expanding to multi-factor extensions. The sample data for the construction of the factors and test portfolios is then explained. This is followed by a section on the testing carried out. A section on the results, conclusion to these findings and references end this thesis.

Additionally an Appendix is added which is split into three sections. Appendix A contains mathematical derivations corresponding to equations and theories stated in the main body of the text. This set-up is designed to allow for continuity and prevent deviation from the main focus. Appendix B contains extracts from the sample data used in the statistical tests. Appendix C contains the Fulfilment of Thesis Objectives.

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9 Alpha - Defined in Section 6.3.1.
10 Beta - Defined in Section 3.3.
11 Gamma - Defined in Section 6.3.2.
12 Chi-squared - Explained in Section 6.3.
3 Background

3.1 History of the UK Market

The United Kingdom has a long and distinguished place in the history of Finance. The world’s second oldest central bank (second only to Sveriges Riksbank), the Bank of England was established in 1694. The banks of Barclays and Bank of Scotland also date back to the 17th century.

The UK’s primary stock market is the London Stock Exchange (hereafter LSE). It’s roots can be traced back to the late 17th century coffeehouses of Change Alley. Due to the rowdiness of the stockbrokers, they were not allowed access to the city’s Royal Exchange where goods were traded. In 1698 at Jonathan’s Coffee-House, John Castaing published “The Course of the Exchange and other things”. It is the earliest evidence of organised trading in marketable securities in the UK.

The Exchange experienced the South Sea Bubble in 1720, one of the earliest financial crises in recorded history. In 1773 the stockbrokers moved to a newly constructed building with a dealing room and named it The Stock Exchange. It became regulated in 1801 and was officially founded, marking the birth of the modern stock exchange.

In the 19th century the Exchange relocated again several times and made amendments to the initial regulations. Regional exchanges were also set up in Manchester and Liverpool.

During the early 20th century the Exchange shut on a number of occasions. Trading ceased for several months due to The Great War but only closed for 7 days during the entire Second World War.

The ”Big Bang” of 1986 was a significant moment for the LSE. The market was largely deregulated and the LSE became a private limited company. The face-to-face trading on the market floor was also moved to separate dealing rooms where dealers would trade using telephones and computers.

In 1995 the Alternative Investment Market (AIM) was launched for smaller growing companies and in 1997 the LSE moved to its flagship electronic order book SETS.

The LSE moved to its current home at Paternoster Square in 2004 and acquired Borsa Italiana in 2007. The group has since added the companies of MillenniumIT and Russell Investments with the latter making it one of the world’s largest providers of index services.

Due to its heritage and reputation the LSE is one of the most recognised stock markets across the world. The LSE is currently ranked as the 3rd largest stock exchange by Market Capitalisation behind the NYSE and NASDAQ US exchanges.\textsuperscript{14}

The well known indices of the LSE are the FTSE 100 and FTSE 250, which were established in 1984. These represent the 100 largest 'Blue Chip' companies and the next 250 respectively, measured by market capitalisation. Another popular index is the FTSE All-Share Index. It is an amalgamation of the FTSE 100, FTSE 250 and FTSE Small Cap indices and represents over 98% of the total UK market capitalisation.\textsuperscript{15}

3.2 Modern Portfolio Theory

Historically investors considered assets individually to assess what was a good investment. In 1952 Harry Markowitz published an article that revolutionised investing. In this article he introduced a model linking risk and return on a portfolio of assets.\textsuperscript{16}

Markowitz defined risk as standard deviation ($\sigma$) of an assets return. He found that the total risk on a portfolio of assets was less than the weighted sum of risk for all the individual assets contained in the portfolio. i.e.

$$\sigma_p < \sum_{i=1}^{N} X_i \sigma_i, \quad (1)$$

where $N$ is the number of assets contained in portfolio $p$, $X_i$ represents the weight of asset $i$ within portfolio $p$ and $\sum_{i=1}^{N} X_i = 1$.

He found that this was a result of varying covariance between assets across the market and the weighted market average.\textsuperscript{17}

Return was defined between investment periods as $R_t = \frac{(P_t+D_t-P_{t-1})}{P_{t-1}}$, where $P$ represents price and $D$ total dividends received.

Markowitz believed investors act rationally. This means they are only willing to take on more risk if there is a reward of greater expected return for bearing this extra risk. He is credited with developing the first Mean-Variance model, marking the birth of Modern Portfolio Theory.

\textsuperscript{14}http://www.world-exchanges.org/statistics/monthly-reports - Last visited September 24\textsuperscript{th} 2015.

\textsuperscript{15}http://www.ftse.com/products/indices/uk - Last visited September 24\textsuperscript{th} 2015.

\textsuperscript{16}Explained further in Section 3.4.

\textsuperscript{17}Later defined as $R_m$ - The Market Portfolio.
3.3 Risk

It is obvious to any rational investor that before any decision can be made, the risk on their investment must be considered. Risk on an investment can be split into two main categories: Systematic and Unsystematic risk.

Systematic risk affects all companies and sectors. It cannot be removed by simply diversifying investments in a portfolio. An example of Systematic risk would be an economic recession or a large scale problem that would cause virtually all stocks on the market to decline simultaneously.

Unsystematic risk is a company or industry specific issue that does not affect the whole market. Also known as Diversifiable, Idiosyncratic and Specific risk, examples could be a new competitor on the market, shortage of raw materials required for production, or any major management or regulatory changes. Essentially Unsystematic risk is any risk that is specific to a company or sector.

The total risk (Systematic + Unsystematic) is measured by the standard deviation, i.e., $\sigma$. Often referred to as volatility, it can be calculated for an individual or portfolio of securities. This is the traditional risk measure used.

Systematic risk is measured by beta ($\beta$). An assets $\beta$ is calculated by dividing the covariance of the asset and the market by the variance of the market, i.e. $\beta_i = \frac{\sigma_{i,m}}{\sigma_m^2}$. $\beta$ can be viewed as Correlated Relative Volatility.

One way to remove Unsystematic risk and improve mean-variance efficiency is through diversification. This can be achieved by investing in several stocks over different sectors. This is explained by the law of large numbers. As you increase the number of investments, spreading out the invested wealth, the random error begins to average out to zero i.e. $n \to \infty, \epsilon \to 0$. Investing in other types of securities such as bonds and treasuries will also help to remove unsystematic risk from an investor’s portfolio.

As Unsystematic risk can be removed from a portfolio simply by diversification, investors are not generally rewarded for taking this extra unnecessary risk. As a result, the asset pricing models considered in this thesis are based around the risk measure of $\beta$.

3.4 The Investment Portfolio

It is obvious that an investor potentially has a whole world full of possible assets to invest in. These assets are held in something defined as an Investment Portfolio. An Investment Portfolio holds $n$ assets, where $1 \leq n \leq N$. Here $N$ represents all available investments. As you can see, an Investment Portfolio can hold any number of assets.$$^{18}$$ Note that although $1 \leq n \leq N$, holding all assets available on all markets would be unrealistic due to the availability and costs of trading too large a number of assets, amongst other reasons.
The fundamental considerations for an investor are - what assets to hold and what proportion of the investor’s total portfolio will each asset be assigned.

3.4.1 Portfolio Return

The return on a portfolio of assets is defined as the weighted sum of each individual asset’s return contained in the portfolio.

The formula for the expected return on a portfolio can therefore be written as

\[ \bar{R}_p = \sum_{i=1}^{n} (X_i \bar{R}_i), \] (2)

where \( \bar{R}_i \) represents the expected return and \( X_i \) the specific weight invested in the investment \( i \) respectively, within the portfolio \( p \). Note that \( \sum_{i=1}^{n} X_i = 1 \), where \( n \) is the number of holdings in \( p \). It can also be easily observed that changing the weights of the individual holdings within the portfolio can greatly change the total return.

3.4.2 Portfolio Variance

The variance on a portfolio is given by the sum of the squared weight of the security variances plus the covariance of those weighted securities. The formula is given as

\[ \sigma_p^2 = \sum_{j=1}^{n} (X_j^2 \sigma_j^2) + \sum_{j=1}^{n} \sum_{k=1}^{n} (X_j X_k \sigma_{jk}), \] (3)

where \( \sigma_j^2 \) denotes the variance of investment \( j \) while \( \sigma_{jk} \) is the covariance between security \( j \) and \( k \). If all assets are independent our covariance term would become zero. However this is not the case in worldwide financial markets and covariance between assets tends to be positive.

3.5 The Efficient Frontier

The securities considered in this thesis are risky assets traded on the LSE. They are risky in the sense that their prices fluctuate, resulting in varying and unknown levels of return.

When considering the risk-return space (where risk is standard deviation denoted by \( \sigma \) and return is expected return denoted \( \bar{R} \)), all the risky assets lie to the right of the vertical axis, i.e., \( \sigma > 0, \forall \) risky assets.

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19 A derivation of Portfolio Return can be found in Appendix A - Section 9.1.1.
20 A derivation of Portfolio Variance can be found in Appendix A - Section 9.1.2.
21 Sample information is explained in Section 5.1.
The covariance of asset returns varies between all these risky assets, i.e., \( \sigma_{ij} \) is non-constant \( \forall i, j \) assets. This is down to the fact that price movements on markets do not all move in perfect correlation\(^{22}\). It is therefore possible to calculate portfolios of minimum risk for given levels of return.

Constructing a line of all the possible minimum risk portfolios results in a hyperbola shaped curve, seen in Figure 1. Intuitively, portfolios on the upper half of the curve are optimal. This curve is defined as the *Efficient Frontier*\(^{23}\).

### 3.6 The Capital Market Line (CML)

Now introduce the existence of a risk-free investment (denoted by \( R_f \)\(^{24}\) in this thesis). An investor now has the choice of allocating their investments in any proportion between the risk-free and risky asset. Assuming that the investor is rational (in the sense that their portfolio of risky securities lies on the *Efficient Frontier*), the optimal portfolios will range from the 0 - 100% in \( R_f \). The line connecting \( R_f \) and the *Efficient Frontier* is the *Capital Market Line* (CML) shown by the blue line in Figure 1.

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\(^{22}\)Correlation is a measure of how closely prices move together. This is defined in Section 6.1.6.

\(^{23}\)Mathematical Derivation can be found in Appendix A Section 9.2.

\(^{24}\)This thesis uses 3-month treasury bills as the risk-free rate.
The line can be extended beyond the efficient frontier once the option of short selling is considered\textsuperscript{25}. The equation of a line takes the form $y = c + mx$ where $c$ is the intercept of the $y$ axis and $m$ is the gradient of the slope between $x$ and $y$. The CML cuts the $y$ axis at $R_f$ and the gradient of the line is the slope connecting $R_f$ and the tangent of the Efficient Frontier portfolios. This is also the Sharpe ratio\textsuperscript{26}. The Sharpe ratio of the portfolio tangent to the CML can be represented by $\left(\frac{R_m - R_f}{\sigma_m}\right)$. As a result we can write the formula for the expected return on a portfolio $p$ on the CML as

$$\hat{R}_p = R_f + \sigma_p \left(\frac{R_m - R_f}{\sigma_m}\right).$$

(4)

The point at which the CML meets the Efficient Frontier is called the Tangency Portfolio and is shown by the orange point on Figure 1. This happens to be the total combination of all the assets on the market and hence is also known as the market portfolio (denoted by $R_m$). Market Indexes are used as proxies for $R_m$ when using asset pricing models\textsuperscript{27}.

\textsuperscript{25}Short Selling is trading with borrowed assets. Generally the idea is to sell borrowed assets at a high price and buy the assets back at a lower price to gain a profit. Although this is common on large-cap main market stocks many short selling restrictions have been introduced over the years to prevent company sell-offs.

\textsuperscript{26}The Sharpe ratio, also known as the reward-to-variability ratio, is a risk-reward measure used to compare assets and portfolios. It is calculated by $\theta_i = \frac{\mu_i - R_f}{\sigma_i}$, where $\mu_i$ is the expected mean return on asset (or portfolio) $i$, $\sigma_i$ is the standard deviation (volatility) of $i$ and $R_f$ is the risk-free rate.

\textsuperscript{27}This thesis uses the FTSE All-Share Index mentioned in Section 3.1.
4 Asset Pricing Models

The purpose of this thesis is to test three commonly used asset pricing models. Before the testing and analysing, however, it is important to understand the concept and development theory of each of the models. The asset pricing models used in this thesis can be split into two types – Single-Index and Multi-Factor models.

4.1 Single-Index (Market) Model

When looking generally at stock market indices and the individual components of a specific index, an obvious pattern can be observed. Most stock values tend to rise and fall with the market index. This correlation in stock price movements may be a result of common responses to market changes. Relating the return on stocks with the market can allow for a useful measure of stock correlation. By applying the Single-Index model, the return on a stock can be written as

\[ R_i = a_i + \beta_i R_m, \]  

(5)

where \( a_i \) is the part of security \( i \)'s return independent of the market, \( R_m \) is the return on the market and \( \beta_i \) is the measure of the rate of expected change in \( i \)'s return given the change in \( R_m \). Also note that \( \beta_i \) is a constant while \( a_i \) and \( R_m \) are random variables.

We can see that a stock's return can be split into both market dependent and independent return. \( \beta_i \) expresses \( i \)'s sensitivity to the market. The higher the \( \beta \), the larger the stock price movements relative to the market. A negative \( \beta \) would give inverse stock price movements relative to the market and \( \beta = 0 \) would show stock price movements completely independent of the market.

The term \( a_i \) can be broken down into two components: \( \alpha_i \) - which denotes the expected value of \( a_i \), and \( e_i \) - the random disturbances from \( a_i \). Also referred to as 'noise', \( e_i \) has an expected value of zero. Applying this gives the return on a stock as

\[ R_i = \alpha_i + \beta_i R_m + e_i. \]  

(6)

As both \( \bar{R}_m \) and \( e_i \) are random variables, they have a probability distribution, mean and standard deviation. If \( e_i \) truly is random then it should not be correlated with the market. Denoting the standard deviation of the market as \( \sigma_m \) and the standard deviation of the random error as \( \sigma_{e_i} \), the co-variance of movements between market return and random error is

\[ \sigma_{m,e_i} = E[(e_i - 0)(R_m - \bar{R}_m)] = 0. \]  

(7)

Therefore Equation (6) is said to describe stock returns provided \( e_i \) and \( R_m \) are uncorrelated.

\[ \text{Disturbances in this context are deviations from the expected value.} \]
The Time-Series regression analysis carried out as part of this thesis guarantees this independence in estimation.\footnote{More on this in Section 6.3.}

All these previously mentioned points hold by construction but note that the Single-Index model holds by the assumption that \( E(e_i e_j) = 0, \forall i, j \) observed securities. This assumption implies that a common co-movement with the market is the only reason for systematic price movements and other factors play no part.\footnote{Multi-factor Models discussed in Section 4.2 suggest an alternative viewpoint.}

Therefore, when the Single-Index Model is used to represent the joint movement of securities, the expected return on security \( i \) is given by

\[
E(R_i) = \alpha_i + \beta_i \bar{R}_m, \tag{8}
\]

the variance on \( i \)’s return is given by

\[
\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2, \tag{9}
\]

And the co-variance of returns between \( i \) and \( j \) is given by\footnote{Derivations of Equation 8, 9 and 10 are included in Appendix A - Section 9.3.}

\[
\sigma_{ij} = \beta_i \beta_j \sigma_m^2. \tag{10}
\]

### 4.1.1 Factor 1: Market (RMRF)

The theory of the single-index model looks at one factor that affects stock price movements - the Market.

