EFFECTIVE TEACHING THROUGH PROBLEM-SOLVING BY SEQUENCING AND CONNECTING STUDENT SOLUTIONS

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How can researchers support teachers in the complexity of orchestrating productive mathematical whole-class discussions based on students’ solutions to challenging problems? This study has two aims: Firstly, to study how the teacher can select, sequence and connect different student solutions in order to effectively orchestrate mathematical whole-class discussions. Secondly, to critically reflect on the role of the researchers in this intervention project. Analyses of audio recorded interviews and video recorded whole-class discussions result in suggestions for how student solutions can be sequenced in this particular case to set the scene for connecting them with each other and with key mathematical ideas. We further critically reflect on how we as researchers can improve our work in supporting practicing teachers.

INTRODUCTION

Many researchers within the field agree that whole-class discussions emanating from students’ solutions to mathematically challenging problems that can be solved in multiple ways are key ingredients of mathematics teaching that, if productively conducted, serve to help students in their mathematics learning (e.g., Franke, Kazemi & Battey, 2007; Kilpatrick, Swafford & Findell, 2001; Stein, Boaler & Silver, 2003). Effectively orchestrating productive whole-class discussions based on the students’ ideas and representations is however challenging for the teacher (Lampert, 2001; Stein, Engle, Smith & Hughes, 2008) due to that the teacher must be able to handle a wide spectrum of student solutions in a way that makes the whole class advance. Teaching mathematics through problem solving builds to a high extent on students’ thinking and reasoning, which implies high “mathematics-in-action problem-solving demands on the teacher” (p. 274, Adler & Davis, 2006) and in this paper we focus on how we as researchers could support teachers in the complexity of such teaching.

We are here particularly interested in processes of how the teacher orchestrates whole-class discussions to make explicit connections between mathematical representations. Making connections is a key ingredient in frameworks for mathematical competencies (e.g., Kilpatrick et al., 2001; NCTM, 2000; Niss & Jensen, 2002). Kilpatrick et al. (2001) state: “How learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it in problem solving” (p. 117). Mathematical practices, such as making connections, are of significant importance for future research (Boaler, 2002).

Stein et al. (2008) propose a five practices model that emphasizes those aspects that can be planned for in advance in orchestrating mathematical discussions. The model is designed to make it more manageable for teachers to productively orchestrate
whole-class discussions that are both based on students’ ideas and responsive to the discipline. The balance between student participation and mathematical content control (Emanualsson & Sahlström, 2008; Ryve, Larsson & Nilsson, in press) is critical here. The five practices are: anticipating student responses, monitoring student responses during the explore phase, selecting student responses for whole-class discussion, sequencing student responses and connecting student responses to each other and to powerful mathematical ideas. The practice of making connections builds on and benefits from the practices that precede it.

The aim of this study is twofold: Firstly, to study how the teacher – based on anticipated and monitored student solutions – can select, sequence and connect different student solutions in order to effectively orchestrate mathematical whole-class discussions. Secondly, to critically reflect on the role of the researchers in the process of establishing the practice of making connections.

THE INTERVENTION PROJECT AND ITS THEORETICAL GROUNDS

The project was carried out as an intervention project, influenced by design experiments methodology (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003), in which the teacher and the researcher (first author) cooperated closely. We chose to conduct a design experiment to be able to analyze and reflect on both the teacher’s and the researcher’s practice. The five practices from Stein et al. (2008) served as a framework for both implementing and analyzing the intervention project. The teacher chosen for this study has a solid mathematical knowledge and is used to conduct shorter whole-class discussions of solutions to mathematical problems. In this project, the discussions have been deeper and based on multiple solutions to rich problems (Larsson, 2007; Taflin, 2007). The teacher is not very used to orchestrate such deep whole-class discussions of student’s different solutions. The researcher made an overall plan for the thought learning trajectory of the students (Cobb et al., 2003) according to the instructional goals of the study, which were to develop the key mathematical ideas of patterns, variables and generalized algebraic formulas.

The researcher suggested problems for the lesson series, all rich pattern problems in which a general formula is sought. Rich problems were chosen to create opportunities for the students to discuss multiple mathematical ideas and representations. Each problem consists of several specific questions with an increasing level of difficulty before a direct formula for the nth case is asked for. For every problem, the students were asked to formulate their own, similar problem and solve it. After the third of totally six problems there was a seven weeks pause before the problem that is in focus here, “The tower” (Taflin, 2007), was treated:
a) How many cubes are needed to build the tower in the picture?
b) How many cubes are needed to build a similar tower whose height is 12 cubes?
c) How many cubes are needed to build a similar tower whose height is \( n \) cubes?
d) Formulate an own, similar problem. Solve it.

