ORCHESTRATING MATHEMATICAL WHOLE-CLASS DISCUSSIONS IN THE PROBLEM-SOLVING CLASSROOM
THEORIZING CHALLENGES AND SUPPORT FOR TEACHERS

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Abstract

Promising teaching approaches for developing students’ mathematical competencies include the approach of teaching mathematics through problem solving. Orchestrating a whole-class discussion of students’ ideas is an important aspect of teaching through problem solving. There is a wide consensus within the field that it is very challenging for the teacher to conduct class discussions that both build on student ideas and highlight key mathematical ideas and relationships. Further, fostering argumentation in the class, which is important for students’ participation, is also a grand challenge. Teachers need support in these challenges. The aim of the thesis is to characterize challenges and support for mathematics teachers in orchestrating productive problem-solving whole-class discussions that focus on both mathematical connection-making and argumentation. In particular, it is investigated how Stein et al.’s (2008) model with five practices – anticipating, monitoring, selecting, sequencing and connecting student solutions – can support teachers to handle the challenges and what constitutes the limitations of the research-based and widely-used model. This thesis builds on six papers. The papers are based on three intervention studies and on one study of a mathematics teacher proficient in conducting problem-solving class discussions. Video recordings of observed whole-class discussions as well as audio-recorded teacher interviews and teacher meetings constitute the data that are analyzed. It is concluded in the thesis that the five practices model supports teachers’ preparation before the lesson by the practice of anticipating. However, making detailed anticipations, which is shown to be both challenging and important to foster argumentation in the class, is not explicitly supported by the model. Further, the practice of monitoring supports teachers in using the variety of student solutions to highlight key mathematical ideas and connections. Challenging aspects not supported by the monitoring practice are, however, how to interact with students during their exploration to actually get a variety of different solutions as a basis for argumentation. The challenge of selecting and sequencing student solutions is supported for the purpose of connection-making, but not for the purpose of argumentation. Making mathematical connections can be facilitated by the last practice of connecting, with the help of the previous practices. However, support for distinguishing between different kinds of connections is lacking, as well as support for creating an argumentative classroom culture. Since it is a great challenge to promote argumentation among students, support is needed for this throughout the model. Lastly, despite the importance and challenge of launching a problem productively, it is not supported by the model. Based on the conclusions on challenges and support, developments to the five practices model are suggested. The thesis contributes to research on the theoretical development of tools that support teachers in the challenges of orchestrating productive problem-solving whole-class discussions.
To Joakim, Ida & Isak
This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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Preface

The journey of writing a doctoral thesis in mathematics education has not always been easy, but it has certainly been both worthwhile and enjoyable. One thing that I have learned is that the more knowledgeable you become within a field, the clearer you also realize what you need to learn more about. This process never ends and I will continue the journey after my dissertation. I was lucky to have the opportunity to choose a subject for the thesis that really interests me. My firm interest in the subject has certainly contributed to making the process of writing this dissertation both worthwhile and enjoyable.

During the process of this thesis, many people have helped me. I would like to give my warmest thanks to my supervisor Andreas Ryve with whom I’ve had many discussions over the years. One of your specialties is to ask counter questions that evoke new, deeper thoughts. I remember some of my frustration in the very beginning of my doctoral studies that there weren’t always straight answers to my questions, but instead more questions to delve into. In course of time, I’ve become more and more used to that. Thanks also to my supervisor Kimmo Eriksson who has always been encouraging and responded quickly.

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I am greatly indebted to the teachers and students who let me into their classrooms. Without you, this thesis wouldn’t have been possible to write. Your courage is impressive and I do my very best to take care of your trust.

My family and friends make me think about other things in life than research, thanks for being there for me all the time. My parents and brother have always been encouraging and believed in me. You’ve always helped me when needed, as is also true for my parents-in-law. My friends, from childhood and forward, help me focus on important aspects of life. Camilla, Elin and Helén, thanks for being my friends since first grade. It feels good to ground our current lives in our history together since we were children. Åsa, thanks for deep conversations ever since we worked together as teachers. Nina, our lunches have been invaluable and inspiring over a broad range of issues, including research and reflections on life.

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Maria Larsson
1 Introduction

In many mathematics classrooms around the world, e.g. Sweden and the U.S., the teaching of procedural skills dominates classroom practice (Bergqvist, Bergqvist, Boesen, Helenius, Lithner, Palm, & Palmberg, 2009; Franke, Kazemi, & Battey, 2007; Hiebert & Grouws, 2007; Skolinspektionen, 2009). Limitations with such teaching are well-known (Hiebert & Grouws, 2007; Stein, Boaler, & Silver, 2003). Procedural skills are necessary but not sufficient for mathematical competence (Niss, 2003).

To create long-term opportunities for students to develop their mathematical competencies1 (Lithner, Bergqvist, Bergqvist, Boesen, Palm, & Palmberg, 2010; NCTM, 2000; Niss & Jensen, 2002; NRC, 2001), teachers need to ensure that they teach mathematics in ways that develop students’ procedural skills as well as the other strands of mathematical proficiency (NCTM, 2000). That is, students need opportunities to develop their conceptual understanding, their competency to justify their claims with mathematical arguments, their competency to communicate with mathematical language and different representations, their competency to make mathematical connections, and their competency to solve and pose different kinds of mathematical problems.

This calls for other ways of working in the classroom besides practicing procedures demonstrated by the teacher (Lester & Lambdin, 2004). Stein, Smith, Henningsen, and Silver (2009) accentuate results showing that students “having the opportunity to work on challenging tasks in a supportive classroom environment translated into substantial learning gains on an instrument specifically designed to measure student thinking, reasoning, problem solving, and communication” (p. 17). Research studies show that well enacted inquiry-based2 problem-solving teaching advances students’ problem-solving ability and conceptual understanding more than traditional approaches (Cobb & Jackson, 2011; Jaworski, 2006; Samuelsson, 2010) and that this is possible to accomplish without the expense of procedural skills (Boaler, 2002a; Cobb & Jackson, 2011; Samuelsson, 2010).

1 In this thesis, a distinction is made between the notions of competence and competency/competencies in line with Niss (2003). That is, a mathematical competency is a constituent of mathematical competence as a whole. In other words, mathematical competence consists of several mathematical competencies.

2 In inquiry-based teaching, questions, problems or scenarios are posed to the students, as opposed to the teacher demonstrating procedures that the students practice. The notion of inquiry-based or inquiry-oriented teaching is seen to include problem-solving teaching (Silver, 1997).
However, the demands are high on the teacher to ensure that all students develop understanding of key mathematical ideas in a problem-solving approach to mathematics (Stein et al., 2003). There is a wide consensus within the field that it is a big challenge for the teacher to highlight important mathematical ideas and relationships when orchestrating whole-class discussions based on students’ ideas to problems (Adler & Davis, 2006; Boaler & Humphreys, 2005; Bray, 2011; Franke et al., 2007; Grant, Kline, Crumbaugh, Kim & Cengiz, 2009; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010; Lester & Lambdin, 2004; Stigler & Hiebert, 1999; Sherin, 2002; Stein, Engle, Smith, & Hughes, 2008).

Despite the big challenge, Stein et al. (2008) conclude that there is still little guidance for teachers to learn how to orchestrate whole-class discussions of students’ different solutions and at the same time to pay attention to key mathematical ideas. In order to handle considerations of how to make important mathematics explicit in discussions about students’ different ideas that they share, teachers need support.

A challenge for the mathematics education research community is to find out how teachers can be supported to teach mathematics through problem-solving in fruitful ways. The field needs to develop more knowledge of detailed pedagogical practices (Boaler & Humphreys, 2005) and detailed teaching actions in classrooms that are effective for students’ learning, since the details of implementation seem to make the difference (e.g. Bieda, 2010; Brown, Pitvorec, Ditto, & Randall Kelso, 2009; Franke et al., 2007).

Besides developing teachers’ mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), supportive tools (Cobb & Jackson, 2012) have the potential to help teachers in the demanding endeavor of orchestrating productive problem-solving whole-class discussions. One such supportive tool is Stein et al.’s (2008) five practices model. Stein et al.’s model aims at supporting teachers over time to plan and conduct productive mathematical whole-class discussions that both: (1) account for students’ different mathematical ideas, and (2) advance key mathematical ideas and relationships. The focus of this thesis is how the five practices model can support teachers in handling the challenges of orchestrating whole-class discussions in a teaching mathematics through problem-solving approach.

The five practices in Stein et al.’s (2008) model include: (1) anticipating student responses, (2) monitoring student responses during the exploration phase, (3) selecting student responses for whole-class discussion, (4) sequencing student responses purposefully, and (5) connecting student responses to one another and to powerful mathematical ideas. Each practice builds on and benefits from the practices that precede it.

Connecting mathematical ideas is thus the ultimate focus of the five practices model. According to Smith and Stein (2011), it is important to encourage students to evaluate their own and other students’ mathematical ideas. However, little explicit support regarding the evaluation of ideas is provided in
their five practices model. Wood, Williams and McNeal (2006) explicitly attend to argumentation in identifying productive interaction patterns to establish an inquiry/argument classroom culture, in which students collaborate to reach understanding by evaluating one another’s solutions, asking one another questions and indicating disagreement. In line with Franke et al. (2007), I view it as promising to connect the five practices model by Stein, Smith and colleagues to inquiry/argument interaction patterns by Wood et al. (2006). This doctoral thesis contributes to the research area of developing support for teachers’ orchestration of productive problem-solving whole-class discussions that focus on both connection-making and argumentation.

1.1 Aim and research questions

The aim of this thesis is to characterize the challenges that mathematics teachers encounter and the support that they need for orchestrating productive problem-solving whole-class discussions. As described in the previous section, two important aspects of a productive problem-solving whole-class discussion are: (1) that it highlights and advances key mathematical ideas and relationships, and (2) that it is based on mathematical arguments that students provide for their different solutions to a problem. The research questions are:

1. What characterizes the challenges encountered by teachers in planning and orchestrating productive problem-solving whole-class discussions?
2. How can the model by Stein et al. (2008) support teachers to handle these challenges and what are the limitations of the model to support teachers?

In this thesis, the five practices model (hereafter: the 5P model) is considered to be constituted of what is written about the five practices in Stein et al. (2008), Smith and Stein (2011), and Smith, Hughes, Engle and Stein (2009).

1.2 How the studies and papers relate to one another

To help the reader to get an overview of the empirical studies and papers in the thesis, Figure 1 is provided. The figure illustrates how the studies and papers are related to one another. The six papers, which build upon one another, are based on three intervention studies and one study of a mathematics teacher proficient in terms of conducting problem-solving discussions. Papers I and II are based on the first intervention study. Paper III is based on the second intervention study. The teacher in the first as well as in the second intervention (both teaching grades 7-9) participated because they wanted to
learn how to orchestrate mathematical whole-class discussions based on students’ solutions to problems. Papers IV and V are based on the study of a teacher (working in grades 6-9) proficient in terms of teaching problem-solving whole-class discussions. Finally, Paper VI is based on both the study of the proficient teacher and on the third intervention study (see Figure 1) (involving all mathematics teachers in grades 6-9 at one school).

The challenges for teachers in teaching complex problems are beginning to surface in Paper I. Making explicit aspects of the problem-solving process in the whole-class discussions was found more challenging for the teacher than making explicit procedural and conceptual aspects. Paper II builds on the same intervention project as Paper I (see Figure 1) and continues to delve into the challenges with teaching complex mathematical problems. Among the conclusions from Paper II is the importance of explicitly introducing appropriate frameworks and vocabulary to create a productive discussion between researcher and teachers. The 5P model is such a framework that stands close to practice.

The 5P model is introduced to the teacher in the second intervention (see Paper III). Important aspects of the 5P model are selecting student solutions for the class discussion and deciding upon the order of the solutions. In Paper III, suggestions for selecting and sequencing are initiated. These are continued more in depth in Paper IV, in which a proficient teacher’s actions in relation to the five practices are in focus (in particular the practices of selecting and sequencing). With the purpose of elaborating on the connecting practice to take into account argumentation, Paper V focuses on interaction patterns that the proficient teacher uses to promote argumentation. Findings from Papers IV and V (regarding the practices of selecting, sequencing and connecting) feed into the suggested developments to Stein et al.’s (2008) 5P model that are made in Paper VI.

![Figure 1. Overview of empirical studies and papers.](image)

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1.3 Purpose and structure of the kappa in this thesis

Papers I-VI provide the grounds for the thesis. The kappa, which consists of chapters 1-9, gives opportunities to go deeper into the background of my research as presented in Papers I-VI. Recall that in this thesis, the process of coming to grips with challenges and support in the orchestration of whole-class discussions is central. In this, an important purpose of the kappa is to show how the three intervention projects build on one another in the process of carving out challenges and support (see section 3.3), as a ground for making suggestions for developments to the 5P model. An additional purpose of the kappa is to further elaborate on rationales for methodological choices. The kappa also gives possibilities to go beyond Paper VI in the efforts to develop support for teachers to conduct productive problem-solving whole-class discussions. The respective chapters in this kappa are organized as follows:

Chapter 2 conceptualizes productive problem-solving whole-class discussions by the introduction of key concepts and their relations. The key concepts include problem-solving, interaction, norms, connections, argumentation, authority, autonomy, instructional strategies and talk moves. Challenges and support for teachers in conducting problem-solving whole-class discussions are delineated. Also, the three different roles that Stein et al.’s (2008) five practices model plays in this thesis are made explicit. The 5P model has the roles of: (1) pedagogical tool to support teachers, (2) analytical tool to analyze teaching, and (3) object of study to be analyzed in itself and further developed.

Chapter 3 contains methodological considerations. Firstly, basic assumptions on mathematics knowledge and learning, classroom interaction and the teacher’s role in whole-class discussions are made explicit. Secondly, a framework for interpreting mathematical whole-class discussions is described. Thirdly, rationales for overarching methodological choices as well as choices made in the four empirical studies are explained. Lastly, ethical considerations are elaborated on and trustworthiness is discussed.

Chapter 4 provides summaries of the six papers in this thesis. In the summaries, the results have the most prominent position.

Chapter 5 draws conclusions based on the results of the six papers in order to answer the research questions. The challenges for teachers as well as support from and limitations of the 5P model that are found in the empirical studies are organized into the plan, launch, explore and discuss-and-summarize phases.

Chapter 6 discusses how Stein et al.’s (2008) model can be developed to face up to teachers’ challenges and support teachers, building on the answers to the research questions as well as on suggestions on developments to the model from Paper VI. More precisely, it is discussed how the 5P model can be developed to face up to the two main challenges in the actual orchestration of a whole-class discussion: (1) to create an argumentative classroom climate, and (2) to make important mathematical connections visible. By combining a
focus on argumentation with a focus on connection-making, a model for third generation practice emerges. The chapter closes with contributions to research and practice, a critical reflection on the researcher’s role and the methodology of the thesis, as well as suggestions for future research.
2 Conceptualizing problem-solving whole-class discussions

In this chapter, a conceptualization is made of productive problem-solving whole-class discussions. After a short delineation of how problem solving is defined in this thesis, the creation of a problem-solving classroom is elaborated on. In this elaboration, aspects of teaching mathematics through problem solving are first outlined. Then, the relationship between whole-class interaction and classroom norms is described, as well as the roles of argumentation, authority and autonomy in whole-class discussions. Finally, an outline of productive instructional strategies and moves leads on to an elaboration of challenges and support for teachers in orchestrating problem-solving whole-class discussions.

2.1 Problem solving

As Schoenfeld (1992) states, the notions of ‘problem’ and ‘problem solving’ have been used with many different meanings in literature. In this thesis, a mathematical problem is seen in line with Schoenfeld’s (1985) view:

> Being a ‘problem’ is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person. […] If one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem. (p. 74).

This definition implies that whether a task is a problem or not is related to the problem solver; what is a problem for one person might be a routine task for another person and what is a problem for a person today might be a routine task tomorrow. In this thesis, problem solving is seen as “engaging in a task for which the solution method is not known in advance” (NCTM, 2000, p. 51). This resonates with Schoenfeld’s (1985) above definition of a problem as a task for which the solver does not have a readily accessible solution schema, or procedure.

Problem solving is related to other mathematical competencies (Lithner et al., 2010; NCTM, 2000; Niss & Jensen, 2002; NRC, 2001). Firstly, Lester and Lambdin (2004) view problem solving ability and conceptual understanding
as the overarching goals of mathematics teaching. Their relationship is symbiotic according to Lester and Lambdin; conceptual understanding enhances problem solving ability at the same time as problem solving develops understanding.

Crucial for the development of problem solving ability and conceptual understanding is mathematical representation and connection-making. NRC (2001) states that “How learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it in problem solving” (p. 117). Conceptual understanding can be operationalized as representation and connection practices (Lithner et al., 2010). The view that understanding something means seeing how it is connected to other things we already know (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray et al., 1997) resonates with this. Making mathematical connections in problem-solving whole-class discussions is central in this thesis.

Continually interacting with both problem solving and conceptual understanding is procedural fluency (NRC, 2001). As procedural fluency gets better, problem solving is facilitated which also leads to deeper conceptual understanding. At the same time, as conceptual understanding deepens, procedures can be recalled and used more flexibly to solve problems (NRC, 2001). In teaching mathematics through problem solving, concepts and procedures are embedded into the problems (see section 2.2.2).

Integral to a problem-solving approach to mathematics are reasoning and communication competencies. Reasoning and argumentation are elaborated on in section 2.2.4 and communication in section 3.1.2. Next, important aspects of the creation of a problem-solving classroom are elaborated.

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3 As Boaler emphasize in the foreword in Brodie (2010), making connections between mathematical ideas by sense making, reasoning and discussions can counteract students’ belief “that mathematics is a set of isolated facts and methods that need to be remembered” (p. v). By contrast, merely practicing procedures demonstrated by the teacher may lead to this belief (Lester & Lambdin, 2004).

4 NRC (2001) defines procedural fluency as “knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently” (p. 121).

5 In resonance with NRC (2001), Hatano and Inagaki (1986) view procedural and conceptual knowledge as interacting; performing procedures with some variation constructs conceptual knowledge and this conceptual knowledge enables flexible and adaptive use of procedures. The notion of adaptive expertise (Hatano & Inagaki, 1986) is about not only being able to perform procedures efficiently, but also understanding the meaning of the procedures. Performing a procedure with understanding includes being able to explain why it works.
2.2 Creation of a problem-solving classroom

To promote students’ conceptual understanding, Hiebert and Grouws (2007) identify two key features of teaching⁶ that are effective across many contexts: (1) letting students struggle with important mathematics⁷ (as opposed to students merely practicing demonstrated procedures), and (2) explicitly attending to concepts/conceptual underpinnings by making connections among mathematical ideas and representations⁸. Teaching actions that Hiebert and Grouws (2007) give as examples that take the two features into account is “posing problems that require making connections and then working out these problems in ways that make the connections visible for students” (p. 391). Hiebert and Grouws stress that with struggle they mean students’ endeavor to understand something that is not evident. They say: “The struggle we have in mind comes from solving problems that are within reach and grappling with key mathematical ideas that are comprehensible but not yet well formed” (p. 387). However, classroom practices that enable students to engage in such struggle are neither well researched (Hiebert & Grouws, 2007) nor common in Swedish practice (Boesen, Helenius, Bergqvist, Bergqvist, Lithner, Palm, & Palmberg, 2014). A potentially productive way to promote struggle in Hiebert and Grouws’s (2007) terms is teaching mathematics through problem solving, which will be elaborated on after a brief overview of the Swedish context.

