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Asymptotic expansion of the expected discounted penalty function in a two-scale stochastic volatility risk model

by

Mahamadi Ouoba

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DIVISION OF APPLIED MATHEMATICS
MÄLARDALEN UNIVERSITY
SE-721 23 VÄSTERÅS, SWEDEN

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Author:
Mahamadi Ouoba

Supervisor:
Anatoliy Malyarenko

Reviewer:
Ying Ni

Examiner:
Linus Carlsson

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Abstract

In this Master thesis, we use a singular and regular perturbation theory to derive an analytic approximation formula for the expected discounted penalty function. Our model is an extension of Cramer–Lundberg extended classical model because we consider a more general insurance risk model in which the compound Poisson risk process is perturbed by a Brownian motion multiplied by a stochastic volatility driven by two factors- which have mean reversion models. Moreover, unlike the classical model, our model allows a ruin to be caused either by claims or by surplus' fluctuation.

We compute explicitly the first terms of the asymptotic expansion and we show that they satisfy either an integro-differential equation or a Poisson equation. In addition, we derive the existence and uniqueness conditions of the risk model with two stochastic volatilities factors.

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Contents

Chapter 1

Introduction

A Swedish actuary Filip Lundberg grounded collective risk theory in his PhD thesis [16] in 1903. Harald Cramér developed Lundberg's contribution into a rigourous mathematical theory in two monographs published in 1930 and 1955 and republished in his collected works [20]. They defined the classical risk model as follows:

Let $u \geq 0$ be the initial surplus of the insurer. Assume that the premiums are received continuously at a rate c per unit time. The aggregate claims constitute the compound Poisson process

$$S_t = \sum_{i=1}^{N(t)} Y_i,$$

where $\{ Y_i \colon i \geq 1 \}$ is a sequence of independent and identically distributed positive random variables representing the claim size, and $N(t)$ is a Poisson process with parameter $\lambda > 0$, independent of the process $\{ Y_i \colon i \geq 1 \}$. Then

$$U_t = u + ct - S_t$$

is the surplus at time t or the classical risk model . The time of ruin is defined as

$$\tau = \inf\{ t \geq 0 \colon U_t \leq 0 \}$$

with $\tau = +\infty$ if the set is empty.

Dufresne and Gerber [6] considered the risk process perturbed by a multiple of the standard Brownian motion W_t:

$$U_t = u + ct - S_t + \sigma W_t,$$

where the additional term may represent the future uncertainty of either aggregate claims or premium incomes, or the fluctuation of investment of surplus.

Gerber and Shiu [15] introduced an Expected Discounted Penalty Function (EDPF) that depends on the time of ruin, the surplus immediately before ruin, and the deficit at ruin:

$$\Psi(u) = \mathsf{E}[e^{-\Delta\tau}\omega(U_\tau)I(\tau < \infty) \,|\, U_0 = u],$$

where $\Delta > 0$ is the force of interest, $I(B)$ is the indicator function of the B, and $\omega(x)$, $x \le 0$, is a non-negative *penalty function* defined on $(-\infty, 0]$.

In a different line of research, Black and Scholes [2] proposed their famous model of the price process of a risky security:

$$dS_t = \mu\, dt + \sigma\, dW_t,$$

with constant volatility σ. Fouque et al [9] generalized Black–Scholes model by modelling volatility as a function of an underlying Ornstein–Uhlenbeck process $\{Y_t, t \ge 0\}$:

$$dY_t = \alpha(m - Y_t)\, dt + \beta\, dZ_t,$$

where $\alpha > 0$ is the rate of mean reversion, m is the long-run mean level of Y_t, Z_t is a standard Brownian motion correlated with W_t:

$$\mathsf{E}(W_s Z_t) = \rho \min\{s, t\}, \qquad \rho \in [-1, 1].$$

Chi et al [3] combined the models by Dufresne and Gerber [6] and by Fouque et al [9] into the following model:

$$\begin{aligned} dU_t &= c\, dt - dS_t + f(Y_t)\, dW_t, \qquad U_0 = u, \\ dY_t &= \alpha(m - Y_t)\, dt + \beta\, dZ_t, \qquad Y_0 = y_0. \end{aligned}$$

They introduced their own expected discounted penalty function

$$\Phi(u, y) = \mathsf{E}[e^{-\Delta\tau}\omega(U_\tau)I_{\tau<\infty} \,|\, U_0 = u, Y_0 = y],$$

and proved that it satisfies the integro-differential equation

$$\begin{aligned} (\mathcal{L} - \Delta)\Phi(u, y) &= 0, \quad u > 0, \\ \Phi(u, y) &= \omega(u), \quad u \le 0, \end{aligned} \tag{1.1}$$

where $\mathcal{L}$ is a very complicated integro-differential operator which we do not reproduce here. Instead of trying to find the exact solution of (1.1), they determine the first few terms of an asymptotic expansion of $\Phi(u, y)$ in powers of $\sqrt{\varepsilon}$, where $\varepsilon = 1/\alpha$.

The aim of this thesis is to realise the program of Chi et al [3] for the model similar to that of Fouque et al [10] with *two* stochastic volatilities (see Equation 2.1 below). In the literature, there are many research papers about either the multiscale stochastic volatility model for pricing financial instruments or the EDPF approximation using methods such as the Laplace transform or others. However, our approach is quite new in term of using the regular and singular theory to study the asymptotic expansion of the EDPF. We compute explicitly the existence and uniqueness conditions of our risk model. In addition, the main result of our thesis is the asymptotic expansion of the EDPF we obtained, and the IDE or the Poisson equation that the first terms of the expansion satisfy.

The rest of the paper is organized as follows. In the section 2, we formulate the risk model with the multiscale stochastic volatility. In section 3, we firstly present the existence and uniqueness conditions of the SDE that our risk model satisfies. Secondly, we use the regular and singular perturbation method to perform the asymptotic expansion of the EDPF and, finally a conclusion and a summary of the thesis's objectives are presented.

Chapter 2

Model formulation

Let $\{ Y_i \colon i \geq 1 \}$ denote the sequential individual claim amounts, which are independent and identically distributed positive random variables with common distribution μ and continuous probability density function g. Assume also that $\eta = \mathsf{E}[Y_1] = \int_0^\infty sg(s)ds < \infty$, and let ν be the measure corresponding to Y_1. Define the total number of individual claims, with any size, up to time t as $N(t,.)$, a Poisson process with parameter $\lambda > 0$, with Lèvy measure ν such that $\nu(B) = \mathsf{E}[N(1,B)]$, which is the average number of claims per unit of time, and with size belonging to the Borel set $B \subseteq \mathbb{R}$. $N(t,.)$ is independent of the individual claims amounts $\{ Y_i \colon i \geq 1 \}$.

Let u be initial surplus and $\theta > 0$ be the relative safety loading (the minimum value such that the received premium assures the survival of the insurance company) and let $c = (1+\theta)\lambda\eta$ be the premium rate.

By denoting $N(dt, ds)$, the number of jumps of size ds occurring during the infinitesimal time interval dt, let $\tilde{N}(dt, ds) = N(dt, ds) - \nu(ds)\,dt$ be the compensated Poisson measure of the process $S_t = \sum_{i=1}^{N(t)} Y_i$, which is the aggregate amount of claims.