A pre- and a post-test were made in the beginning and in the end of the intervention project. The 16 students, who were in the end of their 7th and 8th grade and hence 13-15 years old, were previously somewhat familiar with variables and algebraic expressions, but not with creating their own algebraic formulas for patterns. They were not very used to be taught mathematics through problem solving in this manner with extensive whole-class discussions of their different ideas as the main focus.

Each problem was typically given one 80 minutes lesson. In order to buy time for the teacher to think about the different student solutions and to discuss the student solutions with the researcher, we strived for the following set-up:

The first half of each lesson, the teacher orchestrated a whole-class discussion, which was video recorded with a focus on the chalkboard, based on the students’ different ideas and representations of the problem from the previous lesson. The second half of each lesson, the teacher introduced a new problem and then monitored the students working on the problem individually and discussing their different solutions in small groups. The student solutions were collected and the teacher and the researcher had a meeting a few days later (before the next lesson) when they discussed the different student solutions and planned the next whole-class discussion in terms of selecting, sequencing and connecting the student solutions (cf. Stein et al., 2008). The teacher and the researcher also discussed the outcome of the previous whole-class discussion and decided which problem to begin working with the next lesson. The researcher interviewed the teacher about these issues and audio recorded the interviews. Right before the next lesson, the researcher interviewed the teacher about what kinds of student solutions the teacher anticipated for the new problem.

However, due to organizational constraints, some of the problems had to be treated in total within the same lesson, among them the problem in focus here, “The tower”. This meant that the time before the whole-class discussion to think about and discuss student solutions consisted of minutes instead of hours (cf. Stein et al., 2008).

**ANALYSIS AND DISCUSSION OF RESULTS**

Now we invite you to follow the researcher’s and the teacher’s journey through each of Stein et al.’s (2008) five practices – anticipating, monitoring, selecting, sequencing and connecting student solutions – in the case of this intervention project. We take-off by taking a look at the teacher’s anticipated student solutions and then we take a flight by examining the actual student solutions that were selected
and sequenced by the teacher on the base of the monitored solutions and further by examining the actual connections made. In the next section, we bring down the plane by proposing an alternative for how student solutions could be selected, sequenced and connected in order to further facilitate effective mathematical discussions.

**Actual student solutions in the light of a journey through five practices**

When anticipating the students’ solutions, the teacher states in an interview just before the problem-solving lesson that maybe some graphical solution will come up, rearranging the cubes, but he is not sure about that. The teacher considers it more likely that solutions will emanate from the sequence of numbers. He further believes that some of the students will be able to pose a general formula for the number of cubes in a tower with height \( n \); at least they have the capacity to do so according to the teacher. The teacher thinks that difficulties may be related partly due to the fact that it was a long time since the previous lesson was held related to finding general algebraic formulas for patterns.

<table>
<thead>
<tr>
<th>1. Adam’s solution</th>
<th>2. Bertil’s solution</th>
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<tbody>
<tr>
<td>(counting – seeing a pattern)</td>
<td>(graphical – drawing a triangle)</td>
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<tr>
<td><img src="image" alt="Adam's solution" /></td>
<td><img src="image" alt="Bertil's solution" /></td>
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<tr>
<td>(+1) (+2) (+3) (+4)</td>
<td>(n) (n)</td>
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<tr>
<td>(n + 4(n^2/2 - n/2) = n + 2n^2 - 2n = 2n^2 - n = n(2n - 1))</td>
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<tr>
<th>3. Cecilia’s solution</th>
<th>4. David’s solution</th>
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</thead>
<tbody>
<tr>
<td>(graphical – rearranging part of the cubes to form a rectangle)</td>
<td>(graphical – rearranging all cubes to form a rectangle)</td>
</tr>
<tr>
<td><img src="image" alt="Cecilia's solution" /></td>
<td><img src="image" alt="David's solution" /></td>
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<tr>
<td>(1) (n - 1)</td>
<td>(2n - 1)</td>
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<tr>
<td>(n + 2n(n - 1) = n + 2n^2 - 2n = 2n^2 - n = n(2n - 1))</td>
<td>(n(2n - 1))</td>
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**Table 1:** Actual solutions presented during whole-class discussion, in sequential order.
Stein et al. (2008) write that “the goal of monitoring is to identify the mathematical learning potential of particular strategies or representations used by the students” (p. 326). Asked to compare the actual student solutions with the anticipated student solutions, the teacher expresses in the interview made after the whole-class discussion that he expected the students to have difficulties with graphical methods, but that the students in fact used graphical methods when they solved the problem. The teacher also expresses in the same interview that he discerned three main ideas among the student solutions that are all graphical. In Table 1 above, these three graphical solutions and one additional solution based on counting the cubes are shown, which were the actual solutions selected by the teacher for discussion in whole-class. These solutions cover the main ideas represented in the class. The four solutions are shown in the same sequential order as they were discussed in whole class.