2.2.1 The Swedish mathematics classroom

For many years, the dominating teaching approach in Swedish mathematics classrooms has been students individually practicing teacher-demonstrated tasks in their textbooks (Bergqvist et al., 2009; Skolverket, 2003, Skolinspektionen, 2009). This teaching that focuses on procedural skills gives little opportunities for students to develop other mathematical competencies than procedural fluency, such as problem solving and reasoning competencies (Skolinspektionen, 2009).

In current Swedish steering documents (Skolverket, 2011a), mathematical competencies are explicitly accentuated. Creating opportunities for students to develop their ability to pose and solve problems and also evaluate chosen strategies and methods is emphasized. The view in the steering documents of what constitutes a mathematical problem resonates with the view held in this

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⁶ Closely related to Hiebert and Grouws’s (2007) two key features are Stigler and Hiebert’s (1999) two features common for high-achieving countries (although the mathematics teaching varies considerably in other respects among the successful countries), namely they: (1) let students do important mathematical work, and (2) focus on important mathematical relationships.

⁷ Cf. Polya’s (1945/57) emphasis that a natural aspect of doing mathematics is to struggle with central mathematical ideas.

⁸ Cf. Stein et al.’s (2008) focus on connection-making in their five practices model.
thesis. Further, creating possibilities for students to develop their conceptual understanding, their procedural fluency, their reasoning ability and communication ability is stressed in current steering documents. In the previous steering documents, the competencies are not as explicitly emphasized as in the current ones (Skolverket, 2003; 2011b).

In line with the emphasis on competencies in current steering documents, ongoing professional development efforts strive for a mathematical teaching that creates opportunities for students to develop all their mathematical competencies. The Boost for Mathematics (in Swedish: Matematiklyftet) is a national professional development program that builds on collegial discussions in which the mathematics teachers collaborate in the planning of and reflection upon mathematics lessons (Boesen, Helenius, & Johansson, 2015). Central ingredients in the Boost for Mathematics are mathematical competencies (among them problem solving), classroom interaction and classroom norms (Skolverket, 2012).

In this thesis, problem solving plays a key role, both as a competency to be developed in itself and as a means to develop other mathematical competencies.

2.2.2 Teaching mathematics through problem solving

In teaching mathematics through problem solving, the idea is that students learn while they try to solve problems in their own ways (Cai, 2003). They do this with the help of their previous knowledge and they justify their ideas by mathematical arguments. Through solving problems, students learn concepts and procedures, and through the learning of concepts and procedures, they develop their problem-solving competency (Cai & Lester, 2010). By teaching mathematics through problem solving, students hence get opportunities to develop their problem-solving competency as well as other mathematical competencies (Wyndhamn, Riesbeck, & Schoultz, 2000). Concepts and procedures to be learned are embedded into the mathematical problems (Cai & Lester, 2010) as opposed to that concepts and procedures are taught separately to problem solving. Two contrasting approaches to teaching mathematics through problem solving are denoted teaching mathematics for or about problem solving (Wyndhamn et al., 2000). In teaching mathematics for problem solving, the view is that if students first master the necessary procedural skills, then they will also be able to solve problems (Wyndhamn et al., 2000). Concepts and procedures are hence taught separately prior to students solving

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9 A mathematical problem is seen as situations or tasks which the students do not immediately know how to solve (Skolverket, 2011b). This view resonates with the view of problem solving in this thesis (see section 2.1). A routine task on the other hand is a task for which the student knows a solution method (Skolverket, 2011b). This view of a mathematical problem implies that a problem can be seen as a relation between the task and the student (Skolverket, 2011b) as was accentuated in section 2.1.
problems. In teaching mathematics about problem solving, a separate focus is put on how to go about to solve problems, e.g. by choosing rules of arithmetic (Wyndhamn et al., 2000) or by using general strategies such as “draw a picture” or “guess and check” (Cai & Lester, 2010). Separating problem solving from the learning of concepts and procedures has shown not to be beneficial by a large amount of research (Cai & Lester, 2010). Instead, considering problem solving as a driving force for the learning and understanding of mathematics, which is the view held in teaching mathematics through problem solving (Cai & Lester, 2010; Wyndhamn et al., 2000), is endorsed by research on problem solving teaching (e.g. Cai & Lester, 2010; Lester & Lambdin, 2004; Stein et al., 2003).

As Cai and Lester (2010) point out, teaching mathematics through problem solving is a natural setting for discussing students’ different ideas and solutions. Typically, a problem-solving lesson consists of the following lesson phases: introduction (or launching) of the problem to the class, students’ exploration of the problem individually and/or in small groups, and whole-class discussion of students’ different solutions (Lampert, 2001; Shimizu, 1999; Stein et al., 2008).

This thesis focuses on orchestrating whole-class discussions in a teaching of mathematics through problem solving approach. The concluding whole-class discussion serves to make mathematical representations, strategies and connections visible in students’ different ways of thinking (cf. Hiebert & Grouws, 2007) and is important for advancing students’ thinking (Fraivillig, Murphy, & Fuson, 1999). Based on the background on teaching mathematics through problem solving provided here, important aspects of whole-class interaction are now elaborated on, including classroom norms, argumentation, and talk moves.

2.2.3 Classroom norms and whole-class interaction

Establishing, maintaining and negotiating productive norms for classroom interactions with students is necessary for teaching mathematics through problem solving productively, in particular for conducting problem-solving whole-class discussions (Franke et al., 2007). For students, participating in collaborative problem-solving discussions includes explaining and justifying one’s own mathematical thinking in detail, listening to other’s well-detailed ideas, reasoning about different solution strategies and representations, making important mathematical connections and generalizing. In this work, classroom norms are crucial.

Classroom norms can be seen as the collective expectations and beliefs that have been negotiated in the classroom (Weber, Radu, Mueller, Powell, & Maher, 2010). The teacher needs to explicitly negotiate norms with the students (Franke et al., 2007; Jackson & Cobb, 2010), at the same time assuring that the norms are in line with the teaching goals. It is clear that the quality of
whole-class interactions rests upon productive norms (Boaler & Humphreys, 2005), but at the same time whole-class interactions establish classroom norms. That is, “norms are interactively constituted” (Yackel & Cobb, 1996, p. 458), i.e. they are interactively established. This means that whole-class interaction and classroom norms mutually affect each other. Classroom norms develop in whole-class interaction, but whole-class interaction is at the same time guided and constrained by prevailing classroom norms (Franke et al., 2007). Norms can be defined as the “interlocking networks of obligations and expectations that exist for both the teacher and students [that] influence the regularities by which student and teacher interact and create opportunities for communication to occur between the participants” (Wood, 1998, p. 170). Norms are also shaped by interactional regularities (Franke et al., 2007). Classroom norms hence enable and constrain interactional patterns at the same time as interactional patterns establish classroom norms.

Classroom norms are negotiated by the teacher and the students in their ongoing interactions (Franke, et al., 2007) and they “become the taken-for-granted ways of interacting that constitute the culture of the classroom” (Wood, 1998, p. 170). Classroom culture is thus constituted by the aggregated, agreed-upon ways of interacting that have been negotiated to be the prevailing norms in the classroom.

This thesis follows Yackel and Cobb (1996) in distinguishing between two types of classroom norms: social and sociomathematical norms. Social norms are not subject-specific. In inquiry-oriented, problem-solving classrooms, social norms encompass the responsibility to explain and justify one’s solutions, to try to understand other students’ reasoning, to ask questions if you do not understand, and to challenge arguments that you do not agree with10 (Gravemeijer, 2014). When these social norms become agreed-upon ways of interacting in a classroom, the classroom is starting to approach an inquiry/argument culture in which students collaborate to reach consensus (cf. Wood et al., 2006). However, sociomathematical norms are also needed to reach an inquiry/argument classroom culture, in order to take into account the mathematical quality of explanation, justification and argumentation. Examples of sociomathematical norms are what constitutes a different, efficient or sophisticated solution to a mathematical problem, and what constitutes an acceptable mathematical explanation (Cobb, Stephan, McClain, & Gravemeijer, 2001; Yackel & Cobb, 1996). For students to be able to understand one another’s explanations during the class discussion (cf. the goal of accessibility stated by Stein et al., 2008), “the norms or standards for what counts as an acceptable mathematical explanation that are established in the classroom appear to be crucial” (Jackson & Cobb, 2010, p. 24). Another example is the sociomathematical norm of how mathematical correctness is established (Harel & Rabin,

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10 By contrast, students learn to follow procedures and not trust their own reasoning in traditional classrooms (Gravemeijer, 2014).
The two sociomathematical norms of (1) making details explicit in explanations and questions (Franke et al., 2007), and (2) deciding upon correctness based on valid mathematical arguments (Harel & Rabin, 2010), serve as guiding principles for carrying out the intervention projects of this thesis.

Summing up, classroom norms guide and constrain the interaction among students and teacher in the classroom at the same time as the interaction establishes classroom norms. With changing classroom norms and interaction patterns, new roles are emerging for teachers and students in European and U.S. classrooms (Singer & Moscovici, 2008):

- the **learner** as an *autonomous thinker* and explorer who expresses his/her own point of view, asks questions for understanding, builds arguments, exchanges ideas and cooperates with others in problem solving – rather than a *passive recipient* of information that reproduces listened/written ideas and work in isolation;

- the **teacher** as a *facilitator* of learning, a coach as well as a partner who helps the student to understand and explain – rather than a ‘*knowledgeable authority*’ who gives lectures and imposes standard points of view;

- classroom **learning** that aims at *developing competences* and is based on *collaboration* – instead of *developing factual knowledge* focused on only validated examples and based on *competition* in order to establish *hierarchies* among students. (p. 1613).

Central notions in the above quote are argumentation, authority and autonomy, notions that will be explicated now.

### 2.2.4 Argumentation, authority and autonomy

Building arguments is central for students in a problem-solving classroom. By releasing responsibility and authority to students to justify ideas by valid mathematical arguments, students’ intellectual autonomy can be developed (Yackel & Cobb, 1996).

This thesis distinguishes between an argument and a reason in line with Lithner (personal communication, 7th November, 2014). Argument is seen as a narrower concept than reason in that a reason may be an argument, but it may also be affective, intuitive, or grounded on guesses. Reasoning\(^{11}\) is defined as “the line of thought that is adopted to produce assertions and reach conclusions when solving tasks” (Bergqvist & Lithner, 2012, p. 253) and contains explicit or implicit reasons (Lithner, personal communication, 7th November, 2014).

\(^{11}\) As Bergqvist and Lithner (2012) underline, “Reasoning can be seen as thinking processes, as the product of these processes, or as both” (p. 253). The same is true for argumentation.
Argumentation, defined as using arguments or as justification by mathematical arguments, “is the substantiation of the reasoning that aims at convincing oneself or someone else that the reasoning is appropriate” (Palm, Boesen, & Lithner, 2011). When the notion of argumentation is used in this thesis, it is referred to verbalized (oral and written) argumentation only, since nothing can be said about students’ inner thoughts based on the data (cf. Bergqvist & Lithner, 2012). In the context of this thesis, students’ oral argumentation is visible in the mathematical whole-class discussions, and students’ written argumentation is visible through students’ written solutions to the problems.

In a problem-solving classroom, mathematical argumentation, or logical proof as Lampert (2001) puts it, “must replace the authority of the teacher deciding what is right and what is wrong” (p. 26). That is, claims have to be justified by valid mathematical arguments. In Toulmin’s (1958/2003) pattern of an argument, a claim, or conclusion, is supported by data, or facts. A warrant is a general statement that allows the step being taken from data to claim (Toulmin, 1958/2003). In other words, a warrant is an explanation of “why one should deduce the claim being made from the data being presented” (Weber, Maher, Powell, & Lee, 2008, p. 248). An argument hence consists of the three necessary parts claim, data and warrant12. Challenging one another’s claims in whole-class discussions can contribute to making students more explicit about the validity of the warrant that they use (Weber et al., 2008).

The sociomathematical norm of how mathematical correctness is established (Harel & Rabin, 2010) is crucial for argumentation in the whole-class setting. Nonauthoritative argumentation (Harel & Rabin, 2010), i.e. that correctness is established by making valid mathematical arguments rather than asking the teacher to decide upon correctness, can promote students’ intellectual autonomy. Instead of relying on the teacher or textbook as authority, students who possess intellectual autonomy make their own mathematical judgments when they participate in classroom practices. The process of jointly negotiating sociomathematical norms can hence foster development of students’ intellectual autonomy (Franke et al., 2007).

Development of students’ intellectual autonomy (Yackel & Cobb, 1996) serves as one guiding principle in carrying out the three intervention projects of this thesis. To develop students’ intellectual autonomy, an important aspect is that the teacher deliberately releases authority to students for them to provide arguments for and against their own and one another’s mathematical ideas. Wood (2002) states that an underlying assumption is that “increased responsibility for student thinking is an increase in student autonomy in learning” (p. 65). Since Wood et al. (2006) have found that an inquiry/argument classroom culture means higher responsibility for student thinking, it can be

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12 In addition, a support for the warrant, a backing, is needed when the listeners question the validity of the warrant (Toulmin, 1958/2003; Weber at al., 2008).
inferred that an argumentative classroom culture is essential to achieve higher student autonomy. Since development of student autonomy is viewed as important in this thesis, the establishment of an inquiry/argument classroom culture (Wood et al., 2006) is consequently seen as critical in the orchestration of mathematical whole-class discussions.

There are many challenges in teachers’ work of planning and conducting productive whole-class discussions that will be elaborated on in section 2.3.1 below. A description of how teachers can get support related to these challenges is at focus in section 2.3.2. Before we go into challenges and support for teachers, research on instructional strategies and interactional moves in whole-class discussions are delineated.

2.2.5 Instructional strategies and interactional whole-class moves

An important question is how teachers can interact with students in problem-solving whole-class discussions in ways that stimulate and advance students’ thinking while – simultaneously – ensure students’ authority over their own ideas and promote students’ autonomy as thinkers (Fraivillig et al., 1999; Hiebert et al., 1997).

Addressing this issue, Fraivillig et al. (1999) articulate effective instructional strategies to advance students’ thinking in an inquiry-based, problem-solving approach to mathematics. They divide instructional strategies into three components: eliciting, supporting and extending. Effective instructional strategies for eliciting students’ thinking include eliciting multiple solution methods for one problem, including erroneous ones, and deciding which methods that should be discussed, or which students that need to speak, in the whole-class setting. Supporting students’ conceptual understanding includes asking classmates to explain their peers’ solutions, letting the whole class help one another in clarifying their methods, and providing immediate teacher-led replays of students’ methods. Extending students’ thinking includes encouraging students to draw generalizations and to discuss interrelationships (i.e. connections) between concepts, and promoting use of more efficient solution methods for all students.

While the eliciting and supporting components has to do with already familiar solution methods, the extending component has to do with further development and challenging of student thinking (Cengiz, Kline, & Grant, 2011). Therefore, instructional strategies related to extending students’ thinking have a prominent role in order to advance students’ thinking.

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13 In Polya’s (1945/57) words: if a teacher “challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them solve their problems with stimulating questions, he (sic) may give them a taste for, and some means for, independent thinking” (p. v).
The Extending student thinking framework by Cengiz and colleagues (Cengiz, 2007; Cengiz et al., 2011) builds on the extending component in Fraivillig et al.’s (1999) framework outlined above. To the instructional strategies that are seen to extend students’ thinking in Fraivillig et al.’s framework, Cengiz and colleagues add instructional strategies related to justification and argumentation, namely: encouraging students to consider the reasonableness/validity of a claim, encouraging students to offer a justification for their solution/claims, and encouraging students to engage with one another’s justifications.

Taken together, instructional strategies involved in extending students’ ideas include encouraging students to justify claims by mathematical arguments, to make connections, and to draw generalizations. These three aspects (argumentation, connection-making and generalization) are visible in Smith, Bill and Hughes’ (2008) lesson planning protocol (included in Smith and Stein, 2011). One key point is which specific questions the teacher will ask in whole-class discussion so that students will:

1. make sense of the mathematical ideas that you want them to learn?
2. expand on, debate, and question the solutions being shared?
3. make connections among the different strategies that are presented?
4. look for patterns?
5. begin to form generalizations? (Smith et al., 2008, p. 134)

The above goals of teacher questions resonate with the view held in this thesis: that the core of conducting whole-class discussions is to advance and extend students’ thinking through a focus on students’ argumentation, connection-making and generalization. Taking into account argumentation besides connection-making in Stein et al.’s (2008) model could provide an important, additional support for teachers to conduct productive mathematical whole-class discussions.

So, how can instructional strategies employed to elicit, support and extend student’s thinking be operationalized by a teacher’s interactional moves in the classroom? Routines for interactional moves14 have great potential to support teachers’ actions in the classroom (Cobb & Jackson, 2011; 2012; Franke et al., 2007). Below, talk moves will be related to the instructional strategies that they operationalize.

A talk move that operationalizes the instructional strategy of eliciting multiple solution methods for one problem is asking “What do you think?” and

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14 Examples of frameworks that focus on teachers’ interactional moves in whole-class discussions include Chapin, O’Connor and Anderson’s (2003; 2009) five productive talk moves, Boaler and Humphreys’ (2005) productive teacher moves, Boaler and Brodie’s (2004) different types of teacher questions, Mason’s (2000) ways of asking questions, Brodie’s (2010) categories for teacher follow-up responses (elicit, press, insert, maintain and confirm), and Wood et al.’s (2006) different kinds of interaction patterns.
following up with “Why?” or “Can you explain that?” after the response. Finally, the teacher can ask “What do other people think?” to the whole class (Sherin, 2002).

Talk moves that operationalize instructional strategies employed to support students’ thinking are for instance revoicing and restating (Chapin, O’Connor, & Anderson, 2003; 2009). Revoicing operationalizes the teaching strategy of providing teacher-led replays (Fraivillig et al., 1999). Revoicing means that the teacher restates an unclear statement in the teacher’s own words and then asks the originator if the teacher’s revoicing is correct. It is important that the teacher does not change the underlying idea (Smith & Stein, 2011). Restating operationalizes the teaching strategy of asking students to explain their peer’s solutions (Fraivillig et al., 1999). Restating means that the teacher asks other students to repeat or rephrase a student’s reasoning and then checks back with the student if it is correctly stated. Revoicing is also included in Lampert’s (2001) productive talk moves. In another talk move, adding on, the teacher prompts others to contribute. This move operationalizes the supporting teaching strategy of letting the whole class help clarifying the methods (Fraivillig et al., 1999).

A talk move that operationalizes instructional strategies employed to extend students’ thinking is for instance the reasoning move. In this, the teacher asks students to apply their own reasoning to another student’s contribution, by asking “Do you agree or disagree?” and following up by asking “Why?” (Chapin et al., 2003; 2009). This move operationalizes strategies related to justification and argumentation by Cengiz and colleagues. A similar talk move is to ask “So, is what you’re saying the same as Tina? What do you think?” (Sherin, 2002, p. 219) after which students often state whether they agree or disagree. In line with Chapin et al. (2003; 2009) and Sherin (2002), Lampert (2001) also highlights the talk move of asking why something makes sense and if other students agree or disagree with it.

Using wait time is a talk move which is applicable over a broad range of instructional strategies. In this, the teacher gives students time to think mathematically, both before calling on a student and after having called on a particular student to answer. The move of using wait time establishes the norm that mathematical thinking is important and that deep thinking takes time.

