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Invariant Random Fields on Spaces with a Group Action

Springer

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To my soulmate, supporter, and love —
my wife Iryna

Foreword

Random functions of several variables, or random fields, appear in the statistical theory of turbulence, meteorology, quantum physics, computer graphics, pattern recognition, system identification, etc. The most important case is when the finite-dimensional distributions of the field under consideration are invariant under an action of some transformation group. The precise links between the spectral theory of invariant random fields, the theory of invariant positive-definite functions, and the theory of group representations have been known since the middle of the last century.

New links have been discovered in recent applications. In particular, Differential Geometry plays a significant role in the spectral theory of invariant random sections of vector and tensor bundles. This link was described in several physical papers where the statistical properties of the Cosmic Microwave Background were studied. Rigourous mathematical results appeared very recently. This theory was poorly represented in monograph form. I am aware of only one book by D. Marinucci and G. Peccati (2011), *Random fields on the sphere*, vol. 389 of London Math. Soc. Lect. Note Ser., Cambridge University Press, Cambridge. However, this book concentrates more on spherical random fields rather than invariant random fields on general spaces with a group action.

The present monograph fills the above gap in the literature. This monograph describes the spectral theory of invariant random fields in vector bundles in a general form. Previously known classical results appear as special cases when the vector bundle is a trivial one.

Apart from the spectral theory, the author considers inversion formulae, differential equations with random fields, linear functionals of random fields, sample path properties of Gaussian random fields. Most results were proved by the author; some results are new and appear here for the first time.

Two special features of the book extend its audience. First, a separate chapter is devoted to applications in Approximation Theory, Earthquake Engineering, and Cosmology. Second, preliminary knowledge of Differential Geometry, Lie Groups and Algebras, Abstract Harmonic Analysis and some other parts of mathematics outside the scope of Probability and Statistics are not required. The necessary results are formulated in the Appendix (Chap. 6), mostly without proofs. Extensive bibliographical remarks are included in each chapter.

Malyarenko's monograph on this highly complex topic is written in a way that makes material very accessible. This is a book which is both technically interesting and a pleasure to read. The presentation is clear and the book should be useful for almost anyone who uses stochastic processes and random fields as tools for describing the real world.

The book may be interesting for all mathematicians, graduate and postgraduate students as a self-contained source of recent results in the theory of invariant random fields. Both specialists in the theory of random fields and scientists working in related applied areas that utilise this theory will no doubt find a great deal of new and useful information in this book.

Cardiff, United Kingdom Nikolai Leonenko

Preface

Random fields appeared for the very first time in the 1920s in classical research papers about turbulence written by A.A. Friedmann, J. Kampé de Fériet, L.P. Keller, A.N. Kolmogorov, T. von Kármán, A.M. Obukhov, among others. Later on, the mathematical theory of random fields was developed and successfully applied in computer graphics, earthquake engineering, medicine, quantum physics, statistical mechanics, etc. The above theory uses such mathematical tools as Abstract Harmonic Analysis, Special Functions, Abelian and Tauberian theorems, etc.

At the end of the last millennium, new applications appeared. At that time, cosmology transformed into a predictive science whose predictions were tested against precise observations. In particular, tiny fluctuations of the cosmic microwave background were discovered. In order to build the rigourous mathematical model of the above fluctuations, one has to construct an isotropic random section of a special tensor bundle over the two-dimensional sphere. Therefore, the set of actively used mathematical tools for a specialist in random fields has to be extended with Differential Geometry, Lie Groups and Lie Algebras, just to mention a few.

New applications generated new challenges. In particular, currently there exists no single book written for specialists in probability that describes the current state of the spectral theory of invariant random fields and includes all necessary material from the above-mentioned non-probabilistic parts of mathematics. Our book originated from an attempt to fill this gap in the literature. The contents of the book are described in detail in Chap. 1. Here, we just present the book's scope.

Most random fields that appear in applied areas are invariant under an action of some group G. The group G acts either on the parametric space T of a random field or in some space of sections of a vector or tensor bundle over T. In Chap. 2 we describe a unified approach to spectral expansions of such fields based on the theory of induced group representations.

We divide the remaining part of the theory of random fields into two areas. In the first area, the so-called L^2 theory, a random field is considered as a function on T with values in the Hilbert space of square integrable random vectors. Some aspects of this theory are presented in Chap. 3. In the second area, some restrictions on the finite-dimensional distributions of the random field under consideration are imposed. In Chap. 4 we consider some questions of the theory of Gaussian random fields.

In Chap. 5 we consider applications of the above-described material to approximation theory, cosmology, and earthquake engineering. Finally, in the Appendix (Chap. 6) we consider mathematical tools outside the scope of probability and statistics, that are necessary for a specialist in random fields. Bibliographical remarks conclude each chapter and the Appendix (Chap. 6). Throughout the book, we use Halmos' abbreviation iff for "if and only if".

The book is intended for specialists in the theory of random fields, for mathematicians who would like to study the above theory, as well as for specialists in applied areas. It may be useful for graduate and postgraduate students in probability, statistics, functional analysis, cosmology, earthquake engineering, etc.

Västerås, Sweden Anatoliy Malyarenko

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Västerås, Sweden Anatoliy Malyarenko

Contents

List of Tables