We assume that the surplus process $\{U_t; t \geq 0\}$ follows the model:

$$\begin{cases} dU_t = c\,dt + f(Y_t, Z_t)d\widetilde{W}_{0,t} - \displaystyle\int_{-\infty}^{\infty} s\tilde{N}(dt, ds), \qquad U_0 = u, \\ dY_t = \left(\dfrac{1}{\varepsilon}\alpha(Y_t) - \dfrac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda_1(Y_t, Z_t)\right)dt + \dfrac{1}{\sqrt{\varepsilon}}\beta(Y_t)d\widetilde{W}_{1,t}, \qquad Y_0 = y_0, \\ dZ_t = \left(\delta c(Z_t) - \sqrt{\delta}g(Z_t)\Lambda_2(Y_t, Z_t)\right)dt + \sqrt{\delta}g(Z_t)d\widetilde{W}_{2,t}, \qquad Z_0 = z_0. \end{cases} \tag{2.1}$$

The volatility $\sigma_t = f(Y_t, Z_t)$ of the surplus process $\{U_t; t \geq 0\}$ is driven by two factors, Y_t as a fast factor and Z_t as a slow factor, which are mean reversion factors. The idea of mean reversion model is that any very high or low value (relative to the long term mean) of the process is just temporary. The process always converges to its long term mean value. In other words, with higher probability, there is a tendency for the insurance surplus volatility to drop when it is high and to rise when it is low.

The function $f(y,z)$ is a smooth, positive, increasing function in the second variable and the square of $f(y,z)$ is integrable with respect to the distribution of Y.

The positive parameters $\frac{1}{\varepsilon}$ and δ are respectively the mean reversion rate of the fast volatility factor Y_t and the slow varying volatility factor Z_t. Their inverse(ε,$\frac{1}{\delta}$) correspond to the time scale of the processes Y_t and Z_t.

Λ_1 and Λ_2 are two variables functions representing the total risk premiums, which are the market price volatility risks from the two sources of randomness.

Under the real probability measure, $\alpha(y)$ and $c(z)$ represent the drift coefficients and $\beta(y)$, $g(z)$ are the diffusions coefficients.

The correlation matrix of the three Wiener processes driven, respectively by the collective risk process and the volatility factors is:

$$\begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_{12} \\ \rho_2 & \rho_{12} & 1 \end{pmatrix}$$

To ensure that the correlation matrix is positive definite we require $|\rho_{12}| < 1$, $|\rho_2| < 1$, $|\rho_1| < 1$ and $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$.

Using Cholesky decomposition (as our correlation matrix is positive definite), Brownian motions $\widetilde{W}_{0,t}$, $\widetilde{W}_{1,t}$, and $\widetilde{W}_{2,t}$ may be expressed as a linear combination of independent Brownian motions as follows:

$$\begin{pmatrix} \widetilde{W}_{0,t} \\ \widetilde{W}_{1,t} \\ \widetilde{W}_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \widetilde{\rho}_{12} & \sqrt{1-\rho_2^2-\widetilde{\rho}_{12}^2} \end{pmatrix} \begin{pmatrix} W_{0,t} \\ W_{1,t} \\ W_{2,t} \end{pmatrix}$$

where $\widetilde{\rho}_{12} = \frac{\rho_{12}-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}$. Now, we can re-write our model in terms of three independent standard Brownian motions $(W_{0,t}, W_{1,t}, W_{2,t})$ as

$$
\begin{cases}
\qquad dU_t = c\,dt + f(Y_t, Z_t)dW_{0,t} - \displaystyle\int_{-\infty}^{\infty} s\tilde{N}(dt, ds), \qquad U_0 = u, \\
\qquad dY_t = \left(\dfrac{1}{\varepsilon}\alpha(Y_t) - \dfrac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda_1(Y_t, Z_t)\right) dt \\
\qquad\qquad + \dfrac{1}{\sqrt{\varepsilon}}\beta(Y_t)\left(\rho_1\, dW_{0,t} + \sqrt{1-\rho_1^2}\, dW_{1,t}\right), \qquad Y_0 = y_0 \\
\qquad dZ_t = \left(\delta c(Z_t) - \sqrt{\delta} g(Z_t)\Lambda_2(Y_t, Z_t)\right) dt \\
\qquad\qquad + \sqrt{\delta} g(Z_t)\left(\rho_2\, dW_{0,t} + \widetilde{\rho}_{12}\, dW_{1,t} + \sqrt{1-\rho_2^2-\widetilde{\rho}_{12}^2}\, dW_{2,t}\right), \\
Z_0 = z_0.
\end{cases}
\tag{2.2}
$$

Let

$$
\tau = \inf\{\, t \geq 0 \colon U_t \leq 0 \,\}
$$

be the time of ruin, $\Delta > 0$ be the force of interest,and ω be a penalty function (non-negative function defined on $(-\infty, 0]$). The *expected discounted penalty function* is

$$
\Phi(u, y, z) = \mathsf{E}[\mathrm{e}^{-\Delta\tau}\omega(U_\tau)\mathbf{I}(\tau < \infty) \,|\, U_0 = u, Y_0 = y, Z_0 = z].
$$

The first step is to find the existence and uniqueness conditions for a solution of our model. Then we calculate the infinitesimal generator of $(U_t, Y_t, Z_t)^\top$. Finally, we use the singular and regular perturbation theory to get an asymptotic expansion of the EDPF.

Chapter 3

Solution

3.1 General integro-differential equation

Theorem 3.1.1 ([18]). ***Existence and uniqueness of solutions of Lévy Stochastic Differential Equations(SDEs).*** *Assume that our Lévy SDE in $\mathbb{R}^n$ with $X(0) = x_0 \in \mathbb{R}^n$ is defined on a filtered probability space (Ω, $\mathfrak{F}$,$(\mathfrak{F}_t)_{t\geq 0}, \mathsf{P}$) and has the following differential form:*

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t) + \int_{\mathbb{R}^n} \gamma(t, X(t^-), s)\tilde{N}(dt, ds) \quad (3.1)$$

where $\alpha : [0,T]\times \mathbb{R}^n \to \mathbb{R}^n$,$\sigma : [0,T]\times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\gamma : [0,T]\times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{n\times l}$ are measurable functions satisfying the following conditions

1. *There exists a constant $C_1 < \infty$ such that*

$$\| \sigma(t,x)\|^2 + |\alpha(t,x)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{\ell} |\gamma_k(t,x,s_k)|^2 \nu_k(ds_k) < C_1(1 + |x|^2)$$

   for all $x \in \mathbb{R}^n$ with $t \in [0,T]$.

2. *There exists a constant $C_2 < \infty$ such that*

$$\begin{aligned}&\|\sigma(t,x) - \sigma(t,y)\|^2 + |\alpha(t,x) - \alpha(t,y)|^2 \\ &\quad + \sum_{k=1}^{\ell} \int_{\mathbb{R}} |\gamma_k(t,x,s_k) - \gamma_k(t,y,s_k)|^2 \nu_k(ds_k) < C_2(|x-y|^2)\end{aligned}$$

   for all $x, y \in \mathbb{R}^n$ and $t \in [0,T]$.

Then there exists a unique càdlàg (right continuous with left limits) adapted solution $X(t)$ such that

$$\mathsf{E}[|X(t)|^2] < \infty$$

for all t.

When the three measurable functions are time-independent or time homogeneous, that is, $\sigma(t,x) = \sigma(x)$, $\alpha(t,x) = \alpha(x)$ and $\gamma(t,x,s) = \gamma(x,s)$, the solutions of Lévy SDEs are called jumps diffusions (or Lévy diffusions).

The two conditions of the above theorem are very important because the first one guarantees us that the solution of the SDE is finite, i.e, $|X(t,w)| < \infty$ for all $t \in [0,T]$ and the second one, i.e the Lipschitz condition ensures the uniqueness of the solution. By uniqueness, we mean that if there exists two continuous stochastic processes $Y_1(t,w)$ and $Y_2(t,w)$ satisfying the two conditions of the theorem and also solution of the SDE, then $Y_1(t,w)$= $Y_2(t,w)$ for all $t \in [0,T]$ almost surely(a.s).

