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CONSTRUCTION OF n -LIE ALGEBRAS AND n -ARY HOM-NAMBU-LIE ALGEBRAS

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ABSTRACT. We present a procedure to construct $(n + 1)$ -Hom-Nambu-Lie algebras from n -Hom-Nambu-Lie algebras equipped with a generalized trace function. It turns out that the implications of the compatibility conditions, that are necessary for this construction, can be understood in terms of the kernel of the trace function and the range of the twisting maps. Furthermore, we investigate the possibility of defining $(n + k)$ -Lie algebras from n -Lie algebras and a k -form satisfying certain conditions.

1. INTRODUCTION

Lie algebras and Poisson algebras have played an extremely important role in mathematics and physics for a long time. Their generalizations, known as n -Lie algebras and “Nambu algebras” [23, 42, 43] also arise naturally in physics and have, for instance, been studied in the context of “M-branes” [14, 25]. Moreover, it has recently been shown that the differential geometry of n -dimensional Riemannian submanifolds can be described in terms of an n -ary Nambu algebra structure on the space of smooth functions on the manifold [8].

A long-standing problem related to Nambu algebras is their quantization. For Poisson algebras, the problem of finding an operator algebra where the commutator Lie algebra corresponds to the Poisson algebra is a well-studied problem, e.g. in the context of matrix regularizations [3, 4, 5, 6, 7]. For higher order algebras much less is known and the corresponding problem seems to be difficult to study. A Nambu-Lie algebra is defined in general by an n -ary multilinear multiplication which is skew-symmetric and satisfies an identity extending the Jacobi identity for the Lie algebras. For $n = 3$ this identity is

$$[x_1, x_2, [x_3, x_4, x_5]] = [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] + [x_3, x_4, [x_1, x_2, x_5]].$$

In Nambu-Lie algebras, the additional freedom in comparison with Lie algebras is mainly limited to extra arguments in the multilinear multiplication. The identities of Nambu-Lie algebras are also closely resembling the identities for Lie algebras. As a result, there is a close similarity between Lie algebras and Nambu-Lie algebras in their appearances in connection to other algebraic and analytic structures and in the extent of their applicability. Thus it is not surprising that it becomes unclear how to associate in meaningful ways ordinary Nambu-Lie algebras with

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the important in physics generalizations and quantum deformations of Lie algebras when typically the ordinary skew-symmetry and Jacobi identities of Lie algebras are violated. However, if the class of Nambu-Lie algebras is extended with enough extra structure beyond just adding more arguments in multilinear multiplication, the natural ways of association of such multilinear algebraic structures with generalizations and quantum deformations of Lie algebras may become feasible. Hom-Nambu-Lie algebras are defined by a similar but more general identity than that of Nambu-Lie algebras involving some additional linear maps. These linear maps twisting or deforming the main identities introduce substantial new freedom in the structure allowing to consider Hom-Nambu-Lie algebras as deformations of Nambu-Lie algebras (n -Lie algebras). The extra freedom built into the structure of Hom-Nambu-Lie algebras may provide a path to quantization beyond what is possible for ordinary Nambu-Lie algebras. All this gives also important motivation for investigation of mathematical concepts and structures such as Leibniz n -ary algebras [16, 23] and their modifications and extensions, as well as Hom-algebra extensions of Poisson algebras [40]. For discussion of physical applications of these and related algebraic structures to models for elementary particles, and unification problems for interactions see [1, 27, 28, 29, 30].

The general Hom-algebra structures arose first in connection to quasi-deformation and discretizations of Lie algebras of vector fields [24, 32]. These quasi-deformations lead to quasi-Lie algebras, quasi-Hom-Lie algebras and Hom-Lie algebras, which are generalized Lie algebra structures with twisted skew-symmetry and Jacobi conditions. The first motivating examples in physics and mathematics literature are q -deformations of the Witt and Virasoro algebras constructed in the investigations of vertex models in conformal field theory [2, 17, 18, 19, 20, 21, 22, 26, 34, 35, 36]. Motivated by these and new examples arising as applications of the general quasi-deformation construction of [24, 31, 32] on the one hand, and the desire to be able to treat within the same framework such well-known generalizations of Lie algebras as the color and super Lie algebras on the other hand, quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras were introduced in [24, 31, 32, 33, 37]. In Hom-Lie algebras, skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as for Lie algebras, reducing to the Jacobi identity for ordinary Lie algebras when the linear twisting map is the identity map.