Two actual connections were made during the whole-class discussions. Firstly, the teacher highlighted that both Bertil’s and Adam’s solution methods resulted in a total of 276 cubes for a tower with height 12. Secondly, the teacher highlighted that the algebraic formula \( n(2n - 1) \) for the number of cubes in a tower with height \( n \) that David’s solution resulted in is the same formula as Cecilia’s solution resulted in. The teacher stated: “that formula was the same we just recently had”. As suggested below, the relatively low number of connections could both be connected to the sequencing of the solutions and to the researcher’s process of initiating the teacher in a professional discourse about teaching mathematics through problem solving.

**Suggestions for selecting, sequencing and connecting student solutions to facilitate effective mathematical discussions**

In order to create maximum opportunities to connect the student solutions and thereby facilitating effective mathematical discussions that both build on students’ ideas and develop important mathematical ideas (Stein et al., 2008), we suggest the following sequencing of the student solutions:

1. Adam: counting and seeing a pattern (each wing increases with 1, 2, 3, 4 … cubes as the height of the tower increases)
2. Extended Adam: counting and triangular numbers (each wing has \( 1 + 2 + 3 + … + (n - 1) \) cubes, which is an arithmetic sum)
3. Cecilia: graphical (middle pile and two wings are rearranged to form a rectangle)
4. a) Erik b) David: graphical (all cubes are rearranged to form a rectangle)
5. Bertil: graphical (each wing is surrounded with a triangle)

As can be seen in the list above, solutions 1 – 2 are related to the strategy of counting, and solutions 3 – 5 are graphical. We believe that starting off the whole-class discussion, as the teacher actually did, with Adam’s method is beneficial since counting the number of cubes and seeing a pattern is a commonly used strategy that
many students begin with and is also easy to understand – which supports “the goal of accessibility” (p. 330, Stein et al., 2008) and may help understand more unique solutions later on in the discussion.

We further see it as productive to explicitly discuss advantages and disadvantages of different solutions. However, instead of just stating that Adam’s solution method has the short-coming of consuming a lot of time when the height of the tower increases to say 100 cubes (which was the case during the whole-class discussion), we suggest that it would be beneficial to dig deeper into the pattern of the tower to make use of the observation that the number of cubes in each wing is $1 + 2 + 3 + \ldots + (n - 1)$. This extension is completely in line with Adam’s thoughts since he has seen that the number of cubes in each wing increases by 1, 2, 3, 4 … as the height of the tower increases. This also gives a natural opportunity to discuss the sum of an arithmetic series of numbers and to make a connection to Cecilia’s graphical solution (see below) as well as to the classical hand-shaking problem which was discussed the previous lesson. For the hand-shaking problem, the class concluded that for n persons, the total number of hand-shakes are $1 + 2 + 3 + \ldots + (n - 1) = \frac{n(n - 1)}{2}$ if every person shakes hand with everyone else.

By connecting Adam’s extended solution with Cecilia’s solution, it can be shown graphically that the arithmetic sum $1 + 2 + 3 + \ldots + (n - 1)$ equals $n(n - 1)/2$. It is clear from the Cecilia’s picture to the left in Figure 1 that two times the arithmetic sum $1 + 2 + 3 + \ldots + (n - 1)$ equals $n(n - 1)$. (By disconnecting the two wings in Cecilia’s picture and putting them together as in the picture to the right in Figure 1, there is an even clearer connection to the classical method for summing up the terms by adding them pair wise from both ends, resulting in $n(n - 1)/2$ cubes for one wing.)

![Figure 1: Connection between Adam’s extended solution and Cecilia’s graphical solution.](image)

We further suggest that the three main graphical solutions are discussed consecutively after each other, starting with Cecilia’s solution, followed by Erik’s and David’s almost identical solutions and rounded off with Bertil’s unique solution. In the interview made after the whole-class discussion, the teacher states that “Cecilia and David’s solutions belong more together, so it was probably good that they were grouped together”. In agreement with the teacher, we suggest that it is beneficial to group those two graphical solutions together in order to facilitate comparison (cf. Stein et al., 2008). We suggest placing Bertil’s graphical solution, which is a unique solution, last in the sequence, as an extra twist to add an
alternative, unique way of finding a general formula for the number of cubes. This is in line with Stein et al.’s (2008) suggestion to place unique or complex solutions last, for the sake of accessibility.

By connecting Cecilia’s rectangle with Erik’s rectangle (see Figure 2), it can be shown graphically that the base length of Erik’s rectangle is \( 1 + (n - 1) + (n - 1) \), which equals \( 2n - 1 \), exactly what Erik claimed.

![Figure 2: Connection between Cecilia’s and Erik’s graphical solutions.](image)

Furthermore, Erik’s method for determining that the length of the base of the rectangle is \( 2n - 1 \) cubes may be connected to David’s method for determining the same by testing numerical examples. David sees for example that for \( n = 12 \), the number of cubes in the base of the rectangle is 23, which equals \( 2 \cdot 12 - 1 \).