The move of asking students if they agree or disagree and why (Chapin et al., 2009; Lampert, 2001; Sherin, 2002) is about having students to consider the reasonablenessvalidity of a claim and to engage with one another’s justifications (Cengiz et al., 2011). This move is consequently about extending students’ thinking. The move is related to Wood and colleagues’ (Wood, 2002; Wood et al., 2006) inquiry/argument classroom culture and is of particular interest for this thesis. In an inquiry/argument culture, students ask one question...
another questions for clarification as well as indicate disagreement and challenge one another’s ideas to be justified (Wood et al., 2006). The teacher regularly checks students for consensus (i.e. do they agree or not?) and students reason and collaborate together to reach shared meaning; they build consensus.

Students who indicate disagreement and challenge one another’s thinking to be justified – the hallmark of an inquiry/argument classroom culture (Wood, 2002) – possess a certain degree of intellectual autonomy. Development of students’ intellectual autonomy is seen as central in this thesis (see section 2.2.4). The teacher’s role is to advance and extend students’ mathematical thinking in inquiry-based, problem-solving classrooms without undermining their intellectual autonomy, as Fraivillig et al. (1999) stress. This is very challenging for teachers.

2.2.6 Guiding principles derived from literature

As a basis for going into challenges and support for teachers, the guiding principles for this thesis, derived from literature, are summarized in Table 1. In the table, each guiding principle is operationalized into teaching actions.

Table 1. Guiding principles and operationalization of principles.

<table>
<thead>
<tr>
<th>Guiding principle</th>
<th>Operationalization of principle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Development of students’ mathematical competencies on a broad front (e.g. NRC, 2001)</td>
<td>Teach mathematics through problem solving with class discussions as central feature</td>
</tr>
<tr>
<td>Struggle with important mathematics (Hiebert &amp; Grouws, 2007)</td>
<td>Encourage students to solve problems and grapple with key mathematical ideas</td>
</tr>
<tr>
<td>Explicitness of conceptual underpinnings (Hiebert &amp; Grouws, 2007)</td>
<td>Conduct whole-class discussions that highlight key mathematical ideas and connections based on students’ different ideas</td>
</tr>
<tr>
<td>Accessibility for students (Stein et al., 2008)</td>
<td>Establish norms of making details explicit in explanations</td>
</tr>
<tr>
<td>Nonauthoritative argumentation (Harel &amp; Rabin, 2010)</td>
<td>Establish norms of deciding upon correctness based on valid mathematical arguments</td>
</tr>
<tr>
<td>Development of intellectual autonomy (Yackel &amp; Cobb, 1996)</td>
<td>Release authority for students to provide mathematical arguments</td>
</tr>
<tr>
<td>Advancement of students’ thinking (Fraivillig et al., 1999)</td>
<td>Elicit, support and extend students’ thinking by using interactional moves</td>
</tr>
</tbody>
</table>

The above principles guide the intervention projects in this thesis. They are also used to distinguish analytical concepts that guide the suggested developments of the 5P model.
2.3 Challenges and support for teachers

In this section, challenges and support for teachers to manage the complexity of whole-class discussions are delineated from literature. The challenges and support are based on the conceptualization of problem-solving whole-class discussions made so far in this chapter.

2.3.1 Challenges for teachers

The act of managing a productive class discussion is complicated (e.g. Boaler & Humphreys, 2005). In whole-class discussions, teachers must both see to the needs of all individual students and to the class as a whole (Lampert, 2001). One substantial challenge is that the teacher has less control over the lesson (Stein et al., 2008) when it builds to a high extent on students’ thinking and reasoning instead of well-rehearsed procedures. By anticipating student solutions before the lesson (Shimizu, 1999, Stein et al., 2008, Stigler & Hiebert, 1999), the teacher can regain control. However, novice teachers generally do not anticipate student solutions, in particular common errors and misconceptions, in such detail as expert teachers (Bray, 2011).

When teachers elicit and use students’ thinking as the center of the lesson, it requires extensive knowledge about the students as well as about common solution strategies, both correct and incorrect solution strategies (Fraivillig et al., 1999). Using students’ thinking as the center of the lesson means encouraging students to connect elicited student responses to key mathematical ideas, to generalize from them, and to justify them by mathematical arguments. In short, it means to advance and extend their thinking.

As concluded in section 2.2.5, the teacher’s role is to advance and extend students’ mathematical thinking without undermining their intellectual autonomy. Related to this view of the teacher’s role is the statement from Hiebert et al. (1997, p. 29-30) of what they view as the largest teaching dilemma:

We begin our discussion with the biggest problem for teachers: how to assist students in experiencing and acquiring mathematically powerful ideas but refrain from assisting so much that students abandon their own sense-making skills in favor of following the teacher’s directions (Ball, 1993b; Lampert, 1991; Wheat, 1941) To put it another way, how do teachers handle the tension between supporting the initiative and problem-solving abilities of students and, at the same time, promoting the construction of mathematically important concepts and skills? Or, using Ball’s words, how do teachers develop “a practice that respects the integrity both of mathematics as a discipline and of children as mathematical thinkers” (376)?

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16 Eliciting and extending students’ thinking seems to be a greater challenge for teachers than supporting students’ thinking (Fraivillig et al., 1999).
This dilemma is about balancing content and participation (cf. Ball, 1993; Emanuelsson & Sahlström, 2008; Sherin, 2002). The pivotal tension between content and participation is acknowledged by many researchers using similar notions of this balance act. Sherin (2002) denotes it a balance between content and process of mathematical discourse. Process “refers to the way that the teacher and students participate in class discussions. This involves how questions and comments are elicited and offered, and through what means the class comes to consensus” (p. 206). Participation is hence closely related to Wood et al.’s (2006) inquiry/argument interaction patterns, such as Inquiry, Argument, Check for consensus and Build consensus. Engle and Conant (2002) frame it as a balance act between accountability to the discipline of mathematics (i.e. content) and student authority over their own ideas (i.e. participation).

Stein et al. (2008) situate their 5P model in Engle and Conant’s (2002) theoretical frame. In Stein et al.’s (2008) words: “At the heart of the challenge associated with student-centered practice is the need to strike an appropriate balance between giving students authority over mathematical work and ensuring that this work is held accountable to the discipline” (p. 332). Engle and Conant’s (2002) framing is central in the conceptualization of productive whole-class discussions in this thesis. To ensure accountability to the discipline of mathematics, the teacher has to make sure that key mathematical ideas are represented in the whole-class discussion. At the same time, to ensure student authority the teacher has to make sure that the discussion is based on students’ ideas. Returning to Hiebert et al.’s (1997) dilemma, they view it as important to realize that the tension will always be there and if teachers are open to it, they “can remain sensitive to both the subject and the students” (p. 30).

In order to teach so that key mathematical ideas are represented in ways that harmonizes with both mathematical content and student thinking (Ball, 1993), asking good questions is crucial. This is also challenging for the teacher. In order to extend students’ thinking, the questions need to connect students’ ideas with key mathematical ideas. Smith and Stein (2011) write:

> Connecting may in fact be the most challenging of all the five practices because it calls on the teacher to craft questions that will make the mathematics visible and understandable. Hence, the questions must go beyond merely clarifying and probing what individual students did and how. Instead, they must focus on mathematical meaning and relationships and make links between mathematical ideas and representations. (p. 49-50).

Smith and Stein (2011) emphasize that questions have to start with what students currently know and connect students’ present thinking (including errors and misconceptions) with the mathematics to be learned. In other words, questions must begin with eliciting and supporting current student thinking and use it for extending students’ thinking further (cf. Cengiz et al., 2011) towards mathematical ideas that are key to the discipline.
A particular challenge is to handle and incorporate errors and misconceptions within the classroom practice. The role of errors and misconceptions in students’ thinking is crucial for whole-class discussions. Bray (2011) accentuate that a grand challenge for teachers is “mobilizing a community of learners to address errors” (p. 33) in whole-class discussions. Bray connects this to the challenge of establishing an inquiry/argument classroom culture in Wood et al.’s (2006) terms. Silver, Ghousseini, Gosen, Charalambous and Font Strawhun (2005) point out teachers’ concerns that displaying errors in whole-class discussions might confuse students.

By purposefully selecting and sequencing student solutions – including erroneous ones – for whole-class discussion, the teacher can ensure that the mathematics that is related to the goals of the instruction is highlighted. How to select and sequence both correct and incorrect student solutions for whole-class discussion is seen as a challenging issue for the teachers in Silver et al.’s (2005) study as well as for the teachers in the third intervention project (see Paper VI). The teachers in Silver et al. (2005) express uncertainty about rationales for sequencing solutions; going from simpler to more complicated or from common to more unusual solutions.

Purposeful selection and sequencing of solutions requires a certain amount of variety to choose among. Limited variety among student solutions is perceived as challenging to handle by the teachers in Silver et al.’s (2005) study. Telling students too much can limit the variety of solutions available for selection. Telling students how to do difficult aspects of the problem is a common factor associated with decline of high-level cognitive demands (Stein, Grover & Henningsen, 1996; Stein et al., 2009).

Research findings indicate (Stein et al., 2003) that the more demanding problem, the more challenging it is to implement the problem the way it was intended with maintained cognitive level. A challenge with launching a problem is to “strike a balance between providing sufficient guidance for students to enter the task but not so much that all the challenge has been removed” (Smith & Stein, 2011, p. 81). This challenge is emphasized by the teachers in the third intervention project (see Paper VI). A major challenge when students work on the problem is to refrain from telling students how to do the difficult aspects of the problem when they struggle (Stein et al., 1996). Sometimes students also press the teacher to specify procedures, which can be challenging to keep from (Stein et al., 1996). The critical question is how teachers can be supported in handling these challenges.

2.3.2 Tools to support teachers

Introducing and using carefully designed tools is one type of support for learning in teacher professional development (Cobb & Jackson, 2012). Examples of tools include textbooks, curriculum guides, tasks, student written work, and video recordings from classrooms. Of particular interest for this thesis, Cobb
and Jackson (2012) refer to several studies indicating that using such tools seems valuable to support teachers’ development of instruction that builds on students’ reasoning in order to attain “a significant disciplinary agenda” (p. 495).

Stein et al.’s (2008) pedagogical model can be introduced and used as one such tool to support teachers (see further in section 2.3.3). Like the Japanese teaching approach (Stigler & Hiebert, 1999), the five practices model makes a significant contribution to the research area of developing support for teachers to gradually gain more proficiency in the challenging handicraft of orchestrating class discussions that both build on students’ ideas and are responsive to the discipline of mathematics.

Likewise, Sherin’s (2002) model focuses on the teacher’s role to balance content and participation in whole-class discussions. The model can support teachers by offering “an account of how process and content goals are coordinated through a cycle of filtering of students’ ideas” (p. 228). The model consists of three components that make explicit teachers’ work in using student ideas to highlight key mathematical ideas in whole-class discussions: (1) idea generation, (2) comparison and evaluation, and (3) filtering. These teaching processes are used to advance the mathematical content. Firstly, multiple ideas are elicited from the students. Secondly, these ideas are compared and evaluated. Thirdly, the teacher filters out the key mathematical ideas that relate to the lesson goals. These ideas are discussed more in detail. After the discussion of the filtered ideas, many ideas are again elicited from the students and the cycle continues. The filtering to balance content and participation is hence made in the midst of the whole-class discussion according to Sherin’s (2002) model, before opening up again for new student ideas. Inherent in the 5P model is that the filtering is made before the whole-class discussion by selecting and sequencing student solutions for discussion. The filtering approach in Sherin’s (2002) model requires the teacher to take many decisions during the whole-class discussion. However, when applying a problem-solving approach, it is extremely difficult for teachers to make important decisions on-the-fly about how to use student work to promote key mathematical ideas. The challenge of building on students’ ideas (see section 2.3.1) and still be in control over the content can be reduced by careful planning. That is why teachers need tools for planning as much as possible in advance.

A key to regain control by planning is anticipating likely student solutions (Shimizu, 1999, Stein et al., 2008, Stigler & Hiebert, 1999; Silver et al., 2005) – both correct and incorrect ones. Stigler and Hiebert (1999) describe the Japanese script for teaching (lesson study), in which careful lesson planning is a key to reduce the need for improvisation and on-the-fly decisions, by anticipating student responses and “planning the class discussion to build on students’ responses and highlight the major points” (p. 158). Stein et al.’s (2008) model emphasizes those aspects that the teacher can plan for in advance. The model makes explicit the role of anticipation of student solutions in order to
plan which solutions to build the whole-class discussion on, and in which order to discuss them, to accentuate key mathematical ideas and connections. Silver et al. (2005) put forward that “[…] by attending carefully to the mathematical ideas embedded in students’ responses, a teacher could influence which ideas are likely to be discussed in class, and in what order, thereby improving their chances of meeting the mathematical goals for the lesson” (p. 297-298). These ideas are similar to the ideas of the 5P model of purposefully selecting and sequencing student responses. However, in Stein et al.’s model, the teacher’s anticipation of student ideas is a key to be able to plan for a selection and sequencing of student solutions that contributes to achieving the lesson goals. Below, we will go further into Stein et al.’s model as support for teachers.

2.3.3 Stein et al.’s five practices model as support for teachers

Stein et al.’s 5P model is designed to make it more manageable for (novice and experienced) teachers to productively orchestrate whole-class discussions. The five practices\textsuperscript{17} are:

1. anticipating student responses,
2. monitoring student responses during the explore phase,
3. selecting student responses for whole-class discussion,
4. sequencing student responses purposefully,
5. connecting student responses to one another and to powerful mathematical ideas.

When a mathematical problem has been chosen in relation to the lesson goals\textsuperscript{18}, the teacher anticipates likely student solutions (both correct and incorrect) to the problem for his or her particular students. This is done prior to the lesson. After introducing the problem to the students, the teacher monitors students’ work during their individual and small-group work. During, or after, the practice of monitoring, the teacher decides which student solutions to select for whole-class discussion and how to sequence them in order for the lesson goals to be achieved. In Lamperts’ (2001) words, “Because I knew what many students had been doing in the first part of class as they worked independently with their peers, I could exercise the option to call on someone in

\textsuperscript{17} The five practices have clear connections to Silver et al.’s (2005) pedagogical choices that teachers make when using students’ multiple solutions. Also, teaching practices in Japan, where teachers often organize a complete lesson around students’ various solutions to a single problem in a whole class setting (Shimizu, 1999), are closely related to Stein et al.’s (2008) five practices.

\textsuperscript{18} Choosing problem is an important part of the planning phase. Smith and Stein (2011) acknowledge Setting goals and choosing problem as “practice 0”.

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the class to get a particular piece of mathematics on the table” (p. 146). The teacher also asks the students questions during monitoring when they need specific support or challenges. During whole-class discussion, the teacher helps students to connect student solutions to one another and to key mathematical ideas.

Each practice builds on the practices that precede it. The practice of making connections hence builds on and benefits from the four preceding practices and may be seen as the ultimate practice in the model. Making connections is a key ingredient in frameworks for mathematical competencies (e.g. Lithner et al., 2010; NCTM, 2000). Ball (1993) stresses that teachers need opportunities to analyze mathematical ideas and make mathematical connections during lessons, which is what the 5P model focuses on. The model is designed for helping teachers to learn to conduct mathematical whole-class discussions with the practice of connection-making in a fundamental position. By making explicit the five practices, teachers get support to reflect upon how these instructional practices can contribute to the orchestration of productive whole-class discussions.

The 5P model can contribute to help teachers move from first-generation practice of show-and-tell towards second-generation practice in which the teacher purposefully guides the whole-class discussion of students’ ideas towards important mathematical ideas and connections. The content aspect of the balance between content and participation is thus strongly supported by Stein et al.’s (2008) model through its emphasis on mathematical connection-making. However, second-generation practice as depicted in Stein et al. (2008) says little about the detailed interaction patterns that strengthen the participation aspect during whole-class discussions. Wood et al. (2006) on the other hand, explicitly attend to this. In Paper VI, it is suggested how practices in Stein et al.’s (2008) model can be sensitized to challenging aspects that are crucial for argumentation. Taking Wood et al.’s (2006) inquiry/argument interaction patterns (see Paper V) into account in the 5P model aims at further strengthening the support for teachers in handling the delicate balance act between content and participation.

2.3.4 The roles of Stein et al.’s five practices model in this thesis

It is important to emphasize that in this thesis, the 5P model plays three different roles. The model is used as:

1. pedagogical tool to support teachers,
2. analytical tool to analyze teaching,
3. object of study to be analyzed in itself and further developed.
To support teachers in planning and conducting productive whole-class discussions, the 5P model was introduced and used as a pedagogical tool. That was the case in the second and third intervention projects (see Papers III and VI). In the first intervention project (see Papers I and II), the researcher (myself) was not yet familiar with Stein et al.’s (2008) five practices. In the second intervention (see Paper III), the 5P model was introduced as pedagogical tool although the deeper rationales underlying the five practices were not explicitly discussed with the teacher. In the third intervention with all mathematics teachers at one school during two years (see Paper VI), Stein et al.’s 5P model served as the basic pedagogical tool and the rationales underlying the five practices were explicitly discussed with the teachers. In addition, the teachers and researcher also used and discussed other tools such as students’ written solutions, curriculum materials with problems and suggestions for solutions, and video recordings of problem-solving whole-class discussions.

Analyzing the proficient teacher’s actions and rationales for her actions constitutes the main use of Stein et al.’s (2008) model as analytical tool (see Papers IV and VI). In addition, the 5P model is used as analytical tool to analyze the teacher’s actions regarding selecting, sequencing and connecting in the second intervention project (see Paper III).

Analyzing the teacher’s actions in the second intervention project, we start touching upon looking at Stein et al.’s (2008) model as object of study to be analyzed in itself (see Paper III). Looking upon the 5P model as object of study to be further developed is though done primarily in relation to the proficient teacher and to the third intervention project which stretches over two years (see Papers IV and VI). In Paper IV, the focus is on studying and developing the practices of selecting and sequencing based on analysis of the proficient teacher’s practices. In Paper VI, developments to several practices are suggested on basis of the analysis of both the proficient teacher’s practices and the challenges that the teachers experience in the third intervention project. Paper V focuses on argumentation and serves as input for Paper VI.

2.4 Summary of chapter 2

In chapter 2, central notions on problem solving teaching, whole-class interaction, classroom norms, argumentation, authority and autonomy have been derived from literature and used to formulate guiding principles for the intervention projects in this thesis. Furthermore, literature on challenges and supportive tools for teachers to plan and conduct problem-solving whole-class discussions have been introduced to lay the foundation for discussing conclusions on challenges and support.
3 Methodological considerations

Within this thesis, four case studies have been conducted during the course of several years, involving one or several teachers. Firstly, basic assumptions in this thesis on mathematics knowledge and learning, classroom interaction and teacher’s role in class discussions are described. Then, a framework for interpreting mathematical whole-class discussions is elaborated on that constitutes the ground for the understandings of mathematical activity in whole-class discussions. Thereafter, the methodology used to uncover challenges and support across all three interventions (see Figure 2 in section 3.3) is explicated. A large section is then dedicated to a description of methodological choices. Finally, ethical considerations and trustworthiness of this thesis are delineated.

3.1 Basic assumptions

Here, the basic assumptions underlying this thesis are articulated. The basic assumptions guide both the research aims and the developmental part of the intervention projects. The basic assumptions are divided into assumptions on mathematics knowledge and learning, on classroom interaction and on teacher’s role in problem-solving whole-class discussions.