In this paper, we will be concerned with n -Hom-Nambu-Lie algebras, a class of n -ary algebras generalizing n -ary algebras of Lie type including n -ary Nambu algebras, n -ary Nambu-Lie algebras and n -ary Lie algebras [10, 9]. In [9], a method was demonstrated of how to construct ternary multiplications from the binary multiplication of a Hom-Lie algebra, a linear twisting map, and a trace function satisfying certain compatibility conditions; and it was shown that this method can be used to construct ternary Hom-Nambu-Lie algebras from Hom-Lie algebras. In this article we extend the results and the binary-to-ternary construction of [9] to the general case of n -ary algebras. This paper is organized as follows. In Section 2 we review basic concepts of Hom-Lie, and n -Hom-Nambu-Lie algebras. In Section 3 we provide a construction procedure of $(n + 1)$ -Hom-Nambu-Lie algebras starting from an n -Hom-Nambu-Lie algebra and a trace function satisfying certain compatibility conditions involving the twisting maps. To this end, we use the ternary bracket introduced in [11]. In Section 4, we investigate how restrictive the compatibility

conditions are. The mutual position of kernels of twisting maps and the trace play an important role in this context. Finally, in Section 6, we investigate the possibility to define $(n + k)$ -Lie algebras starting from an n -Lie algebra and a k -form satisfying certain conditions.

2. PRELIMINARIES

In [10], generalizations of n -ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced. These generalizations include n -ary Hom-algebra structures generalizing the n -ary algebras of Lie type including n -ary Nambu algebras, n -ary Nambu-Lie algebras and n -ary Lie algebras, and n -ary algebras of associative type including n -ary totally associative and n -ary partially associative algebras.

Definition 2.1. A *Hom-Lie algebra* $(V, [\cdot, \cdot], \alpha)$ is a vector space V together with a skew-symmetric bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a linear map $\alpha : V \rightarrow V$ satisfying

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]]$$

for all $x, y, z \in V$.

Definition 2.2. A *n -Hom-Nambu-Lie algebra* $(V, [\cdot, \dots, \cdot], \alpha_1, \dots, \alpha_{n-1})$ is a vector space V together with a skew-symmetric multilinear map $[\cdot, \dots, \cdot] : V^n \rightarrow V$ and linear maps $\alpha_1, \dots, \alpha_{n-1} : V \rightarrow V$ such that

$$\begin{aligned} & [\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]] \\ &= \sum_{k=1}^n [\alpha_1(y_1), \dots, \alpha_{k-1}(y_{k-1}), [x_1, \dots, x_{n-1}, y_k], \alpha_k(y_{k+1}), \dots, \alpha_{n-1}(y_n)] \end{aligned}$$

for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in V$. The linear maps $\alpha_1, \dots, \alpha_{n-1}$ are called the *twisting maps* of the Hom-Nambu-Lie algebra. A n -Lie algebra is an n -Hom-Lie algebra with $\alpha_1 = \alpha_2 = \dots = \alpha_{n-1} = \text{id}_V$.

3. CONSTRUCTION OF HOM-NAMBU-LIE ALGEBRAS

In [9], the authors introduced a procedure to induce 3-Hom-Nambu-Lie algebras from Hom-Lie algebras. In the following, we shall extend this procedure to induce a $(n + 1)$ -Hom-Nambu-Lie algebra from an n -Hom-Nambu-Lie algebra. Let us start by defining the skew-symmetric map that will be used to induce the higher order algebra. In the following, \mathbb{K} denotes a field of characteristic 0, and V a vector space over \mathbb{K} .

Definition 3.1. Let $\phi : V^n \rightarrow V$ be an n -linear map and let τ be a map from V to \mathbb{K} . Define $\phi_\tau : V^{n+1} \rightarrow V$ by

$$(3.1) \quad \phi_\tau(x_1, \dots, x_{n+1}) = \sum_{k=1}^{n+1} (-1)^k \tau(x_k) \phi(x_1, \dots, \hat{x}_k, \dots, x_{n+1}),$$

where the hat over \hat{x}_k on the right hand side means that x_k is excluded, that is ϕ is calculated on $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1})$.

We will not be concerned with just any linear map τ , but rather maps that have a generalized trace property. Namely

Definition 3.2. For $\phi : V^n \rightarrow V$ we call a linear map $\tau : V \rightarrow \mathbb{K}$ a ϕ -trace if $\tau(\phi(x_1, \dots, x_n)) = 0$ for all $x_1, \dots, x_n \in V$.

