

Probability and Its Applications

Published in association with the Applied Probability Trust

Editors: S. Asmussen, J. Gani, P. Jagers, T.G. Kurtz



Probability and Its Applications

The *Probability and Its Applications* series publishes research monographs, with the expository quality to make them useful and accessible to advanced students, in probability and stochastic processes, with a particular focus on:

- Foundations of probability including stochastic analysis and Markov and other stochastic processes
- Applications of probability in analysis
- Point processes, random sets, and other spatial models
- Branching processes and other models of population growth
- Genetics and other stochastic models in biology
- Information theory and signal processing
- Communication networks
- Stochastic models in operations research

For further volumes:

www.springer.com/series/1560

Anatoliy Malyarenko

Invariant
Random Fields
on Spaces with
a Group Action

 Springer

Anatoliy Malyarenko
School of Education, Culture,
and Communication
Mälardalen University
Västerås, Sweden

Series Editors

Søren Asmussen
Department of Mathematical Sciences
Aarhus University
Aarhus, Denmark

Peter Jagers
Mathematical Statistics
Chalmers University of Technology
and University of Gothenburg
Gothenburg, Sweden

Joe Gani
Centre for Mathematics and its Applications
Mathematical Sciences Institute
Australian National University
Canberra, Australia

Thomas G. Kurtz
Department of Mathematics
University of Wisconsin–Madison
Madison, WI, USA

ISSN 1431-7028 Probability and Its Applications

ISBN 978-3-642-33405-4

ISBN 978-3-642-33406-1 (eBook)

DOI 10.1007/978-3-642-33406-1

Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012952289

Mathematics Subject Classification (2010): 60G60, 60G17, 60G22, 85A40

© Springer-Verlag Berlin Heidelberg 2013

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

*To my soulmate, supporter, and love —
my wife Iryna*

Foreword

Random functions of several variables, or random fields, appear in the statistical theory of turbulence, meteorology, quantum physics, computer graphics, pattern recognition, system identification, etc. The most important case is when the finite-dimensional distributions of the field under consideration are invariant under an action of some transformation group. The precise links between the spectral theory of invariant random fields, the theory of invariant positive-definite functions, and the theory of group representations have been known since the middle of the last century.

New links have been discovered in recent applications. In particular, Differential Geometry plays a significant role in the spectral theory of invariant random sections of vector and tensor bundles. This link was described in several physical papers where the statistical properties of the Cosmic Microwave Background were studied. Rigorous mathematical results appeared very recently. This theory was poorly represented in monograph form. I am aware of only one book by D. Marinucci and G. Peccati (2011), *Random fields on the sphere*, vol. 389 of London Math. Soc. Lect. Note Ser., Cambridge University Press, Cambridge. However, this book concentrates more on spherical random fields rather than invariant random fields on general spaces with a group action.

The present monograph fills the above gap in the literature. This monograph describes the spectral theory of invariant random fields in vector bundles in a general form. Previously known classical results appear as special cases when the vector bundle is a trivial one.

Apart from the spectral theory, the author considers inversion formulae, differential equations with random fields, linear functionals of random fields, sample path properties of Gaussian random fields. Most results were proved by the author; some results are new and appear here for the first time.

Two special features of the book extend its audience. First, a separate chapter is devoted to applications in Approximation Theory, Earthquake Engineering, and Cosmology. Second, preliminary knowledge of Differential Geometry, Lie Groups and Algebras, Abstract Harmonic Analysis and some other parts of mathematics outside the scope of Probability and Statistics are not required. The necessary re-

sults are formulated in the Appendix (Chap. 6), mostly without proofs. Extensive bibliographical remarks are included in each chapter.

Malyarenko's monograph on this highly complex topic is written in a way that makes material very accessible. This is a book which is both technically interesting and a pleasure to read. The presentation is clear and the book should be useful for almost anyone who uses stochastic processes and random fields as tools for describing the real world.

The book may be interesting for all mathematicians, graduate and postgraduate students as a self-contained source of recent results in the theory of invariant random fields. Both specialists in the theory of random fields and scientists working in related applied areas that utilise this theory will no doubt find a great deal of new and useful information in this book.