3.1.1 Basic assumptions on mathematics knowledge and learning

According to the traditional view of school mathematics, mathematics is a fixed collection of concepts and procedures and the goal is that students can perform these procedures skillfully and in the right way (Ball, 1990). In contrast, mathematical knowledge can be seen as continuously constructed by humans (Ball, 1990). Mathematics is a “domain of inquiry” (Ball, 1990, p. 256) and mathematical activity is about formulating and solving problems, conjecturing, and justifying claims by mathematical arguments (cf. NCTM, 2000; Niss & Jensen, 2002; NRC, 2001). Diverse ways of thinking is central in this latter view which underlies this thesis. Cobb et al. (2001) additionally emphasize that mathematical learning is supported by students’ diverse ways of participating.
The suggestion to discuss both correct and incorrect student ideas builds on the view that errors are a feature of the learning process (Bray, 2011), i.e. that errors and misconceptions are “a normal part of coming to a correct conception” (Brodie, 2010, p. 14). In this, thinking critically of one another’s mathematical ideas is crucial, as opposed to uncritically accepting teacher’s or peer’s claims (Bray, 2011). Research findings show that when students reflect upon rival ideas they learn a great deal, which is true also when some of the ideas are incorrect (Boaler & Humphreys, 2005). The main idea is that student errors are used for learning. Faulty reasoning and misunderstandings “are the raw material with which teachers can work to guide students’ mathematical learning. And in the process, students become more confident in their ability to make sense of concepts, skills, and problems.” (Chapin et al., 2009, p. 19).

3.1.2 Basic assumptions on classroom interaction

Smith and Stein (2011) state that we learn through using others as resources in social interaction, sharing our ideas and participating in co-construction of knowledge (cf. Cobb, 2000). This is in line with the assumptions held in this thesis and with Cobb et al.’s (2001) key assumption that knowledge19 is co-constructed by teacher and students in the course of their ongoing classroom interactions. The point of reference is hence the classroom, and not the wider societal context.

The view of teaching as “classroom interactions among teachers and students around content directed toward facilitating students’ achievement of learning goals” (Hiebert & Grouws, 2007, p. 372) underlies this thesis. Looking at teaching this way seizes the two-way relationship between teacher and students in their interactions. Teacher and students co-construct knowledge in their ongoing interactions (Cobb et al., 2001) around mathematical content. Whole-class interaction and classroom norms exist in interplay and mutually affect each other. Classroom norms develop in whole-class interaction, but whole-class interaction is at the same time guided and constrained by prevailing norms.

In problem-solving whole-class discussions, students express their thoughts by communicating them. At the same time, communicating their ideas helps students to construct their thinking. A basic assumption underlying this thesis is that thought and language are mutually dependent of each other. This relation of mutual dependence means that language or communication is not merely expressed thinking, but also a means for constructing, or shaping, the thinking (Ryve, 2011; Skott, Jess, Hansen, & Lundin, 2010). Hence, a basic assumption is the importance for students’ mathematical thinking and

19 By knowledge is here meant mathematical content knowledge related to topic areas such as number, algebra and geometry as well as engagement in processes such as representing, connecting, visualizing, conjecturing, extending, justifying and generalizing.
learning that students get opportunities to articulate their thinking verbally, both orally and in writing. Different forms of representations (e.g. graphical, verbal, arithmetic and algebraic) play an important role in communication. In problem-solving whole-class discussions, teachers can create opportunities for students to articulate their mathematical thinking as a means to construct their thinking.

3.1.3 Basic assumptions on teacher’s role in class discussions

To support students in constructing their mathematical knowledge, the teacher plays a crucial role. The teacher’s role is to create opportunities for students to develop all their mathematical competencies. As NRC (2001) states, “opportunity to learn is widely considered the single most important predictor of student achievement” (p. 334). A basic assumption in this thesis is the importance of the teacher’s active role to guide problem-solving whole-class discussions towards key mathematical ideas. Supporting teachers to handle the big teaching dilemma stated by Hiebert et al. (1997) – being sensitive to both students’ ideas and authority, and to the subject of mathematics – is critical. A basic assumption in this thesis, shared with Fraivillig et al. (1999) and Hiebert et al. (1997), is that it is feasible and important to intervene in ways that stimulate and advance students’ thinking and, simultaneously, promote students’ autonomy. The intervention projects have aimed to provide support for teachers to intervene productively by using Stein et al.’s (2008) 5P model together with certain interactional moves that promote students’ argumentation.

3.2 Interpreting mathematical whole-class discussions

To interpret what happens in a classroom when a teacher conducts a mathematical whole-class discussion based on students’ different solutions, Cobb and colleagues’ interpretative framework is useful. The framework coordinates a social and psychological perspective with the pragmatic aim of understanding classroom interactions. It is important to stress that it is coordination (and not integration) of social and psychological perspectives, which are seen as “two alternative ways of looking at and making sense of what is going on in classrooms” (Cobb et al., 2001, p. 122).

Cobb and colleagues’ framework has an extraordinary strong relationship between the social and the psychological perspective that implies that “neither perspective exists without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other perspective” (Cobb et al. 2001, p. 122). The social and the psychological perspective are reflexively related to each other which means that they mutually affect and constitute each other.
Cobb et al.’s (2001) framework is found useful for the purposes in this thesis since it needs to be accounted for both the social and the psychological perspective in the investigations of the teacher’s role in orchestrating productive whole-class discussions. In this thesis, the social perspective constitutes the foreground against the background of the psychological perspective. The teacher’s role is central for establishing and maintaining norms for students’ mathematical activity (Yackel & Cobb, 1996). The normative activities during the whole-class interactions (social perspective) are taken as foreground in the analyses in this thesis, but they are always interpreted against the background of the diversity in the students’ ways of reasoning (psychological perspective), both orally and in writing.

In problem-solving whole-class discussions, we have both the multiplicity of different students’ individual reasoning about the problem at hand and the communal class discussion of students’ different solutions in interplay. Students develop their individual understanding and other mathematical competencies when participating in problem-solving whole-class discussions and they contribute to the class discussions based on their different ideas and solutions to the problem. Thus, individual students’ reasoning and the whole-class discussion are mutually affected and constituted by each other (Cobb et al., 2001). For the purposes of this thesis, Cobb et al.’s (2001) two alternative ways of making sense of what happens in a classroom shed light on this interplay, although the focus is predominantly on the social perspective as foreground.

Summing up, Cobb and colleagues’ interpretative framework takes into account social and sociomathematical norms which are crucial for conducting productive mathematical class discussions. In order to understand the teacher’s role in whole-class discussions, Cobb and colleagues offer a framework that shed light on how students’ individual reasoning is related to whole-class interaction and classroom norms. In turn, Stein, Smith and colleagues offer a guiding framework that shed light on how teachers can plan and conduct productive class discussions that focus on mathematical connections (see section 2.3.2). Wood et al. (2006) offer a focus on interactional aspects in order to reach an inquiry/argument culture in whole-class discussions, which is integral to the development of students’ intellectual autonomy (see section 2.2.4). Taken together, these three frameworks offer lenses to analyze challenges and support for teachers learning to orchestrate high-quality problem-solving whole-class discussions that focus on connection-making as well as consensus-building through inquiry and argumentation.

3.3 Methodology to uncover challenges and support

This section focuses on overall aspects of the methodology to uncover challenges and support for teachers due to the overall aim of this thesis. Figure 1
(see section 1.2) provides an overview of the empirical studies that I have conducted in collaboration with mathematics teachers. To read more about each study, see section 3.4 and the respective papers. The overall methodology to uncover challenges and support for teachers to orchestrate productive problem-solving whole-class discussions is illustrated in Figure 2.

**Figure 2.** Methodology to uncover challenges and support.

In the first intervention project, I as researcher collaborated with one teacher during a period of five weeks, involving eleven problem-solving lessons. Representations and connections were in focus in the discussions between teacher and researcher, both to plan for lessons and to reflect on the outcomes. Different ways of solving each problem with different representations as well as possible mathematical connections between different solutions were discussed. Representations and connections hence functioned as support for teacher’s orchestration of problem-solving whole-class discussions (see Figure 2).

Starting to approach the data from the first intervention, it was noted that teaching one problem (Mosaic pattern), that can be seen as being of higher complexity (both mathematically and in an instructional sense) than the other problems, resulted in specific challenges. Paper I finds that in teaching this more complex problem, aspects of the problem-solving process were made less explicit by the teacher than procedural and conceptual aspects. (See Paper I for a related discussion about balance between content and participation.) Moreover, Paper II finds that the proportion of the mathematical practices of representing, connecting and justifying were lower for the problem with higher complexity than for the rest of the problems. The teacher falls back on showing procedures for how to solve this complex problem and builds to a less extent on students’ ideas and suggestions. It was hence challenging aspects for the teacher to refrain from having a procedural focus and to build on...
students’ ideas (see Figure 2). A conclusion from Paper II is that it is important that the researcher explicitly introduces appropriate frameworks, productive vocabulary and distinctions from literature in discussions with teachers in intervention projects, e.g. regarding social and sociomathematical norms (Yackel & Cobb, 1996). Introducing explicit frameworks that stand close to practice, such as Stein al.’s (2008) model, are of particular importance to support teachers in orchestrating productive whole-class discussions.

The second intervention involved six problem-solving lessons during a period of three months. Based on the conclusions from the first intervention, I introduced the 5P model to the teacher with whom I collaborated in the second intervention project. The model served as supportive framework (see Figure 2), or pedagogical tool in the project (see section 2.3.4 for different roles that the 5P model plays in this thesis). Starting to approach the data from the second intervention, it was noted (see Paper III) that relatively few and limited mathematical connections were made during the whole-class discussions (see Figure 2). This was especially true when the whole-class discussion was held during the same lesson as the students worked on the problem20. Moreover, as in the first intervention, the connections actually made were mainly connections between student solutions within the same form of representation (mostly between different algebraic expressions).

Since each of the practices in Stein et al.’s (2008) model builds on previous practices, the low number of connections could be related to the prior practice of sequencing student solutions for whole-class discussion. It could also be related to the researcher’s role in establishing a professional discourse about important components of teaching mathematics through problem solving (see Paper III). Drawing on both interventions, such important components include frameworks for different forms of representations, frameworks for mathematical competencies and frameworks for teaching practices, such as the 5P model. Although the latter was used as pedagogical tool in the second intervention (cf. Cobb & Jackson, 2012), it could have been more deeply incorporated, with rationales for the practices more explicitly stated by the researcher (e.g. that purposefully sequencing student solutions is a way to optimize mathematical connection-making). Underlying guiding principles for whole-class discussions (such as the importance of accessibility to students) could also have been more explicitly stated by the researcher.

In the third intervention, the 5P model was more profoundly incorporated as supportive tool (see Figure 2), and during a considerably longer period of time, namely two years (see Paper VI). In this project, I collaborated with all

20 This setup gives very limited time for the teacher to think through possible connections to make and for the researcher and teacher to discuss possible mathematical connections based on the actual student solutions. Although we strived for not having the whole-class discussion until the next lesson (the lesson after the students worked on the problem), this could only be achieved for half of the problems (for three out of six problems) due to organizational constraints.
mathematics teachers in grade 6-9 in one school. As was elaborated on in section 3.4.3.3, the main focus in the cycles of the third intervention was key issues of Stein et al.'s (2008) five practices. The teachers read Stein et al. (2008) and we discussed the five practices thoroughly in relation to the teachers’ planning, enactment and reflection on their respective lessons. Further literature (see Paper VI) that the researcher recommended was also read by the teachers during the course of the intervention and reflected upon during our meetings in relation to the teachers’ enacted lessons. Importantly, additional aspects that emerged through the researcher’s collaboration with one teacher proficient in conducting problem-solving whole-class discussions (see Papers IV, V and VI) were incorporated as support into the third intervention (see Figure 2). The teachers in the third intervention were hence supported by the proficient teacher’s practices through the researcher. Challenges for the teachers in the third intervention constitute the ground for suggestions on how the 5P model can be further developed (see Paper VI and section 6.1). Among these challenges is the challenge of going from a dyadic interaction between the teacher and the explaining student, towards an interaction that builds on inquiry, argument, collaboration and consensus-building among the students (cf. Wood et al., 2006). This was a major challenge throughout all three intervention projects.

3.4 Methodological choices

I will here give an account of the methodological choices that I have made in data construction and data analysis. Transparency of the processes of construction and analysis of data is important for credibility (Clarke, Keitel, & Shimizu, 2006). Firstly, I elaborate on overarching rationales for conducting case studies and for making interventions inspired by design research methodology. Then, I delve into rationales for choices that I made in the three intervention projects as well as in the study of the proficient teacher.

3.4.1 Rationales for conducting case studies

All four empirical studies are case studies in that a detailed study of one specific case is made. I chose to conduct case studies to gain insights into the complexity of teaching mathematics through problem solving and conducting whole-class discussions. To unravel some of this complexity, I as a researcher strive for investigating the details of the teachers’ work. To do this, case studies can provide the depth that is needed.

The cases in the three intervention studies can be considered to be of the form of typical cases (Yin, 2003). That is, the mathematics teachers in the three intervention projects can be seen as exemplifying the broader category of Swedish mathematics teachers in grade 6-9 striving towards a teaching that
gives students opportunities to develop not only their procedural skills, but their mathematical competencies on a broad front. Additionally, the third intervention project is a case study with a longitudinal element. On the other hand, the case of the proficient teacher can be considered to be a unique case (Yin, 2003) in the context of Swedish mathematics classrooms (see section 2.2.1). That is, this case is unusual but interesting enough to be worth studying closely.

The case studies are all focused on the teacher’s orchestration of problem-solving whole-class discussions. In all studies, the focus is therefore on problem-solving lessons. A mathematical problem is first introduced to the students and thereafter the students work on the problem before the teacher orchestrates a whole-class discussion based on the students’ different strategies and representations.

All case studies were performed in collaboration with teachers who teach grade 6-9. This choice was originally made because I have my own teaching experience in these grades and therefore have a firm ground regarding both the mathematical content taught in these grades and the students.

Below, I first elaborate on the rationales for making interventions inspired by design research methodology, before I make explicit my rationales for the choices made in the three intervention projects as well as in the study of the proficient teacher.

3.4.2 Rationales for making interventions inspired by design research methodology

The three intervention projects were inspired by design research methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Salient features for design research approaches are that they are: (1) collaborative between teachers and researcher, (2) based on intervention, (3) iterative with recurring cycles in which refinement of theory and practice interact (Cobb et al., 2003).

The rationales for the choice to conduct research inspired by design research methodology can be placed on two levels. Firstly, to get hold of challenges and support for teachers starting to teach mathematics through problem solving with whole-class discussions as a key feature, interventions with supportive tools are needed. Secondly, it is well-established that teachers learn to teach mathematics through problem solving by getting opportunities to practice in cycles (Cobb & Jackson, 2011; Lampert et al., 2010) by enacting lessons. Teachers need opportunities to reflect upon their practices together with one another and with researchers (cf. Kazemi & Franke, 2004). Opportunities to analyze lessons and ideas with colleagues is important because it is concrete and contextual (Lester & Lambdin, 2004).

I now turn to rationales for choices made in the three intervention projects as well as in the study of a teacher proficient in problem-solving discussions.
3.4.3 Rationales for choices in the three intervention projects

Here, rationales for the following choices are elaborated on for the three intervention projects: Choice of (1) participants, (2) mathematical problems, and (3) design and data collection.

3.4.3.1 Choice of participants

All the teachers participating in the three intervention projects were experienced teachers (with 5-30 years of teaching experience when the projects started). However, they had little experience of conducting problem-solving whole-class discussions which is common for mathematics teachers in Sweden (see section 2.2.1). The teachers wanted to learn how to teach mathematics through problem solving with whole-class discussions as a cardinal component. Studying experienced teachers who are new to teaching mathematics through problem solving makes it more probable that the challenges they encounter are specifically tied to this new way of teaching, and not to other elements of being a mathematics teacher. This together with their strive to learn to teach through problem solving makes it especially worthwhile to study the particular kinds of challenges that these teachers encounter in the orchestration of problem-solving discussions and how the 5P model can support them.

The decision to focus on teachers in grades 6-9 was made since I have my own teaching experience in these grades, which provides a foundation for making productive teaching interventions. The teacher in the first intervention project (see Papers I and II) showed interest in collaborating with me as a researcher when we met with the mathematics developer in the municipality. The teacher in the second intervention project (see Paper III) asked our university for a researcher to collaborate with and the team of teachers in the third intervention project (see Paper VI) approached me directly after having gotten my name from the mathematics developer in their municipality. In both these two latter cases, the teachers were looking for a collaboration with a researcher in their respective developmental projects focusing on problem solving, funded by the Swedish National Agency for Education. I seized the opportunity to combine their developmental purposes with my research purposes which resulted in combined research- and developmental projects.

All three schools where the teachers in the three intervention projects work are communal schools in mid-Sweden. The first school is located in a middle-sized town, the second is located outside a small town and the third is located in a small community outside a medium-sized town. The area where the first school is located has larger social problems than the average in the whole town. The proportion of students with at least one parent having college education is below average in the municipality. The proportion of students with foreign background\(^2\) is considerably higher compared to the average in the

\(^2\) Foreign background here and below means that the student is born out of Sweden, or that both parents are born out of Sweden.
whole municipality. In the second school, the proportion of students with at least one parent having college education is slightly above average in the municipality, but still below 50%. The proportion of students with foreign background is low. In the third school, the proportion of students with at least one parent having college education is considerably lower compared to the average in the whole municipality. The proportion of students with foreign background is low.

3.4.3.2 Choice of mathematical problems

In the first two intervention projects, the mathematical problems selected for the lessons focused on finding algebraic formulas for geometric growth patterns. The rationales for choosing pattern problems were several: (1) pattern problems are accessible for all students from different entry points, (2) the relationships between the quantities in pattern problems can be expressed with different forms of representations among which connections can be made, (3) pattern problems can help students develop algebraic reasoning, (4) pattern problems are usually visualized in multiple ways which gives opportunities to justify expressions and sends the signal that different ways of thinking is valued, and (5) important classroom norms can be established by engaging in pattern problems (Smith, Hillen, & Catania, 2007). Since the students in the first two intervention projects were not used to participating in whole-class discussions based on their different ways of thinking about mathematical problems, social and sociomathematical norms for this kind of mathematical work needed to be established in the class. Since the students in the classes varied considerably regarding prior mathematical knowledge, pattern problems were suitable for exploration from different views and entry points, with representations, connections and justifications as key mathematical practices (see Paper II). In the first two intervention projects, the researcher suggested pattern problems and discussed them with the teacher, using the outcome of the previous lesson as input for choosing the next problem. All problems had several sub questions with an increasing level of difficulty (see Papers I and II for examples).

In the third intervention project, the problems that were chosen varied more regarding mathematical content according to the wishes from the teachers in the project. The teachers though used several geometric pattern problems (both linear and quadratic)\(^{22}\) as well as problems focusing on combinatorics\(^{23}\), statistics\(^{24}\), geometry\(^{25}\) and rational numbers\(^{26}\). The chosen problems were rich mathematical problems from primarily Larsson (2007) and Hagland, Hedrén

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\(^{22}\) Bushes in a row (Larsson, 2007), Houses of cards (Larsson, 2007) and Stone slabs (Hagland et al., 2005)

\(^{23}\) Shaking hands (Larsson, 2007) and The bus queue (Larsson, 2007)

\(^{24}\) Anna’s siblings (Larsson, 2007)

\(^{25}\) Tangram (Hagland et al., 2005)

\(^{26}\) A real feast (Larsson, 2007)
and Taflin. (2005). Important characteristics of a rich problem is that it introduces important mathematical ideas and relationships and that it can be solved with many different strategies and representations which can form the base for a class discussion. Every student in a class shall be able to work with a rich problem at the same time as it is not solely a routine task for any student. To fulfill these two latter criteria, the problems we used consisted of a number of sub problems with increasing level of difficulty.