In our model (see Equation 2.2 above), we have $n = 3$, $m = 3$, $l = 1$,

$$X_t = \begin{pmatrix} U_t \\ Y_t \\ Z_t \end{pmatrix}$$

and

$$x = \begin{pmatrix} u \\ y \\ z \end{pmatrix}.$$

Therefore the diffusion, drift and jump coefficients have respectively the following forms:

1. $\sigma(x) = \begin{pmatrix} f(y,z) & 0 & 0 \\ \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y) & \frac{1}{\sqrt{\varepsilon}}\sqrt{1-\rho_1^2}\beta(y) & 0 \\ \sqrt{\delta}g(z)\rho_2 & \sqrt{\delta}g(z)\widetilde{\rho}_{12} & \sqrt{\delta}g(z)\sqrt{1-\rho_2^2-\widetilde{\rho}_{12}^2} \end{pmatrix}$

2. $\alpha(x) = \begin{pmatrix} c \\ \frac{1}{\varepsilon}\alpha(y) - \frac{1}{\sqrt{\varepsilon}}\beta(y)\Lambda_1(y,z) \\ \delta c(z) - \sqrt{\delta}g(z)\Lambda_2(y,z) \end{pmatrix}$

3. $\gamma(x,s) = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$

$$
\begin{aligned}
\|\sigma(t,x)\|^2 &= \sum_{i,j} \sigma_{i,j}^2 \\
&= f^2(y,z) + \left(\frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y)\right)^2 + \left(\frac{1}{\sqrt{\varepsilon}}\sqrt{1-\rho_1^2}\beta(y)\right)^2 + \left(\sqrt{\delta}g(z)\rho_2\right)^2 \\
&\quad + \left(\sqrt{\delta}g(z)\widetilde{\rho}_{12}\right)^2 + \left(\sqrt{\delta}g(z)\sqrt{1-\rho_2^2-\widetilde{\rho}_{12}^2}\right)^2 \\
&= f^2(y,z) + \frac{1}{\varepsilon}\beta^2(y) + \delta g^2(z) \\
|\alpha(t,x)|^2 &= c^2 + \left[\frac{1}{\varepsilon}\alpha(y) - \frac{1}{\sqrt{\varepsilon}}\beta(y)\Lambda_1(y,z)\right]^2 + \left[\delta c(z) - \sqrt{\delta}g(z)\Lambda_2(y,z)\right]^2 \\
&= c^2 + \frac{1}{\varepsilon^2}\alpha^2(y) + \delta^2 c(z)^2 + \frac{1}{\varepsilon}\beta^2(y)\Lambda_1^2(y,z) + \delta g^2(z)\Lambda_2^2(y,z) \\
&\quad - 2\left[\frac{1}{\varepsilon^{3/2}}\alpha(y)\Lambda_1(y,z) + \delta^{3/2}c(z)g(z)\Lambda_2(y,z)\right] \\
&\int_{\mathbb{R}} \sum_{k=1}^{\ell} |\gamma_k(t,x,z_k)|^2 \nu_k(dz_k) = \int_{\mathbb{R}} z^2 \nu(dz) = \lambda \mathsf{E}[Y^2]
\end{aligned}
$$

If the first condition of the above theorem is satisfy, then there exists a $C_1 < \infty$ such that

$$
\begin{aligned}
f^2(y,z) + \frac{1}{\varepsilon}\beta^2(y) + \delta g^2(z) + c^2 + \frac{1}{\varepsilon^2}\alpha^2(y) + \delta^2 c(z)^2 + \frac{1}{\varepsilon}\beta^2(y)\Lambda_1^2(y,z) \\
+ \delta g^2(z)\Lambda_2^2(y,z) \\
-2\left[\frac{1}{\varepsilon^{3/2}}\alpha(y)\Lambda_1(y,z) + \delta^{3/2}c(z)g(z)\Lambda_2(y,z)\right] + \lambda \mathsf{E}[Y^2] \leq C_1(1+u^2+y^2+z^2)
\end{aligned}
$$

for all $u, y, z \in \mathbb{R}^n$.

By denoting

$$
x_1 = \begin{pmatrix} u_1 \\ y_1 \\ z_1 \end{pmatrix}
$$

and

$$
x_2 = \begin{pmatrix} u_2 \\ y_2 \\ z_2 \end{pmatrix}
$$

we have

1. $\sigma(t,x_1)-\sigma(t,x_2)$ is an lower triangular matrix with

$$\begin{aligned}
\sigma_{11}(t,x_1)-\sigma_{11}(t,x_2) &= f(y_1,z_1)-f(y_2,z_2),\\
\sigma_{21}(t,x_1)-\sigma_{21}(t,x_2) &= \frac{1}{\sqrt{\varepsilon}}\rho_1(\beta(y_1)-\beta(y_2)),\\
\sigma_{22}(t,x_1)-\sigma_{22}(t,x_2) &= \frac{1}{\sqrt{\varepsilon}}\sqrt{1-\rho_1^2}(\beta(y_1)-\beta(y_2)),\\
\sigma_{31}(t,x_1)-\sigma_{31}(t,x_2) &= \sqrt{\delta}\rho_2(g(z_1)-g(z_2)),\\
\sigma_{32}(t,x_1)-\sigma_{32}(t,x_2) &= \sqrt{\delta}\widetilde{\rho}_{12}(g(z_1)-g(z_2)),\\
\sigma_{33}(t,x_1)-\sigma_{33}(t,x_2) &= \sqrt{\delta}\sqrt{1-\rho_2^2-\widetilde{\rho}_{12}^2}(g(z_1)-g(z_2)).
\end{aligned}$$

2. $\alpha(t,x_1)-\alpha(t,x_2)=$

$$\begin{pmatrix} 0 \\ \frac{1}{\varepsilon}\Big(\alpha(y_1)-\alpha(y_2)\Big)-\frac{1}{\sqrt{\varepsilon}}\Big(\beta(y_1)\Lambda_1(y_1,z_1)-\beta(y_2)\Lambda_1(y_2,z_2)\Big) \\ \delta\Big(c(z_1)-c(z_1)\Big)-\sqrt{\delta}\Big(g(z_1)\Lambda_2(y_1,z_1)-g(z_2)\Lambda_2(y_2,z_2)\Big) \end{pmatrix}$$

3. $\gamma(t,x_1,s)-\gamma(t,x_2,s)=\begin{pmatrix}0\\0\\0\end{pmatrix}$

Therefore

$$\|\sigma(t,x)-\sigma(t,y)\|^2+|\alpha(t,x)-\alpha(t,y)|^2 < C_2((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2).$$

Now, since we know the conditions that ensure the existence and uniqueness of our SDE solution, our next natural step is to calculate the infinitesimal generator of the solution of the same SDE. But, first of all, let us remind ourselves of a definition of an infinitesimal generator. An infinitesimal generator $\mathcal{L}_X$ of the jump diffusion process X_t is defined as follows:

$$\mathcal{L}_X H(x)=\lim_{t\to 0}\frac{E_x[H(X_t,t)]-H(x,0)}{t}$$

if the limit exists , where

- $x\in\mathbb{R}^n$
- E_x is the expectation with the condition that $X(0)=x$
- $H:\mathbb{R}^n\to\mathbb{R}$ function and we denote D_A, the set of all functions for which the above limits exits

Theorem 3.1.2 ([18]). ***Infinitesimal generator of the diffusion process*** X_t
Assume that $H \in C_0^2(\mathbb{R}^n)$ *(therefore* $H \in \mathrm{D}_A$*) and consider the same SDE*

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t) + \int_{\mathbb{R}^n} \gamma(t, X(t^-), s)\tilde{N}(dt, ds)$$

with α, σ, γ *satisfying the linear growth and Lipschitz continuity conditions above Then we have:*

$$\begin{aligned}\mathcal{L}_X H(x) &= \sum_{k=1}^{n} \alpha_k(x)\frac{\partial H}{\partial x_k}(x) + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}(\sigma\sigma^T)_{kj}(x)\frac{\partial^2 H}{\partial x_k \partial x_j}(x) \\ &+ \sum_{k=1}^{\ell}\int_{\mathbb{R}}\Big[H(x+\gamma_k(x,s)) - H(x) - \nabla H(x).\gamma_k(x,s)\Big]\nu_k(ds_k)\end{aligned}$$

where

$$\nabla H(x) = \begin{pmatrix} \frac{\partial H}{\partial x_1}(x) \\ \frac{\partial H}{\partial x_2}(x) \\ \vdots \\ \frac{\partial H}{\partial x_n}(x) \end{pmatrix} \text{ is the gradient of the function } H \text{ .}$$

As the drift and diffusion coefficients of the SDE's solution do not depend explicitly on time, our infinitesimal generator is also time-homogeneous and give us the behaviour of the SDE during a very small time interval.