Finally, we propose that Bertil’s unique solution method to inscribe one wing of the tower in a triangle is connected to Cecilia’s solution in the following way: the dark sections of each figure each equals \( n \) cubes and the whole square equals \( n \) times \( n \) cubes (see Figure 3). Hence, since the number of cubes in the two wings in Bertil’s and Cecilia’s graphical solutions equals each other, it follows that \( n^2 - n \) (Bertil’s solution) equals \( n(n - 1) \) (Cecilia’s solution). A simple check shows that this equivalence holds.

![Figure 3: Connection between Bertil’s and Cecilia’s graphical solutions](image)

The triangular numbers in Adam’s solution and Bertil’s idea to inscribe one wing in a triangle could also be connected. Of course, additional connections are also possible, but we limit our suggestions to embrace the above proposed connections since there are always time constraints during lessons to take into consideration. Note that Cecilia’s solution may be seen as the node to which the other student solutions are connected. Further note that the connections we propose are not merely connections between algebraic formulas for the whole pattern, but connections between different forms of representations – numerical, algebraic and graphical – for one wing as well as for the whole pattern. Of course, connecting the algebraic
formulas for the whole pattern, like the teacher actually did during the whole-class discussion, is also beneficial to the mathematical discussions. To sum up, as a consequence of selecting and sequencing the student solutions as we suggested here, the scene is set for connecting the solutions in ways that facilitate effective mathematical discussions.

CRITICAL REFLECTION ON THE RESEARCHERS’ ROLE

We regard it as important to consider what we as researchers could improve in order to create even better opportunities for the teacher to teach, thereby creating better affordances for the students to engage in the problem solving process, to discern key mathematical ideas and to connect these key ideas to different strategies and representations. Therefore, we will here in the final discussion reflect upon the researchers’ role. Our reflection draws on the project presented in this paper as well as from another similar design project. Below we suggest several aspects that might be worth considering for researchers in collaborative projects.

Firstly, throughout the project we suggest that it is the researchers’ responsibility to establish a professional discourse of how to denote important components of teaching mathematics through problem solving (cf. Adler & Davis, 2006; Ryve, 2007). For instance, different forms of mathematical representations (graphical, algebraic, arithmetical etc.), the practices of Stein et al. (2008), and strands of mathematical proficiency (Kilpatrick et al., 2001) are of importance both for creating an effective communication (cf. Sfard, 2008) between the researcher and the teacher and between the teacher and the students. Hence, explicit frameworks constituted by technical terms taken from research on problem solving and mathematics for teaching (e.g. Ball, Thames, & Phelps, 2008) need to be introduced and used by the researchers. For example, introducing the picture of Stein et al.’s (2008) five practices to the teacher would make the model more explicit to the teacher. More generally, it is likely that the researcher and teacher should use multiple forms of representations when introducing and discussing frameworks and principles with each other.

Secondly, the researchers have got a significant responsibility to introduce and discuss underlying assumptions guiding the project (cf. Cobb et al., 2003). For instance, the example of sequencing elaborated in this paper takes its departure from the principle of connections between different representations as an essential component of mathematical knowing. Hence, ways of sequencing the problems should be understood as ways of optimizing possibilities to make mathematical connections. Further, another benchmark is that as many students as possible should be able to follow and contribute to the whole-class discussion which might affect the sequencing of problems. During the project the researcher could have been more explicit about such underlying guiding principles.
Thirdly, the handling of procedures – in this case simplifying algebraic expressions – could have been integrated differently through both design projects. Stein et al. (2003) propose that we need to learn much more about how procedural skills can be effectively taught in problem solving approaches to mathematics. One suggestion is to put more emphasis on letting the students simplify their own and each other’s algebraic formulas as an integrated part of the problem solving process. Thereby, the teacher does not need to take as much time to show simplifications during whole-class discussions and the students will probably learn more when they do it themselves.

Finally, the researchers have the responsibility to arrange and discuss important aspects of the three phases launch, explore and discuss-and-summarize (Stein et al., 2008). Creating enough time to think about and discuss student solutions before the whole-class discussion is important for the teacher in order to be able to purposefully select and sequence student solutions (cf. Stein et al., 2008). The teacher in the current study expresses that when the problem is treated in total within the same lesson, “I don’t have time to think through all the possible connections that can be made” and what sticks out the most is then emphasized. Important aspects of the launch phase such as clearly stating the aim of the lesson and which kinds of products the students are expected to produce (Stein et al., 2008) needs to be explicitly discussed with the students. The researchers play an important role in being explicit about these aspects in the discussions with the teacher, in order for the teacher to be able to be explicit to the students.

REFERENCES


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