In theory, the Market factor is defined as the value-weighted return on the total market. As mentioned by Roll (1977) in reality it is impossible to quantify the total market and thus a proxy must be taken to represent market return. In studies of the US market the usual proxy is the Standard and Poor’s S&P 500. This is a value weighted average return of the 500 largest US firms. UK studies tend to use the FTSE All-Share Index which comprises roughly 1000 UK firms.

Recalling Markowitz proposal of rational investors and that stocks are more risky than fixed income investments, a good measure to consider is the Market Risk Premium. This essentially is the amount investors are rewarded extra for holding stocks rather than ‘risk-free’ assets. The Market Risk Premium is defined as \( R_m - R_f \). For the purpose of testing the UK market this thesis uses the return on the FTSE All-Share for the market return \( R_m \) and the return on 3-month treasury bills for the risk-free return \( R_f \).

The Market Risk Premium is included as a factor in all three of the asset pricing models tested in this paper. The factor will be represented as RMRF \( (R_m - R_f) \).
4.1.2 The Capital Asset Pricing Model (CAPM)

This model developed by William Sharpe (1964), John Lintner (1965) and Jan Mossin (1966) marks the birth of asset pricing theory.

The formula for the CAPM is given by

$$\bar{R}_i = R_f + \beta_i(\bar{R}_m - R_f),$$  \hspace{1cm} (11)

where $R_f$ is the risk-free rate, $\beta_i$ is the covariance of security $i$ and the market divided by the variance of the market (previously defined as Systematic risk) i.e. $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$, $\bar{R}_m$ is the expected return on the market and $\bar{R}_i$ is the expected return on security $i$.

The Capital Asset Pricing Model is still widely used today. It is used to price an individual security or portfolio. When considering investment opportunities it can be used to estimate a firm’s cost of capital and when evaluating the performance of a managed portfolio against what was expected.

In general the model suggests that investors should be compensated in two ways:- for the time value of their investment period and for any additional risk incurred during the time of the investment.

Many of the empirical tests conducted on the CAPM show poor results. However, its attraction remains strong due to the simple framework used to realise the relationship between expected risk and return. It is a parsimonious model in the sense that for many it delivers a desired level of explanation by using only one predicting variable.

The concept of the *Equity Risk Premium* is an important one when considering the CAPM. The theory is that an investor who takes on additional risk should be compensated. Therefore the larger the risk, the larger the risk premium will be.

As a result it can be proposed that the return on the market is equal to the risk-free rate plus an equity risk premium.

When economies are suffering from increased levels of uncertainty, the risk premiums tend to increase. This explains why developed economies like Europe and North America will have smaller market risk premiums compared with emerging market economies. According to Dimson, Marsh and Staunton (2011), the world’s average market risk premium is around 4.5% per year. \(^{33}\)

Although the CAPM is a great intuitive starting point, it has many flaws. As it is a basic equilibrium model it cannot take into consideration real world conditions that affect asset pricing.

\(^{32}\)A derivation of CAPM can be found in Appendix A - Section 9.5.
\(^{33}\)This was calculated relative to Treasury bills using a 19-country *World Index* between 1900-2011.
CAPM ASSUMPTIONS:

• No transaction, trading or taxation costs.

• All assets are infinitely divisible.

• No individual investor can affect prices via their actions.

• All investor decision are based solely on $E(r)$ and $\sigma$

• All investors are rational - A desire to maximise $E(r)$ and minimise $\sigma$.

• No restrictions on short selling.

• Unlimited lending and borrowing available at $R_f$.

• All investors have the same investment time horizon.

• All investors have identical expectations for $E(r)$, $\sigma$ and $\rho$.

• All assets are marketable.

These are all recognised issues concerning the CAPM model.

$^{34}$ $\rho$ represents the correlation coefficient between asset price movements.
4.2 Multi-Factor Models

Like the Single-Index model, Multi-Factor models can be used to explain the price of an individual security or a portfolio of securities. The idea of the Single-Index model is to consider only a single factor to explain stock price movements on the market. The CAPM is a linear model of market risk and implies there are no other factors affecting stock prices.

Multi-Factor models look at other factors to further explain price movements. In this paper two models that extend the CAPM theory for other risk factors are considered and analysed. The Fama-French 3-Factor Model (FF3F) includes factors for Size and Value as well as the Market factor of the CAPM. The Carhart 4-Factor Model extends even further the FF3F adding a factor of Momentum.

One of the first suggestions of alternative factors affecting stock movements comes from the work of Benjamin King (1966). He found evidence of industry influences
causing groups of stocks to 'co-move'. He suggested these factors contributed to stock price changes independent of market co-variance.

Multi-Factor models can be split into two groups - models depending on specific industries and models depending on specific economic factors. This thesis will concentrate on the latter.

The generalised formula for return on a Multi-Factor model can be written as

\[ R_i = a_i + \beta_{i,1}I_1 + \beta_{i,2}I_2 + \ldots + \beta_{i,L}I_L + c_i, \]  

(12)

where \( R_i \) is the return on investment \( i \), \( a_i \) is the return independent of all \( L \) factors, \( \beta_{i,j} \) is the rate of change of \( i \)'s return relative to factor \( j \), \( I_j \) is the return of factor \( j \) and \( c_i \) is the random sampling error.

As \( E[c_i] = 0 \), the expected return of the generalised Multi-Factor model is

\[ E(R_i) = a_i + \beta_{i,1}I_1 + \beta_{i,2}I_2 + \ldots + \beta_{i,L}I_L, \]  

(13)

with variance

\[ \sigma_i^2 = \beta_{i,1}^2\sigma_{I_1}^2 + \beta_{i,2}^2\sigma_{I_2}^2 + \ldots + \beta_{i,L}^2\sigma_{I_L}^2 + \sigma_{ci}^2, \]  

(14)

and co-variance\(^{35}\)

\[ \sigma_{ij} = \beta_{i,1}\beta_{j,1}\sigma_{I_1}^2 + \beta_{i,2}\beta_{j,2}\sigma_{I_2}^2 + \ldots + \beta_{i,L}\beta_{j,L}\sigma_{I_L}^2. \]  

(15)

4.2.1 Factor 2: Size (SMB)

In the paper *The Relationship Between Return and Market Value of Common Stocks* - Banz (1981), an anomaly in the single factor theory of the CAPM model was spotted. Stocks with lower Market Capitalisation appeared to significantly outperform larger stocks over the period of 1936-1975 on the US market.

These findings contributed to a move away from the standard CAPM model. New alternatives to the idea that stock correlation to the market solely determines stock price movements arose. Two resulting models are the Multi-Factor models of Fama-French and Carhart discussed later.

In this thesis the Size factor will be represented by SMB. SMB (Small Minus Big) accounts for the spread in returns between small- and large- sized firms. This is based on the market capitalisation\(^{36}\) sizes of the firms.

---

\(^{35}\)Derivations of equations 12, 13 and 14 are given in Appendix A Section 9.4.

\(^{36}\)Market Capitalisation = Market price per share \( \times \) Number of outstanding (total number of) shares.
The factor can be referred to as the small firm effect and comes from the theory that small firms often tend to outperform larger ones over time. However it is important to consider that the factor does not account for the higher risk of default of smaller firms.

Within the models, SMB attempts to explain some of the excess return on a portfolio. Incorporating SMB can show if investing in stocks with lower market values achieves abnormal higher returns.

4.2.2 Factor 3: Value (HML)

Before the discovery of the Size effect was the discovery of another significant investment factor. The idea of Value investing was first suggested long before the Value effect and even the Mean-Variance portfolio theories of Markowitz.

Benjamin Graham and David Dodd wrote the book Security Analysis (1934). The book was published after the great stock market crash of 1929 and the consequent bear market dubbed the Great Depression. The sensible ideas of thorough consideration of investments made sense after a period of catastrophic decline for the American and Worldwide stock markets.

Graham and Dodd noticed the 'short sightedness' of pre-crash investors and their lack of ability to take longer term investment views. They proposed alternative investment strategies. This involved looking deeper into the true value of a company deciding and whether the stock market accurately reflected this. Their theories had been developed from the late 20’s and early 30’s classes they taught at Columbia Business School.

One way to measure a Value factor is to use something called the Book-to-Market ratio.

This ratio is known for comparing the growth status of companies. It takes the book value of a company and divides it by the market value. Book value is calculated by looking at the historical cost or accounting value of a firm. Market value is determined by the current market capitalisation value for a firm on the stock market.

\[
\text{Book-to-Market Ratio} = \frac{\text{Book Value of firm}}{\text{Market Value of firm}}
\]

The Book-to-Market ratio can be used to identify under or over-valued security prices that may have fallen or become inflated on the financial markets.

This ratio is the inverse of the commonly used investment metric P/B (Price/Book).\footnote{Considered to be the most severe worldwide economic decline on record.}
In basic terms, a Book-to-Market ratio above 1 indicates an undervalued stock and a ratio below 1 indicates it is overvalued. Undervalued stocks are commonly referred to as *value* stocks and overvalued ones as *growth* stocks.

It is believed that this can be due to what investors believe will happen to the future earnings of a firm and as a result the current market value is seen to reflect the current investor confidence in the firm’s capacity for potential growth.

HML (High minus Low) accounts for the spread in returns between value and growth stocks. It is understood that firms with higher Book-to-Market ratios tend to outperform firms with lower ratios i.e. $\bar{R}_{\text{Value}} > \bar{R}_{\text{Growth}}$.

This effect is often referred to as the *Value Premium*.

Like SMB, within the models HML looks to explain some of the excess return in the portfolio. It can show how much of the abnormal return was attributable to investing in the *Value Premium*.

Buying only *Value* stocks results in a positive HML factor.

### 4.2.3 Fama-French 3-Factor Model (FF3F)

The FF3F model was developed by Eugene Fama and Kenneth French in 1993 off the back of their 1992 paper *The Cross-Section of Expected Stock returns*. In this paper, Fama & French first propose that if CAPM is to hold, then expected returns are a positive linear function of $\beta$, i.e., the slope of the regression is $\beta$. They believed that other risk factors could play a role in describing stock price movements.

Earlier work by Banz (1981) proposed an apparent *Size effect*. He found returns on small stocks appeared abnormal relative to their $\beta$’s. A *Value effect* was also found in the papers of Stattman (1980) and Rosenberg, Reid & Lanstein (1985).

Fama and French followed on from these findings and others and decided to test the relation of stock prices to the factors of Market Capitalisation, Book-to-Market Ratio, Earnings-to-Price Ratio and Leverage along with the market $\beta$.

They performed the cross-sectional regressions of Fama & MacBeth (1973) on portfolios of US stocks arranged according to the factors from 1963-1990. They found no evidence to support the earlier theory of CAPM – the average stock returns were not positively related to the market $\beta$ for the period.

But they did find that stock prices were positively related to Book-to-Market Ratio and negatively related to Market Capitalisation. They also found evidence for Earnings-to-Price Ratio and Leverage effects. However, these appeared to be absorbed in the effects of Book-to-Market Ratio and Market Capitalisation.
They believed that the 'Value effect' was down to 'irrational market whims', i.e., that prices later correct previous investor irrationality.

In their follow on 1993 paper 'Risk Factors in the Return of Stocks and Bonds' they proposed and tested a three factor extension of the CAPM.

They perform a Time-Series regression on 25 portfolios arranged according to both their Book-to-Market Ratio and Market Capitalisation.

The Fama-French 3-Factor Model:

\[ R_i = R_f + \beta_{i,Rm}(RMRF) + \beta_{i,SMB}(SMB) + \beta_{i,HML}(HML). \] (16)

4.2.4 Factor 4: Momentum (UMD)

In Jegadeesh & Titman’s 1993 paper, they find evidence suggesting that stocks with a recent positive return performance will continue to outperform stocks that have recently performed negatively.

UMD (Up minus Down) accounts for the spread in returns between stocks with High and Low momentum over the last 2-12 months. It is proposed that stocks that have had a positive yearly return will outperform stocks that have had a negative return over that period for the following 12-months.

This effect is often referred to as the 'Momentum factor'.

4.2.5 Carhart 4-Factor Model (C4F)

This was followed by the work of Mark M. Carhart (1997). He proposed a model that further extended the FF3F model by adding the factor of Momentum. The factor was calculated by looking at performance of assets over the previous 2-12 months.

Carhart performed the Cross-Sectional Regression method of Fama & MacBeth (1973) to test his model. He found evidence to support the claim that adding a Momentum factor to the FF3F model further explains stock price movements.

The Carhart 4-Factor Model:

\[ R_i = R_f + \beta_{i,Rm}(RMRF) + \beta_{i,SMB}(SMB) + \beta_{i,HML}(HML) + \beta_{i,UMD}(UMD). \] (17)
5  Formulation

5.1  Data

The data from this thesis was taken from the website of Xfi Centre for Finance and Investment at the University of Exeter. They have created an equivalent to the Kenneth French data library for the UK market and allow the data to be used to aid the greater research community.

The data is compiled using various sources which are stated in Gregory, Tharyan & Christidis (2013) (hereafter GTC). Following the work of Dimson, Nagel & Quigley (2003) (hereafter DNQ), the factors and test portfolios are constructed using only London Stock Exchange main market stocks and exclude everything else i.e., AIM, ISDX etc. They also exclude stocks with missing or negative book values.

The starting point of September 1980 was chosen and 896 valid stocks were found for the sample. By 2010 the number of stocks left to analyse had fallen to 513.

5.2  Construction of Factors

GTC mention the importance of correctly selecting break points for the UK market to mimic the one used in the Kenneth French data library.