Cardiff, United Kingdom

Nikolai Leonenko

Preface

Random fields appeared for the very first time in the 1920s in classical research papers about turbulence written by A.A. Friedmann, J. Kampé de Fériet, L.P. Keller, A.N. Kolmogorov, T. von Kármán, A.M. Obukhov, among others. Later on, the mathematical theory of random fields was developed and successfully applied in computer graphics, earthquake engineering, medicine, quantum physics, statistical mechanics, etc. The above theory uses such mathematical tools as Abstract Harmonic Analysis, Special Functions, Abelian and Tauberian theorems, etc.

At the end of the last millennium, new applications appeared. At that time, cosmology transformed into a predictive science whose predictions were tested against precise observations. In particular, tiny fluctuations of the cosmic microwave background were discovered. In order to build the rigorous mathematical model of the above fluctuations, one has to construct an isotropic random section of a special tensor bundle over the two-dimensional sphere. Therefore, the set of actively used mathematical tools for a specialist in random fields has to be extended with Differential Geometry, Lie Groups and Lie Algebras, just to mention a few.

New applications generated new challenges. In particular, currently there exists no single book written for specialists in probability that describes the current state of the spectral theory of invariant random fields and includes all necessary material from the above-mentioned non-probabilistic parts of mathematics. Our book originated from an attempt to fill this gap in the literature. The contents of the book are described in detail in Chap. 1. Here, we just present the book's scope.

Most random fields that appear in applied areas are invariant under an action of some group G . The group G acts either on the parametric space T of a random field or in some space of sections of a vector or tensor bundle over T . In Chap. 2 we describe a unified approach to spectral expansions of such fields based on the theory of induced group representations.

We divide the remaining part of the theory of random fields into two areas. In the first area, the so-called L^2 theory, a random field is considered as a function on T with values in the Hilbert space of square integrable random vectors. Some aspects of this theory are presented in Chap. 3. In the second area, some restrictions on the finite-dimensional distributions of the random field under consideration are

imposed. In Chap. 4 we consider some questions of the theory of Gaussian random fields.

In Chap. 5 we consider applications of the above-described material to approximation theory, cosmology, and earthquake engineering. Finally, in the Appendix (Chap. 6) we consider mathematical tools outside the scope of probability and statistics, that are necessary for a specialist in random fields. Bibliographical remarks conclude each chapter and the Appendix (Chap. 6). Throughout the book, we use Halmos' abbreviation iff for "if and only if".

The book is intended for specialists in the theory of random fields, for mathematicians who would like to study the above theory, as well as for specialists in applied areas. It may be useful for graduate and postgraduate students in probability, statistics, functional analysis, cosmology, earthquake engineering, etc.

Västerås, Sweden

Anatoliy Malyarenko

Acknowledgements

This book would never have been written without the influence of two people. My teacher, Professor M.Ī. Yadrenko, showed me the beauty of Probability. He was both the very best teacher and the most benevolent and responsive person I have ever met in my life.

Love, care, and attention of my wife and friend, Iryna Yelsukova, played a crucial role in writing this book.

I am grateful to my university teachers Professors V.V. Buldygin, N.V. Kartashov, V.S. Korolyuk, Yu.V. Kozachenko, N.N. Leonenko, M.P. Moklyachuk, O.I. Ponomarenko, D.S. Silvestrov, A.V. Skorokhod, and A.F. Turbin.

Some results described in this book were proved due to a fruitful collaboration with my co-authors Lambros Katafygiotis, Andriy Olenko, Constantinos Papadimitriou, and Aspaziya Zerva.

I would like to thank my colleagues Professors Nicholas Bingham, Anatoliy Klimyk, Domenico Marinucci, and Carl Mueller for useful discussions on Tauberian and Abelian theorems, representation theory, cosmology, and functional limit theorems.

The remarks of the three anonymous referees of this book helped to improve its structure and exposition.

Finally, I am grateful to my colleagues from the Division of Applied Mathematics of the Mälardalen University for providing such a friendly working and research environment.