Since the notion of maintaining the cognitive demand of a task (Stein et al., 2009) is important in this thesis (see section 5.1.2, 5.1.3 and Paper VI), it is crucial to look at the cognitive demand of the tasks (problems) used in the interventions. Since none of the tasks neither rests on mere reproduction of facts, rules, formulas or definitions, nor are algorithmic in its character, they all have a high level of cognitive demand according to the task analysis framework in Stein et al. (2009). That means, they would be classified either as “procedures with connections” or as “doing mathematics” (Stein et al., 2009). According to the criteria in Stein et al. (2009), the tasks would be classified as possessing the highest level of cognitive demand, as “doing mathematics” tasks, since “there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example” (p. 6), and the tasks “require students to explore and understand the nature of mathematical concepts, processes, or relationships (p. 6).

3.4.3.3 Choice of design and data collection

The first intervention spanned over eleven lessons during a period of five weeks. Each problem was typically given one 50-60 minutes lesson (see Papers I and II). As in the two other intervention projects, the lessons were organized according to the phases launch, explore and discuss-and-summarize. The teacher and I had a meeting before each lesson (planning the lesson) and after each lesson (discussing the outcome of the lesson, in particular the whole-class discussion). We prepared for each lesson by choosing problem and talking about possible ways of solving it (with different strategies and representations) and discussing possible mathematical connections that could be made in the whole-class discussion. After the lesson, we discussed representations that were made and mathematical connections that were drawn, in relation to students’ actual solutions.

Data from the first intervention includes video-recorded whole-class discussions with a focus on the teacher (to capture the teacher’s role and actions), video-recorded small-group discussions of two groups, audio-recorded teacher interview, student pre- and posttests and collected student solutions. Student solutions were always collected since I view whole-class interaction and students’ diverse reasoning as mutually affected and constituted by each other (Cobb et al., 2001). The whole-class interactions must then always be interpreted against the background of students’ individual reasoning (see section 3.2).
At the time of the first intervention, spring 2008 (see Figure 1 in section 1.2), I was not yet familiar with Stein et al.’s (2008) 5P model, but the mathematical practices of representing, making connections and justifying claims by mathematical arguments were already central aspects (see Paper II). No explicit model for teaching mathematics through problem solving was however used.

The 5P model came to my awareness before the second intervention project held in spring 2010 (see Figure 1), in which the five practices of anticipating, monitoring, selecting, sequencing and connecting were introduced and referred to in collaboration with the teacher (see Paper III). Introduction of the 5P model was made due to the conclusion from the first intervention project that it is crucial that the researcher introduces teachers to appropriate frameworks and vocabulary (see Paper II).

The second intervention spanned over six lessons over a period of three months. Each problem was given 80 minutes, sometimes during one lesson and sometimes divided into two lessons (see Paper III for details). Dividing the teaching of a problem into two lessons gives the teacher possibility to collect the student solutions after the first lesson and use them to plan the class discussion, i.e. it gives the teacher more time to think about how to use students’ ideas for class discussion in terms of selecting, sequencing and connecting the solutions (cf. Stein et al., 2008).

We chose problem during a meeting a few days before the lesson to get more time to think about possible and likely ways of solving the problem. Right before each lesson, we talked about which kind of solutions the teacher anticipated from his students. After the lesson, we discussed the outcome in terms of Stein et al.’s (2008) practices anticipating, monitoring, selecting, sequencing and connecting. For example, the teacher was asked to compare the actual student solutions to the anticipated ones, and to reflect on the selection and sequencing of student solutions.

Data from the second intervention includes video-recorded whole-class discussions with a focus on the teacher, student pre- and posttests and collected student solutions. However, instead of recording two small groups in their discussions as in the first intervention, the teacher carried around an audio recorder when circulating among the small groups during the explore phase. The reason for this change was to better capture the teacher’s role and interactions. In addition, the dialogue between teacher and researcher before and after every lesson was audio recorded to seize the teacher’s reflections on the five practices.

In the third intervention project – a two-year research- and development project – the 5P model functioned as the backbone, or the primary tool for the project (cf. Cobb & Jackson, 2012). Other tools that we used were students’ written solutions to problems, curriculum materials with problems and suggestions for solutions, and video recordings of problem-solving whole-class discussions (both video recordings with teachers in the project and other video
recordings). The choice to use the 5P model as the primary tool to support teachers rests on that teachers benefit from concrete instructional practices to guide their actions (Cobb & Jackson, 2011), that planning is important for conducting productive whole-class discussions (Stigler & Hiebert, 1999), and that mathematical connections are crucial for conceptual understanding and problem solving (NRC, 2001).

The 5P model serves as the basis for the cyclic design of the third intervention. In Figure 3, the cyclic structure is shown for each monthly iterative cycle during the first project year. When we met for two hours every month, we first evaluated the outcomes (in particular challenging aspects) of the previous whole-class discussions. We based our discussions on actual student solutions and reflections that the teachers brought from their lessons. That is, the teachers brought data from their own practice and reflected upon their own teaching (cf. Ponte, 2012). In relation to the five practices, we discussed in particular how the teachers monitored the students, which student solutions the teachers selected, how these solutions were sequenced for whole-class discussion, and which connections that could be made. The reasons for their choices were also discussed. Then we chose and adapted a problem for the next cycle and anticipated student solutions. All teachers enacted the same (sometimes adapted) mathematical problem in their classes. In some cases, the teachers used the same problem in several of their classes (up to four times).

As researcher, I routinely posed certain questions to the teachers regarding key issues (cf. Cobb & Jackson, 2011; Coburn & Russell, 2008) during the meetings. For example, I posed questions on how teachers reasoned when they selected certain student solutions from the solutions in the class, why they sequenced them in certain ways, and which strategies and representations they anticipated in a class. Coburn and Russell (2008) have shown that when leaders press teachers on key issues, teachers subsequently start to press one another on these key issues. This makes it worthwhile to focus on key issues of practices that teachers could use regularly in the teaching of mathematics, such as Stein et al.’s (2008) five practices. By teachers’ continuous press on key issues during planning, enactment and reflection, the practices can not only survive after a project has ended, but also be further refined by the teachers themselves in their work over a longer period of time.

Data from the first year includes audio-recorded monthly teacher discussions. It also includes audio-recorded teacher group interviews in the start of the project. One day in the spring both the teachers and I observed three teachers’ lessons. I took field notes and interviewed each teacher before and after their lesson (see Appendix A). The same day we had a meeting in which we discussed the lessons in relation to Stein et al.’s (2008) five practices. On another occasion in the spring, one teacher’s lesson was observed and discussed in a meeting the next day.
During the second year, we ran through cycles with a different character than during the first year. Teachers in the project held problem-solving lessons which we discussed with stimulated recall the next meeting (in particular the introductions and the whole-class discussions). Focus in these discussions were challenging aspects and the practices of Stein et al.’s (2008) model. In the end of the project, a concluding meeting was held with a particular focus on challenges during the two years and support from the five practices.

Data from the second year includes video-recorded lesson observations of six lessons. Pre- and postlesson interviews with the teachers were audio recorded. The meetings in which we discussed the lessons with stimulated recall were audio recorded.

Several features of high-quality professional development as articulated in literature were salient in the design of the third intervention project. Desimone (2009) identifies the following five critical features for professional development: (1) content focus, (2) collective participation, (3) duration, (4) active learning, and (5) coherence with teachers’ previous knowledge and skills. Similarly, Bell, Wilson, Higgins and McCoach (2010) (referring to National Academy of Education, 2009) as well as Cobb and Jackson (2011; 2012) accentuate features such as focus on issues central to teaching such as subject matter knowledge, participation by teachers from the same school, sustainment over a longer period of time, support from actively engaged educators, and building on what teachers know and are able to do. In resonance with the above features, Wayne, Yoon, Zhu, Cronen and Garet (2008) state that promising best practices include “intensive, sustained, job-embedded PD focused on the content of the subject that teachers teach” (p. 470) as well as active learning, coherence and collective participation.
3.4.4 Rationales for choices in the study of the proficient teacher

Here, the following choices are elaborated on for the observational study of the proficient teacher: Choice of: (1) participants, (2) lessons and mathematical problems, and (3) design and data collection.

3.4.4.1 Choice of participants

The analysis of classrooms where there is an effective use of problem-solving approaches is a crucial research area (Stein et al., 2003). Against this background, I searched for a teacher who skillfully teaches mathematics through problem solving regularly, in particular who conducts productive whole-class discussions based on students’ different solutions to challenging mathematical problems. When I got the opportunity, in another matter, to observe the proficient teacher’s problem solving lessons I was struck by the argument culture that seemed to pervade the classroom climate.

Moreover, it was apparent that the teacher, without previously having read anything about the 5P model, already planned and conducted her problem solving lessons very much in line with the five practices. Towards the end of her students’ independent work with the problem, she selected some student solutions for public display in the whole-class discussion. The selected solutions were discussed and justified in a certain order, decided by the teacher during monitoring their work. Moreover, the selected student solutions were compared, contrasted and connected to one another and to central mathematical ideas. The students evaluated their peers’ solutions in terms of correctness, effectiveness and accessibility. They also actively asked one another questions for clarification, indicated disagreement and challenged one another’s solutions to be justified and defended. The students helped one another to clarify explanations and justify claims.

This initial picture of the proficient teacher’s practices was deepened during subsequent lesson observations and interviews. The teacher is also known in the community of mathematics teachers as a skillful teacher and she regularly holds seminars for other mathematics teachers regarding problem solving and whole-class discussions in particular, but also regarding other areas of mathematics teaching. The proficient teacher’s problem-solving practices share many features with Fraivillig et al.’s (1999) instructional strategies for effectively eliciting, supporting and extending students’ thinking. For example, she elicits many solution methods for a single problem from the class and decides which solutions to discuss in the whole-class setting, she supports students by asking also other students than the author(s) to explain and justify a solution to the class, and she extends students’ thinking by pushing them to go beyond initial solution methods and by focusing on connections and justifications.
The school where the teacher works is an independent school located in a large city in Sweden. The proportion of students with at least one parent having college education is considerably higher compared to the average in whole the municipality. The proportion of students with foreign background is considerably lower compared to the average in the whole municipality.

### 3.4.4.2 Choice of lessons and mathematical problems

The choice of lessons to observe were made based on which lessons that the proficient teacher had selected to be problem-solving lessons according to her overall planning. She regularly teaches mathematics through problem solving with the phases launch, explore and discuss-and-summarize. During one or two lessons, one problem is introduced, worked on individually and in pairs, and discussed in the whole-class setting. I did not intervene into the choices of which mathematical problems that should be worked on. The problems chosen were both geometric pattern problems and problems from other mathematical areas. The problems have in common that they are rich (Hagland et al., 2005; Larsson, 2007) in the sense that they can be solved with many different strategies and representations as well as on different levels by different students in a class, that they thereby are suitable for whole-class discussion, and that they introduce important mathematical ideas and relationships (cf. section 3.4.3.2). The problems used by the proficient teacher would be classified as having high-level cognitive demands (Stein et al., 2009) for the same reasons as the problems used in the third intervention (see section 3.4.3.2).

### 3.4.4.3 Choice of design and data collection

My purpose with observing the proficient teacher was to gain a deep understanding of her current teaching practices. Therefore, I observed and interviewed the teacher recurrently without making interventions during the lessons; I tried to be as passive and neutral as possible. If the students asked me a question I referred to the teacher.

I visited the school during eight days during one school year, with the majority in the autumn. I followed the teacher’s mathematics lessons with one 6th grade and one 7th grade class during these days. Since I view whole-class interaction and individual students’ reasoning as mutually affected and constituted by each other (Cobb et al., 2001), whole-class interactions must be interpreted against the background of the diversity in students’ ways of reasoning (both orally and in writing), see section 3.2. Therefore, I always collected student solutions that were related to the lessons I observed.

The classes allowed for far-reaching generalizations for being 6th and 7th grades (see Papers IV and V). The teachers in grade 6-9 at the school work collaboratively with problem solving and whole-class discussions. They have

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27 Foreign background here means that the student is born out of Sweden, or that both parents are born out of Sweden.
regular teacher meetings in which they select or construct problems, anticipate student responses and discuss outcomes of problem solving lessons. The teachers in grade 1-5 also work regularly with problem solving and whole-class discussions in mathematics, which means that when students begin sixth grade, they are used to working with challenging problems and discussing their different solutions in the whole-class setting.

In extensive teacher interviews (both before and after each lesson and in the end of each day), the teacher’s rationales for her decisions could be uncovered to gain understanding of her actions. In the interviews, I particularly focused on the underlying rationales for why the teacher made certain choices related to Stein et al.’s (2008) five practices, e.g. why she selected certain student solutions, why she sequenced them in a certain way and why she had certain ways of interacting with the students during monitoring and in whole-class discussion. To gain some understanding of the students’ views of their teaching, I also interviewed three students from each class.

Data for the study was collected during eight school days in one 6th grade and one 7th grade taught by the proficient teacher. Due to the aim of this thesis, the whole-class discussions and the introductions of problems were video recorded with a primary focus on the teacher. Data also includes audio-recorded teacher interviews (before and after every lesson and in the end of each of the eight days) as well as collected student solutions connected to the lessons. Additionally, six teacher meetings and three student interviews from each class were audio recorded.

### 3.5 Ethical considerations

A reflected, ethical stance and acting as researcher is essential (Vetenskapsrådet, 2011). Here, ethical considerations are described that I have made in relation to my research.

First of all, the great challenges in teachers’ work of planning and orchestrating whole-class discussions are important to accentuate. Everyone who has tried to lead a mathematically productive whole-class discussion based on students’ different responses can probably witness on the many challenges it encompasses. Acknowledging that it is challenging to learn to orchestrate problem-solving whole-class discussions also implies that this important work has to be viewed over time; teachers gradually develop more and more proficiency by practicing and reflecting upon their practice. Through this thesis, I have tried to accentuate and nuance the kinds of challenges in this work to do justice to the teachers, who are all accomplished and experienced mathematics teachers.

As researcher I need to be humble with my interpretations of the processes; I only have the possibility to describe, explain, or understand a limited part of the complex and multifaceted course of events in a classroom. The theoretical
frameworks that I use function as lenses with which I as researcher can focus on aspects that I would otherwise overlook (Skærbæk, 2012). Still, it is important to stress that I make interpretations that are affected by who I am as a person.

An ethical aspect to consider when doing research together with teachers is what premises our collaboration builds on. I strive for a climate of collaboration between teachers and researcher that builds on mutual respect and confidence; we both have our areas of expertise and we learn from each other (cf. Ponte, 2012). Together, we can contribute with something novel to research and practice. As researcher in intervention projects, it is my responsibility to provide useful tools for teachers that give them good possibilities to engage in the demanding work of teaching mathematics through problem solving and lead whole-class discussions, so they can in turn give their students good possibilities to engage in the problem solving process and discussions. An open, critical reflection on my role as researcher in the intervention projects is important (see section 6.5.1). As doctoral student, I have developed over time in my role as researcher. What I learned from my collaboration with teachers in my two earlier intervention projects and with the proficient teacher has come to use in the third intervention project.

Focus in my thesis is on the teacher’s role. Therefore, students’ reasoning (oral and written) function as a background against which the teacher’s actions and reasoning is interpreted (see section 3.2). Due to the focus on the teacher’s role, I do not go deeply into individual students’ thinking and reasoning in my research. Nevertheless, it is always important to be clear with the research aims to both teachers and students when getting the opportunity to observe lessons.

The four ethical principles on information, consent, confidentiality and utilization (Bryman, 2011) have been followed carefully regarding both teachers and students. Regarding information, both I as researcher and the teacher informed the students orally about the research project and the students had opportunities to ask questions about their participation. Articulating the information to the students in a language that is accessible to them was given particular consideration as well as answering their questions so they understand what their participation in the project means. That the participation is voluntarily and that the students can withdraw anytime during the project just by saying so to me or their teacher was emphasized. We explained that the material will be used for research purposes only and that results will be published in journals, without names of persons, schools etc. Regarding informed consent, a letter of consent with information about the research project was sent to the parents when the students were not yet 15 years of age (which was the case most of the time). The parents had the option to approve or not approve for their child’s participation and were asked to sign. When the students were 15 years old their own signature was sufficient. In the letter of consent, all
four ethical principles were taken into account: (1) information about the research project, its aim and its parts, including voluntariness, was given, (2) consent for participation was asked for (see above), (3) confidentiality was promised, and (4) utilization of the material was clarified, i.e. that the data will only be used for the specified research purposes.

3.6 Trustworthiness

The world can be described in many different ways. I have tried, throughout the research process, to take Lincoln and Guba’s (1985) and Guba and Lincoln’s (1994) aspects of trustworthiness into consideration. The four aspects for assessing trustworthiness in qualitative research are credibility, dependability, confirmability and transferability (Bryman, 2011).

First of all, I have studied teaching, both in the third intervention project and in the case of the proficient teacher, during long periods of time with recurrent observations. To strengthen the credibility, I have triangulated data by combining teacher interviews and lesson observations. In relation to observed lessons, I have always conducted a pre- and a postinterview in direct adherence to the lesson (see Appendix A). Both interviews and lesson observations have been recorded (with sound and video, respectively) to strengthen the credibility since I was the only observer. Moreover, I have discussed research findings and articles with my respondents on several occasions, i.e. made respondent validation. I had a meeting with the teachers in the third intervention project in which we read and discussed Paper VI in which they figure. I started with presenting the paper and the conclusions drawn from it. Then, we together went through the paper part by part. The teachers read and commented on each part of the content, in particular the sections with quotations from the teacher’s utterances. The teachers found that the paper harmonized with their view of events that played out during the project. There were also much recognizing laughter and many smiles during the reading of the paper. Also, I discussed findings and papers on several occasions with the proficient teacher.

Regarding dependability, I have tried to explain and justify my research design and the rationales for my choices in as much detail as possible. Further, I have discussed research results with fellow researchers during the research process. Respondent validation and triangulation (see above) have also contributed to strengthening the dependability.

Regarding confirmability, I have tried to nourish my awareness of my preconceptions by constantly reflecting on and discussing research results with fellow researchers and with in-service mathematics teachers in different settings. I often teach mathematics through problem solving to in-service teachers, with a particular focus on whole-class discussions. The structure I use in
these in-service courses is similar to the cyclic structure of the third intervention project, with the difference that we only have two or three meetings in these courses. Similarly, the problem-solving module for grade 7-9 in a nation-wide professional development called the Boost for Mathematics (in Swedish: Matematiklyftet) is built upon a cyclic structure with eight cycles. The 5P model serves as the backbone for the problem-solving module for grade 7-9, which I have been responsible for developing (see Skolverket, 2012). My experiences from this widely-used professional development module and in-service teacher courses continuously help me to view challenges that teachers experience and the support that they need through teacher’s eyes in a wide variety of settings in Sweden.