According to the research of DNQ and GTC the UK market appears to have a large illiquid tail of small-cap stocks. They suggest that these stocks would generally not be considered as part of the tradable universe of the many institutional investors involved in the UK market.

As a result, the break point is selected to include the top 350 UK stocks. This happens to represent a combination of the FTSE 100 and FTSE 250, the large-cap and mid-cap stock indexes for the UK.

To calculate the factors of Size, Value and Momentum, portfolios are created and subtracted from one another. This creates the factor risk premiums that will be incorporated in the statistical analysis of the models.

The stocks in the sample are first sorted by market capitalisation into two groups: small "S" and big "B". The break point used by GTC is the median of the 350 stocks as at the beginning of the sample. This Size factor break point differs from the 70th percentile of the valid stocks on the market used in the paper of DNQ.

The stocks are then sorted by Book-to-Market ratio into three groups: high "H", medium "M" and low "L". GTC use the 30th and 70th percentiles for breakpoints, the same as many studies mentioned in the paper of Michou, Mouselli & Stark (2014), Al-Horani, Pope & Stark (2003) (Hereafter APS) for example.

However unlike APS and others analysing the UK market, the percentiles used by GTC are only on the largest 350 stocks and not their whole valid sample of stocks. This again differs from DNQ who use the 40th and 60th percentiles of their total sample.

Finally the stocks are also sorted by Momentum into three groups up "U", medium "M" and down "D". For the Momentum break points the paper of GTC follows the methods used on the Kenneth French data library. The stocks are ordered by their return over the prior 2-12 months and the break points are the 30th and 70th percentiles as on the Kenneth French data library. Like the other two factors only the top 350 stocks are considered.

The SMB and HML factors are calculated using 6 portfolios ordered on Size and Book-to-Market ratio SH, SM, SL, BH, BM, BL.

The UMD factor is calculated using 6 different portfolios ordered by Size and Momentum SU, SM, SD, BU, BM, BD.

The factors are calculated as follows:

\[
SMB = \frac{SL + SM + SH}{3} - \frac{BL + BM + BH}{3}, \tag{18}
\]

\[
HML = \frac{SH + BH}{2} - \frac{SL + BL}{2}, \tag{19}
\]

\[
UMD = \frac{SU + BU}{2} - \frac{SD + BD}{2}. \tag{20}
\]

5.3 Construction of Test Portfolios

In order to test the risk factors on the market, the stocks are again ordered according to their market capitalisation, book-to-market ratio and past 2-12 month return. An alternative group of portfolios are formed using an element independent of the test factors. For this the past 12 month standard deviation is used to sort the stocks.

The sorting range for Size is Small, 2, 3, 4 and Big. The ranges for Value and Momentum are Low, 2, 3, 4 and High. For standard deviation the range is SD1, SD2, ..., SD25. Here SD1 represents the portfolio of stocks with the lowest Standard Deviation and SD25 the highest.

This thesis tests 4 sets of portfolios:
First, 25 portfolios with the stocks arranged by Market Capitalisation and Book-to-Market ratio, i.e., 5 size x 5 value groups. 5 size portfolios - 4 portfolios formed from the largest 350 firms + 1 portfolio formed from the rest intersected with 5 B/M portfolios - based on the largest 350 firms.

Second, 25 portfolios with the stocks arranged by Market Capitalisation and past 2-12 month returns, i.e., 5 size x 5 momentum groups. 5 size portfolios - 4 portfolios from the largest 350 + 1 portfolio from the rest intersected with 5 Momentum portfolios - based on the largest 350 firms.

Third, 27 portfolios with the stocks arranged by Market Capitalisation, Book-to-Market ratio and past 2-12 month returns, i.e., 3 size x 3 value x 3 momentum groups. 3 Size portfolios - 2 portfolios formed from the largest 250 firms + 1 group from the rest, then within each size group we create 3 B/M groups and within each of these 9 portfolios we form 3 momentum groups.

Finally, 25 portfolios arranged by Standard Deviation over the past 12 months. This group of portfolios is included as an alternative to using portfolios grouped by the factors.
6 Statistical Tests

To check the suitability of the Asset Pricing models mentioned in this paper, it is essential to run statistical analysis on the data gathered.

"The objective of statistics is to make an inference about a population based on the information contained in a sample from that population and to provide an associated measure of goodness for the inference”

- Wackerly, Mendenhall & Scheaffer

Within the scope of this thesis the statistics are to check the statistical significance of the CAPM, FF3F and C4F models as appropriate asset pricing models for the UK market.

6.1 Descriptive Statistics

First to be assessed are the descriptive statistics of the samples under consideration. Below are formulas for statistics given in this thesis

6.1.1 Sample Mean

The mean of a sample containing \( n \) measurements given by \( y_1, y_2, ..., y_n \) is defined as

\[
\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i.
\]  
(21)

6.1.2 Sample Variance

The variance of a sample of measurements \( y_1, y_2, ..., y_n \) can be defined as the sum of the differences between the measurements and their mean, divided by \( n - 1 \), i.e.,

\[
s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2.
\]  
(22)

Here the divisor \( n - 1 \) is used instead of \( n \) as it provides an unbiased estimator of the true population variance.

6.1.3 Sample Standard Deviation

The standard deviation of a sample of measurements is given by the square root of the sample variance, i.e.,

\[
s = \sqrt{s^2}.
\]  
(23)

The corresponding population mean, variance and standard deviation are defined as \( \mu, \sigma^2 \) and \( \sigma \) respectively.
6.1.4 Sample Skewness

The skewness of a sample can be defined as

\[ SK = \frac{1}{(n-1)s^3} \sum_{i=1}^{n} (y_i - \bar{y})^3. \] (24)

6.1.5 Sample Kurtosis

The kurtosis of a sample can be defined as

\[ K = \frac{1}{(n-1)s^4} \sum_{i=1}^{n} (y_i - \bar{y})^4. \] (25)

The skewness and kurtosis are the third and fourth normalised moments of the sample data.

6.1.6 Factor Correlation

This thesis uses the Pearson correlation coefficient formula

\[ \rho_{x,y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2} \sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}. \] (26)

where \( x_i \) and \( y_i \) are the \( i \)th values of factors \( x \) and \( y \), \( \bar{x} \) and \( \bar{y} \) are the sample means for the factors. The formula is used to calculate the correlation between the factors.

6.1.7 (Adjusted) \( R^2 \)

\( R^2 \) (R-squared) is a statistical measure of how close a sample of data fits a line of regression. Also known as the *Coefficient of determination*, the formula for \( R^2 \) is

\[ R^2 = 1 - \frac{\sum_{i=1}^{n} (y_i - f(x_i))^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}, \] (27)

where \( y_i \) is the \( i \)th value of the dependent variable, \( f(x_i) \) is the predicted value of the \( i \)th dependent variable and \( \bar{y} \) is the mean of observed \( y_i \)'s.

For samples with more than one independent variable \( \bar{R}^2 \) (adjusted R-squared) is a more accurate measure. \( \bar{R}^2 \) is calculated by using the adjustment formula

\[ \bar{R}^2 = 1 - \frac{(1 - R^2)(N-1)}{N-p-1}. \] (28)

Here \( p \) represents the number of independent variables and \( N \) represents the total number of observations in the sample.
6.2 Hypothesis Testing

A hypothesis test is used to assess whether a data sample supports a given hypothesis for a population. To draw inferences from the data it is essential to first propose a hypothesis.

A null hypothesis ($H_0$) is the assumed position of a population. For the purpose of this thesis $H_0$ will propose the model to be suitable for the population, in this case the UK market.

The other option is the alternative hypothesis ($H_A$). If $H_A$ is accepted then the assumed theory for a population is rejected. In the case of this thesis $H_A$ would suggest the model is not appropriate for the UK market.

Quantitative analysis to test theories about markets, investing strategies, or economies depends on null hypotheses to decide if ideas are true or false.

A test statistic is used to support or reject a hypothesis. This test statistic has a critical level. Below this level $H_0$ is accepted. Above this level is defined as the Rejection Region ($RR$). Here $H_0$ is rejected and $H_A$ is accepted. The confidence level and degrees of freedom determine the critical level. In this thesis a confidence level of 95% is used for quoted statistics in the results section. The degrees of freedom are quoted for each test.

It is possible that the wrong conclusion can be drawn from a hypothesis test. There are two distinct types of errors:

Type I error - $H_0$ rejected when $H_0$ is TRUE.

Type II error - $H_0$ accepted when $H_0$ is FALSE.

A Type I error is also known as a ‘false positive’ and a Type II error a ‘false negative’. The probability of a Type I error occurring depends on the level of the test. For a test of 95% confidence level the probability of Type I error is 5%. The probability of a type II error decreases as the number of observations in the sample increases.

6.3 Empirical Tests

6.3.1 Black, Jensen & Scholes (1972)

The first test performed in this thesis follows Black, Jensen & Scholes (1972) (hereafter BJS). They conducted the first in-depth time-series test of the CAPM. BJS tested the US market by grouping their sample of assets according to their $\beta$ estimates.

\[\text{40} \text{Taken from the University of Chicago Center for Research in Security Prices (CRSP), they analysed all securities listed on the New York Stock Exchange (NYSE) from January 1926 - March 1966.}\]
BJS noticed an issue in standard time-series tests of CAPM using a cross-section of asset returns. Tests of $\alpha$ distributions assume $e_{i,j} = 0$ but this is not the case. Their decision to group assets into portfolios was taken so the issue of residual covariance could be mitigated when regressing over a large sample time-series.

The BJS time-series regression model is

$$R_{it} - R_{ft} = \alpha_i + \beta_i(R_{Mt} - R_{ft}) + e_{it}, \quad (29)$$

where $R_{it}$ is the return at time $t$ for portfolio $i$, $R_{ft}$ is the risk-free rate at time $t$ and $R_{Mt}$ is the market return at time $t$. $\beta_i$ is calculated as the slope of this regression line. $e_{it}$ is the sampling error. Over time this has a distribution mean of zero meaning $E[e_{it}] = 0$.

The formula for $\alpha_i$ can be arranged for a test equation of the CAPM model

$$\alpha_i = R_{it} - R_{ft} - \beta_i,R_{Mt}RMRF_t. \quad (30)$$

Fama & French (1993) used this model as a basis for testing multi-factor models. This thesis adopts this approach to get the following $\alpha$ equations:

**FF3F model**

$$\alpha_i = R_{it} - R_{ft} - \beta_i,R_{Mt}RMRF_t - \beta_i,SMB_{it}SMB_t - \beta_i,HML_{it}HML_t, \quad (31)$$

**C4F model**

$$\alpha_i = R_{it} - R_{ft} - \beta_i,R_{Mt}RMRF_t - \beta_i,SMB_{it}SMB_t - \beta_i,HML_{it}HML_t - \beta_i,UMD_{it}UMD_t. \quad (32)$$

For each model the hypotheses proposed are:

- $H_0$: $\alpha_i = 0$
- $H_A$: $\alpha_i \neq 0$

The t-statistic is used to calculate a critical level using the formula

$$t = \frac{\alpha_i}{s/\sqrt{T}}, \quad (33)$$

Note the the denominator of this formula is the Standard Error. This is quoted in the tables for results of the BJS tests.

For a 95% confidence level with $\infty$ degrees of freedom\(^{42}\) the critical level is 1.96.

---

\(^{41}\)BJS use a value weighted portfolio of all assets on the NYSE.

\(^{42}\)This is used as the sample contains 408 time steps, far higher than the next degrees of freedom level of 150.
6.3.2 Fama-Macbeth (1973)

The second test performed in this thesis is a two-stage cross-sectional test in the style of Fama-MacBeth (1973) (hereafter FM). FM like BJS used portfolios of grouped assets to test the CAPM on the US market.

They noticed that as the residuals of a cross-section of stock portfolios are correlated, a simple one-stage cross-sectional test would not be suitable. FM got around this issue by first calculating \( \hat{\beta} \) estimates from the first-stage regression in the style of BJS. They then proceeded by running the following second-stage regression

\[
R_{it} - R_{ft} = \gamma_0 t + \gamma_1 t \hat{\beta}_{i,Rm} + e_{it},
\]

where \( R_{it} \) represents the return on test portfolio \( i \), \( R_{ft} \) denotes the risk free return, \( \gamma_0 t \) is the constant, \( \gamma_1 t \) is the cross-sectional regression coefficient and \( \hat{\beta}_{i,Rm} \) is the estimated factor loading from the first-stage regression.

This regression was run for each time period in their sample to obtain a time-series for \( \gamma_0 t \) and \( \gamma_1 t \). The mean of these time-series is calculated by the formulas

\[
\bar{\gamma}_0 = \frac{1}{T} \sum_{t=1}^{T} \gamma_{0t}, \quad \bar{\gamma}_1 = \frac{1}{T} \sum_{t=1}^{T} \gamma_{1t}.
\]

(35)

to obtain values for \( \bar{\gamma}_0 \) and \( \bar{\gamma}_1 \).

This regression equation can be extended for multi-factor models. The regression equations for the FF3F and C4F models are:

**FF3F model**

\[
R_{it} - R_{ft} = \gamma_0 t + \gamma_1 t \hat{\beta}_{i,Rm} + \gamma_2 t \hat{\beta}_{i,SMB} + \gamma_3 t \hat{\beta}_{i,HML} + e_{it},
\]

(36)

**C4F model**

\[
R_{it} - R_{ft} = \gamma_0 t + \gamma_1 t \hat{\beta}_{i,Rm} + \gamma_2 t \hat{\beta}_{i,SMB} + \gamma_3 t \hat{\beta}_{i,HML} + \gamma_4 t \hat{\beta}_{i,UMD} + e_{it},
\]

(37)

where \( \hat{\beta}_i \)'s are the corresponding factor loadings calculated in the first-stage regression. Over a large time sample \( E[e_{it}] = 0 \) as \( t \to 0 \).

For each model the hypotheses proposed are:

\[
H_0: \bar{\gamma}_0 = 0 \\
H_A: \bar{\gamma}_0 \neq 0
\]
The t-statistic is used to calculate a critical level using the formula
\[ t = \frac{\gamma_0}{s/\sqrt{T}}. \] (38)

Critical levels are the same mentioned in the previous section and cross-sectional \( R^2 \) and \( \bar{R}^2 \) are also calculated for goodness of fit for each model.