The first part of the generator is equal to the standard inner product between the drift coefficient and the gradient of the function H and is given by:

$$\begin{aligned}\alpha(x)\cdot\nabla H(x) &= \sum_{k=1}^{n}\alpha_k(x)\frac{\partial H}{\partial x_k}(x) \\ &= c\frac{\partial H}{\partial u} + \left(\frac{1}{\varepsilon}\alpha(y) - \frac{1}{\sqrt{\varepsilon}}\beta(y)\Lambda_1(y,z)\right)\frac{\partial H}{\partial y} \\ &+ \left(\delta c(z) - \sqrt{\delta}g(z)\Lambda_2(y,z)\right)\frac{\partial H}{\partial z}.\end{aligned}$$

The second part of the generator is a multiplication of the Hessian matrix of the function H with an another matrix, which itself is obtained by matrix product of our SDE diffusions coefficients matrix and its transpose. First of all, let us calculate $\sigma(x)\sigma(x)^\top$. By definition,

$$\sigma(x)\sigma(x)^\top = \begin{pmatrix} f(y,z) & 0 & 0 \\ \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y) & \frac{1}{\sqrt{\varepsilon}}\sqrt{1-\rho_1^2}\beta(y) & 0 \\ \sqrt{\delta}g(z)\rho_2 & \sqrt{\delta}g(z)\widetilde{\rho}_{12} & \sqrt{\delta}g(z)\sqrt{1-\rho_2^2-\widetilde{\rho}_{12}^2} \end{pmatrix}$$
$$\times \begin{pmatrix} f(y,z) & \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y) & \sqrt{\delta}g(z)\rho_2 \\ 0 & \frac{1}{\sqrt{\varepsilon}}\sqrt{1-\rho_1^2}\beta(y) & \sqrt{\delta}g(z)\widetilde{\rho}_{12} \\ 0 & 0 & \sqrt{\delta}g(z)\sqrt{1-\rho_2^2-\widetilde{\rho}_{12}^2} \end{pmatrix}.$$

Using the fact that $\widetilde{\rho}_{12} = \frac{\rho_{12}-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}$, we can rewrite the above product as

$$\sigma(x)\sigma(x)^\top = \begin{pmatrix} f^2(y,z) & \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y)f(y,z) & \sqrt{\delta}\rho_2 g(z)f(y,z) \\ \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y)f(y,z) & \frac{1}{\varepsilon}\beta^2(y) & \frac{1}{\sqrt{\varepsilon}}\sqrt{\delta}\rho_{12}\beta(y)g(z) \\ \sqrt{\delta}\rho_2 g(z)f(y,z) & \frac{1}{\sqrt{\varepsilon}}\sqrt{\delta}\rho_{12}\beta(y)g(z) & \delta g^2(z) \end{pmatrix}.$$

As $H \in C_0^2(\mathbb{R}^n)$, it is twice continuously differentiable , we can define its first and second partial derivatives and they are continuous.

with $x = \begin{pmatrix} u \\ y \\ z \end{pmatrix}$ the explicit expression of the Hessian matrix is:

$$\left(\frac{\partial^2 H}{\partial x_k \partial x_j}(x)\right)_{k,j} = \begin{pmatrix} \frac{\partial^2 H}{\partial^2 u}(x) & \frac{\partial^2 H}{\partial y\partial u}(x) & \frac{\partial^2 H}{\partial z\partial u}(x) \\ \frac{\partial^2 H}{\partial u\partial y}(x) & \frac{\partial^2 H}{\partial^2 y}(x) & \frac{\partial^2 H}{\partial z\partial y}(x) \\ \frac{\partial^2 H}{\partial u\partial z}(x) & \frac{\partial^2 H}{\partial y\partial z}(x) & \frac{\partial^2 H}{\partial^2 z}(x) \end{pmatrix}.$$

.

Thus, the second part (without the constant) of our infinitesimal generator is as follows:

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{n}(\sigma\sigma^T)_{kj}(x)\frac{\partial^2 H}{\partial x_k\partial x_j}(x) &= f^2(y,z)\frac{\partial^2 H}{\partial^2 u}(x) + \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y)f(y,z)\frac{\partial^2 H}{\partial y\partial u}(x) \\ &+ \sqrt{\delta}\rho_2 g(z)f(y,z)\frac{\partial^2 H}{\partial z\partial u}(x) + \frac{1}{\sqrt{\varepsilon}}\rho_1\beta(y)f(y,z)\frac{\partial^2 H}{\partial u\partial y}(x) + \frac{1}{\varepsilon}\beta^2(y)\frac{\partial^2 H}{\partial^2 y}(x) \\ &+ \frac{1}{\sqrt{\varepsilon}}\sqrt{\delta}\rho_{12}\beta(y)g(z)\frac{\partial^2 H}{\partial z\partial y}(x) + \sqrt{\delta}\rho_2 g(z)f(y,z)\frac{\partial^2 H}{\partial u\partial z}(x) \\ &+ \frac{1}{\sqrt{\varepsilon}}\sqrt{\delta}\rho_{12}\beta(y)g(z)\frac{\partial^2 H}{\partial y\partial z}(x) + \delta g^2(z)\frac{\partial^2 H}{\partial^2 z}(x).\end{aligned}$$

Since $\nu(ds) = \lambda\mu(ds)$ and also using the fact that $\ell = 1$ in our model, the last part of the generator is given by,

$$\sum_{k=1}^{\ell} \int_{\mathbb{R}} \left[H(x+\gamma_k(x,s)) - H(x) - \nabla H(x).\gamma_k(y,s) \right] \nu_k(ds_k)$$
$$= \int_{\mathbb{R}} \left[H(x+\gamma(x,s)) - H(x) - \nabla H(x).\gamma(x,s) \right] \nu(ds)$$
$$= \lambda \int_{\mathbb{R}} \left[H(u-s,y,z) - H(u,y,z) - s\frac{\partial H}{\partial u}(x) \right] \mu(ds)$$
$$= \lambda \int_{\mathbb{R}} \left[H(u-s,y,z) - s\frac{\partial H}{\partial u}(x) \right] \mu(ds) - \lambda H(u,y,z)$$
$$= \lambda \int_{\mathbb{R}} H(u-s,y,z)\mu(ds) - \lambda \left(\eta \frac{\partial H}{\partial u}(x) + H(x) \right)$$

Now, if we assume that the Expected Discounted Penalty Function (EDPF), $\Phi(u,y,z)$ is twice differentiable($\Phi \in C_0^2(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$), we can rewrite the infinitesimal generator as:

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3, \tag{3.2}$$

where

$$\mathcal{L}_0 = \alpha(y)\frac{\partial}{\partial y} + \frac{1}{2}\beta^2(y)\frac{\partial^2}{\partial^2 y},$$
$$\mathcal{L}_1 = -\beta(y)\Lambda_1(y,z)\frac{\partial}{\partial y} + \rho_1\beta(y)f(y,z)\frac{\partial^2}{\partial y\partial u},$$
$$\mathcal{L}_2 H(x) = (c-\lambda\eta)\frac{\partial H}{\partial u}(x) + \frac{1}{2}f^2(y,z)\frac{\partial^2 H}{\partial^2 u}(x)$$
$$+ \lambda \int_{\mathbb{R}} H(u-s,y,z)\mu(ds) - \lambda H(x),$$
$$\mathcal{M}_1 = -g(z)\Lambda_2(y,z)\frac{\partial}{\partial z} + \rho_2 g(z)f(y,z)\frac{\partial^2}{\partial z\partial u},$$
$$\mathcal{M}_2 = c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial^2 z},$$
$$\mathcal{M}_3 = \rho_{12}\beta(y)g(z)\frac{\partial^2}{\partial y\partial z}.$$