The fourth aspect of trustworthiness, transferability, has been strengthened by applying several of the proficient teacher’s ideas, practices and moves in the context of the third intervention project. This context is more of an average Swedish context which also makes it more probable that much of the challenges experienced by these teachers are more broadly experienced in Sweden. Since the proportion of students with another mother tongue than Swedish is low in both cases, I do not claim to cover challenging aspects related to Swedish language learners. In order to help the reader to judge how transferable my results are to other contexts, I have tried to provide detailed descriptions of, and rationales for, the proficient teacher’s interaction patterns and her practices in relation to the 5P model.

3.7 Summary of chapter 3

In chapter 3, basic assumptions have been articulated that guide this thesis. Further, an interpretative framework has been outlined that constitutes the ground for understanding mathematical activity in whole-class discussions. Rationales for choices regarding the four studies have been described with the purpose of making it clearer for the reader on what grounds certain choices were made. The overall methodology to uncover challenges and support (in order to answer the research questions) has been explicated. Finally, ethical considerations and trustworthiness in relation to the empirical studies in this thesis have been described.

In the next chapter, a summary of each paper is provided as the foundation upon which to draw conclusions on challenges and support for teachers.

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28 After making some important distinctions from literature and introducing frameworks such as Stein et al.’s (2008) model, the in-service teachers plan problem-solving lessons that they implement in their respective classes. The next meeting, based on teachers’ concrete student solutions and reflections, we reflect upon challenging aspects from the lessons and support from the five practices model.
4 Summary of papers

Here, a brief summary of each of the Papers I-VI is provided with emphasis on the results of each paper.

4.1 Paper I

Title: Analyzing content and participation in classroom discourse: Dimensions of variation, mediating tools and conceptual accountability


Paper I builds on the first intervention project (see Figure 1). The aim of Paper I is to reflect on the analytical approach taken for conceptualizing and analyzing mathematical classroom interaction and how this approach helps us understand how opportunities are created for students to discern content and participate in whole-class interaction. Analytical constructs that were used from start include frameworks for mathematical proficiency (NRC, 2001) and variation theory (e.g. Marton & Tsui, 2004). Analytical constructs that emerged during the process of analysis include mediating tools (e.g. Säljö, 2000; Wertsch, 2007) and the interactional constructs of conceptual accountability (Sfard, 2008), meta-shifts (Cobb, Boufi, McClain, & Whitenack, 1997) and repairs (e.g., Emanuelsson & Sahlström, 2008). Analysis was made of one transcribed, video-recorded whole-class discussion of a pattern problem in 7th grade. The teacher had not previously taught mathematics through problem solving. The results show that both content and students’ participation are affected by the mediating tool of a table that the teacher draws on the board; the table steers students’ possibilities to contribute to the discussion. Further, the teacher’s conceptual accountability in using technical terms is high regarding procedural and conceptual strands of mathematical proficiency, but low regarding the problem-solving process. Dimensions of variation related to procedural and conceptual aspects of the pattern problem (in deriving arithmetic and algebraic expressions) become explicit through the teacher’s relatively precise use of technical terms, different forms of mediating tools (different
forms of representations) and meta-shifts. By contrast, dimensions of variation related to the problem-solving process become less explicit due to use of few mediating tools (only verbal representation) and absence of meta-shifts to elaborate on technical terms used to denote dimensions of variation. It is concluded that analytical constructs such as mediating tools, conceptual accountability, technical terms and meta-shifts allow for articulation of important aspects of how dimensions of variation become more or less explicit in the enacted object of learning in classroom interaction.

4.2 Paper II

Title: Balancing on the edge of competency-oriented versus procedural-oriented practices: Orchestrating whole-class discussions of complex mathematical problems


Paper II also builds on the first intervention project (see Figure 1). The aim is to investigate how and why the establishment of key mathematical practices of representing, connecting and justifying varies in an intervention project on teaching mathematics through problem solving. Both a quantitative and qualitative analysis are made of six transcribed video-recorded whole-class discussions regarding the establishment of representations, connections and justifications. Conclusions from Paper II are that there is a shift in the focus of whole-class discussions to finding the correct answer instead of focusing on processes such as representing, connecting and justifying. This shift occurs in the middle of the lesson series when the more complex problem Mosaic pattern is discussed. Dismissing several other possible factors that can contribute to the decline of task complexity (Henningsen & Stein, 1997), the most plausible reason found for the focus to find the correct answer in a procedural manner during the Mosaic problem discussions is that the teacher’s knowledge is balancing on the edge in relation to the complexity of the problem. This leads to little focus on the mathematical practices of representing, connecting and justifying, which creates limited opportunities for the students to develop their mathematical competencies on a broad front. The lower proportion of representations, connections and justifications is hence related to the higher complexity of the problem for the teacher (both in a mathematical and an instructional sense). To handle this complexity, the teacher’s practices fall back on well-known procedures. Nevertheless, the teacher seems to develop
through the work with the demanding Mosaic problem, suggested by the considerably higher proportion of representations and connections in the last two lessons compared to the first two (which are of similar complexity).

One conclusion regarding the researcher’s role is that it is crucial that the researcher explicitly introduces appropriate frameworks and vocabulary to create a productive discussion with teachers, so that teachers can create a productive discussion with their students. Stein et al.’s (2008) 5P model is such a framework that stands in close connection to practice.

4.3 Paper III

Title: Effective teaching through problem solving by sequencing and connecting student solutions


Paper III builds on the second intervention project (see Figure 1). The paper has two aims: (1) studying the selection, sequencing and connection of student solutions, and (2) critically reflecting on the researcher’s role in this intervention project. Stein et al.’s (2008) model serves as a framework for both implementation and analysis. Audio-recorded teacher interviews and video-recorded whole-class discussions in a class with 7th and 8th graders are analyzed. Results of Paper III focus on how teachers can select and sequence student solutions to facilitate mathematical connections and effective mathematical discussions. By using data from one lesson (collected student solutions, transcribed whole-class discussion and pre- and postlesson teacher interviews), suggestions are made on how student solutions can be purposefully selected and sequenced to set the scene for connecting them productively in a mathematical discussion. The actual sequencing of solutions for class discussion is compared to an alternative sequencing of student solutions aiming to maximize possibilities to make connections between the solutions. In this case, it is suggested that it is beneficial that the teacher begins the whole-class discussion with an incomplete, easy-to-understand solution that many students started with. Completing this incomplete solution together in the class is suggested to be a good way of continuing the whole-class discussion. Two different strategies with the same easily accessed form of representation (in this case graphical) is suggested to follow, making connections between them and also
to the first completed strategy. A more unique, complex solution is suggested to be placed last in the sequence since it might be harder to understand for many students in the class. Connecting it to the previous more accessible solutions may facilitate understanding the more complex solution.

Paper III concludes with a critical reflection of the researcher’s role that draws on this intervention as well as the previous intervention reported in Papers I and II. That the researcher explicitly denotes key components and guiding principles of teaching mathematics through problem solving in discussions with teachers is accentuated.

4.4 Paper IV

Title: Incorporating the practice of arguing in Stein et al.’s model for helping teachers plan and conduct productive whole-class discussions

Author: Larsson, M. (2015a).


Paper IV builds on video-recorded lessons and audio-recorded interviews with a proficient teacher regarding problem-solving whole-class discussions (see Figure 1). The aim is to suggest developments to Stein et al.’s (2008) model in order to take into account argumentation as well as connection-making in the model. The first four practices are not only crucial for connection-making, but also for argumentation. Important aspects of anticipating, monitoring, selecting and sequencing are elaborated on to take into consideration for productive whole-class argumentation. The findings suggest that it is crucial that the teacher does not disclose to students during monitoring whether their solution is correct or not, but instead asks questions to activate students as owners of their own learning or as instructional resources for one another (cf., Wiliam, 2007). The reasons for this are related to both the problem-solving process and to the quality of argumentation in the whole-class setting. Also, some aspects of sequencing student solutions suggested by Stein et al. (2008) that can be problematic for argumentation are discussed. In particular, beginning the whole-class discussion with a correct solution that a majority of the students have made may compromise argumentation. Based on analyses of whole-class discussions and interviews with the proficient teacher, principles are suggested for sequencing student solutions that can facilitate argumentation as well as connection-making in the whole-class setting. For example,
starting the discussion with an incorrect or incomplete solution can serve as a springboard for argumentation, which is illustrated in the paper. A suggestion is made to broaden the last practice in Stein et al.’s (2008) model to incorporate the practice of arguing besides connection-making. It is concluded that the ongoing work will continue to develop Stein et al.’s five practices model and also to explore teachers’ moment-to-moment decisions in classroom interaction to promote students’ reflection and arguments.

4.5 Paper V

Title: Exploring a framework for classroom culture: A case study of the interaction patterns in mathematical whole-class discussions

Author: Larsson, M. (in press).

To be published in: K. Krainer & N. Vondrová (Eds.), Proceedings of CERME9, 9th Congress of European Research in Mathematics Education, Prague, February, 4-8, 2015. Prague: PedF UK v Praze and ERME.

Paper V builds on data from the study of a proficient teacher (see Figure 1). The aim of the paper is to explore Wood et al.’s (2006) framework for classroom culture, with the overarching goal of supporting teachers to conduct whole-class discussions that focus on argumentation as well as connection-making. Wood et al.’s framework distinguishes between conventional and reform classroom cultures, but they state that the most important findings of their study are the differences between the two kinds of reform-oriented classroom cultures: strategy-reporting and inquiry/argument culture. In an inquiry/argument culture, there is a “major shift in participation from an emphasis on the child reporting her/his different strategies to the children as listeners taking over the role of the teacher in questioning, clarifying, and validating mathematical ideas” (Wood et al., 2006, p. 235). Analyses of interaction patterns in the proficient teacher’s whole-class discussions result in articulation of some difficulties to make a clear distinction between certain reform interaction patterns in Wood et al.’s (2006) framework for classroom culture. For example, some interaction patterns (e.g. Check for consensus and Inquiry) would need clearer criteria for their beginning and end. The difficulties found are however related to making clear distinctions between certain interaction patterns within different classroom cultures and hence only affect the relative distribution of interaction patterns within a specific classroom culture. Therefore, the findings suggest that Wood et al.’s (2006) framework can be useful for characterizing interaction in terms of classroom culture and for looking into the details of how teachers can interact with students to establish an inquiry/argument classroom culture. Since Wood et al. (2006) have shown that
an inquiry/argument culture is closely related to higher cognitive levels of students’ verbalized mathematical thinking in class discussion, it is contended that it is desirable for teachers to strive for such a culture. In line with Franke et al.’s (2007) suggestion, inquiry/argument interaction patterns – which are illustrated in this paper – can be connected to Stein et al.’s (2008) 5P model to support teachers in conducting whole-class discussions that focus on mathematical argumentation in addition to connection-making.

4.6 Paper VI
Title: Sensitizing Stein et al.’s five practices model to challenges crucial for argumentation in mathematical whole-class discussions


Paper VI builds on data from the third intervention project together with data from the proficient teacher (see Figure 1). The paper suggests developments to Stein et al.’s (2008) model. The aim of this study is to further nuance Stein et al.’s (2008) model to support argumentation in addition to connection-making, in order to face up to challenges crucial for argumentation met by teachers learning to orchestrate problem-solving whole-class discussions. Analyses of transcripts from teacher meetings, lessons and teacher interviews in the third intervention project result in two overarching challenges related to argumentation: (1) the challenge of getting sufficient variety among student solutions for argumentation, regarding both correct and incorrect solutions, and (2) the challenge of using the variety for argumentation, in particular incorporating erroneous student solutions productively in the whole-class discussion. The first overarching challenge is related to the challenges of not telling students too much during launching and monitoring, i.e. maintaining the cognitive demand of the problem (Stein & Smith, 1998). The second overarching challenge is related to the challenges of understanding students’ incorrect reasoning during monitoring and incorporating erroneous solutions into whole-class discussion by selecting and sequencing student solutions.

Based on data from the proficient teacher on how to handle the challenges encountered by teachers in the third intervention project, it is suggested that it is important for argumentation to:

- sensitize the Monitoring practice to “teaching practices that promote nonauthoritative argumentation, in particular referring students to themselves and one another as resources” (p. 32),
- sensitize the Selecting and Sequencing practices to “the roles of incorrect, common and elegant student solutions for productive argumentation” (p. 32), and
- sensitize the Connecting and consensus-building practice to "interaction patterns aiming at having students collaboratively make sense of one another’s ideas and reach consensus by inquiry and argumentation” (p. 32).

Teaching actions that operationalize the sensitizations are also suggested. In addition, it is discussed how launching is both challenging and critical for argumentation in whole-class discussions. The challenge and importance of making detailed anticipations is also discussed and the paper concludes with pointing to an important area for future research: how support for teachers regarding anticipating and launching can be designed to facilitate for teachers to foster students’ argumentation.
5 Conclusions

By contextualizing the results from Papers I-VI in the research literature, conclusions are drawn here that answer the two research questions.

5.1 Challenges for teachers and support from Stein et al.’s model

The first research question is:

- What characterizes the challenges encountered by teachers in planning and orchestrating problem-solving whole-class discussions?

To characterize challenging aspects for teachers who are new to teaching mathematics through problem solving, conclusions are drawn here based on the empirical results from all six papers. The challenging aspects include:

1. Detailing anticipations of likely student solutions (Paper VI)
2. Getting sufficient variety among student solutions (Paper VI)
3. Maintaining the cognitive level of the problem during launching (Papers III and VI)
4. Maintaining the cognitive level of the problem during students’ exploration (Papers IV and VI)
5. Deciding upon how to select and sequence student solutions for the class discussion (Papers III and VI)
6. Building upon mathematically complex suggestions from students in the class discussion (Papers I and II)
7. Refraining from falling back on procedural-oriented practices in the class discussion (Paper II)
8. Making connections in the class discussion (Papers II and III)
9. Creating an argumentative classroom culture in the class discussion (Papers IV, V and VI)
10. Balancing the tension between content and participation (Papers, I, II, III, IV, V and VI)
Each challenge is related to either the plan phase or to one of the launch, explore, or discuss-and-summarize lesson phases (see Table 2).

Table 2. Challenges related to phases.

<table>
<thead>
<tr>
<th>Challenge(s)</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Challenge 1</td>
<td>Plan</td>
</tr>
<tr>
<td>Challenges 2-3</td>
<td>Launch</td>
</tr>
<tr>
<td>Challenges 2,4-5</td>
<td>Explore</td>
</tr>
<tr>
<td>Challenges 6-10</td>
<td>Discuss-and-summarize</td>
</tr>
</tbody>
</table>

It is important to note that handling the challenges may also involve earlier lesson phases. For example, making detailed anticipations in the planning phase (challenge 1) lays the ground for handling the challenges in subsequent lesson phases. In Figure 4, the challenges and their relationships to one another are shown. The vertical order of the challenges in Figure 4 follows the phases. The challenges of maintaining the cognitive demand during launching and monitoring (challenges 3 and 4) are connected to the overarching challenge of getting variety among student solutions (challenge 2), see Paper VI. The rest of the challenges (5-10) are related to using the actual variety, in which deciding upon selection and sequencing is a challenge in itself (challenge 5). The challenges of building upon mathematically complex suggestions and refraining from falling back on procedural-oriented practices (challenges 6 and 7) are connected to the overarching challenge of balancing content and participation in the class discussion (challenge 10), see Papers I and II. Making connections visible in the whole-class discussion (challenge 8) is important for the content scale in the balance act between content and participation, while creating argumentation in the whole-class discussion (challenge 9) is important for the participation scale, see Papers V and VI.

Figure 4. Challenges and their relationships to one another.
The challenges that teachers encounter when they plan and orchestrate problem-solving discussions can thus be embodied by and organized into the plan, launch, explore and discuss-and-summarize phases. The second research question is:

- How can the model by Stein et al. (2008) support teachers to handle these challenges and what are the limitations of the model to support teachers?

In Table 3 below, conclusions on challenges, support and limitations are summarized for each phase.

Table 3. Summary of challenges, support and limitations for each phase.

<table>
<thead>
<tr>
<th>Phase and challenges</th>
<th>Support from model</th>
<th>Limitations of model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plan (challenge 1)</td>
<td>Make general</td>
<td>Make detailed</td>
</tr>
<tr>
<td>Launch</td>
<td>anticipations</td>
<td>anticipations not</td>
</tr>
<tr>
<td>(challenges 2-3)</td>
<td>Launching aspects</td>
<td>supported</td>
</tr>
<tr>
<td>Explore</td>
<td>briefly mentioned</td>
<td>Launching not</td>
</tr>
<tr>
<td>(challenges 2, 4-5)</td>
<td></td>
<td>included as a practice</td>
</tr>
<tr>
<td>Discuss-and-summarize (challenges 6-10)</td>
<td>Make mathematical connections</td>
<td>Distinguish between different kinds of connections not supported</td>
</tr>
</tbody>
</table>

To go deeper into the challenges and to answer the second research question for each challenge in Figure 4, each phase – plan, launch, explore and discuss-and-summarize – is now elaborated on for challenges as well as support from and limitations of Stein et al.’s (2008) model. The next chapter discusses how the model can be further developed to support teachers to handle the challenges they encounter.

5.1.1 Plan phase – Challenges, support and limitations

It is evident from teachers’ actual anticipations in the three interventions (see Papers I, II, III and VI) that teachers often anticipate likely student solutions but do that in general terms. This means that teachers anticipate which general strategies (Polya, 1945/57) or which forms of representations the students are likely to use (see Paper VI) and do not go into details of specific strategies or representations. Incorrect solutions are most often not anticipated at all. These findings resonate with Bray’s (2011) findings that novice teachers (the teach-
ers in the interventions can be considered as novices regarding problem-solving discussions) generally do not anticipate student solutions, in particular common errors and misconceptions, in such detail as expert teachers.

These kind of general (or shallow) anticipations are of some, but not substantial, help to prepare how to ask students questions during the explore phase without lowering the cognitive demand (Stein et al., 2009) or to prepare how to make use of different solutions, both correct and incorrect, for connection-making and argumentation. That novice teachers often anticipate likely solutions in general terms suggests that it is a challenge for teachers to make detailed anticipations and that support is needed.

The teacher’s preparation before the lesson is supported in Stein et al.’s (2008) model by the practice of anticipating. However, making detailed anticipations, which is needed to support teachers to interact in the classroom, is not explicitly supported by the model. In the context of planning whole-class discussions, general strategies are not detailed enough to help teachers prepare for how to make use of student solutions (in particular incorrect or incomplete solutions) for argumentative purposes. Lesh and Zawojewski (2007) refer to Schoenfeld (1992) who concludes that although Polya’s widely known heuristics (or general problem-solving strategies) are recognizable when they are being used, they are not detailed enough to help students solve problems. In this thesis, it is argued that they are neither detailed enough to help teachers anticipate student solutions. How detailed anticipation of problem-specific strategies can help teachers interact in the classroom is discussed in section 6.1.1.

5.1.2 Launch phase – Challenges, support and limitations
The challenge of maintaining the cognitive demand of the problem during launching in order to get variety among student solutions (see Figure 4) is evident in the analysis of data, exemplified in the following statement (from the final meeting) in the third intervention project, agreed-upon by several teachers: “I still think that the most difficult thing is to introduce the problem in such a way that all understand without steering” (see Paper VI). This challenge is related to “strike a balance between providing sufficient guidance for students to enter the task but not so much that all the challenge has been removed” (Smith & Stein, 2011, p. 81). If the teacher “steers” students towards a solution strategy during launching, the variety of student solutions risks being decreased. To maintain the cognitive level of the problem, teachers have to refrain from telling students too much during launching.