6.3.3 Gibbons, Ross & Shanken (1989)

The final test performed in this thesis is the multivariate test of Gibbons, Ross & Shanken (1989) (hereafter GRS). They constructed an equation to test if \( \alpha \) intercepts are jointly equal to zero i.e., for the time-series and along the cross-section. This can also be thought of as a panel test on panel data.

To get around the issue of errors being correlated across assets i.e., \( E(\epsilon_i\epsilon_j) \neq 0 \), they proposed the following \( F \) test formula with test statistic \( \omega \)
\[ \omega = \frac{T - N - K}{N} \left( 1 + E_T(f)'\hat{\Omega}^{-1}E_T(f) \right)^{-1} \hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim F_{N,T-N-K}, \] (39)

were \( T \) is the number of time-step observations, \( N \) is the number of test assets (or portfolios) and \( K \) is the number of factors in the model being tested.

\( E_T(f) \) is the column vector of \( K \) factor mean returns
\[ E_T(f) = [\hat{f}_1 \ \hat{f}_2 \ \cdots \ \hat{f}_K]' . \]

\( \hat{\Omega} \) is the \( K \times K \) covariance matrix of factor returns
\[ \hat{\Omega} = \begin{bmatrix} \sigma^2_{\hat{f}_1} & \sigma_{\hat{f}_1,\hat{f}_2} & \cdots & \sigma_{\hat{f}_1,\hat{f}_K} \\ \sigma_{\hat{f}_2,\hat{f}_1} & \sigma^2_{\hat{f}_2} & \cdots & \sigma_{\hat{f}_2,\hat{f}_K} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\hat{f}_K,\hat{f}_1} & \sigma_{\hat{f}_K,\hat{f}_2} & \cdots & \sigma^2_{\hat{f}_K} \end{bmatrix} . \]

\( \hat{\alpha} \) is the column vector of \( N \) estimated intercepts
\[ \hat{\alpha} = [\hat{\alpha}_1 \ \hat{\alpha}_2 \ \cdots \ \hat{\alpha}_N]' . \]

The matrix \( \hat{\Sigma} = \sigma_{\hat{\epsilon},\hat{\epsilon}} \), where \( \hat{\epsilon} \) is the \( N \times T \) matrix of residual returns
\[ \hat{\epsilon} = \begin{bmatrix} \epsilon_{1,1} & \epsilon_{2,1} & \cdots & \epsilon_{N,1} \\ \epsilon_{1,2} & \epsilon_{2,2} & \cdots & \epsilon_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_{1,T} & \epsilon_{2,T} & \cdots & \epsilon_{N,T} \end{bmatrix} . \]

\( ^{43}\)In the sense that the data is a 2-dimensional matrix with \( T \times N \) rows and columns.
The resulting $N \times N$ residual covariance matrix $\hat{\Sigma}$ is then given by

$$
\hat{\Sigma} = \begin{bmatrix}
\sigma_{\hat{\epsilon}_1}^2 & \sigma_{\hat{\epsilon}_1,\hat{\epsilon}_2} & \cdots & \sigma_{\hat{\epsilon}_1,\hat{\epsilon}_N} \\
\sigma_{\hat{\epsilon}_1,\hat{\epsilon}_2} & \sigma_{\hat{\epsilon}_2}^2 & \cdots & \sigma_{\hat{\epsilon}_2,\hat{\epsilon}_N} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{\hat{\epsilon}_1,\hat{\epsilon}_N} & \sigma_{\hat{\epsilon}_N,\hat{\epsilon}_2} & \cdots & \sigma_{\hat{\epsilon}_N}^2
\end{bmatrix}.
$$

For one factor models like CAPM the GRS $F$ statistic test takes the form

$$
\omega = \frac{T - N - K}{N} \left[ 1 + \left( \frac{E_T(f)}{\hat{\sigma}(f)} \right)^2 \right]^{-1} \hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim F_{N,T-N-1},
$$

(40)

where $\hat{\sigma}(f)^2$ denotes factor variance and $E_T(f)$ factor mean. The term $\left( \frac{E_T(f)}{\hat{\sigma}(f)} \right)^2$ is the Sharpe ratio of the factor.

The following $\chi^2$ test for measure of good fit is an alternative to the GRS $F$ test with test statistic $\delta$

$$
\delta = T \left[ 1 + \left( \frac{E_T(f)}{\hat{\sigma}(f)} \right)^2 \right]^{-1} \hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} \sim \chi^2_{N}.
$$

(41)

However this $\chi^2$ test requires that the $\epsilon$ errors are normally distributed, uncorrelated and homoskedastic.

The GRS $F$ and $\chi^2$ tests are run for each of the three models on the 4 test portfolio groups. The hypotheses for both tests:

- $H_0$: $\hat{\alpha} = 0$
- $H_A$: $\hat{\alpha} \neq 0$

For the $\chi^2$ test the critical value for $\delta$ at the 95% confidence level is 40.65 for groups of 25 test portfolios. For 27 portfolios the value is 43.19.

For the $F$ test the critical value for $\omega$ at the 95% confidence level is 1.91 for groups of 25 test portfolios. For 30 portfolios the value is 1.79 so for 27 portfolios it is $\approx 1.85$. 408 time steps is taken as $\infty$ for degrees of freedom.

\[\text{Having equal statistical variances.}\]
7 Results

All tests and statistics apart from correlation coefficients in Table 2 were calculated using Excel functions and formulas. The correlation coefficients were calculated in the programming software $R$ and the results were transferred into an MS Excel spreadsheet.

Here is a list of the Excel functions used

- **MMULT** - Calculates the product of two matrices.
- **LINEST** - Calculates statistics of a line by using the least squares method.
- **INDEX** - Returns specific value of an array.
- **TRANSPOSE** - Converts a column array to a row (or vice versa)

7.1 Descriptive Statistics

Descriptive statistics are displayed in this section for total returns on factor and test portfolios i.e., before $R_f$ has been subtracted. The colour system is set up to show the lowest values white and highest values red for Mean, Var (variance), SD (standard deviation), Kurt (kurtosis), Max (maximum) and Median of returns. For Skew (skewness) white is zero and red are the largest absolute values and for Min (minimum) the largest number i.e closest to zero is white and smallest value is red.

7.1.1 Factors

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Var</th>
<th>SD</th>
<th>Skew</th>
<th>Kurt</th>
<th>Max</th>
<th>Median</th>
<th>Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMRF</td>
<td>0.528%</td>
<td>0.20%</td>
<td>3.143%</td>
<td>-0.99%</td>
<td>3.681</td>
<td>13.230%</td>
<td>0.99%</td>
<td>-27.05%</td>
</tr>
<tr>
<td>SMB</td>
<td>0.199%</td>
<td>0.09%</td>
<td>3.060%</td>
<td>0.11%</td>
<td>2.257</td>
<td>15.607%</td>
<td>0.027%</td>
<td>-11.47%</td>
</tr>
<tr>
<td>HML</td>
<td>0.321%</td>
<td>0.10%</td>
<td>3.214%</td>
<td>-0.556</td>
<td>6.519</td>
<td>12.287%</td>
<td>0.320%</td>
<td>-18.608%</td>
</tr>
<tr>
<td>UMD</td>
<td>0.870%</td>
<td>0.18%</td>
<td>4.385%</td>
<td>-0.973</td>
<td>5.894</td>
<td>16.044%</td>
<td>1.083%</td>
<td>-25.638%</td>
</tr>
</tbody>
</table>

Table 1: Descriptive statistics of Factors

Table 1 shows that factor portfolio UMD obtained the highest mean return over the sample period by a significant margin. The second highest return was obtained by RMRF. These two factor portfolios also had the largest variance and standard deviation, skewness, median and minimum monthly returns. UMD and HML had the largest values for Kurtosis and UMD and SMB had the highest maximum monthly return of the 4 factor portfolios. SMB obtained the lowest mean return along with the smallest values for variance and standard deviation, skewness, kurtosis, median and minimum monthly returns.

All the factor portfolios have relatively small correlation with each other, except 1. HML and UMD have a fairly large negative correlation of -0.513. All the rest are between -0.138 and 0.048.
### Table 2: Correlation of Factors

<table>
<thead>
<tr>
<th></th>
<th>RMRF</th>
<th>SMB</th>
<th>HML</th>
<th>UMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMRF</td>
<td>1.000</td>
<td>0.007</td>
<td>0.048</td>
<td>-0.138</td>
</tr>
<tr>
<td>SMB</td>
<td>0.007</td>
<td>1.000</td>
<td>-0.059</td>
<td>-0.075</td>
</tr>
<tr>
<td>HML</td>
<td>0.048</td>
<td>-0.059</td>
<td>1.000</td>
<td>-0.513</td>
</tr>
<tr>
<td>UMD</td>
<td>-0.138</td>
<td>-0.075</td>
<td>-0.513</td>
<td>1.000</td>
</tr>
</tbody>
</table>

### 7.1.2 Portfolio group 1 - Size and Value

Table 3: Descriptive statistics of portfolios sorted by Size and Value

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Mean</th>
<th>Var</th>
<th>SD</th>
<th>Skew</th>
<th>Kurt</th>
<th>Max</th>
<th>Median</th>
<th>Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>0.907%</td>
<td>0.453%</td>
<td>6.731%</td>
<td>-0.127</td>
<td>3.756</td>
<td>38.088%</td>
<td>1.145%</td>
<td>-25.503%</td>
</tr>
<tr>
<td>S2</td>
<td>1.173%</td>
<td>0.315%</td>
<td>5.612%</td>
<td>0.310</td>
<td>-1.985</td>
<td>22.483%</td>
<td>1.470%</td>
<td>-23.639%</td>
</tr>
<tr>
<td>S3</td>
<td>1.307%</td>
<td>0.291%</td>
<td>5.397%</td>
<td>0.096</td>
<td>5.423</td>
<td>35.614%</td>
<td>1.543%</td>
<td>-21.084%</td>
</tr>
<tr>
<td>S4</td>
<td>1.530%</td>
<td>0.279%</td>
<td>5.281%</td>
<td>0.202</td>
<td>2.987</td>
<td>24.332%</td>
<td>1.738%</td>
<td>-21.323%</td>
</tr>
<tr>
<td>S5</td>
<td>1.811%</td>
<td>0.278%</td>
<td>5.272%</td>
<td>0.467</td>
<td>5.593</td>
<td>35.329%</td>
<td>1.735%</td>
<td>-20.573%</td>
</tr>
<tr>
<td>S2L</td>
<td>0.847%</td>
<td>0.426%</td>
<td>6.526%</td>
<td>-0.599</td>
<td>2.480</td>
<td>26.503%</td>
<td>1.248%</td>
<td>-27.635%</td>
</tr>
<tr>
<td>S2H</td>
<td>1.197%</td>
<td>0.379%</td>
<td>6.155%</td>
<td>-0.836</td>
<td>2.903</td>
<td>23.689%</td>
<td>1.622%</td>
<td>-28.368%</td>
</tr>
<tr>
<td>S2L</td>
<td>1.165%</td>
<td>0.302%</td>
<td>5.498%</td>
<td>-0.409</td>
<td>2.315</td>
<td>24.298%</td>
<td>1.425%</td>
<td>-22.896%</td>
</tr>
<tr>
<td>S2H</td>
<td>1.290%</td>
<td>0.347%</td>
<td>5.809%</td>
<td>-0.238</td>
<td>1.710</td>
<td>20.164%</td>
<td>1.403%</td>
<td>-24.934%</td>
</tr>
<tr>
<td>S2H</td>
<td>1.465%</td>
<td>0.543%</td>
<td>7.365%</td>
<td>-1.966</td>
<td>17.675</td>
<td>62.841%</td>
<td>1.644%</td>
<td>-27.359%</td>
</tr>
<tr>
<td>M3L</td>
<td>0.946%</td>
<td>0.463%</td>
<td>6.806%</td>
<td>-1.188</td>
<td>5.463</td>
<td>30.731%</td>
<td>1.696%</td>
<td>-36.155%</td>
</tr>
<tr>
<td>M3L</td>
<td>0.869%</td>
<td>0.382%</td>
<td>6.182%</td>
<td>-0.777</td>
<td>3.117</td>
<td>24.002%</td>
<td>1.377%</td>
<td>-28.938%</td>
</tr>
<tr>
<td>M3H</td>
<td>1.120%</td>
<td>0.352%</td>
<td>5.935%</td>
<td>-1.063</td>
<td>4.692</td>
<td>17.871%</td>
<td>1.534%</td>
<td>-33.512%</td>
</tr>
<tr>
<td>M3H</td>
<td>1.221%</td>
<td>0.401%</td>
<td>6.403%</td>
<td>-0.213</td>
<td>3.034</td>
<td>32.834%</td>
<td>1.440%</td>
<td>-26.646%</td>
</tr>
<tr>
<td>M3H</td>
<td>1.574%</td>
<td>0.559%</td>
<td>6.777%</td>
<td>0.324</td>
<td>6.925</td>
<td>48.428%</td>
<td>1.780%</td>
<td>-28.683%</td>
</tr>
<tr>
<td>B4L</td>
<td>1.032%</td>
<td>0.380%</td>
<td>6.216%</td>
<td>-0.786</td>
<td>3.967</td>
<td>26.456%</td>
<td>1.655%</td>
<td>-32.882%</td>
</tr>
<tr>
<td>B4H</td>
<td>1.069%</td>
<td>0.335%</td>
<td>5.786%</td>
<td>-0.559</td>
<td>5.268</td>
<td>31.776%</td>
<td>1.056%</td>
<td>-30.974%</td>
</tr>
<tr>
<td>B4H</td>
<td>1.273%</td>
<td>0.324%</td>
<td>5.695%</td>
<td>-0.643</td>
<td>2.714</td>
<td>18.840%</td>
<td>1.362%</td>
<td>-27.743%</td>
</tr>
<tr>
<td>B4H</td>
<td>1.312%</td>
<td>0.415%</td>
<td>6.442%</td>
<td>-0.252</td>
<td>3.125</td>
<td>34.775%</td>
<td>1.755%</td>
<td>-27.640%</td>
</tr>
<tr>
<td>B4H</td>
<td>1.477%</td>
<td>0.471%</td>
<td>6.968%</td>
<td>-0.269</td>
<td>3.401</td>
<td>34.152%</td>
<td>1.888%</td>
<td>-33.012%</td>
</tr>
<tr>
<td>B1</td>
<td>0.850%</td>
<td>0.313%</td>
<td>5.593%</td>
<td>-1.168</td>
<td>4.708</td>
<td>19.407%</td>
<td>1.164%</td>
<td>-32.888%</td>
</tr>
<tr>
<td>B2</td>
<td>0.976%</td>
<td>0.265%</td>
<td>5.149%</td>
<td>-0.975</td>
<td>3.694</td>
<td>14.624%</td>
<td>1.457%</td>
<td>-29.190%</td>
</tr>
<tr>
<td>B3</td>
<td>1.085%</td>
<td>0.263%</td>
<td>5.130%</td>
<td>-0.615</td>
<td>2.807</td>
<td>16.362%</td>
<td>1.148%</td>
<td>-26.834%</td>
</tr>
<tr>
<td>B4</td>
<td>1.185%</td>
<td>0.273%</td>
<td>5.228%</td>
<td>-0.686</td>
<td>3.098</td>
<td>19.292%</td>
<td>1.363%</td>
<td>-28.934%</td>
</tr>
<tr>
<td>B1</td>
<td>1.371%</td>
<td>0.326%</td>
<td>5.711%</td>
<td>-0.445</td>
<td>2.079</td>
<td>20.388%</td>
<td>1.638%</td>
<td>-22.081%</td>
</tr>
</tbody>
</table>

The high and low values for the descriptive statistics for portfolios sorted by size and value are spread quite evenly across the portfolios. The stand-out portfolio is ‘S2H’. This is the portfolio with stocks in the second lowest quintile for market capitalisation and highest quintile of book-to-market ratio. Although its mean monthly return of 1.405% is not huge the returns are positively skewed and very highly concentrated around the mean with a kurtosis of 17.675. This portfolio also had a maximum monthly return of 62.841%.