Lemma 3.3. Let $\phi : V^n \rightarrow V$ be a totally skew-symmetric n -linear map and τ a linear map $V \rightarrow \mathbb{K}$. Then ϕ_τ is a $(n+1)$ -linear totally skew-symmetric map. Furthermore, if τ is a ϕ -trace then τ is a ϕ_τ -trace.

Proof. The $(n+1)$ -linearity property of ϕ_τ follows from n -linearity of ϕ and linearity of τ as it is a linear combination of $(n+1)$ -linear maps

$$\tau(x_k)\phi(x_1, \dots, \hat{x}_k, \dots, x_{n+1}), \quad 1 \leq k \leq n+1.$$

To prove total skew-symmetry one simply notes that

$$\phi_\tau(x_1, \dots, x_{n+1}) = -\frac{1}{n!} \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \tau(x_{\sigma(1)}) \phi(x_{\sigma(2)}, \dots, x_{\sigma(n+1)}),$$

which is clearly skew-symmetric. Since each term in ϕ_τ is proportional to ϕ and since τ is linear and a ϕ -trace, τ will also be a ϕ_τ -trace. \square

The extension of Theorem 3.3 in [9] can now be formulated as follows:

Theorem 3.4. Let $(V, \phi, \alpha_1, \dots, \alpha_{n-1})$ be a n -Hom-Nambu-Lie algebra, τ a ϕ -trace and $\alpha_n : V \rightarrow V$ a linear map. If it holds that

$$(3.2) \quad \tau(\alpha_i(x))\tau(y) = \tau(x)\tau(\alpha_i(y))$$

$$(3.3) \quad \tau(\alpha_i(x))\alpha_j(y) = \alpha_i(x)\tau(\alpha_j(y))$$

for all $i, j \in \{1, \dots, n\}$ and all $x, y \in V$, then $(V, \phi_\tau, \alpha_1, \dots, \alpha_n)$ is a $(n+1)$ -Hom-Nambu-Lie algebra. We shall say that $(V, \phi_\tau, \alpha_1, \dots, \alpha_n)$ is induced from $(V, \phi, \alpha_1, \dots, \alpha_{n-1})$.

Proof. Since ϕ_τ is skew-symmetric and multilinear by Lemma 3.3, one only has to check that the Hom-Nambu-Jacobi identity is fulfilled. This identity is written as

$$\begin{aligned} & \sum_{s=1}^{n+1} \phi_\tau(\alpha_1(u_1), \dots, \alpha_{s-1}(u_{s-1}), \phi_\tau(x_1, \dots, x_n, u_s), \alpha_s(u_{s+1}), \dots, \alpha_n(u_{n+1})) \\ & - \phi_\tau(\alpha_1(x_1), \dots, \alpha_n(x_n), \phi_\tau(u_1, \dots, u_{n+1})) = 0. \end{aligned}$$

Let us write the left-hand-side of this equation as $A - B$ where

$$A = \sum_{s=1}^{n+1} \phi_\tau(\alpha_1(u_1), \dots, \alpha_{s-1}(u_{s-1}), \phi_\tau(x_1, \dots, x_n, u_s), \alpha_s(u_{s+1}), \dots, \alpha_n(u_{n+1}))$$

$$B = \phi_\tau(\alpha_1(x_1), \dots, \alpha_n(x_n), \phi_\tau(u_1, \dots, u_{n+1})).$$

Furthermore, we expand B into terms B_{kl} such that

$$\begin{aligned} B_{kl} &= (-1)^{k+l} \tau(\alpha_k(x_k)) \tau(u_l) \times \\ & \quad \times \phi(\alpha_1(x_1), \dots, \widehat{\alpha_k(x_k)}, \dots, \alpha_n(x_n), \phi(u_1, \dots, \hat{u}_l, \dots, u_{n+1})) \end{aligned}$$

$$B = \sum_{k=1}^n \sum_{l=1}^{n+1} B_{kl},$$

taking into account that τ is a ϕ -trace form. We expand A as

$$\begin{aligned} A &= (-1)^{n+1} \sum_{s=1}^{n+1} \tau(u_s) \phi_\tau(\alpha_1(u_1), \dots, \phi(x_1, \dots, x_n), \dots, \alpha_n(u_{n+1})) \\ &\quad + \sum_{s=1}^{n+1} \sum_{k=1}^n (-1)^k \tau(x_k) \phi_\tau(\alpha_1(u_1), \dots, \phi(x_1, \dots, \hat{x}_k, \dots, x_n, u_s), \dots, \alpha_n(u_{n+1})) \\ &\equiv (-1)^{n+1} A^{(1)} + A^{(2)}. \end{aligned}$$