Lesh and Zawojewski’s (2007) interpretation of Polya’s heuristics is that “the strategies are intended to help problem solvers think about, reflect on, and interpret problem situations, more than they are intended to help them decide what to do when “stuck” during a solution attempt” (p. 768).
The 5P model does not support launching. Stein et al. (2008) and Smith and Stein (2011) mention launching, but only with a few sentences. For example, Stein et al. (2008) write that “During the ‘launch phase’, the teacher introduces the students to the problem, the tools that are available for working on it, and the nature of the products they will be expected to produce” (p. 315). Smith and Stein (2011) mention launching in the Thinking Through a Lesson Protocol from Smith et al. (2008):

How will you introduce students to the activity so as to provide access to all students while maintaining the cognitive demands of the task? How will you ensure that students understand the context of the problem? What will you hear that lets you know students understand what the task is asking them to do? (p. 134)

Speaking for the importance of launching, Jackson, Garrison, Wilson, Gibbons and Shahan (2011; 2013) have found that launching is highly related to the quality of the whole-class discussion. To launch the problem to the class, the teacher can conduct a whole-class discussion (Boaler, 2002b; Jackson & Cobb, 2010) with the aim that all the students understand what the problem asks them to do so that they can begin to explore the problem independently. Against the background of the importance and the challenge of launching, it is argued here that it deserves more explicit attention in the model (see further in section 6.1.2 about launching).

5.1.3 Explore phase – Challenges, support and limitations

It is a great challenge to maintain the cognitive demand of the problem during students’ exploration in order to get variety among student solutions (see Figure 4). As in launching, this challenge is related to refraining from telling students too much. If the teacher tells students how to do difficult aspects of a problem while they explore the problem – which is a great challenge to refrain from when students are struggling (Stein et al., 1996) – the variety of student solutions risks being decreased. As stated in Paper VI: “This includes asking questions in ways that support and challenge students’ thinking without steering or funneling them, as well as responding to students’ questions during monitoring in ways that do not take away the challenge of the problem” (p. 16). This is in line with Schoenfeld’s (1992) teaching actions during problem-solving – adapted from Lester, Garofalo and Kroll (1989) - which include both observing and questioning, providing hints as needed and challenging students to generalize.

Support for teachers through the 5P model in how to handle these interactional aspects in ways that maintain the cognitive demand of the problem during the explore phase is currently lacking, despite that Smith et al. (2009) acknowledge the teacher’s work of asking questions to support and challenge
students during their exploration. Interactional aspects involve handling students’ questions on how to do the problem and their craving for the teacher’s confirmation of whether their answer is correct or not. Detailed anticipations and productive launching (see the two previous sections) facilitate for the teacher to handle the challenge of maintaining the cognitive demand during students’ exploration. One particular challenge during the explore phase is to release authority to the students to use themselves and one another as resources to decide upon correctness (Wiliam, 2007) by justifying claims with mathematical arguments (see Papers IV and VI). This challenge is related to establishing “sociomathematical norms of how correctness is established in mathematics” (Harel & Rabin, 2010, p. 14).

Among the variety of solutions, the teacher decides during the explore phase how to select and sequence student solutions for class discussion. This selection and sequencing is a challenge for the teachers in the intervention studies (see Papers III and VI) as well as in Silver et al.’s (2005) study. Using the variety of different strategies and representations for connection-making is strongly emphasized in the 5P model. The model puts less emphasis on using the variety of student solutions for the purpose of argumentation (see section 6.1.4). Although some guidelines are given by Stein et al. (2008) on how to select and sequence solutions, these guidelines are limited and only take connection-making into consideration.

5.1.4 Discuss-and-summarize phase – Challenges, support and limitations

Papers I and II show that when the teacher conducts a whole-class discussion of a more complex problem, both in a mathematical and an instructional sense, it is challenging to build upon mathematically complex suggestions from students, although the suggestions are insightful. Instead of building upon these insightful suggestions, the observed teacher falls back on procedural-oriented classroom practices and shows students a method of how to think.

These challenges could largely be avoided if the teacher makes anticipations of likely student solutions when planning the lesson (cf. Shimizu, 1999; Silver et al., 2005; Smith & Stein, 2011; Stein et al., 2008; Stigler & Hiebert, 1999). In this intervention project (the first one), the 5P model was not used since the researcher (myself) was not yet familiar with the model. Although we discussed different ways of solving the problem before each lesson, a more explicit emphasis on anticipating (as the 5P model supports) could have helped the teacher to build to a higher extent on mathematically complex student suggestions.

Several intervention projects in this thesis aiming at teaching mathematics through problem solving (see Papers II and III) indicate that teachers who are
new to this teaching approach tend to make limited kinds of connections\(^\text{30}\). Most common in the three intervention projects are connections between different equivalent algebraic formulas, i.e. connections between different strategies with the same form of representation. Other kinds of connections, e.g. between different forms of representations for the same strategy, or between different problems or sub-problems, also exist but appear less frequent. That the teachers in the intervention projects tend to make limited kinds of connections suggests that it is a challenge to make different kinds of mathematical connections.

The 5P model can facilitate for teachers to make mathematical connections, with the help of the previous practices of anticipating, monitoring, selecting and sequencing. However, support is lacking in distinguishing between different kinds of connections that can be made. Stein et al. (2008) make an effort to give some input on different kinds of connections in relation to their example, but they talk mainly in terms of “connections among different solution methods” (p. 319) and “connections between different students’ responses and between students’ responses and the key ideas” (p. 321). A more comprehensive distinction between different kinds of connections is currently lacking in the model.

Analyses of whole-class discussions conducted by teachers in the third intervention suggest that it is a great challenge for teachers to establish an argumentative classroom culture (see Paper VI). 1 ½ years into the third intervention project, one teacher can be seen as approaching a collaborative inquiry/argument culture in which students act as a community of learners (cf. Bray, 2011). Apart from this, dyadic interaction between the teacher and the explaining student (cf. Wood et al., 2006) dominates the classrooms in all three intervention projects in this thesis. In line with this, the teacher in Bray’s (2011) study who mobilized students as a community of learners to resolve errors – and hence was closest to an inquiry/argument classroom culture – saw this as the most challenging aspect that she still struggled with.

Support for creating an argumentative classroom culture is not explicitly provided by the 5P model, neither in the discuss-and-summarize phase nor in the earlier phases. Since it is a great challenge for teachers to establish an argumentative classroom culture, it is argued here that support is needed for this throughout the practices in the 5P model (see Paper VI). Although Bray (2011) accentuates that “teachers particularly need support with envisioning how students’ errors can be productively used as springboards for inquiry in the context of class discussion.” (p. 35), there is no explicit support for this in the 5P model. Support for establishing an inquiry/argument classroom culture is important because in such a culture there is a “major shift in participation from

\(^{30}\) This is true especially when the whole-class discussion is held during the same lesson as the students work on the problem. This gives very limited time for the teacher to think through possible connections to make.
an emphasis on the child reporting her/his different strategies to the children as listeners taking over the role of the teacher in questioning, clarifying, and validating mathematical ideas” (Wood et al., 2006, p. 235). The nature of students’ participation is altered. Participation “involves how questions and comments are elicited and offered, and through what means the class comes to consensus” (Sherin, 2002, p. 206) and is thus closely related to inquiry/argument interaction patterns.

Providing support for teachers to create an argumentative classroom culture can hence strengthen students’ participation as validators of key mathematical ideas and better balance content and participation. In the next chapter, developments are suggested to the 5P model that support teachers in this.

5.2 Summary of chapter 5

In chapter 5, conclusions on challenges, support and limitations have been drawn based on the empirical studies in this thesis. The challenges, support and limitations have been organized into the phases plan, launch, explore and discuss-and-summarize. Taking departure from these conclusions, it is next discussed how the 5P model can be developed to further support teachers in the challenges they encounter.
6 Discussion

Developments to the 5P model are discussed here. The basis for the discussion are the conclusions on challenges and support that were drawn in the previous chapter, based on empirical results from all four case studies and six papers. From the proposed developments, the possible emergence of a model for third-generation classroom practice is discussed. Contributions to research and practice are then discussed after which a critical reflection is made on the researcher’s role and methodology. Lastly, suggestions are made for future research.

6.1 Developments to Stein et al.’s model

The 5P model is found to be a useful tool in many regards to support teachers in planning and conducting problem-solving whole-class discussions. Still, there are a number of ways their model can be developed to face up even better to teachers’ challenges.

The overarching challenge of balancing content and participation is related to the two main challenges in the actual orchestration of a whole-class discussion, namely (1) to create an argumentative classroom climate, and (2) to make visible important mathematical connections (see the previous chapter). Since each practice builds on the previous practices, we will now move through the model to discuss how earlier practices can be developed to support later practices in fostering (1) argumentation, and (2) connection-making. First of all, the following five practices are included in the developed model:

1. Anticipating
2. Launching
3. Monitoring
4. Selecting and sequencing
5. Connecting and consensus-building

Due to its importance for subsequent practices and its challenging nature (see section 5.1.2), Launching is proposed to be included in the model as a practice of its own (see above). Further, Selecting and sequencing are proposed to be merged into one practice due to their intertwined nature in teachers’ work. This merging has already been made by other scholars (e.g. Sztajn, Confrey,
Wilson, & Edgington, 2012). Finally, the last practice is about Connecting, but equally important about having students build consensus by inquiry and argumentation (cf. Wood et al., 2006). Therefore, the last practice in the developed model is labelled Connecting and consensus-building.

The instructional practices in the developed model correspond to the phases in which the practices are enacted in the way outlined in Table 4.

Table 4. Correspondence between practices in the developed model and phases.

<table>
<thead>
<tr>
<th>Practice</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anticipating</td>
<td>Plan</td>
</tr>
<tr>
<td>Launching</td>
<td>Launch</td>
</tr>
<tr>
<td>Monitoring</td>
<td>Explore</td>
</tr>
<tr>
<td>Selecting and sequencing</td>
<td>Explore</td>
</tr>
<tr>
<td>Connecting and consensus-building</td>
<td>Discuss-and-summarize</td>
</tr>
</tbody>
</table>

Each practice is discussed below. The challenges in each phase as well as the support and limitations in Stein et al.’s (2008) model to handle the challenges (concluded upon in section 5.1) serve as a departure for the following discussion of how each practice can be developed to support teachers in the challenges.

6.1.1 Practice 1: Anticipating

Both argumentation and connection-making can be facilitated by sensitizing the Anticipating practice to detailed anticipations, as is suggested in section 5.1.1 and Paper VI. Detailed anticipation lays the foundation for understanding students’ different solutions during monitoring and for deciding how to use them productively in whole-class discussion. It involves both deciding upon how to select and sequence solutions and figuring out which questions the teacher can ask in the class discussion to promote students’ argumentation and connection-making. For argumentation it is particularly important to anticipate likely errors and misconceptions in detail.

Anticipating also lays the foundation for being able to handle interactively how to support and challenge students during monitoring, as discussed in section 6.1.3. Detailed anticipation facilitates preparation of questions to ask students during monitoring to support them without lowering the cognitive demand of the problem and without limiting the variety of student solutions. This speaks for the need to emphasize detailed anticipations of problem-specific strategies in the 5P model. This way, teachers can get support from the model to take the step beyond anticipations in terms of general strategies. Schoenfeld (1992) recommends that, since Polya’s problem-solving strategies are “too general” to be useful for prescriptive purposes, each general strategy should be broken down into a number of more specific strategies, each tied to a class
of problems. This recommendation, originally made for the teaching of problem-solving strategies (cf. teaching mathematics about problem solving), can be applied to teachers’ anticipation of student solutions when teaching mathematics through problem solving. Lesh and Zawojewski (2007) conclude:

The dilemma we see is that, on the one hand, short lists of descriptive processes (i.e. the conventional problem-solving strategies) appear to be too general to be meaningful for instructional purposes; on the other hand, long lists of prescriptive processes (i.e. more detailed and specific problem-solving strategies for classes of problems) tend to become so numerous that knowing when to use them becomes the heart of understanding them. (p. 769).

Clearly, it seems neither feasible nor productive that students are expected to learn long lists of specific problem-solving strategies for different classes of problems. Even so, it is argued here that anticipating specific problem-solving strategies could help teachers prepare for how to ask productive questions and make teachers better equipped for distinguishing between and categorizing different kinds of strategies for class discussion. Teachers can get support from teaching materials that include specific problem-solving strategies for classes of problems. How to support teachers more exactly in how to go about anticipating problem-specific strategies is suggested as an important area for future research (see section 6.6). Detailed anticipation of students’ difficulties in understanding the problem facilitates preparation of how to launch the problem productively.

6.1.2 Practice 2: Launching

Launching a problem without lowering the cognitive demand (Jackson et al., 2013) is crucial for both argumentation and connection-making. Making sure that all students understand what the problem asks you to do, how they should work and what they are expected to produce (Boaler & Staples, 2008; Smith et al., 2008), without telling them how to do the problem, lays the foundation for a variety of interesting solutions (both correct and incorrect) that can be compared, connected and argued for and against.

The most challenging aspect for the teachers in the third intervention project is to introduce a problem “in such a way that all understand without steering”. Based on data from many lesson observations and interviews in different projects, it is suggested here that a Launching practice is included in Stein et al.’s (2008) model. One rationale for including the Launching practice is to ensure that all students have the understanding to start working on the problem. Launching a problem in a purposeful way can reduce re-introduction of the task when students start to work on it (Jackson et al., 2013), allowing time for the teacher to monitor students’ work, which in turn puts the teacher in a
better position to make well-founded decisions about how to select and sequence student solutions for class discussion. Jackson et al.’s (2013) findings suggest that, “the quality of the setup appears to be related to students’ opportunities to learn in the concluding whole-class discussion” (p. 677). As Jackson et al. (2011) point out, the launching practice has implications for equity in students’ learning opportunities since launching is important for supporting all students to participate in high-quality ways. Including a Launching practice in the model can hence be justified by quality reasons as well as equity reasons (Jackson & Cobb, 2010).

The challenge and importance of launching for subsequent practices speaks for the need to include the Launching practice in Stein et al.’s (2008) model. Further, among the lesson phases launch, explore, discuss-and-summarize (e.g. Jackson et al., 2011; Stein et al., 2008), the explore phase is clearly related to the practice of Monitoring and the discuss-and-summarize phase is clearly related to the practice of Connecting. The launch phase, however, has no counterpart among the five practices in Stein et al.’s (2008) original model.

Jackson et al.’s (2013) four key aspects for high-quality launching of complex mathematical tasks are suggested to have a central role in the Launching practice. Jackson et al.’s (2013) four aspects can be seen as complementary to suggestions from prior research (Boaler & Staples, 2008; Smith et al., 2008) that effective launching clarifies both the expected final product and how students should work. The four aspects are:

1. Key contextual features of the task scenario are explicitly discussed.
2. Key mathematical ideas and relationships, as represented in the statement, are explicitly discussed.
3. Common language is developed to describe contextual features, mathematical ideas and relationships, and any other vocabulary central to the task statement that might be confusing or unfamiliar to students.
4. The cognitive demand of the task is maintained over the course of the setup. (p. 652).

By enacting a whole-class discussion that develops a common language for contextual features and mathematical relationships and that also maintains the cognitive demand of the problem, the scene is set for students’ exploration of the problem.

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31 Equity is defined here in line with Jackson and Cobb (2010) who states: “Equity, in this context, means that all students can participate substantially in all phases of mathematics lessons (e.g. individual work, small group work, whole class discussion), but not necessarily in the same ways.” (p. 3).
6.1.3 Practice 3: Monitoring

Getting a variety of student solutions is challenging (see section 5.1.3) and is important for both connection-making and argumentation in the whole-class discussion. A limited set of solutions to select among limits the possibilities to make mathematical connections in the concluding discussion. It also limits the possibilities for argumentation in the class. For argumentation, it is of particular importance also to discuss incomplete or incorrect solutions.

During monitoring, a critical aspect of the teacher’s work is to monitor students’ progress (see Paper VI) without lowering the cognitive demand of the problem (Stein et al., 2009). As Paper VI argues, this aspect is related to getting variety among student solutions, which is not self-evident to exist. If the teacher falls into the trap of telling students too much during monitoring, in particular regarding the correctness of their solutions, the variety in the class risks becoming limited. However, the interactional aspects of monitoring students’ progress by supporting and challenging them in ways that maintain the cognitive demand of the problem are not supported in the 5P model. The interactional challenges imposed on the teacher seem to be higher in the monitoring of the students’ progress compared to the monitoring of the students’ variety of solutions (the latter of which is strongly supported by the model). While monitoring how to use the students’ variety of solutions certainly includes asking questions to probe the students’ thinking in order to understand and figure out how to make use of their different ideas in class discussion, monitoring students’ progress includes asking questions to support and challenge students’ thinking and responding to their questions without telling them too much.

Teaching actions to support teachers in the challenge of monitoring students’ progress without lowering the cognitive demand include (see Paper VI):

1. responding to students’ questions “by probing their reasoning or asking them to check the correctness themselves”, and
2. allowing students “to debate and resolve disagreements” (Harel & Rabin, 2010, p. 18).

The above teaching actions are regularly used by the proficient teacher (see Papers IV and VI) and are likely to promote nonauthoritative argumentation (Harel & Rabin, 2010), i.e. that argumentation is based on mathematical rigor and not on the authority of the teacher.

6.1.4 Practice 4: Selecting and sequencing

Facilitation of both argumentation and connection-making needs to be taken into account when deciding upon how to select and sequence student solutions for class discussion. The emphasis in Stein et al.’s (2008) model is on selecting
and sequencing with the aim of facilitating connection-making. Regarding argumentation, it is suggested in Paper VI that the Selecting and sequencing practice is sensitized to the roles of incorrect, elegant and common student solutions for the purpose of argumentation. More precisely, the teacher can start the discussion with an incorrect or incomplete solution as a springboard for argumentation (cf. Bray, 2011). Further, the teacher can preferably save an elegant solution to be discussed last, as a concluding highlight of the discussion (Shimizu, 2006) since an early introduction of an elegant solution may make the rest of the solutions appear dull in comparison. In addition, Stein et al.’s (2008) suggestion to start with a solution that the majority of the class has used is problematized because an early confirmation that what most students in class have done is correct may compromise argumentation.

Regarding connection-making, the importance of considering the potential to generalize to key mathematical ideas for each solution that is selected for discussion is emphasized here (see Paper IV). Moreover, if solutions are sequenced from easier towards more difficult to understand (cf. goal of accessibility in Stein et al., 2008), it can facilitate the connection of later, more complex solutions to earlier, easier-to-understand solutions. A related suggestion that is made in Silver et al. (2005) is to sequence student solutions based on progression in sophistication (from lower to higher sophistication). In Paper IV, the following principles for sequencing student solutions are suggested based on analyses of lessons and interviews with the proficient teacher (p. 103):

1. an incorrect solution that seems reasonable that gives rise to argumentation (cf. common misconception in Stein et al., 2008)
2. a correct solution that is well structured with each step written where you can easily follow the whole line of thought (cf. goal of accessibility in Stein et al., 2008)
3. different solutions that show variety among solution strategies and representations with the potential to generalize to key mathematical ideas carefully considered, sequenced as more and more difficult to understand
4. (a solution that a majority of the students have made)
5. an elegant solution that makes the problem appear easy

The above suggestions for selecting and sequencing aim at facilitating both argumentation and connection-making in the subsequent whole-class discussion. Stein et al. (2008) state that principles for sequencing need much more research and that this is a fruitful area for future research (see section 6.6).
6.1.5 Practice 5: Connecting and consensus-building

The focus in Stein et al.’s (2008) last practice is on making mathematical connections. In Papers IV, V and VI, it is argued for the need to take into account argumentation in addition to connection-making in the last practice (and hence in all previous practices since the practices build on one another). Argumentation among students is important in order to build consensus in the class on ways to solve a mathematical problem correctly. Building consensus in the class seems to be a key for creating opportunities for all students to participate in the whole-class discussion. Therefore, it is suggested that the last practice is labelled *Connecting and consensus-building*.