### 7.1.3 Portfolio group 2 - Size and Momentum

For the portfolios sorted by size and momentum there appears to be a trend of higher mean monthly returns for small market capitalisation portfolios. Alternatively the large size and high momentum portfolios appear to have the largest minimum monthly return.
Table 4: Descriptive statistics of portfolios sorted by Size and Momentum

7.1.4 Portfolio group 3 - Size, Value and Momentum

The small and medium sized portfolios with high book-to-market ratios and low momentum have the most extreme values of the 27 portfolios arranged by the 3 factors. SVL had the highest mean monthly return of 2.316%. Along with portfolio MVL their returns are highly skewed with very large kurtosis values of 19.514 and 28.383 respectively. They also had a maximum monthly return higher than 80%. The descriptive statistics of the other portfolios appear fairly evenly distributed although there does seem to be a return premium for small stocks.
Table 5: Descriptive statistics of portfolios sorted by Size, Value and Momentum

### 7.1.5 Portfolio group 4 - Standard Deviation

For stocks arranged into portfolios according to their prior 12 month standard deviation, it is no surprise that a near perfect trend appears for standard deviation and variance statistics. Stocks in the largest 3 standard deviation groups have the largest values across the statistics, whilst the 2 smallest standard deviation portfolios attain mean monthly returns relative to their low reported risk.
### Descriptive Statistics

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Mean</th>
<th>Var</th>
<th>SD</th>
<th>Skew</th>
<th>Kurt</th>
<th>Max</th>
<th>Median</th>
<th>Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD1</td>
<td>1.597%</td>
<td>0.134%</td>
<td>3.667%</td>
<td>-0.300</td>
<td>2.951</td>
<td>16.755%</td>
<td>1.728%</td>
<td>-16.908%</td>
</tr>
<tr>
<td>SD2</td>
<td>1.427%</td>
<td>0.147%</td>
<td>3.829%</td>
<td>-0.694</td>
<td>2.476</td>
<td>13.134%</td>
<td>1.657%</td>
<td>-15.996%</td>
</tr>
<tr>
<td>SD3</td>
<td>1.550%</td>
<td>0.192%</td>
<td>4.384%</td>
<td>-0.421</td>
<td>3.459</td>
<td>24.438%</td>
<td>1.753%</td>
<td>-20.596%</td>
</tr>
<tr>
<td>SD4</td>
<td>1.165%</td>
<td>0.203%</td>
<td>4.500%</td>
<td>-0.658</td>
<td>3.880</td>
<td>22.229%</td>
<td>1.609%</td>
<td>-22.404%</td>
</tr>
<tr>
<td>SD5</td>
<td>1.170%</td>
<td>0.217%</td>
<td>4.651%</td>
<td>-0.582</td>
<td>2.979</td>
<td>19.065%</td>
<td>1.483%</td>
<td>-20.976%</td>
</tr>
<tr>
<td>SD6</td>
<td>1.293%</td>
<td>0.202%</td>
<td>4.489%</td>
<td>-0.645</td>
<td>2.719</td>
<td>17.074%</td>
<td>1.652%</td>
<td>-20.121%</td>
</tr>
<tr>
<td>SD7</td>
<td>1.286%</td>
<td>0.225%</td>
<td>4.748%</td>
<td>-0.771</td>
<td>4.177</td>
<td>22.174%</td>
<td>1.775%</td>
<td>-22.320%</td>
</tr>
<tr>
<td>SD8</td>
<td>1.223%</td>
<td>0.268%</td>
<td>5.178%</td>
<td>-0.259</td>
<td>5.988</td>
<td>32.782%</td>
<td>1.660%</td>
<td>-23.503%</td>
</tr>
<tr>
<td>SD9</td>
<td>1.288%</td>
<td>0.250%</td>
<td>4.998%</td>
<td>-0.865</td>
<td>3.342</td>
<td>19.059%</td>
<td>1.686%</td>
<td>-24.244%</td>
</tr>
<tr>
<td>SD10</td>
<td>1.319%</td>
<td>0.295%</td>
<td>5.429%</td>
<td>0.220</td>
<td>8.152</td>
<td>40.405%</td>
<td>1.587%</td>
<td>-24.900%</td>
</tr>
<tr>
<td>SD11</td>
<td>1.276%</td>
<td>0.248%</td>
<td>4.983%</td>
<td>-0.678</td>
<td>2.571</td>
<td>19.717%</td>
<td>1.535%</td>
<td>-23.173%</td>
</tr>
<tr>
<td>SD12</td>
<td>1.315%</td>
<td>0.327%</td>
<td>5.718%</td>
<td>-0.132</td>
<td>5.221</td>
<td>35.169%</td>
<td>1.530%</td>
<td>-25.775%</td>
</tr>
<tr>
<td>SD13</td>
<td>1.327%</td>
<td>0.300%</td>
<td>5.559%</td>
<td>0.145</td>
<td>6.569</td>
<td>37.953%</td>
<td>1.651%</td>
<td>-23.466%</td>
</tr>
<tr>
<td>SD14</td>
<td>1.559%</td>
<td>0.358%</td>
<td>5.984%</td>
<td>0.624</td>
<td>8.834</td>
<td>45.378%</td>
<td>1.685%</td>
<td>-22.879%</td>
</tr>
<tr>
<td>SD15</td>
<td>1.369%</td>
<td>0.322%</td>
<td>5.674%</td>
<td>0.003</td>
<td>4.546</td>
<td>35.636%</td>
<td>1.282%</td>
<td>-23.277%</td>
</tr>
<tr>
<td>SD16</td>
<td>1.383%</td>
<td>0.322%</td>
<td>5.670%</td>
<td>-0.171</td>
<td>3.327</td>
<td>29.565%</td>
<td>1.521%</td>
<td>-24.968%</td>
</tr>
<tr>
<td>SD17</td>
<td>1.235%</td>
<td>0.326%</td>
<td>5.706%</td>
<td>-0.460</td>
<td>1.680</td>
<td>20.212%</td>
<td>1.735%</td>
<td>-22.115%</td>
</tr>
<tr>
<td>SD18</td>
<td>1.384%</td>
<td>0.338%</td>
<td>5.810%</td>
<td>-0.167</td>
<td>5.970</td>
<td>37.830%</td>
<td>1.546%</td>
<td>-28.825%</td>
</tr>
<tr>
<td>SD19</td>
<td>1.482%</td>
<td>0.374%</td>
<td>6.118%</td>
<td>-0.162</td>
<td>1.367</td>
<td>20.381%</td>
<td>1.626%</td>
<td>-28.902%</td>
</tr>
<tr>
<td>SD20</td>
<td>1.315%</td>
<td>0.346%</td>
<td>6.604%</td>
<td>-0.027</td>
<td>2.634</td>
<td>28.658%</td>
<td>1.427%</td>
<td>-26.708%</td>
</tr>
<tr>
<td>SD21</td>
<td>1.477%</td>
<td>0.456%</td>
<td>6.755%</td>
<td>-0.264</td>
<td>2.951</td>
<td>31.956%</td>
<td>1.406%</td>
<td>-27.288%</td>
</tr>
<tr>
<td>SD22</td>
<td>1.662%</td>
<td>0.458%</td>
<td>6.366%</td>
<td>-0.136</td>
<td>1.979</td>
<td>25.418%</td>
<td>1.665%</td>
<td>-28.937%</td>
</tr>
<tr>
<td>SD23</td>
<td>1.861%</td>
<td>0.661%</td>
<td>8.233%</td>
<td>0.666</td>
<td>4.395</td>
<td>45.266%</td>
<td>1.714%</td>
<td>-24.864%</td>
</tr>
<tr>
<td>SD24</td>
<td>1.938%</td>
<td>0.761%</td>
<td>8.725%</td>
<td>1.512</td>
<td>3.304</td>
<td>67.946%</td>
<td>1.367%</td>
<td>-26.075%</td>
</tr>
<tr>
<td>SD25</td>
<td>2.361%</td>
<td>0.850%</td>
<td>9.219%</td>
<td>1.234</td>
<td>6.081</td>
<td>61.509%</td>
<td>1.879%</td>
<td>-25.127%</td>
</tr>
</tbody>
</table>

Table 6: Descriptive statistics of portfolios sorted by Standard Deviation

### 7.2 Black, Jensen & Scholes (1972) - Test Results

These results have been adapted to show absolute values. This is for clarity in determining statistical significance. The red-white colour for cells of $\alpha$ (alpha) and Standard Error (SE) in the below figures indicates the lowest (white) value to the highest (red) value. For the t-statistic cells any value above 1.96 is red. This is due to it being the critical level for t-tests with over 150 degrees of freedom\(^{45}\) at a 95% level.

#### 7.2.1 Portfolio group 1 - Size and Value

For the CAPM model the test portfolios of size and value have 2 portfolios with statistically significant $\alpha$’s. These are portfolios SH and S4H. These portfolios $\alpha$’s were 0.888% and 0.598%. These $\alpha$’s reduce when using both FF3F and C4F models but the t-statistics are still both highly significant. The FF3F model has a total of 3 significant $\alpha$ portfolios whilst the C4F model reports 5.

---

\(^{45}\)Degrees of freedom for t-tests are equal to the number of observations. These tests have 408 monthly time steps.
7.2.2 Portfolio group 2 - Size and Momentum

For portfolios arranged by size and momentum there is a clear trend. All the portfolios in the lowest size quintile have statistically significant $\alpha$'s. The FF3F model also reports significant $\alpha$'s for 4 out of 5 portfolios in both the lowest and 2nd highest momentum quintiles. In total CAPM reports 6, FF3F 11 and C4F 6 significant $\alpha$ portfolios.

7.2.3 Portfolio group 3 - Size, Value and Momentum

Table 7: BJS t-test for 25 portfolios arranged by the Size and Value

<table>
<thead>
<tr>
<th>Portfolio Group</th>
<th>Alpha</th>
<th>Standard Error</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>3.25%</td>
<td>2.12%</td>
<td>2.33%</td>
</tr>
<tr>
<td>Medium</td>
<td>4.36%</td>
<td>2.34%</td>
<td>2.78%</td>
</tr>
<tr>
<td>High</td>
<td>5.46%</td>
<td>2.60%</td>
<td>3.02%</td>
</tr>
</tbody>
</table>

Table 8: BJS t-test for 25 portfolios arranged by Size and Momentum

<table>
<thead>
<tr>
<th>Portfolio Group</th>
<th>Alpha</th>
<th>Standard Error</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>4.55%</td>
<td>3.12%</td>
<td>4.43%</td>
</tr>
<tr>
<td>Medium</td>
<td>5.65%</td>
<td>3.48%</td>
<td>5.24%</td>
</tr>
<tr>
<td>High</td>
<td>7.10%</td>
<td>4.16%</td>
<td>7.30%</td>
</tr>
</tbody>
</table>

For portfolios arranged by size and momentum there is a clear trend. All the portfolios in the lowest size quintile have statistically significant $\alpha$'s. The FF3F model also reports significant $\alpha$'s for 4 out of 5 portfolios in both the lowest and 2nd highest momentum quintiles. In total CAPM reports 6, FF3F 11 and C4F 6 significant $\alpha$ portfolios.
Table 9: BJS t-test for 27 portfolios arranged by Size, Value and Momentum - CAPM model

<table>
<thead>
<tr>
<th>Absolute Values</th>
<th>Down Momentum</th>
<th>Medium Momentum</th>
<th>Up Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>M</td>
<td>H</td>
</tr>
<tr>
<td>S</td>
<td>0.15%</td>
<td>0.09%</td>
<td>1.13%</td>
</tr>
<tr>
<td>M</td>
<td>0.13%</td>
<td>0.04%</td>
<td>0.91%</td>
</tr>
<tr>
<td>B</td>
<td>0.30%</td>
<td>0.11%</td>
<td>0.08%</td>
</tr>
</tbody>
</table>

Table 10: BJS t-test for 27 portfolios arranged by Size, Value and Momentum - FF3F model

<table>
<thead>
<tr>
<th>Absolute Values</th>
<th>Down Momentum</th>
<th>Medium Momentum</th>
<th>Up Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>M</td>
<td>H</td>
</tr>
<tr>
<td>S</td>
<td>0.24%</td>
<td>0.25%</td>
<td>0.33%</td>
</tr>
<tr>
<td>M</td>
<td>0.20%</td>
<td>0.06%</td>
<td>0.32%</td>
</tr>
<tr>
<td>B</td>
<td>0.13%</td>
<td>0.15%</td>
<td>0.21%</td>
</tr>
</tbody>
</table>

Table 9: BJS t-test for 27 portfolios arranged by Size, Value and Momentum - CAPM model

<table>
<thead>
<tr>
<th>Absolute Values</th>
<th>Down Momentum</th>
<th>Medium Momentum</th>
<th>Up Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>M</td>
<td>H</td>
</tr>
<tr>
<td>S</td>
<td>0.567</td>
<td>1.984</td>
<td>1.455</td>
</tr>
<tr>
<td>M</td>
<td>0.758</td>
<td>0.098</td>
<td>0.034</td>
</tr>
<tr>
<td>B</td>
<td>2.279</td>
<td>0.767</td>
<td>0.386</td>
</tr>
</tbody>
</table>

Table 10: BJS t-test for 27 portfolios arranged by Size, Value and Momentum - FF3F model
Table 11: BJS t-test for 27 portfolios arranged by Size, Value and Momentum - C4F model
For portfolios sorted by Size, Value and Momentum the models of CAPM, FF3F and C4F report 10, 10 and 6 statistically significant $\alpha$ portfolios respectively.