3.2 Asymptotic expansion of the expected discounted penalty function

Chi et al [3] have shown that the EDPF satisfies the following partial IDE with boundary condition.

$$\begin{cases} (\mathcal{L}^{\varepsilon,\delta} - \Delta)\Phi(u,y,z) = 0, u > 0 \\ \Phi(u,y,z) = \omega(u), u \leq 0 \end{cases} \tag{3.3}$$

However, the previous equation does not have an analytical solution, therefore we use asymptotic expansion of the EDPF. As the mean reversion rates δ, ε are small parameters and independent, we expand $\Phi(u,y,z)$ in powers of $\sqrt{\varepsilon}$ and $\sqrt{\delta}$ as follows:

$$\begin{aligned} \Phi^{\varepsilon,\delta} &= \sum_{k=1}^{n}\sum_{j=1}^{n}\left(\sqrt{\varepsilon}\right)^{i}\left(\sqrt{\delta}\right)^{j}\Phi_{i,j} \\ &= \Phi_0 + \sqrt{\varepsilon}\Phi_{1,0} + \sqrt{\delta}\Phi_{0,1} + \sqrt{\varepsilon\delta}\Phi_{1,1} + \varepsilon\Phi_{2,0} + \delta\Phi_{0,2} + \cdots . \end{aligned} \tag{3.4}$$

Here $\Phi_{i,j}$, are functions of (u,y,z). By inserting the above expansion of EDPF into Equation 3.3 and taking into account (3.2), we get

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2' + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3\right)\Phi(u,y,z) = 0$$

where $\mathcal{L}_2' = \mathcal{L}_2 - \Delta$ or

$$(\mathcal{L}^{\varepsilon,\delta} - \Delta)\left(\Phi_0 + \sqrt{\varepsilon}\Phi_{1,0} + \sqrt{\delta}\Phi_{0,1} + \sqrt{\varepsilon\delta}\Phi_{1,1} + \varepsilon\Phi_{2,0} + \delta\Phi_{0,2} + ...\right) = 0.$$

where $\mathcal{L}^{\varepsilon,\delta} - \Delta = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2' + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3$

If we rearrange the equation by putting coefficients with same orders in terms of $\sqrt{\varepsilon}$, $\sqrt{\delta}$, together, we obtain:

$$\begin{aligned}
0 =& \frac{1}{\varepsilon}\mathcal{L}_0\Phi_0 + \frac{1}{\sqrt{\varepsilon}}\Big(\mathcal{L}_0\Phi_{1,0} + \mathcal{L}_1\Phi_0\Big) + \Big(\mathcal{L}_0\Phi_{2,0} + \mathcal{L}_1\Phi_{1,0} + \mathcal{L}_2'\Phi_0\Big)\\
&+ \varepsilon\mathcal{L}_2'\Phi_{2,0} + \sqrt{\varepsilon}\Big(\mathcal{L}_1\Phi_{2,0} + \mathcal{L}_2'\Phi_{1,0}\Big)\\
&+ \sqrt{\delta}\Big(\mathcal{M}_1\Phi_0 + \mathcal{M}_3\Phi_{1,0} + \mathcal{L}_2'\Phi_{0,1} + \mathcal{L}_1\Phi_{1,1}\Big)\\
&+ \delta\Big(\mathcal{M}_1\Phi_{0,1} + \mathcal{M}_2\Phi_0 + \mathcal{M}_3\Phi_{1,1} + \mathcal{L}_2'\Phi_{0,2}\Big) + \delta^2\mathcal{M}_2\Phi_{0,2}\\
&+ \sqrt{\frac{\delta}{\varepsilon}}\Big(\mathcal{L}_0\Phi_{1,1} + \mathcal{L}_1\Phi_{0,1} + \mathcal{M}_3\Phi_0\Big)\\
&+ \sqrt{\delta\varepsilon}\Big(\mathcal{L}_2'\Phi_{1,1} + \mathcal{M}_1\Phi_{1,0} + \mathcal{M}_3\Phi_{2,0}\Big)\\
&+ \delta\sqrt{\varepsilon}\Big(\mathcal{M}_1\Phi_{1,1} + \mathcal{M}_2\Phi_{1,0}\Big) + \delta\sqrt{\delta}\Big(\mathcal{M}_1\Phi_{0,2} + \mathcal{M}_2\Phi_{0,1}\Big)\\
&+ \frac{\delta}{\sqrt{\varepsilon}}\Big(\mathcal{M}_3\Phi_{0,1} + \mathcal{L}_1\Phi_{0,2}\Big) + \varepsilon\sqrt{\delta}\mathcal{M}_1\Phi_{2,0} + \varepsilon\delta\mathcal{M}_2\Phi_{2,0}\\
&+ \sqrt{\varepsilon}\delta^{3/2}\mathcal{M}_2\Phi_{1,1} + \frac{\sqrt{\delta}}{\varepsilon}\mathcal{L}_0\Phi_{0,1} + \frac{\delta}{\varepsilon}\mathcal{L}_0\Phi_{0,2} + \frac{\delta^{3/2}}{\sqrt{\varepsilon}}\mathcal{M}_3\Phi_{0,2}.
\end{aligned}$$

As the order terms $\sqrt{\varepsilon}$, $\sqrt{\delta}$ are positive, we equate each coefficient to zero and get:

1. term of order $\frac{1}{\varepsilon}$: $\mathcal{L}_0\Phi_0 = 0$
   As $\mathcal{L}_0$ is an operator which contains only partial derivatives with respect to y in each of its components, therefore $\mathcal{L}_0\Phi_0 = 0$ means that Φ_0 is independent of y. That is $\Phi_0 = \Phi_0(u, z)$.
   In other words, If we solve the equation $\mathcal{L}_0\Phi_0 = 0$, we get the general solution
   $\Phi_0(u, y, z) = C_1(u, z)\int_{-\infty}^{y} e^{(t-\mu)^2/2\sigma^2}\,\mathrm{d}t + C_2(u, z)$.
   where $C_1(u, z)$ and $C_2(u, z)$ are independent functions of y.
   Since $\int_{-\infty}^{+\infty} e^{(t-\mu)^2/2\sigma^2}\,\mathrm{d}t = \infty$, and from the growth assumption (See Fouque et al [13]),$C_1(u, z)$ must be equal to zero, otherwise Φ_0 will grow exponentially as y goes to ∞.

2. term of order $\frac{1}{\sqrt{\varepsilon}}$: $\mathcal{L}_0\Phi_{1,0} + \mathcal{L}_1\Phi_0 = 0$
   Since $\mathcal{L}_1$ contains also partial derivative with respect to y and Φ_0 does not depend on y, therefore $\mathcal{L}_1\Phi_0 = 0$. Thus $\mathcal{L}_0\Phi_{1,0} + \mathcal{L}_1\Phi_0 = 0 \Rightarrow \mathcal{L}_0\Phi_{1,0} = 0$
   The last equality means that $\Phi_{1,0}$ does not depend on y. $\Phi_{1,0} = \Phi_{1,0}(u, z)$

3. term of order $\frac{\delta}{\varepsilon}$: $\mathcal{L}_0\Phi_{0,2} = 0$
   Similarly, $\mathcal{L}_0\Phi_{0,2} = 0 \Rightarrow \Phi_{0,2} = \Phi_{0,2}(u, z)$

4. term of order δ^2 : $\mathcal{M}_2\Phi_{0,2} = 0$
   $\mathcal{M}_2$ is an operator with first and second derivatives with respect to z and using the fact that $\Phi_{0,2}$ is already a function of two variables(u, z), thus $\mathcal{M}_2\Phi_{0,2} = 0 \Rightarrow \Phi_{0,2} = \Phi_{0,2}(u)$.