Let us now show that $A^{(1)} = 0$. For every choice of integers $k < l \in \{1, \dots, n+1\}$, $A^{(1)}$ contains two terms where one τ involves u_k and the other τ involves u_l . Namely,

$$\begin{aligned} A^{(1)} &= \sum_{k < l = 1}^{n+1} (-1)^l \tau(u_k) \tau(\alpha_{l-1}(u_l)) \times \\ &\quad \times \phi(\alpha_1(u_1), \dots, \alpha_{k-1}(u_{k-1}), \phi(x_1, \dots, x_n), \dots, \widehat{\alpha_{l-1}(u_l)}, \dots, \alpha_n(u_{n+1})) \\ &\quad + \sum_{k < l = 1}^{n+1} (-1)^k \tau(u_l) \tau(\alpha_k(u_k)) \times \\ &\quad \times \phi(\alpha_1(u_1), \dots, \widehat{\alpha_k(u_k)}, \dots, \phi(x_1, \dots, x_n), \alpha_l(u_{l+1}), \dots, \alpha_n(u_{n+1})). \end{aligned}$$

By using relations (3.2) and (3.3) one can write these two terms together as

$$\begin{aligned} A^{(1)} &= \sum_{k < l = 1}^{n+1} (-1)^l \left[\tau(u_k) \tau(\alpha_{l-1}(u_l)) - \tau(u_l) \tau(\alpha_{l-1}(u_k)) \right] \times \\ &\quad \times \phi(\alpha_1(u_1), \dots, \alpha_{k-1}(u_{k-1}), \phi(x_1, \dots, x_n), \dots, \widehat{\alpha_{l-1}(u_l)}, \dots, \alpha_n(u_{n+1})), \end{aligned}$$

and it follows from (3.2) that each term in this sum is zero. Let us now consider the expansion of $A^{(2)}$ via

$$\begin{aligned} A_{kl}^{(2)} &= \sum_{s=1}^{l-1} (-1)^{k+l} \tau(x_k) \tau(\alpha_{l-1}(u_l)) \times \\ &\quad \times \phi(\alpha_1(u_1), \dots, \phi(x_1, \dots, \hat{x}_k, \dots, x_n, u_s), \dots, \widehat{\alpha_{l-1}(u_l)}, \dots, \alpha_n(u_{n+1})) \\ &\quad + \sum_{s=l+1}^{n+1} (-1)^{k+l} \tau(x_k) \tau(\alpha_l(u_l)) \times \\ &\quad \times \phi(\alpha_1(u_1), \dots, \widehat{\alpha_l(u_l)}, \dots, \phi(x_1, \dots, \hat{x}_k, \dots, x_n, u_s), \dots, \alpha_n(u_{n+1})). \end{aligned}$$

One notes that relations (3.2) and (3.3) allows one to swap any α_i and α_j in the expression above. Therefore, $B_{kl} - A_{kl}^{(2)}$ will be proportional to the Hom-Nambu-Jacobi identity for ϕ (acting on the elements $x_1, \dots, \hat{x}_k, \dots, x_n, u_1, \dots, \hat{u}_l, \dots, u_{n+1}$) since one can make sure that α_n , which is not one of the twisting maps of the original Hom-Nambu-Lie algebra, appears outside any bracket. Hence, $A^{(2)} - B = 0$ and the Hom-Nambu-Jacobi identity is satisfied for ϕ_τ . \square

Since τ is also a ϕ_τ -trace, one can repeat the procedure in Theorem 3.4 to induce a $(n+2)$ -Hom-Nambu-Lie algebra from an n -Hom-Nambu-Lie algebra. However, the result is an abelian algebra.

Proposition 3.5. *Let $\mathcal{A} = (V, \phi, \alpha_1, \dots, \alpha_{n-1})$ be a n -Hom-Nambu-Lie algebra. Let \mathcal{A}' be any $(n+1)$ -Hom-Nambu-Lie algebra induced from \mathcal{A} via the ϕ -trace τ . If \mathcal{A}'' is a $(n+2)$ -Hom-Nambu-Lie algebra induced from \mathcal{A}' using the same τ again, then \mathcal{A}'' is abelian.*

Proof. By the definition of ϕ_τ , the bracket on the algebra \mathcal{A}'' can be written as

$$(\phi_\tau)_\tau = \sum_{k=1}^{n+2} (-1)^k \tau(x_k) \phi_\tau(x_1, \dots, \hat{x}_k, \dots, x_{n+2}).$$