The Size portfolios for CAPM show 6, 1 and 3 significant $\alpha$’s for small, medium and large respectively. The Value portfolios show 6, 1 and 3 for low, medium and high book-to-market ratios and the Momentum portfolios show 1, 4 and 5 for low, medium and high past 2-12 month momentum.

For the FF3F model the Size portfolios show 6, 1 and 3 for small, medium and big, 2, 3 and 5 for low, medium and high Value portfolios and 4, 2 and 4 for low, medium and high Momentum.

Finally the C4F model reports Size portfolios with 5, 0 and 1 for small, medium and big, Value portfolios with 0, 3 and 3 for low, medium and high and Momentum portfolios with 2, 2 and 3.

Also worth noting one portfolio has a massive $\alpha$ in relation to the rest. Portfolio SVL has a monthly $\alpha$ return of 1.371%, 1.009% and 1.261% for the models of CAPM, FF3F and C4F respectively.

To summarise it appears all the models are rejected for a large number of the test portfolios but the C4F model seems to be the 'least bad'.

7.2.4 Portfolio group 4 - Standard Deviation
Table 12: BJS t-test for 25 portfolios arranged by Standard Deviation

For portfolios sorted into 25 standard deviation groups, the larger α’s can be seen in the portfolios with the largest standard deviation. The 3 smallest standard deviation portfolios also report significant α’s for the 3 models. The CAPM model shows 15 significant α’s. This reduces to 11 for the FF3F and 12 for the C4F models. The large number of significant α’s show that all the tested models perform badly when predicting stock returns for portfolios of stocks arranged in this way.
7.3 Fama-Macbeth (1973) - Test Results

This section reports the results of the Fama-Macbeth (1973) (FM) tests. It also reports $R^2$ values as $\hat{R}^2$ and the $\bar{R}^2$ as $\text{adjR}^2$ for models with more than one factor.

7.3.1 Portfolio group 1 - Size and Value

<table>
<thead>
<tr>
<th>Fama-MacBeth (1973) test</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAPM</td>
</tr>
<tr>
<td>gamma 0 0.013</td>
</tr>
<tr>
<td>SE 0.040</td>
</tr>
<tr>
<td>t 0.262</td>
</tr>
<tr>
<td>gamma 1 -0.006</td>
</tr>
<tr>
<td>SE 0.047</td>
</tr>
<tr>
<td>t -0.124</td>
</tr>
<tr>
<td>gamma 2 0.001</td>
</tr>
<tr>
<td>SE 0.015</td>
</tr>
<tr>
<td>t 0.046</td>
</tr>
<tr>
<td>gamma 3 0.005</td>
</tr>
<tr>
<td>SE 0.013</td>
</tr>
<tr>
<td>t 0.390</td>
</tr>
<tr>
<td>gamma 4 -0.004</td>
</tr>
<tr>
<td>SE 0.083</td>
</tr>
<tr>
<td>t -0.048</td>
</tr>
<tr>
<td>$R^2$ 0.117</td>
</tr>
<tr>
<td>adj $R^2$ 0.338</td>
</tr>
<tr>
<td>FF3F</td>
</tr>
<tr>
<td>gamma 0 0.012</td>
</tr>
<tr>
<td>SE 0.050</td>
</tr>
<tr>
<td>t 0.247</td>
</tr>
<tr>
<td>gamma 1 -0.007</td>
</tr>
<tr>
<td>SE 0.045</td>
</tr>
<tr>
<td>t -0.148</td>
</tr>
<tr>
<td>gamma 2 0.001</td>
</tr>
<tr>
<td>SE 0.016</td>
</tr>
<tr>
<td>t 0.041</td>
</tr>
<tr>
<td>gamma 3 0.005</td>
</tr>
<tr>
<td>SE 0.013</td>
</tr>
<tr>
<td>t 0.384</td>
</tr>
<tr>
<td>gamma 4 -0.004</td>
</tr>
<tr>
<td>SE 0.083</td>
</tr>
<tr>
<td>t -0.048</td>
</tr>
<tr>
<td>$R^2$ 0.342</td>
</tr>
<tr>
<td>adj $R^2$ 0.179</td>
</tr>
<tr>
<td>CAF</td>
</tr>
<tr>
<td>gamma 0 0.013</td>
</tr>
<tr>
<td>SE 0.050</td>
</tr>
<tr>
<td>t 0.218</td>
</tr>
<tr>
<td>gamma 1 -0.007</td>
</tr>
<tr>
<td>SE 0.057</td>
</tr>
<tr>
<td>$R^2$ 0.373</td>
</tr>
</tbody>
</table>

Table 13: FM test for 25 portfolios arranged by Size and Value

7.3.2 Portfolio group 2 - Size and Momentum

<table>
<thead>
<tr>
<th>Fama-MacBeth (1973) test</th>
</tr>
</thead>
<tbody>
<tr>
<td>CAPM</td>
</tr>
<tr>
<td>gamma 0 0.017</td>
</tr>
<tr>
<td>SE 0.037</td>
</tr>
<tr>
<td>t 0.445</td>
</tr>
<tr>
<td>gamma 1 -0.009</td>
</tr>
<tr>
<td>SE 0.037</td>
</tr>
<tr>
<td>t -0.255</td>
</tr>
<tr>
<td>gamma 2 0.001</td>
</tr>
<tr>
<td>SE 0.015</td>
</tr>
<tr>
<td>t 0.077</td>
</tr>
<tr>
<td>gamma 3 -0.002</td>
</tr>
<tr>
<td>SE 0.019</td>
</tr>
<tr>
<td>t -0.111</td>
</tr>
<tr>
<td>gamma 4 0.006</td>
</tr>
<tr>
<td>SE 0.057</td>
</tr>
<tr>
<td>t 0.105</td>
</tr>
<tr>
<td>$R^2$ 0.130</td>
</tr>
<tr>
<td>adj $R^2$ 0.345</td>
</tr>
<tr>
<td>FF3F</td>
</tr>
<tr>
<td>gamma 0 0.015</td>
</tr>
<tr>
<td>SE 0.037</td>
</tr>
<tr>
<td>t 0.406</td>
</tr>
<tr>
<td>gamma 1 -0.009</td>
</tr>
<tr>
<td>SE 0.033</td>
</tr>
<tr>
<td>t -0.256</td>
</tr>
<tr>
<td>gamma 2 0.001</td>
</tr>
<tr>
<td>SE 0.015</td>
</tr>
<tr>
<td>t 0.094</td>
</tr>
<tr>
<td>gamma 3 -0.001</td>
</tr>
<tr>
<td>SE 0.029</td>
</tr>
<tr>
<td>t -0.020</td>
</tr>
<tr>
<td>gamma 4 0.006</td>
</tr>
<tr>
<td>SE 0.057</td>
</tr>
<tr>
<td>t 0.105</td>
</tr>
<tr>
<td>$R^2$ 0.350</td>
</tr>
<tr>
<td>adj $R^2$ 0.195</td>
</tr>
<tr>
<td>CAF</td>
</tr>
<tr>
<td>gamma 0 0.012</td>
</tr>
<tr>
<td>SE 0.054</td>
</tr>
<tr>
<td>t 0.233</td>
</tr>
<tr>
<td>gamma 1 -0.006</td>
</tr>
<tr>
<td>SE 0.051</td>
</tr>
<tr>
<td>t -0.115</td>
</tr>
<tr>
<td>gamma 2 0.001</td>
</tr>
<tr>
<td>SE 0.016</td>
</tr>
<tr>
<td>t 0.094</td>
</tr>
<tr>
<td>gamma 3 -0.001</td>
</tr>
<tr>
<td>SE 0.029</td>
</tr>
</tbody>
</table>
| t 0.020                 

Table 14: FM test for 25 portfolios arranged by Size and Momentum

7.3.3 Portfolio group 3 - Size, Value and Momentum
It is clear from all the portfolio groups that $H_0$ cannot be rejected for the intercept $\bar{\gamma}_0$. The largest values of $\bar{\gamma}_0$ appear when using the CAPM model on the 25 size and momentum and 27 size, value and momentum test portfolios. However this $\bar{\gamma}_0$ value is just 0.017 for both with $t$ statistics of 0.445 and 0.442 respectively. For all the $\bar{\gamma}_i$ values, nothing is remotely significant with all $t$ statistics lower than 0.400.

It can also be seen that the FF3F and furter C4F models improve on the CAPM model’s ability to explain the cross-section of returns. However with all $R^2$ and $\bar{R}^2$
values under 0.4, all 3 models fail to accurately predict the stock price movements in this sample.

### 7.4 Gibbons, Ross & Shanken (1989) - Test Results

This section reports the results of the Gibbons, Ross & Shanken (1989) (GRS) $F$ and $\chi^2$ tests. Presented are the estimated test statistics $\delta$ and $\omega$. Here the red values indicate above the critical level.

#### 7.4.1 Portfolio group 1 - Size and Value

Table 17: GRS $\chi^2$ and F tests for 25 portfolios arranged by Size and Value

<table>
<thead>
<tr>
<th>Chi-squared test</th>
<th>40.644</th>
<th>C4F</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>48.321</td>
<td>FF3F</td>
</tr>
<tr>
<td></td>
<td>58.120</td>
<td>CAPM</td>
</tr>
<tr>
<td>F-test</td>
<td>1.301</td>
<td>C4F</td>
</tr>
<tr>
<td></td>
<td>1.658</td>
<td>FF3F</td>
</tr>
<tr>
<td></td>
<td>2.005</td>
<td>CAPM</td>
</tr>
</tbody>
</table>

#### 7.4.2 Portfolio group 2 - Size and Momentum

Table 18: GRS $\chi^2$ and F tests for 25 portfolios arranged by Size and Momentum

<table>
<thead>
<tr>
<th>Chi-squared test</th>
<th>44.913</th>
<th>C4F</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>71.872</td>
<td>FF3F</td>
</tr>
<tr>
<td></td>
<td>73.361</td>
<td>CAPM</td>
</tr>
<tr>
<td>F-test</td>
<td>1.537</td>
<td>C4F</td>
</tr>
<tr>
<td></td>
<td>2.466</td>
<td>FF3F</td>
</tr>
<tr>
<td></td>
<td>2.531</td>
<td>CAPM</td>
</tr>
</tbody>
</table>

#### 7.4.3 Portfolio group 3 - Size, Value and Momentum
Table 19: GRS $\chi^2$ and F tests for 27 portfolios arranged by Size, Value and Momentum

7.4.4 Portfolio group 4 - Standard Deviation

<table>
<thead>
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<th>Chi-squared test</th>
<th>F-test</th>
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<td>FF3F</td>
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<td></td>
</tr>
<tr>
<td>2.442</td>
<td>CAPM</td>
<td></td>
</tr>
</tbody>
</table>

Table 20: GRS $\chi^2$ and F tests for 25 portfolios arranged by Standard Deviation

The CAPM model rejects the $H_0$ that $\alpha$’s are jointly equal to zero for all test portfolios for both $\chi^2$ and $F$ tests of GRS.

The $\chi^2$ test also rejects the FF3F model for all test portfolios and 3 out of 4 test portfolios for the $F$ test. However the model cannot be rejected using the $F$ test for Size and Value portfolios.

The C4F model performs best of both $\chi^2$ and $F$ tests. It is accepted for 2 of the 4 portfolio groups using the $\chi^2$ test and 3 of the 4 portfolios when using the $F$ test, only being rejected on standard deviation portfolios.
8 Conclusion

The results of this thesis appear to support the findings of Fama & French (2011) and Gregory, Tharyan & Christidis (2013) (GTC) that in the UK market the Fama-French 3-Factor (FF3F) and Carhart 4-Factor (C4F) asset pricing models have problems in the pricing of portfolios. Unlike GTC this thesis also tests the standard CAPM model but this is found to be an even worse fit for the UK market over the range of tests conducted, hinting at some form of presence of the size, value and momentum market premiums. Of the three models, C4F does however show the highest explanatory power in predicting stock returns.

The pricing issues in all three models can be seen by the large number of significant $\alpha$’s reported by the Black, Jensen & Scholes (1972) (BJS) tests. It appears though that when factors match the criteria for the arrangement of test portfolio, the number of significant $\alpha$’s drops. This is further shown by the largest amount of significant $\alpha$’s appearing in the standard deviation sorted test portfolios for all of the tested models.

The Fama-MacBeth (1973) (FM) test results fail to reject any of the models tested. However, with the $R^2$ and $\bar{R}^2$ values so low for all three models across all the test portfolios these results show the test portfolios do not fit any of the models. However, it is worth noting that there is an increase in explanatory power visible for higher factor models.

For Gibbons, Ross & Shanken (1989) (GRS) $F$ and $\chi^2$ tests the CAPM is rejected across all four test portfolios. FF3F is rejected in all but one portfolio. This happened to be the size and value sorted portfolio using the $F$ test. C4F is accepted for two of the $\chi^2$ and three of the $F$ tests. All models are rejected with both tests for portfolios using standard deviations.

The descriptive statistics show, of the four factors, momentum has the largest monthly returns indicating a presence of a momentum premium for the UK market.

This thesis adopts the methods of GTC for factor and test portfolio construction. It also uses the same sample of data source as GTC but tests the data over an additional 4 years. Michou, Mouselli & Stark (2014) find varying results in their analysis of previous UK studies and suggest that the different ways of constructing the factors of SMB (small minus big - size) and HML (high minus low - value) may impact results.

An important consideration noted by Dimson, Nagel & Quigley (2003) and GTC is that the UK has a large illiquid tail of small stocks. It is quite possible that these stocks are contributing to the poor results across these three asset pricing models.

The author of this paper advises further research and tests may be required when searching for optimal asset pricing techniques to fit the UK data. Two possible avenues are recommended to pursue. First, is the work of Fama & French (2014). Here they
look at a 5-factor asset pricing model, removing the factor of *Momentum* and adding factors for *Profitability* and *Investment* of stock. The other suggestion is the work of Gregory et al (2015). This paper suggests more focus on accurate estimates of market sensitivity ($\beta$).
References


9 Appendix

The purpose of these appendices is to allow for continuity of the main body of the thesis. The information contained here is important for the understanding of the work but is kept separate in order to prevent deviation from the intended direction of the thesis.