5. term of order $\frac{\sqrt{\delta}}{\varepsilon}$: $\mathcal{L}_0\Phi_{0,1} = 0$
   $\mathcal{L}_0\Phi_{0,1} = 0 \Rightarrow \Phi_{0,1} = \Phi_{0,1}(u, z)$

6. term of order $\delta\sqrt{\delta}$: $\mathcal{M}_1\Phi_{0,2} + \mathcal{M}_2\Phi_{0,1} = 0$
   The operator $\mathcal{M}_1$ involves in both of its components a partial derivatives with respect to variable z, but the function $\Phi_{0,2}$ is a function of the variable u only, therefore $\mathcal{M}_1\Phi_{0,2} = 0$ and $\mathcal{M}_1\Phi_{0,2} + \mathcal{M}_2\Phi_{0,1} = 0 \Rightarrow \mathcal{M}_2\Phi_{0,1} = 0$. If the first and the second derivatives of $\Phi_{0,1}$ with respect to z is zero, necessarily $\Phi_{0,1}$ is independent of z. Thus $\Phi_{0,1} = \Phi_{0,1}(u)$.

7. term of order $\sqrt{\varepsilon}\delta^{3/2}$: $\mathcal{M}_2\Phi_{1,1} = 0$
   $\mathcal{M}_2\Phi_{1,1} = 0 \Rightarrow \Phi_{1,1} = \Phi_{1,1}(u, y)$

8. term of order $\sqrt{\frac{\delta}{\varepsilon}}$: $\mathcal{L}_0\Phi_{1,1} + \mathcal{L}_1\Phi_{0,1} + \mathcal{M}_3\Phi_0 = 0$
   Since $\mathcal{L}_1\Phi_{0,1}$ and $\mathcal{M}_3\Phi_0$ are already equal to zero, thus $\mathcal{L}_0\Phi_{1,1} + \mathcal{L}_1\Phi_{0,1} + \mathcal{M}_3\Phi_0 = 0 \Rightarrow \mathcal{L}_0\Phi_{1,1} = 0 \Rightarrow \Phi_{1,1} = \Phi_{1,1}(u)$

9. term of order $\delta\sqrt{\varepsilon}$: $\mathcal{M}_1\Phi_{1,1} + \mathcal{M}_2\Phi_{1,0}$
   $\mathcal{M}_1\Phi_{1,1}+\mathcal{M}_2\Phi_{1,0} \Rightarrow \mathcal{M}_2\Phi_{1,0} = 0$ (because $\Phi_{1,1} = \Phi_{1,1}(u, y) \Rightarrow \mathcal{M}_1\Phi_{1,1} = 0$). Now, $\mathcal{M}_2\Phi_{1,0} = 0 \Rightarrow \Phi_{1,0} = \Phi_{1,0}(u)$

10. term of order $\varepsilon\delta$: $\mathcal{M}_2\Phi_{2,0} = 0$
    $\mathcal{M}_2\Phi_{2,0} = 0 \Rightarrow \Phi_{2,0} = \Phi_{2,0}(u, y)$

    To sum up:

    - $\Phi_{1,0} = \Phi_{1,0}(u)$, $\Phi_{0,1} = \Phi_{0,1}(u)$,$\Phi_{1,1} = \Phi_{1,1}(u)$,$\Phi_{0,2} = \Phi_{0,2}(u)$ do not depend on the current level of volatilities. They are functions only of the collective risk level.
    - $\Phi_0 = \Phi_0(u, z)$ is a function of two variables, which are the collective risk and the slow factor volatility levels.
    - $\Phi_{2,0} = \Phi_{2,0}(u, y)$ depends not only on the company surplus but also on the fast volatility level.

11. term of order ε^0 : $\mathcal{L}_0\Phi_{2,0} + \mathcal{L}_1\Phi_{1,0} + \mathcal{L}_2'\Phi_0 = 0$
As $\Phi_{1,0}$ does not depend on y, then $\mathcal{L}_1\Phi_{1,0} = 0$.
We obtain

$$\mathcal{L}_0\Phi_{2,0} + \mathcal{L}_1\Phi_{1,0} + \mathcal{L}_2'\Phi_0 = 0$$

which implies that

$$\mathcal{L}_0\Phi_{2,0} + \mathcal{L}_2'\Phi_0 = 0$$

The last equality is a Poisson equation for $\Phi_{2,0}$. The source term is $\mathcal{L}_2'\Phi_0$. The solvability condition (Chi et al [3] and therein) requires that the averaging of the source term $\mathcal{L}_2'\Phi_0$ with respect to the invariant distribution of Y_t must be zero.
In other words,

$$\langle \mathcal{L}_2'\Phi_0 \rangle = 0$$

The averaging operator $\langle \Pi \rangle$ for any function Π is defined as

$$\langle \Pi \rangle = \int_{\mathbb{R}} \Pi(y)\chi(dy),$$

where χ is the invariant distribution of Y_t.

$$\begin{aligned}
0 &= \langle \mathcal{L}_2'\Phi_0 \rangle \\
&= \langle \mathcal{L}_2' \rangle \Phi_0, \text{(because } \Phi_0 \text{ does not depend on } y) \\
&= \Big\langle (c - \lambda\eta)\frac{\partial \Phi_0}{\partial u}(x) + \frac{1}{2}f^2(y,z)\frac{\partial^2 \Phi_0}{\partial^2 u}(x) \\
&\quad + \lambda \int_{\mathbb{R}} \Phi_0(u-s,z)\mu(ds) - (\lambda + \Delta)\Phi_0(u,z) \Big\rangle \\
&= (c - \lambda\eta)\frac{\partial \Phi_0}{\partial u}(u,z) + \frac{1}{2}\left\langle f^2(y,z) \right\rangle \frac{\partial^2 \Phi_0}{\partial^2 u}(u,z) \\
&\quad + \lambda \int_{\mathbb{R}} \Phi_0(u-s,z)\mu(ds) - (\lambda + \Delta)\Phi_0(u,z).
\end{aligned}$$

The integro-differential equation(IDE)

$$\begin{aligned}
&(c - \lambda\eta)\frac{\partial \Phi_0}{\partial u}(u,z) + \frac{1}{2}\left\langle f^2(y,z) \right\rangle \frac{\partial^2 \Phi_0}{\partial^2 u}(u,z) \\
&\quad + \lambda \int_{\mathbb{R}} \Phi_0(u-s,z)\mu(ds) - (\lambda + \Delta)\Phi_0(u,z) = 0
\end{aligned}$$

shows that the first term of the asymptotic expansion satisfies the IDE of the classical perturbed compound Poisson process with volatility equal to

$$\left\langle f^2(y,z) \right\rangle = \int_{\mathbb{R}} f^2(y,z))\chi(dy)$$

Here χ is the invariant distribution of Y_t. Using Laplace transform, (Chi et al [3]), gave the probabilistic interpretation as well as the explicit solution. Now, with the fact that

$$\begin{aligned}\mathcal{L}_2'\Phi_0 &= \mathcal{L}_2'\Phi_0 - \langle\mathcal{L}_2'\rangle\,\Phi_0\\ &= \frac{1}{2}\Big(f^2(y,z) - \big\langle f^2(y,z)\big\rangle\Big)\frac{\partial^2\Phi_0}{\partial^2 u}(u,z).\end{aligned}$$

Then the Poisson equation $\mathcal{L}_0\Phi_{2,0} + \mathcal{L}_2'\Phi_0 = 0$ for $\Phi_{2,0}$ has solution given by

$$\begin{aligned}\Phi_{2,0} =& -\frac{1}{2}\mathcal{L}_0^{-1}\Big(\mathcal{L}_2'\Phi_0\Big)\\ =& -\frac{1}{2}\mathcal{L}_0^{-1}\Big(f^2(y,z) - \big\langle f^2(y,z)\big\rangle\Big)\frac{\partial^2\Phi_0}{\partial^2 u}(u,z)\\ =& -\frac{1}{2}\Big(\psi_1(y) + \kappa_1(u,z)\Big)\frac{\partial^2\Phi_0}{\partial^2 u}(u,z)\end{aligned}$$

where ψ_1 is a solution of $\mathcal{L}_0\psi_1 = f^2(y,z) - \langle f^2(y,z)\rangle$ and $\kappa_1(u,z)$ is an arbitrary function independent of y.