Expanding the bracket ϕ_τ , there will be, for every choice of integers $k < l$, two terms which are proportional to $\tau(x_k)\tau(x_l)$. Their sum becomes

$$\tau(x_k)\tau(x_l)\phi(x_1, \dots, \hat{x}_k, \dots, \hat{x}_l, \dots, x_{n+2}) \left((-1)^{k+l} + (-1)^{k+l-1} \right) = 0.$$

Hence, $(\phi_\tau)_\tau(x_1, \dots, x_{n+2}) = 0$ for all $x_1, \dots, x_{n+2} \in V$. \square

Note that one might choose a different ϕ_τ -trace when repeating the construction, and this can lead to non-abelian algebras as in the following example.

Example 3.6. *Let us use an example from [9], where one starts with a Hom-Lie algebra defined on a vector space $V = \text{span}(x_1, x_2, x_3, x_4)$ via*

$$[x_i, x_j] = a_{ij}x_3 + b_{ij}x_4$$

and $\alpha_1(x_i) = x_3$ for $i = 1, \dots, 4$. This will be a Hom-Lie algebra provided a_{ij}, b_{ij} satisfy certain conditions; let us for definiteness choose

$$(b_{ij}) = \begin{pmatrix} 0 & b & b & b+c \\ -b & 0 & 0 & c \\ -b & 0 & 0 & c \\ -b-c & -c & -c & 0 \end{pmatrix}$$

together with $a_{ij} = 1$ for all $i < j$. To construct a 3-Hom-Nambu-Lie algebra we set $\tau(x_3) = \tau(x_4) = 0$, $\tau(x_1) = \tau(x_2) = 1$ and $\alpha_2(x_i) = x_4$ for $i = 1, \dots, 4$. One can easily check that τ is a ϕ -trace and that the compatibility conditions are fulfilled. The induced algebra will then have the following brackets:

$$\begin{aligned} [x_1, x_2, x_3] &= -bx_4 \\ [x_1, x_2, x_4] &= -cx_4 \\ [x_1, x_3, x_4] &= x_3 + cx_4 \\ [x_2, x_3, x_4] &= x_3 + cx_4. \end{aligned}$$

Now, we want to continue this process and find another trace ρ together with a linear map α_3 such that the resulting 4-Hom-Nambu-Lie algebra is non-abelian. By choosing $\rho(x_3) = \rho(x_4) = 0$, $\rho(x_1) = \delta_1$, $\rho(x_2) = \delta_2$ and $\alpha_3 = \alpha_1$, one sees that ρ is a trace and that the compatibility conditions are fulfilled. The induced 4-Hom-Nambu-Lie algebra has only one independent bracket, namely

$$[x_1, x_2, x_3, x_4] = (\delta_2 - \delta_1)(x_3 + cx_4),$$

which is non-zero for $\delta_1 \neq \delta_2$.

4. THE COMPATIBILITY CONDITIONS

Given an n -Hom-Nambu-Lie algebra we ask the question: Can we find a trace and a linear map such that a $(n + 1)$ -Hom-Nambu-Lie algebra can be induced? In the following we shall study the implications of the assumptions in Theorem 3.4; it turns out that the relation between the kernel of τ and the range of α_i is important.

Definition 4.1. Let V be a vector space, $\alpha_1, \dots, \alpha_n$ linear maps $V \rightarrow V$ and τ a linear map $V \rightarrow \mathbb{K}$. The tuple $(\alpha_1, \dots, \alpha_n, \tau)$ is *compatible on V* if

$$(4.1) \quad \tau(\alpha_i(x))\tau(y) = \tau(x)\tau(\alpha_i(y))$$

$$(4.2) \quad \tau(\alpha_i(x))\alpha_j(y) = \alpha_i(x)\tau(\alpha_j(y))$$

for all $x, y \in V$ and $i, j \in \{1, \dots, n\}$. A compatible tuple is *nondegenerate* if $\ker(\tau) \neq V$ and $\ker(\tau) \neq \{0\}$.

We introduce $K = \ker(\tau)$ and $U = V \setminus K$. Note that for a nondegenerate compatible tuple U is always non-empty and K contains at least one non-zero element.