The appendices are split into three parts. Part I contains the mathematical derivations for stated equations not proved in the theoretical sections. Part II contains extracts of the market data in its downloaded form. These were all the files used for the statistical testing. Part III is a section on the fulfilment of thesis objectives written by the author. This is a requirement of the Division of Applied Mathematics at Mälardalen University.

Part I

Appendix A - Mathematical Derivations

9.1 Portfolio Return and Variance

9.1.1 Portfolio Return

The return on a portfolio can be derived in the following way.

The return on a portfolio of assets is the sum of return multiplied by weight of each individual asset contained in the portfolio.

Define $R_{pj}$ as the $j$th return on the portfolio and $X_i$ as the fraction of the funds invested in the $i$th asset, and $N$ the number of assets, then it can be said that

$$R_{pj} = \sum_{i=1}^{N} (X_i R_{ij}).$$

Then taking expected values gives

$$\bar{R}_p = E(R_p) = E\left( \sum_{i=1}^{N} (X_i R_{ij}) \right).$$

The expected value of the sum of returns is equal to the sum of the expected returns, therefore
\[ \bar{R}_p = \sum_{i=1}^{N} E(X_i R_{ij}). \]

Finally, the expected value of a constant (the weights) multiplied by return is equal to the constant multiplied by the expected return. This gives the formula for the expected return on a portfolio

\[ \bar{R}_p = \sum_{i=1}^{N} (X_i \bar{R}_i). \]

Equation 2.

9.1.2 Portfolio Variance

The variance of a portfolio can be derived in the following way.

Let us define the variance on a portfolio as the expectation of squaring the actual return on the portfolio minus expected return on the portfolio

\[ \sigma_p^2 = E[(R_p - E(R_p))^2]. \]

Apply the expressions for return and expected return on a portfolio

\[ \sigma_p^2 = E\left[ \sum_{i=1}^{n} (X_i R_{ij}) - \sum_{i=1}^{n} (X_i \bar{R}_i) \right]^2. \]

Rearranging the expression inside the brackets gives

\[ \sigma_p^2 = E\left[ \sum_{i=1}^{n} X_i (R_{ij} - \bar{R}_i) \right]^2. \]

Recalling that \((X + Y)^2 = X^2 + Y^2 + 2XY\), multiply out the brackets to get

\[ \sigma_p^2 = E\left[ \sum_{i=1}^{n} X_i^2 (R_{ij} - \bar{R}_i)^2 + \sum_{i=1}^{n} \sum_{k=1}^{n} X_i X_k (R_{ij} - \bar{R}_i)(R_{kj} - \bar{R}_k) \right]. \]

As the expected value of a sum is equal to the sum of the expected value and the expected value of a constant multiplied by return is equal to the constant multiplied by the expected return, we now have

\[ \sigma_p^2 = \sum_{i=1}^{n} X_i^2 E(R_{ij} - \bar{R}_i)^2 + \sum_{i=1}^{n} \sum_{k=1}^{n} X_i X_k E[(R_{ij} - \bar{R}_i)(R_{kj} - \bar{R}_k)]. \]
Assigning \( \sigma_i^2 = E(R_{ij} - \bar{R}_i)^2 \) and \( \sigma_{ik} = E[(R_{ij} - \bar{R}_i)(R_{kj} - \bar{R}_k)] \) as the variance and covariance on an asset respectively, we get the formula for the variance on a portfolio as

\[
\sigma_p^2 = \sum_{j=1}^{n} (X_j^2 \sigma_j^2) + \sum_{j=1}^{n} \sum_{k=1}^{n} (X_j X_k \sigma_{jk}).
\]

Equation 3.
9.2 Efficient Frontier: Mean-Variance Efficient Portfolios

Consider a portfolio \( p \) with \( N \) assets each with weight \( X_i \). The sum of the combined asset weights is \( \bar{X} = \sum_{i=1}^{N} \bar{X}_i = 1 \). The expected total return on a portfolio is defined in Section 3.4.1 as the sum of the proportional expected return relative to weight on each asset.

The expected return can be calculated using vector multiplication as follows

\[
E[\tilde{R}_p] = X' R = \begin{bmatrix} X_1 & X_2 & \cdots & X_N \end{bmatrix} \cdot \begin{bmatrix} E[\tilde{R}_1] \\ E[\tilde{R}_2] \\ \vdots \\ E[\tilde{R}_N] \end{bmatrix}.
\]

The variance of a portfolio is dependent on the covariance between assets as mentioned in section 3.4.2. In vector form the covariance is captured by a matrix and the variance of a portfolio is calculated by multiplying the weight vector and its transpose as follows

\[
\sigma_p = X' \Sigma X = \begin{bmatrix} X_1 & X_2 & \cdots & X_N \end{bmatrix} \cdot \begin{bmatrix} \sigma^2 & \sigma_{2,1} & \cdots & \sigma_{N,1} \\ \sigma_{1,2} & \sigma^2 & \cdots & \sigma_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,N} & \sigma_{2,N} & \cdots & \sigma^2 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}.
\]

Define \( \mu \) as a given level of expected return. Now consider that for given levels of \( \mu \), portfolios of minimum risk can be calculated using the following minimisation problem

\[
\min_X X' \Sigma X',
\]

\[s.t. \quad X' R \geq \mu, \quad X' 1 = 1.
\]

Here \( 1 \) is an \( N \) column vector of 1’s.

This problem is then solved using Lagrangian multipliers. This can be written in the following way

\[
\mathcal{L}(X, \lambda_1, \lambda_2) = X' \Sigma X - \lambda_1 (\mu - X' R) - \lambda_2 (1 - X' 1).
\]

Now take partial derivatives and set each equal to zero,

1. \( \frac{\partial \mathcal{L}}{\partial X} = 2 \Sigma X - \lambda_1 R - \lambda_2 1 = \Rightarrow X = \frac{1}{2} \Sigma^{-1} (\lambda_1 R - \lambda_2 1), \]
2. \( \frac{\partial \mathcal{L}}{\partial \lambda_1} = -X R + \mu \Rightarrow \mu = X' R, \]
3. \( \frac{\partial \mathcal{L}}{\partial \lambda_2} = -X 1 + 1 \Rightarrow 1 = X' 1. \]
[1] can be re-written as
\[ X = \frac{1}{2} \Sigma^{-1} \begin{bmatrix} R & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \]

Call this (*)

[2] and [3] can be solved using simultaneous equations as follows
\[ \mu = X'R = R'X; \quad 1 = X'1 = 1'X \Rightarrow \begin{bmatrix} \mu \\ 1 \end{bmatrix} = \begin{bmatrix} R'X \\ 1'X \end{bmatrix} = \begin{bmatrix} R & 1 \end{bmatrix}' \cdot X \]
\[ \Rightarrow [R & 1]' \cdot X = [R & 1]' \cdot \frac{1}{2} \Sigma^{-1} [R & 1] \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \]

Call this (**) 

Now define \( A \equiv [R & 1]' \cdot \Sigma^{-1} \cdot [R & 1]' \).

Multiply together to get a single matrix
\[ A = \begin{bmatrix} R'\Sigma^{-1}R & R'\Sigma^{-1}1 \\ 1'\Sigma^{-1}R & 1'\Sigma^{-1}1 \end{bmatrix}. \]

All the entries in \( A \) are scalars making it a \( 2 \times 2 \) positive definite information matrix. Therefore it has an inverse \( A^{-1} \).

Substitute \( A \) back into (**) to give
\[ \frac{1}{2} A \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix} \Rightarrow \frac{1}{2} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix}. \]

Recall the formula for the optimal weight vector (*). Substituting yields
\[ X = \Sigma^{-1} [R & 1] A^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix}, \]
which is a formula with known variables.

Now define
\[ A^{-1} = \begin{bmatrix} \frac{1}{a} \\ c & -b \\ -b & a \end{bmatrix}. \]
where \( d \) is the determinant of \( A \) and \( d = ac - b^2 \). Substituting this into the formula for \( X \) gives

\[
X = \Sigma^{-1} \begin{bmatrix} R & 1 \end{bmatrix} A^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix} = \begin{bmatrix} \Sigma^{-1} R & \Sigma^{-1} 1 \end{bmatrix} \begin{bmatrix} \frac{1}{d} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \mu \\ 1 \end{bmatrix}.
\]

Simplifying further gives

\[
\Rightarrow \left[ \frac{1}{d} \begin{bmatrix} \Sigma^{-1} R & \Sigma^{-1} 1 \end{bmatrix} \begin{bmatrix} c\mu - b \\ -b\mu + a \end{bmatrix} \right] \Rightarrow \left[ \frac{1}{d} \right] \left[ \Sigma^{-1} R c \mu - \Sigma^{-1} R b \right] + \left[ \frac{1}{d} \right] \left[ \Sigma^{-1} 1 a - \Sigma^{-1} 1 b \mu \right],
\]

which can be rearranged in terms of \( \mu \) to get

\[
\Rightarrow \left[ \frac{1}{d} \right] \left[ \Sigma^{-1} 1 a - \Sigma^{-1} R b \right] + \left[ \frac{1}{d} \right] \left[ \Sigma^{-1} R c - \Sigma^{-1} b \right] \mu.
\]

Defining

\[
G \equiv \left[ \frac{1}{d} \right] \left[ \Sigma^{-1} 1 a - \Sigma^{-1} R b \right],
\]

and

\[
H \equiv \left[ \frac{1}{d} \right] \left[ \Sigma^{-1} R c - \Sigma^{-1} b \right].
\]

We arrive at the simple linear formula

\[
X = G + H \mu.
\]

This shows how to calculate an efficient portfolio for a given level of return. For the Efficient frontier calculate \( \forall \mu \) in the opportunity set.
9.3 Single-Index (Market) Model

9.3.1 Expected return on a security

The expected return on a security can be written as

\[ E(R_i) = E[\alpha_i + \beta_i R_m + e_i]. \]

As the expected value of the sum of random variables is the sum of the expected values,

\[ E(R_i) = E(\alpha_i) + E(\beta_i R_m) + E(e_i). \]

By definition, \( e_i = 0 \) and \( \alpha_i \) and \( \beta_i \) are constants, therefore

\[ E(R_i) = \alpha_i + \beta_i \bar{R}_m. \]

Equation 7.

9.3.2 Variance of return of a security

The variance of a security’s return is defined as

\[ \sigma_i^2 = E(R_i - \bar{R}_i)^2. \]

Now substitute the previously defined expressions of \( R_i \) and \( \bar{R}_i \) to get

\[ \sigma_i^2 = E[(\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m)]^2. \]

Observing that the \( \alpha_i \) terms cancel, re-arranging gives

\[ \sigma_i^2 = E[(\beta_i(R_m - \bar{R}_m + e_i)]^2. \]

Now multiply out the brackets,

\[ \sigma_i^2 = \beta_i^2 E(R_m - \bar{R}_m)^2 + 2\beta_i E[e_i(R_m - \bar{R}_m)] + E(e_i)^2. \]

Recall from Section (3.1) \( E[e_i(R_m - \bar{R}_m)] = 0 \), thus,

\[ \sigma_i^2 = \beta_i^2 E(R_m - \bar{R}_m)^2 + E(e_i)^2. \]

And as \( E(R_m - \bar{R}_m)^2 = \sigma_m^2 \) and \( E(e_i)^2 = \sigma_{e_i}^2 \),

\[ \sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2. \]

Equation 8.
9.3.3 Co-variance between two securities

For any two securities $i$ and $j$, the covariance is defined as

$$\sigma_{ij} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)].$$

Substituting previously defined terms of the return on investment and expected return on investment yields

$$\sigma_{ij} = E\left\{[(\alpha_i + \beta_i R_m + e_i) - (\alpha_i + \beta_i \bar{R}_m)] \cdot [(\alpha_j + \beta_j R_m + e_j) - (\alpha_j + \beta_j \bar{R}_m)]\right\}.$$

Now simplify by cancelling the $\alpha$ terms and combine the $\beta$ terms to get

$$\sigma_{ij} = E\left[\beta_i (R_m - \bar{R}_m) + e_i \right] \cdot \left[\beta_j (R_m - \bar{R}_m) + e_j \right].$$

Multiplying out the brackets gives

$$\sigma_{ij} = \beta_i \beta_j E(R_m - \bar{R}_m)^2 + \beta_j E[e_i (R_m - \bar{R}_m)] + \beta_i E[e_j (R_m - \bar{R}_m)] + E(e_i e_j).$$

By assumption, the last three terms are zero, therefore,

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2.$$

Equation 9.
9.4 Multi-Factor Models

9.4.1 Expected return on a security

The expected return on a security using a multi-factor model can be written as

$$E(R_i) = E(a_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \ldots + \beta_{iL}I_L + c_i).$$

And the expected value of the sum of random variables is the sum of the expected values, giving,

$$E(R_i) = E(a_i) + E(\beta_{i1}I_1) + E(\beta_{i2}I_2) + \ldots + E(\beta_{iL}I_L) + E(c_i).$$

By construction, $E(c_i) = 0$ and as $a$ and $\beta$’s are constants,

$$E(R_i) = a_i + \beta_{i1}\bar{I}_1 + \beta_{i2}\bar{I}_2 + \ldots + \beta_{iL}\bar{I}_L.$$  
Equation 12.