12. term of order $\delta : \mathcal{M}_1\Phi_{0,1} + \mathcal{M}_2\Phi_0 + \mathcal{M}_3\Phi_{1,1} + \mathcal{L}_2'\Phi_{0,2} = 0$
As above, $\mathcal{M}_1\Phi_{0,1} = 0$ and $\mathcal{M}_3\Phi_{1,1} = 0$ implies that

$$\mathcal{M}_2\Phi_0 + \mathcal{L}_2'\Phi_{0,2} = 0$$

This is a Poisson equation for Φ_0 with $\mathcal{L}_2'\Phi_{0,2}$ acting as a source term. If the solution Φ_0 exists, then the averaging operator condition with respect to the invariant distribution Z_t must be satisfied, i.e.,

$$\begin{aligned}0 =& \langle\mathcal{L}_2'\Phi_{0,2}\rangle\\ =& \langle\mathcal{L}_2'\rangle\,\Phi_{0,2}\\ =& \langle\mathcal{L}_2'\rangle\,\Phi_0, (\text{because}\Phi_{0,2}\text{does depend only on}u)\\ =& \Big\langle (c-\lambda\eta)\frac{\partial\Phi_{0,2}}{\partial u}(u) + \frac{1}{2}f^2(y,z)\frac{\partial^2\Phi_{0,2}}{\partial^2 u}(u)\\ &+\lambda\int_{\mathbb{R}}\Phi_{0,2}(u-s)\mu(ds) - (\lambda+\Delta)\Phi_{0,2}(u)\Big\rangle\\ =&(c-\lambda\eta)\frac{\partial\Phi_{0,2}}{\partial u}(u) + \frac{1}{2}\big\langle f^2(y,z)\big\rangle\frac{\partial^2\Phi_{0,2}}{\partial^2 u}(u)\\ &+\lambda\int_{\mathbb{R}}\Phi_{0,2}(u-s)\mu(ds) - (\lambda+\Delta)\Phi_{0,2}(u).\end{aligned}$$

Thus, the correction term $\Phi_{0,2}$ satisfies the following integro-differential equation.

$$(c-\lambda\eta)\frac{\partial\Phi_{0,2}}{\partial u}(u)+\frac{1}{2}\left\langle f^2(y,z)\right\rangle\frac{\partial^2\Phi_{0,2}}{\partial^2 u}(u)$$
$$+\lambda\int_{\mathbb{R}}\Phi_{0,2}(u-s)\mu(ds)-(\lambda+\Delta)\Phi_{0,2}(u)=0,$$

where

$$\left\langle f^2(y,z)\right\rangle=\int_{\mathbb{R}}f^2(y,z))\chi_2(dz)$$

Here χ_2 is the invariant distribution of Z_t

13. term of order $\sqrt{\varepsilon}$:
$\mathcal{L}_0\Phi_{2,0}+\mathcal{L}_2'\Phi_{1,0}=0$ is also a Poisson equation for $\Phi_{1,0}$ with respect to to $\mathcal{L}_0$, and $\mathcal{L}_2'\Phi_{1,0}$ as a source term. As above the centering condition has to be satisfied

$$\begin{aligned}0&=\langle\mathcal{L}_2'\Phi_{1,0}\rangle\\&=\langle\mathcal{L}_2'\rangle\,\Phi_{1,0}\end{aligned}$$

Therefore, $\Phi_{1,0}$ satisfies the following equation, which is similar to the classical IDE

$$(c-\lambda\eta)\frac{\partial\Phi_{1,0}}{\partial u}(u)+\frac{1}{2}\left\langle f^2(y,z)\right\rangle\frac{\partial^2\Phi_{1,0}}{\partial^2 u}(u)$$
$$+\lambda\int_{\mathbb{R}}\Phi_{1,0}(u-s)\mu(ds)-(\lambda+\Delta)\Phi_{1,0}(u)=0,$$

where

$$\left\langle f^2(y,z)\right\rangle=\int_{\mathbb{R}}f^2(y,z))\chi_1(dy)$$

Here χ_1 is the invariant distribution of Y_t

14. From the terms of order $\sqrt{\delta}$ and $\sqrt{\delta\varepsilon}$, we have

$$\begin{aligned}0=&\mathcal{M}_1\Phi_0+\mathcal{M}_3\Phi_{1,0}+\mathcal{L}_2'\Phi_{0,1}+\mathcal{L}_1\Phi_{1,1}\\=&\mathcal{L}_2'\Phi_{1,1}+\mathcal{M}_1\Phi_{1,0}+\mathcal{M}_3\Phi_{2,0}=0\end{aligned}$$

Since

$$\mathcal{M}_3\Phi_{1,0},\mathcal{L}_1\Phi_{1,1}\text{and},\mathcal{M}_1\Phi_{1,0}$$

are equal to zero then $\Phi_{0,1}$ and $\Phi_{1,1}$ are respectively solution of the following equations

$$
\begin{aligned}
&(c-\lambda\eta)\frac{\partial \Phi_{0,1}}{\partial u}(u)+\frac{1}{2}f^2(y,z)\frac{\partial^2\Phi_{0,1}}{\partial^2 u}(u)+\lambda\int_{\mathbb{R}}\Phi_{0,1}(u-s)\mu(ds)\\
&-(\lambda+\Delta)\Phi_{0,1}(u)+\rho_2 g(z)f(y,z)\frac{\partial^2\Phi_0(u,z)}{\partial z\partial u}\\
&-g(z)\Lambda_2(y,z)\frac{\partial\Phi_0(u,z)}{\partial z}=0
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&(c-\lambda\eta)\frac{\partial \Phi_{1,1}}{\partial u}(u)+\frac{1}{2}f^2(y,z)\frac{\partial^2\Phi_{1,1}}{\partial^2 u}(u)\\
&+\lambda\int_{\mathbb{R}}\Phi_{1,1}(u-s)\mu(ds)-(\lambda+\Delta)\Phi_{1,1}(u)=0.
\end{aligned}
\tag{3.6}
$$

To sum up:

- see item 10 at p. 18.
- The asymptotic expansion of the EDPF is:

$$
\begin{aligned}
\Phi^{\varepsilon,\delta}&=\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\sqrt{\varepsilon}\right)^i\left(\sqrt{\delta}\right)^j\Phi_{i,j}\\
&=\Phi_0+\sqrt{\varepsilon}\Phi_{1,0}+\sqrt{\delta}\Phi_{0,1}+\sqrt{\varepsilon\delta}\Phi_{1,1}+\varepsilon\Phi_{2,0}+\delta\Phi_{0,2}+\cdots,
\end{aligned}
$$

where the values of the correction terms Φ_0, $\Phi_{1,0}$, $\Phi_{0,1}$, $\Phi_{1,1}$, $\Phi_{2,0}$, $\Phi_{0,2}$ in the above expansion are obtained by solving respectively the following equations (either an IDE or a Poisson equation):