It is a big challenge to create an inquiry/argument classroom culture in Wood et al.’s (2006) terms in which students ask questions to understand, indicate disagreement and collaborate to reach shared consensus (see section 2.3.1). Norms of collaboratively trying to make sense of one another’s solutions and making details explicit in explanations and questions (Franke et al., 2007) to make them accessible to other members in the classroom community are crucial for an inquiry/argument classroom culture. The sociomathematical norm of what constitutes an acceptable mathematical explanation (Yackel & Cobb, 1996) is tightly and mutually coupled with students’ interaction in asking questions (inquiring) for clarification in order to understand one another’s reasoning. Correspondently, the sociomathematical norm of how mathematical correctness is established (Harel & Rabin, 2010) is tightly and mutually coupled with students’ interaction in making mathematical arguments for and against the correctness of one another’s solutions.

Two interaction patterns – used routinely by the proficient teacher (see Papers V and VI) – that teachers can use regularly in class discussions to reinforce the above norms are:

1. also asking other students than the author(s) of a solution for detailed explanations and justifications (cf. Fraivillig et al., 1999), and
2. checking the class for consensus (cf. Wood et al., 2006) to open up for their questions and comments.

In Paper VI, it is suggested that the last practice in the 5P model is sensitized to these two interaction patterns to facilitate argumentation and collaboration.

It is also a big challenge to make different kinds of mathematical connections. It seems likely that if the last practice is sensitized to different kinds of connections that can be made, it will facilitate for teachers to help their students to draw important connections. Stein et al. (2008) use rather general terms such as “connections among different solution methods” (p. 319). An important point is emphasis on key mathematical ideas, which is evident in their statement “helping the class make mathematical connections between different students’ responses and between students’ responses and the key
ideas” (p. 321). The distinction between (1) connections between individual student responses and key ideas, and (2) connections between student responses is closely related to Ma’s (1999) distinction between (1) connections to more conceptually powerful ideas, and (2) connections to ideas of similar or less conceptual power. The former can be seen as connections with depth and the latter as connections with breadth (Ma, 1999). Elaborating on this distinction together with different kinds of connections in Lithner et al.’s (2010) framework for mathematical competencies result in the framework for different kinds of connections in Figure 5.

![Diagram of different kinds of mathematical connections](image)

*Figure 5. Different kinds of mathematical connections in whole-class discussions.*

The framework builds on the three kinds of connections between representations in Lithner et al.’s (2010) framework: (1) between representations of different entities, (2) between representations of the same entity, and (3) between different parts of one representation. The framework in Figure 5 also builds on the above distinctions made by Ma (1999) and Stein et al. (2008) as well as the latter’s statement that “teachers can help students draw connections between the mathematical ideas that are reflected in the strategies and representations that they use” (p. 330).

The last practice in the developed model is labeled “Connecting and consensus-building” to stress the teachers’ work of facilitating connection-making as well as consensus-building by argumentation in the class.

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32 Lithner et al.’s (2010) two kinds of connections between and within mathematical entities are excluded, since we must represent an abstract mathematical entity externally to be able to discuss it with others.
6.2 Emergence of a model for third-generation practice

The 5P model supports teachers to take whole-class discussions from the 1st generation practice of show-and-tell to the 2nd generation practice in which teachers use student ideas to highlight important mathematical ideas and relationships (Stein et al., 2008). 1st generation practice can also be viewed as strategy-reporting in contrast to an inquiry/argument classroom culture (Wood et al., 2006) in which students collaborate to build consensus on key ideas. Developments in previous research of 1st generation into 2nd generation practice has hence been made in two different directions, focusing on: (1) mathematical connection-making (Stein et al., 2008), and (2) mathematical inquiry and argumentation (Wood et al., 2006). By combining these two developments, a 3rd generation practice is beginning to see the light, see Figure 6. 3rd generation practice hence combines the strengths from 2nd generation practice as depicted by Stein et al. (2008) with the strengths from the inquiry/argument classroom culture as depicted by Wood et al. (2006).

Support for teachers to create an inquiry/argument classroom culture is scant in Stein et al.’s (2008) model. Taking the 5P model one step further towards supporting teachers to create 3rd generation practice is in essence the intention with my suggestions for developments to the model. In classrooms where 3rd generation practice prevails, whole-class discussions (1) focus on key mathematical ideas and relationships, and (2) build on students’ consensus-building of one another’s ideas. The role of students as listeners is as crucial in 3rd generation classroom practice as in inquiry/argument classroom culture. Listeners get opportunities to participate by “taking over the role of the teacher in questioning, clarifying, and validating mathematical ideas” (Wood et al., 2006, p. 235). This change in the role of the listening students is closely related to equity since classroom norms in third-generation practice require all listeners to be actively engaged. The salient feature of inquiry/argument classrooms (Wood et al., 2006) – that the whole class collaborates with the aim of reaching shared understanding and agreement – is prominent in 3rd generation practice. So is mathematical connection-making. By focusing on consensus-building by inquiry and argumentation as well as mathematical connections, the challenge of balancing content and participation can be better handled.
6.3 Research contributions

In research inspired by design research methodology (Cobb et al., 2003), refinements of teaching practice walks hand in hand with refinements of theory development. This thesis contributes to research on the theoretical development of tools (Cobb & Jackson, 2012) – more precisely Stein et al.’s (2008) 5P model – that support teachers in the challenges of orchestrating productive problem-solving whole-class discussions.

These challenges are widely agreed upon to be great (Adler & Davis, 2006; Boaler & Humphreys, 2005; Bray, 2011; Franke et al., 2007; Grant et al., 2009; Lampert et al., 2010; Lester & Lambdin, 2004; Stigler & Hiebert, 1999; Stein et al., 2008). Results from Papers I, II, III and VI in this thesis contribute to further distinguishing the kinds of challenges it encompasses to conduct problem-solving whole-class discussions. Results from Papers IV, V and VI contribute to suggestions on how challenges crucial for argumentation can be further supported in Stein et al.’s (2008) model. The overall research contribution is on development of the 5P model as a tool (Cobb & Jackson, 2012) to support teachers in the planning and enactment of problem-solving whole-class discussions that take into account both mathematical argumentation and connection-making. In other words, this thesis contributes to the development of tools that support 3rd generation practice.
6.4 Contributions to practice

This thesis contributes to refinements of teaching practice by further developing the 5P model as a useful tool for teachers in managing to conduct problem-solving whole-class discussions. Many teachers in Sweden are currently struggling to refine their teaching practice regarding whole-class discussions that are responsive to both students’ ideas and mathematics. With the help of the 5P model, teachers can work collaboratively at their schools over time to gradually refine their planning and enactment of problem-solving discussions that focus on mathematical connections. By using the developed model suggested in this thesis, teachers can gain stronger support regarding how to facilitate students’ argumentation in these whole-class discussions.

Professional development programs can also take into account aspects found crucial for supporting teachers in their challenges. The results from this thesis could be used to develop more hands-on support materials. To strengthen the support further, curriculum developers can work out materials that focus on planning and enactment of problem-solving whole-class discussions according to the developed model. That is, the curriculum materials could detail possible problem-specific solution strategies (both correct and incorrect) for different kinds of mathematical problems. Based on that, the material could provide questions that can be posed to students during monitoring with the aim of both getting variety of solutions and using the variety productively. The material could also suggest mathematical connections that can be made between ideas in these specific strategies and to key mathematical ideas, and suggest ways to facilitate argumentation from the strategies.

6.5 Critical reflection on the studies

This thesis provides insight into the teacher’s role in orchestrating problem-solving whole-class discussions. Here, my role as a researcher and the methodology of this thesis is discussed.

6.5.1 Researcher’s role

My role as a researcher in the intervention studies (see Papers I, II, III and VI) needs to be conceptualized and discussed. The collaborative, combined research- and developmental projects with interventions were inspired by design research methodology (Cobb et al., 2003) and were conducted in cycles in which I as a researcher participated actively. Hence, as a researcher I took an active role in informing practice and was part of the process.

In Paper I we conclude that in collaboration projects “we regard it as important that not only the teacher’s way of acting becomes an explicit object of study but also that the researcher’s role and responsibilities are conceptualized
and analyzed” (p. 112). In Paper III we start to conceptualize the researcher’s role by making three suggestions for the researcher’s responsibility: (1) “to establish a professional discourse of how to denote important components of teaching mathematics through problem solving” (p. 432), e.g. the five practices model, different forms of mathematical representations, and strands of mathematical proficiency (NRC, 2001), (2) “to introduce and discuss underlying assumptions guiding the project” (p. 432), e.g. sequencing student solutions to facilitate access to the discussion and to optimize opportunities for connection-making, (3) to discuss integration of procedures in productive ways, and (4) to “discuss important aspects of the three phases launch, explore and discuss-and-summarize” (p. 433).

One conclusion regarding the researcher’s role made in Paper II is that it is crucial as a researcher in an intervention project to consider the learning of both students and teachers. One question to consider is whether the project should optimize the students’ or the teacher’s learning (see Paper II). We further highlight the importance of the researcher’s explicit introduction of appropriate frameworks and vocabulary to create a productive discussion with teachers, so that teachers can create a productive discussion with their students (see Paper II). The 5P model is such a framework that is close to practice. It is also the responsibility of the researcher to bring classroom social and sociomathematical norms to the fore when discussing mathematics teaching through problem solving, since the norms differ to a great extent from the typical norms of Swedish classrooms (see Paper II).

By reflecting on my role as a researcher in the two initial intervention projects, I also developed as a researcher. In the third intervention with a team of teachers at one school (see Paper VI), I was actually very explicit regarding several of the aspects mentioned above. I managed to establish a professional discourse regarding frameworks and vocabulary for teaching mathematics through problem solving. Components that were explicitly denoted through the teachers’ reading of literature and reflected upon together in our meetings were: (1) the 5P model, (2) aspects important for equity in the launch, explore and discuss-and-summarize phases (Jackson & Cobb, 2010), (3) different forms of representations (Hagland et al., 2005), (4) social and sociomathematical norms (Yackel & Cobb, 1996), and (5) underlying key assumptions in the project such as the importance of not lowering the cognitive demand of the mathematical problem. However, there are always aspects to consider in relation to my role as a researcher in intervention projects. If I as a researcher had pressed the teachers (Coburn & Russell, 2008) more to make detailed anticipations of their students’ specific strategies, it would have been more likely that the teachers would not have stopped at making rather general anticipations. Nevertheless, I argue that the need for another person to point out the importance of detailed anticipations speaks for the benefits of sensitizing the Anticipating practice in Stein et al.’s (2008) model to detailed anticipations of
problem-specific strategies (as suggested in Paper VI). Similarly, if I as a researcher had put even more emphasis on different kinds of mathematical connections that can be made (see Paper IV), it would have been more likely that the teachers would have helped their students to make more kinds of connections in whole-class discussions. I actually did suggest different kinds of connections when we planned lessons together, but if the nature of different kinds of connections had been stated more clearly the teachers would have gained stronger support. This means that if I as a researcher had introduced a vocabulary for different kinds of connections it could have helped the teachers further in making connections. This is in line with the conclusion from Paper II that it is crucial that the researcher explicitly introduces appropriate frameworks and vocabulary to create a productive discussion with teachers. As with anticipations, I argue, however, that the need for another person to point out the different kinds of connections that can be made speaks for the benefits of sensitizing Stein et al.’s (2008) model to different kinds of connections (see section 6.1.5).

As a researcher, I have developed substantially during the course of this doctoral thesis project, which also means that my contributions as researcher in the two earlier intervention studies were different than in the third intervention study. In the third intervention as well as in the study of the proficient teacher, I was more inclined to ask in-depth follow-up questions during interviews with the teachers. I was also more competent at creating a relaxed atmosphere during the interviews. In the classrooms, I was more skilled at handling video- and sound equipment in ways that disturbed teachers and students as little as possible during lessons while ensuring that the necessary data were collected. Regarding support, I gradually became more competent in helping the teachers to grasp the key elements of the 5P model. In particular, observing and interviewing the proficient teacher deepened my own understandings of important aspects of the orchestration of whole-class discussions. I could use these deeper understandings in the third intervention to discuss rationales for different kinds of decisions with the teachers, thereby deepening the support to the teachers.

6.5.2 Methodological discussion
Here, I reflect upon how aspects of the methodological set up affect the results and conclusions of this thesis. To be able to investigate my research questions in depth, I have made case studies of some particular teachers. The analyses of the practices of the proficient teacher reported in Paper IV, V and VI as well as in this kappa should be seen as an examination of a unique case (Yin, 2003), see section 3.4.1. The reasoning and moves of this particular teacher cannot be assumed to function in the same way for all teachers, students and schools. One methodological aspect that strengthens my conclusions about support for teachers is that I have put key practices from research and from
the proficient teacher to use in the third intervention project. In this school, 
the teachers work under different circumstances than the proficient teacher 
does, but still many elements of her teaching practices are found to be useful. 
Nevertheless, the teachers in my intervention projects are also particular 
teachers working in particular circumstances. Although these cases can be 
considered to be rather more typical cases (Yin, 2003) for Swedish teachers 
striving to learn to orchestrate productive problem-solving whole-class dis-
cussions (see section 3.4.1), there are as many subtle variations as there are 
cases. Worth mentioning here is that the teachers in my intervention projects 
are teachers who have an incentive to develop themselves as teachers. They 
are also experienced teachers with 5-30 years of teaching experience. It is pos-
sible that the results would have been different if the teachers in the interven-
tions were aiming less for developing their capacity to conduct problem-solv-
ing discussions. Likewise, it is possible that the results would have been dif-
f erent if challenges and support for recently graduated mathematics teachers 
had been investigated. However, studying experienced teachers who have in-
centives to learn to teach mathematics through problem solving was a delib-
erate choice (see section 3.4.3.1).

The teachers whom I have collaborated with in my different empirical stud-
ies work in different circumstances among themselves. Still, there are similar-
ities in the challenges that they encounter. I believe – like Lampert (2001) – 
that there are problems of teaching practice that are common across differ-
ences in schools. Besides the experiences gained from my research projects, I 
have collaborated with many teachers in many different schools. Anecdotally, 
I have both seen for myself and heard from teachers that the instructional prac-
tices from the 5P model can function similarly in many different settings, with 
similar challenges tied to the practices. Right now, teachers all over Sweden 
are participating in a nation-wide professional development called the Boost 
for Mathematics (in Swedish: Matematiklyftet). The problem-solving module 
for teachers in grade 7-9, which I have been responsible for developing, has 
the 5P model as the backbone for planning, implementing and reflecting upon 
problem-solving lessons. This means that many mathematics teachers in Swe-
den already have used or will get to use the 5P model in their teaching. Like-
wise, teachers from different parts of Sweden have used the 5P model in in-
service courses held by our university. Many of the challenges that they ex-
press are similar to the challenges concluded upon in this thesis. For example, 
they often express that it is very challenging to launch a problem without steer-
ing the students towards certain solution strategies. The challenge of not tell-
ing students too much during monitoring, e.g. whether students’ solutions are 
correct or not, is also commonly expressed. Moreover, it is very common that 
teachers in in-service courses anticipate student solutions in general terms. 
These experiences could indicate that the challenges found in my case studies 
are more broadly experienced by teachers, which would imply that the sug-
gestions for support could be useful for many teachers.
6.6 Future research

An interesting continuation of this research would be to investigate the results on challenges and support from this thesis in other instructional settings in mathematics, besides problem-solving class discussions. That is, in what ways are the results on challenges and support from this thesis applicable for conceptually oriented class discussions, or for class discussions of students’ different strategies for mental calculation? What aspects are useful for each kind of discussion? In other words, how are the challenges that this thesis casts light upon similar to the challenges that teachers encounter when they conduct other kinds of mathematical class discussions? How does the kind of support teachers need differ for the different kinds of class discussions?

Testing the developed model suggested in this thesis in its entirety in different school contexts (cf, Hiebert & Grouws, 2007) could contribute to valuable insights regarding the development of supportive tools for teachers. Investigating teaching practices in different contexts could be worthwhile to differentiate the teaching practices to the surroundings in which they are enclosed (Franke et al., 2007). Researching teachers’ use of the developed model in relation to different cultural views on the teacher’s role would be worthwhile for understanding more about how different cultural contexts afford and constrain the use of the model. Similarly, researching teachers’ use of the model in relation to teachers’ mathematical knowledge for teaching (Ball, Thames & Phelps, 2008) would be interesting, in particular teachers’ knowledge about student thinking (including common misconceptions).

Further research at the level of the different practices in the developed model could be another way of proceeding. Stein et al. (2008) state that principles for sequencing student solutions need much more research. This thesis contributes to this area of research by suggesting principles for selecting and sequencing for the purpose of argumentation besides connection-making. An interesting area for future research would be to investigate and further develop the principles for selecting and sequencing put forward in this thesis. Additionally, looking further into how teachers can be supported in the anticipation of problem-specific strategies, as suggested in this thesis, could be an important area for future research. Likewise, how to proceed in order to support teachers in the productive launching of problems would be a worthwhile continuation. Looking deeper into how teachers can be further supported in the challenging interactional aspects of the monitoring practice or the connecting and consensus-building practice is another suggestion for future research.

We can conclude that the research area of developing supportive tools for teachers to interact productively in the mathematics classroom is relatively young, and that there are still many important questions to answer. This thesis contributes to this emerging research area by theorizing challenges and support for teachers in their orchestration of mathematically productive whole-class discussions in the problem-solving classroom.
utvecklingar av modellen med fem praktiker. Avhandlingen bidrar till forskning om teoretisk utveckling av verktyg som stödjer lärare i utmaningarna att leda givande problemlösningsdiskussioner i helklass.
8 References


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9 Appendix A. Pre- and postobservation interview questions

The pre- and postobservation interview questions that I asked the teachers in the third intervention project as well as the proficient teacher are grounded in the five practices in Stein et al.’s (2008) model (see below). Additionally, I asked questions related to the specific lesson that was observed.

Prior to each lesson I asked the teacher the following questions:
  - What are the goals of today’s lesson?
  - Which mathematical ideas do you view as central in the chosen problem?
  - What kinds of student solutions (strategies, representations) do you anticipate in this class?
  - How do you plan to introduce the problem?

After every lesson I asked the teacher the following questions:
  - Can you please comment on the whole-class discussion that you just had? How would you describe the outcome of the whole-class discussion?
  - How did you go about to create a mental picture of the different student solutions that were represented in the class?
  - Which student solutions (strategies, representations) were represented in the class and how did they relate to the student solutions that you anticipated before the lesson?
  - How did you reason when you selected and sequenced student solutions for whole-class discussion?
  - How did you help the students to connect their ideas to one another’s ideas and to central mathematical ideas and relationships?
  - How did you support students’ argumentation?
  - What do you believe your students learned/did not learn? What did you learn?
  - Additional, specific questions based on what I observed during the lesson.