9.4.2 Variance of return of a security

The variance of a security’s return can be defined as

$$\sigma^2_i = E(R_i - \bar{R}_i)^2.$$  

Substituting gives

$$E(R_i) = E[(a_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \ldots + \beta_{iL}I_L + c_i)$$

$$-(a_i + \beta_{i1}\bar{I}_1 + \beta_{i2}\bar{I}_2 + \ldots + \beta_{iL}\bar{I}_L)]^2.$$  

Cancel the $a_i$’s and re-arrange to get

$$E(R_i) = E[\beta_{i1}(I_1 - \bar{I}_1) + \beta_{i2}(I_2 - \bar{I}_2) + \ldots + \beta_{iL}(I_L - \bar{I}_L) + c_i]^2.$$  

Now square out the brackets. First consider only the terms involving the first index

$$E[\beta_{i1}^2(I_1 - \bar{I}_1)^2 + \beta_{i1}\beta_{i2}(I_1 - \bar{I}_1)(I_2 - \bar{I}_2) + \ldots$$

$$+\beta_{i1}\beta_{iL}(I_1 - \bar{I}_1)(I_L - \bar{I}_L) + \beta_{i1}(I_1 - \bar{I}_1)(c_i)].$$

Recall the expected value of the sum of random variables is the sum of the expected values. Since the $\beta_i$’s are constants,

$$\beta_{i1}^2E(I_1 - \bar{I}_1)^2 + \beta_{i1}\beta_{i2}E[(I_1 - \bar{I}_1)(I_2 - \bar{I}_2)] + \ldots$$

$$+\beta_{i1}\beta_{iL}E[(I_1 - \bar{I}_1)(I_L - \bar{I}_L)] + \beta_{i1}E[(I_1 - \bar{I}_1)(c_i)].$$

67
By construction

\[ E[(I_i - \bar{I}_i)(I_j - \bar{I}_j)] = 0, \]

and

\[ E[(I_1 - \bar{I}_1)(c_i)] = 0. \]

Therefore the higher terms involving index one cancel leaving

\[ \beta^2_{i1}E(I_1 - I_1)^2 = \beta^2_{i1}\sigma^2_{I_1}. \]

\[ E[(I_i - \bar{I}_i)(c_i)] = 0, \forall i \text{ and } E(c_i)^2 = \sigma^2_{ci}. \]

As a result we arrive at

\[ \sigma^2_i = \beta^2_{i1}\sigma^2_{I_1} + \beta^2_{i2}\sigma^2_{I_2} + \ldots + \beta^2_{iL}\sigma^2_{I_L} + \sigma^2_{ci}. \]

Equation 13.

9.4.3 Co-variance between two securities

For any two securities \(i\) and \(j\), the covariance can be defined as

\[ \sigma_{ij} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)]. \]

Substituting expressions for \(R_i\) and \(R_j\) gives

\[
E(R_i) = E \left\{ \left[ (a_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \ldots + \beta_{iL}I_L + c_i) - (a_i + \beta_{i1}\bar{I}_1 + \beta_{i2}\bar{I}_2 + \ldots + \beta_{iL}\bar{I}_L) \right] \right.
\]

\[ \cdot \left[ (a_j + \beta_{j1}I_1 + \beta_{j2}I_2 + \ldots + \beta_{jL}I_L + c_j) - (a_j + \beta_{j1}\bar{I}_1 + \beta_{j2}\bar{I}_2 + \ldots + \beta_{jL}\bar{I}_L) \right] \}.
\]

Cancelling \(a\)'s and combining terms yields

\[
E(R_i) = E \left\{ \left[ \beta_{i1}(I_1 - \bar{I}_1) + \beta_{i2}(I_2 - \bar{I}_2) + \ldots + \beta_{iL}(I_L - \bar{I}_L) + c_i \right] \right.
\]

\[ \cdot \left[ \beta_{j1}(I_1 - \bar{I}_1) + \beta_{j2}(I_2 - \bar{I}_2) + \ldots + \beta_{jL}(I_L - \bar{I}_L) + c_j \right] \}.
\]

Now multiply out the brackets and first only consider the terms involving \(\beta_{i1}\). This gives

\[
E[\beta_{i1}\beta_{j1}(I_1 - \bar{I}_1)^2 + \beta_{i1}\beta_{j2}(I_1 - \bar{I}_1)(I_2 - \bar{I}_2) + \beta_{i1}\beta_{j3}(I_1 - \bar{I}_1)(I_3 - \bar{I}_3)
\]

\[ + \beta_{i1}\beta_{jL}(I_1 - \bar{I}_1)(I_L - \bar{I}_L) + \beta_{i1}(I_1 - \bar{I}_1)c_i].
\]
As the expected value of all terms involving different indexes i.e., \((I_1 - \bar{I}_1)(I_k - \bar{I}_k)\) and \(\beta_1(I_1 - \bar{I}_1)c_j\) are zero by construction, there is only one non-zero term. This leaves

\[ \beta_{ij} \beta_{j1} E(I_1 - \bar{I}_1)^2 = \beta_{i1} \beta_{j1} \sigma_{i1}^2. \]

Here there are two types of error terms. The terms \(\beta_{ik}(I_k - \bar{I}_k)c_j\) are zero by construction and \(c_i c_j\) are zero by assumption, therefore,

\[ \sigma_{ij} = \beta_{i1} \beta_{j1} \sigma_{i1}^2 + \beta_{i2} \beta_{j2} \sigma_{i2}^2 + \ldots + \beta_{iL} \beta_{jL} \sigma_{iL}^2. \]

Equation 14.
9.5 Capital Asset Pricing Model (CAPM)

Consider the risk-free asset $R_f$ and the market portfolio $m$ as points that define the Capital Market Line (CML). Now assign $i$ as any risky asset.

In equilibrium the theory is that all assets are held by investors and prices adjust through supply and demand. Also the market portfolio is made up of all assets on the market with proportions equal to their relative market capitalisations. However, in practice, a stock index is used as a market proxy.

Define portfolio $P$ as a combination of the market and a risky asset with proportions of $x$ and $1 - x$ invested in $i$ and $m$ respectively.

The expected return of $P$ can be written as

$$E(R_p) = xE(R_i) + (1 - x)E(R_m),$$

with the standard deviation of $p$ as

$$\sigma_p = \left[x^2\sigma^2_i + (1 - x)^2\sigma^2_m + 2x(1 - x)\sigma_{im}\right]^{1/2},$$

where $\sigma^2_i$ represents the variance of $i$, $\sigma^2_m$ the variance of $m$ and $\sigma_{im}$ the covariance between $i$ and $m$.

A curve can then be constructed by considering all possible weight proportions $x$ of $i$ (and consequently $(1 - x)$ of $m$).

The tangent to this curve can be calculated as

$$\frac{\partial E(R_p)}{\partial \sigma_p} = \frac{\partial E(R_p)}{\partial x} \cdot \frac{\partial x}{\partial \sigma_p},$$

where

$$\frac{\partial E(R_p)}{\partial x} = E(R_i) - E(R_m),$$

and

$$\frac{\partial \sigma_p}{\partial x} = \frac{2x\sigma^2_i + 2\sigma^2_m(1 - x) + 2\sigma_{im}(1 - 2x)}{2\sigma_p}$$

Substituting the above into the expression for $\frac{\partial E(R_p)}{\partial \sigma_p}$ gives

$$\frac{\partial E(R_p)}{\partial \sigma_p} = \frac{(E(R_i) - E(R_m))\sigma_p}{x(\sigma^2_i + \sigma^2_m - 2\sigma_{im}) + \sigma_{im} - \sigma^2_m}.$$
As \( m \) is the market portfolio, by definition it includes all assets, including \( i \). Therefore \( p \) is in excess of \( m \) by the \( x \) proportion of \( i \). But in equilibrium there is no excess so \( x = 0 \) and consequently \( \sigma_p = \sigma_m \).

As a result the slope from the tangent when the market is in equilibrium can be written as

\[
\frac{\partial E(R_p)}{\partial \sigma_p}(m) = \frac{(E(R_i) - E(R_m))\sigma_p}{\sigma_{im} - \sigma_m^2}.
\]

Now recall that the slope of the market line is given by

\[
b = \frac{E(R_m) - R_f}{\sigma_m}.
\]

At \( m \), the slope of the tangent must equal the market line, therefore,

\[
\frac{(E(R_i) - E(R_m))\sigma_p}{\sigma_{im} - \sigma_m^2} = \frac{E(R_m) - R_f}{\sigma_m}.
\]

This can be re-arranged to yield

\[
E(R_i) = R_f + \left(\frac{E(R_m) - R_f}{\sigma_m^2}\right)\sigma_{im}
\]

Recalling \( \beta_i = \frac{\sigma_{im}}{\sigma_m^2} \), \( \bar{R}_i = E(R_i) \) and \( \bar{R}_m = E(R_m) \), we arrive at

\[
\bar{R}_i = R_f + \beta_i(\bar{R}_m - R_f).
\]

Equation 10 - the CAPM formula.
Part II

Appendix B - Thesis Data

9.6 Market Data

The data for this thesis was taken from the website of The Xfi Centre for Finance and Investment at the University of Exeter. A sample was taken which would be too large to include in this thesis so included in this section are extracts from the data files used.

The full data set can be accessed at their website:


Table 21: Extract of data on monthly returns for Size by Book-to-Market sorted Portfolios

Table 22: Extract of data on monthly returns for Size by Momentum sorted Portfolios
Table 23: Extract of data on monthly returns for Size by Book-to-Market by Momentum sorted Portfolios

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Table 24: Extract of data on monthly returns for Standard Deviation sorted Portfolios

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Table 25: Extract of data on monthly returns for the Factors

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Part III
Appendix C - Fulfilment of Thesis Objectives

This section is included at the request of the Division of Applied Mathematics at Mälardalen University. For a Masters degree there are 6 objectives on which the student is evaluated. The author explains below how each objective was achieved.

**Objective 1:**
For Master degree, student should demonstrate knowledge and understanding in the major field of study, including both broad knowledge in the field and substantially deeper knowledge of certain parts of the area as well as insight into current research and development.

Modern Portfolio Theory is the backbone of this thesis. The author reviewed extensive literature from books, journals and genuine internet websites. These sources were thoroughly analysed to obtain the theory and history of asset pricing models.

Much of the modern literature inclines towards multi-factor models and the search for the most suitable factors to include. This is supported by the recent Fama & French (2014) working paper on a 5-factor asset pricing model.

Testing techniques vary. The author has tried to incorporate a range of robust tests to check the suitability of each model for the UK market.

There exists a range of studies done since the turn of the century on multi-factor asset pricing models for the UK market. However, the author was unable to find a paper supporting a valid asset pricing model to predict stock returns on the LSE.

The choice of the three models used in the thesis was decided using the following criteria:-

CAPM is the starting point in asset pricing models and although simple, it is still extremely popular in the world of business and investing.

The Fama-French three factor model (FF3F) appears in the literature to be the most popular multi-factor model to be used and tested across all markets since its creation. Eugene Fama is also one of the most cited researchers in economics.

The choice of the Carhart four factor model (C4F) was taken as momentum clearly appears to be a prominent factor anomaly in the UK market.
Three models were tested to allow for close comparison of the strength of both factors and suitability of models.

**Objective 2:**
For Master Degree, student should demonstrate deeper methodological knowledge in the major field of study.

This thesis starts with a motivational section on why the author came to study the asset pricing models using UK market data. The author begins with a thorough introduction and background sections designed to bring a reader with limited knowledge of Modern Portfolio Theory to a high enough level of understanding to interpret the objective of the tests carried out.

The author consulted a large number of financial journal articles before deciding which sample data to use and how this data would be constructed to accurately test single and multi-factor asset pricing models on the UK market.

The book of Modern Portfolio Theory and Investment Analysis, 8th edition (2011) was thoroughly studied. A number of other books were also core to the literature gathered for this thesis. Mathematical Statistics with Applications, 7th edition (2007) was crucial in understanding the statistics required to run this study.

From analysing a large range of previous empirical studies of asset pricing models, it appears both time-series and cross-sectional tests are conducted when running robust checks on the effectiveness of the models.

The author decided that the classical tests of Black, Jensen & Scholes (1972) and Fama-MacBeth (1973) still show relevance when considering the range of modern literature. A third multivariate test of Gibbons, Ross & Shanken (1989) was conducted. It effectively treats the excess returns on the grouped test portfolios as panel data. This is in order to test the time-series and cross-sectional aspects of the model simultaneously.

**Objective 3:**
For Master degree, student should demonstrate the ability to critically and systematically integrate knowledge and to analyze, assess and deal with complex phenomena, issues and situations even with limited information.

The author combines complex theories from multiple sources into a consistent study of work. Knowledge and skills learned from the Financial Engineering programme gave the author a mathematical and financial base from where to build. Much of the theory in the background and asset pricing models sections was discussed in the classes on Portfolio Theory I & II. For the tests of BJS, FM and GRS the author was required to expand on knowledge beyond the scope of the courses (particularly in the case of GRS).
The complex theories in GRS are not widely explained on the internet like many more commonly used tests. The author particularly excelled in the successful implementation of this model to the test portfolios in MS Excel with limited guidance in this test.

The author also completed all this whilst being based away from the university (explained in objective 4), thus not having all of its resources at his disposal.

**Objective 4:**

*For Master degree, student should demonstrate the ability to critically, independently and creatively identify and formulate issues and to plan and carry out advanced tasks within specified time frames, thereby contributing to the development of knowledge and to evaluate this work.*

The author spent a considerable time analysing the literature before deciding which area to study. Prior to starting the in-depth research and construction of the thesis, the author took a break from academia to take on an operational role working for a large UK banking and financial services company. The author left Sweden to return to his homeland, Scotland. He later made the decision to return to studies and then set out a plan to write the thesis.

Although the author remained in Scotland during the implementation of the thesis, he remained in close contact with the university and thesis supervisors.

The author determined the direction of study by thoroughly analysing extensive literature on asset pricing theories and models. Having decided which models to test, the author then reviewed the range of asset pricing tests.

Michou, Moselli & Stark (2014) was a key paper in influencing which models and tests were best suited as they researched a range of previous empirical studies. These specifically concerned the UK market.

**Objective 5:**

*For Master degree, student should demonstrate ability in both national and international contexts, orally and in writing to present and discuss their conclusions and the knowledge and arguments behind them, in dialogue with different groups.*

The author reviewed and cites many financial articles concerning markets in America, Europe and Asia. Coming from the UK, the author decided to focus on the UK market but took forward the knowledge attained from the Financial Engineering programme at Mälardalen University, Sweden. The teachings from lectures and recommended literature primarily contain American and Swedish content.

The details of the authors findings will be presented verbally and visually at the presentation of this thesis.
Objective 6:
For Master degree, student should demonstrate ability in the major field of study make judgements taking into account relevant scientific, social and ethical aspects, and demonstrate an awareness of ethical issues in research and development.

There is a vast range of information and theories available on the internet. The author was careful when considering what to reference when studying. The sources used when gathering theories and data were taken from reputable sources within the field.

The data was sourced from the website for Xfi - Centre for Finance and Investment at the University of Exeter, with the personal permission of Professor Alan Gregory. There database is cited in other published articles and was the best compiled sample of the UK market the author could find.

The journal articles and books used as references in this thesis are all published or working papers by respected researchers in the field of Finance. Any other information was taken from websites that are clearly stated in the footnotes.

As with any information that can be publicly accessed, the author understands his duty to write sound theories with accurate data calculations. He must stress a warning though to any reader that, as finance is not an exact science, there are severe financial risks that can be potentially incurred by incorporating these theories in an investment strategy.