$$
\begin{aligned}
&(c-\lambda\eta)\frac{\partial \Phi_0}{\partial u}(u,z)+\frac{1}{2}\left\langle f^2(y,z)\right\rangle\frac{\partial^2\Phi_0}{\partial^2 u}(u,z)\\
&+\lambda\int_{\mathbb{R}}\Phi_0(u-s,z)\mu(ds)-(\lambda+\Delta)\Phi_0(u,z)=0,
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&(c-\lambda\eta)\frac{\partial \Phi_{1,0}}{\partial u}(u)+\frac{1}{2}\left\langle f^2(y,z)\right\rangle\frac{\partial^2\Phi_{1,0}}{\partial^2 u}(u)\\
&+\lambda\int_{\mathbb{R}}\Phi_{1,0}(u-s)\mu(ds)-(\lambda+\Delta)\Phi_{1,0}(u)=0,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&(c-\lambda\eta)\frac{\partial \Phi_{0,1}}{\partial u}(u)+\frac{1}{2}f^2(y,z)\frac{\partial^2\Phi_{0,1}}{\partial^2 u}(u)+\lambda\int_{\mathbb{R}}\Phi_{0,1}(u-s)\mu(ds)\\
&-(\lambda+\Delta)\Phi_{0,1}(u)+\rho_2 g(z)f(y,z)\frac{\partial^2\Phi_0(u,z)}{\partial z\partial u}-g(z)\Lambda_2(y,z)\frac{\partial\Phi_0(u,z)}{\partial z}=0,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&(c-\lambda\eta)\frac{\partial \Phi_{1,1}}{\partial u}(u) + \frac{1}{2}f^2(y,z)\frac{\partial^2 \Phi_{1,1}}{\partial^2 u}(u) + \lambda \int_{\mathbb{R}} \Phi_{1,1}(u-s)\mu(ds) \\
&\quad -(\lambda+\Delta)\Phi_{1,1}(u) = 0,
\end{aligned} \tag{3.10}
$$

$$
\Phi_{2,0} = -\frac{1}{2}\Big(\psi_1(y) + \kappa_1(u,z)\Big)\frac{\partial^2 \Phi_0}{\partial^2 u}(u,z), \tag{3.11}
$$

$$
\begin{aligned}
&(c-\lambda\eta)\frac{\partial \Phi_{0,2}}{\partial u}(u) + \frac{1}{2}\left\langle f^2(y,z)\right\rangle \frac{\partial^2 \Phi_{0,2}}{\partial^2 u}(u) \\
&\quad + \lambda \int_{\mathbb{R}} \Phi_{0,2}(u-s)\mu(ds) - (\lambda+\Delta)\Phi_{0,2}(u) = 0.
\end{aligned} \tag{3.12}
$$

The first terms of the asymptotic expansion are only functions of at most two variables. All of them depend, not surprisingly, of the insurer's surplus level. However, either the fast stochastic volatility factor or the slow one contributes to their level, not both at the same time. The above IDE of the first terms could be solved by choosing first the distribution of the individual claim, second determine the invariant distributions of the two stochastic volatility factors and finally compute each IDE.

Chapter 4

Concluding remarks

4.1 Conclusions

In this thesis, we consider a general approach of the insurance surplus model. Our model contains not only a general form of the compound Poisson risk model but also a multiscale stochastic volatility model, which has a great advantage for providing for more accurate values of the insurance surplus volatility. One of the most challenges in financial mathematics is how to get for any specific financial instrument, a realistic model to capture its stochastic volatility. This work by integrating two factors volatility model provides a good framework to achieve that goal.

As there is no analytical formula for the Expected Discounted Penalty Function, we use an asymptotic approximation among the different approaches that researchers use to overcome this difficulty. Although the method involves solving not obvious differential equations, it presents some advantages such as fast computation, and more accurate stochastic values.

In the paper, we apply our model to an insurance surplus, however, it can be used to price any option either American or European, Asian under a stochastic volatility. In addition, our model can be used more specifically by any insurance company, which faces at any time a non deterministic claims from its clients, and with two stochastic factors for its surplus volatility.

The expected discounted penalty function is very useful in term of pricing perpetual financial securities or securities with finite maturities, therefore our work can be an asset in credit risk modelling.

For the future work, we are thinking about solving the integro-differential equation for some specific risk models. Also, it will be useful to compute by Monte Carlo simulation for instance, in order to compare the accuracy and speed of our method to the classical one.

4.2 Summary of reflection of objectives in the Thesis

Objective 1: Knowledge and understanding

In the introduction section, a review of literature related the insurer's risk model and the expected discounted penalty function has been presented. Moreover, we define the most important mathematical concepts used in our thesis. In term of models, we describe firstly the famous classical risk model as well as the extended one by Lundberg and Gerber et al. Secondly, a brief description of the expected discounted penalty function and the mean reversion model have been presented. Finally, the main theorem we use in our thesis has been presented as well. In the model formulation section, we describe clearly with all the details of the different parts of our model.

Objective 2: Deeper methodological knowledge

In the section about the general integro-differential equation, we start by describing the theorem about the existence and uniqueness of solutions of Lévy stochastic diffusion equation. We find the conditions that the parameters, specifically the drift, the volatility and the initial conditions, of our model should verify in order to guarantee the existence and uniqueness of the solution. In other words, we show that under a specific conditions, there exists an insurer's surplus process with multiscale stochastic volatility and jump-diffusion process which verify those condition. In addition, in the second theorem of the same section, we describe and calculate the infinitesimal generator of the diffusion process. In sum, a deep description of the infinitesimal generator, which we need in order to calculate the integro-differential equation that the expected discounted penalty function satisfy, has been presented.

Objective 3: Ability to critically and systematically integrate knowledge and to analyse

The expected discounted penalty function has been presented at the end of the model formulation section. We give the mathematical definition of the concept. It is obvious that when the force of the interest rate is zero and the penalty function is identity, the expected discounted penalty function becomes the ruin probability. From our model formulation, the penalty function can be seen as the amount payable by the insurance company if a ruin occurs. Moreover, as in general the insurance companies invest their surplus in a risky environment, in financial institutions, their funds are also exposed to all types of risks specially market and credit risks. Therefore, the insurer's surplus is subject to some fluctuation which may cause ruin to occurs. we integrate the volatility driven by two factors to capture more accurately these fluctuations. No comparison has been made because we do not perform numerical illustrations.

Objective 4: Ability to critically, independently determine, formulate problems and carry out advanced tasks

In our formulation section, we describe clearly the model with the definitions of all mathematical terms. As the initial model contains very advanced and complex financial mathematical concepts, we rewrite it in more details by describing each part of the model. The dynamic of the surplus's volatility, which is modelled by two factors volatility, is explained by giving explicitly the Mean Reverting Processes of each factors. In addition, the three dimensional Brownian motion has been transformed to get a Wiener process with three independent components.

Objective 5: Communication in context, both national and international level

Modelling the insurer's surplus is undoubtedly a first step for the insurance or related business to assess it short and long term survival probabilities. The expected discounted penalty function or the present value of the surplus immediately after a ruin- given the initial conditions- is crucial for valuing perpetual financial securities, structural form of credit risk's model, and corporate debt pricing . Therefore, in model formulation section, and asymptotic approximation of the expected discounted function section, we describe our model in a way that any reader with acceptable background in quantitative subjects such as financial mathematics, actuarial science, statistics and even financial economics can read and understand easily. However, to understand the method of singular perturbation technique that we use to derive the analytical approximation, the reader will need to know some advanced concepts in financial mathematics.

Objective 7: Ability to make judgement by taking in account relevant factors: scientific,social, ethical

In the financial mathematics research area, there has been an extensive research papers which analysed the expected discounted penalty function with different approaches or techniques and also the regular-singular perturbation theory which is introduced for the first time by Fouque–Papanicolaou and al has been used to price many financial derivatives with or without mulstiscale stochastic volatility model. However, our model is quite a unifying approach, in which we introduce a volatility driven by two factors- fast and slow mean reverting processes,and use a singular perturbation approach to get an approximation of the expected discounted penalty function. Moreover, our model captures almost the complete structure of the implied volatility. We only derive the integro-differential equations for the first terms without an explicit solution. Although we do not perform any numerical values to compare them with Monte Carlo method or another technique, it has been proved in the literature that the perturbation method is much faster than many other methods.

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