Västerås, Sweden

Anatoliy Malyarenko

Contents

1	Introduction	1
	References	7
2	Spectral Expansions	9
2.1	Random Fields	9
2.1.1	The Scalar Case	9
2.1.2	The Vector Case	11
2.2	Invariant Random Fields on Groups	16
2.2.1	The Scalar Case	16
2.2.2	The Vector Case	24
2.3	Invariant Random Fields on Homogeneous Spaces	28
2.3.1	The Scalar Case	28
2.3.2	The Vector Case	37
2.4	Invariant Random Fields on General Spaces	46
2.4.1	The Scalar Case	46
2.4.2	The Vector Case	55
2.5	Isotropic Random Fields on Commutative Groups	59
2.6	Volterra Isotropic Random Fields	69
2.7	Bibliographical Remarks	77
	References	83
3	L^2 Theory of Invariant Random Fields	91
3.1	Inversion Formulae	91
3.2	Linear Differential Equations with Random Fields	98
3.3	Linear Functionals of Invariant Random Fields	102
3.4	Bibliographical Remarks	112
	References	113
4	Sample Path Properties of Gaussian Invariant Random Fields	115
4.1	Introduction	115
4.2	Elementary Abelian Theorems	116
4.2.1	The Compact Case	116

- 4.2.2 The Noncompact Case 121
- 4.3 Advanced Abelian and Tauberian Theorems 125
 - 4.3.1 The Compact Case 125
 - 4.3.2 The Noncompact Case 132
- 4.4 An Optimal Series Expansion of the Multiparameter Fractional Brownian Motion 136
- 4.5 Functional Limit Theorems for the Multiparameter Fractional Brownian Motion 144
- 4.6 Bibliographical Remarks 164
- References 165
- 5 Applications 171**
 - 5.1 Applications to Approximation Theory 171
 - 5.2 Applications to Cosmology 174
 - 5.2.1 The Cosmic Microwave Background (CMB) 174
 - 5.2.2 The Probabilistic Model of the CMB 177
 - 5.3 Applications to Earthquake Engineering 191
 - 5.4 Bibliographical Remarks 198
 - References 199
- 6 Appendix A: Mathematical Background 203**
 - 6.1 Differential Geometry 203
 - 6.1.1 Manifolds 203
 - 6.1.2 Vector Bundles 204
 - 6.1.3 Differential Operators 206
 - 6.1.4 Invariant Differential Operators on Manifolds 207
 - 6.2 Lie Groups and Lie Algebras 207
 - 6.2.1 Basic Definitions 207
 - 6.2.2 Spherical Functions 210
 - 6.3 Group Actions and Group Representations 210
 - 6.3.1 Group Actions 210
 - 6.3.2 Positive-Definite Functions on Groups and Group Representations 211
 - 6.3.3 Unitary Representations of Commutative Topological Groups 216
 - 6.3.4 Unitary Representations of Compact Topological Groups 219
 - 6.3.5 The Fourier Transform 222
 - 6.3.6 Positive-Definite Functions on Homogeneous Spaces and Induced Representations 223
 - 6.3.7 Irreducible Unitary Representations of Low Dimensional Noncompact Lie Groups by Examples 225
 - 6.3.8 The Spherical Fourier Transform 229
 - 6.3.9 Paley–Wiener Theorems 231
 - 6.4 Special Functions 232
 - 6.4.1 The Gamma Function and the Beta Function 232
 - 6.4.2 The Fox H -function 233

6.4.3	Harmonics	237
6.5	Rigged Hilbert Spaces	241
6.6	Abelian and Tauberian Theorems	243
6.7	Bibliographical Remarks	247
	References	250
Index	257

List of Tables

Table 2.1	Compact two-point homogeneous spaces	29
Table 2.2	Noncompact two-point homogeneous spaces	29
Table 2.3	Enumeration of nonzero associated spherical functions	33
Table 4.1	The regions corresponding to different types of convergence of (4.6) for compact spaces T	127
Table 4.2	The coefficients α' , β' , and b_k for noncompact spaces T	135
Table 4.3	The regions corresponding to different types of convergence of (4.13) for noncompact spaces T	135
Table 5.1	Examples of different notation for temperature expansion coefficients and power spectrum	179
Table 5.2	Examples of different notation for complex polarisation expansion coefficients	183
Table 5.3	Examples of different notation for the fields $E(\mathbf{n})$ and $B(\mathbf{n})$ and their expansion coefficients	187