Lemma 4.2. *If $(\alpha_1, \dots, \alpha_n, \tau)$ is a nondegenerate compatible tuple on V then $\alpha_i(K) \subseteq K$ for $i = 1, \dots, n$.*

Proof. Let x be an arbitrary element of K . Since the tuple is assumed to nondegenerate, there exists a non-zero element $y \in U$. Relation (3.2) applied to x and y gives

$$\tau(y)\tau(\alpha_i(x)) = 0,$$

and since $\tau(y) \neq 0$ this implies that $\tau(\alpha_i(x)) = 0$. □

Lemma 4.3. *Let $(\alpha_1, \dots, \alpha_n, \tau)$ be a nondegenerate compatible tuple on V and assume that there exists an element $u \in U$ such that $\alpha(u) \in K$. Then $\alpha(V) \subseteq K$.*

Proof. Let x be an arbitrary element of V . Relation (3.2) applied to x and u gives

$$\tau(u)\tau(\alpha_i(x)) = 0,$$

and since $\tau(u) \neq 0$ it follows that $\tau(\alpha_i(x)) = 0$. □

Hence, for a nondegenerate compatible tuple it either holds that $\alpha_i(V) \subseteq K$ or $\alpha_i(U) \subseteq U$.

Proposition 4.4. *Let $(\alpha_1, \dots, \alpha_n, \tau)$ be a nondegenerate compatible tuple on V and assume that there exist $i, j \in \{1, \dots, n\}$ such that $\alpha_i(U) \subseteq U$ and $\alpha_j(U) \subseteq U$. Then there exists $\lambda_{ij} \in \mathbb{K} \setminus \{0\}$ such that $\alpha_i = \lambda_{ij}\alpha_j$, where $\lambda_{ij} = \tau(\alpha_i(u))/\tau(\alpha_j(u))$ for any $u \in U$.*

Proof. With $u \in U$ and $x \in V$ equation (3.3) becomes

$$\tau(\alpha_i(u))\alpha_j(x) = \tau(\alpha_j(u))\alpha_i(x).$$

By assumption, $\tau(\alpha_i(u)) \neq 0$ and $\tau(\alpha_j(u)) \neq 0$, which implies that

$$\alpha_i(x) = \frac{\tau(\alpha_i(u))}{\tau(\alpha_j(u))}\alpha_j(x),$$

which proves the statement. □

Proposition 4.5. *Let $(\alpha_1, \dots, \alpha_n, \tau)$ be a nondegenerate compatible tuple on V and assume there exist $i, j \in \{1, \dots, n\}$ such that $\alpha_i(U) \subseteq U$ and $\alpha_j(U) \subseteq K$. Then $\alpha_j(x) = 0$ for all $x \in V$.*

Proof. Assume that $\alpha_i(U) \subseteq U$ and $\alpha_j(U) \subseteq K$ and let $u \in U$ and $x \in V$. Equation (3.3) gives

$$\tau(\alpha_i(u))\alpha_j(x) = 0,$$

which implies that $\alpha_j(x) = 0$ since $\alpha_i(u) \in U$. \square

The above results tell us that given an n -Hom-Nambu-Lie algebra with twisting maps $\alpha_1, \dots, \alpha_{n-1}$, and a ϕ -trace τ , there is not much choice when choosing α_n . Namely, if there is a twisting map α_i such that $\alpha_i(U) \subseteq U$ then either $\alpha_n = 0$ or α_n is proportional to α_i . In the case when $\alpha_i(U) \subseteq K$ for $i = 1, \dots, n-1$ one has slightly more freedom of choosing α_n , but note that unless $\alpha_n(U) \subseteq K$, Proposition 4.5 gives $\alpha_i = 0$ for $i = 1, \dots, n-1$. When $\alpha_i(U) \subseteq K$ for $i = 1, \dots, n$ the compatibility conditions (3.2) and (3.3) are automatically satisfied.

Hence, there are two potentially interesting cases which give rise to non-zero twisting maps:

$$(C1) \quad \alpha_i(U) \subseteq U \text{ for } i = 1, \dots, n$$

$$(C2) \quad \alpha_i(U) \subseteq K \text{ for } i = 1, \dots, n.$$

In case (C1) all the twisting maps will be proportional.

Having considered the case of nondegenerate compatible tuples, let us show that degenerate ones lead to abelian algebras.

Proposition 4.6. *Let $\mathcal{A} = (V, \phi_\tau, \alpha_1, \dots, \alpha_n)$ be a Hom-Nambu-Lie algebra induced by $(V, \phi, \alpha_1, \dots, \alpha_{n-1})$ and a ϕ -trace τ . If $(\alpha_1, \dots, \alpha_n, \tau)$ is a degenerate compatible tuple then \mathcal{A} is abelian.*

Proof. By the definition of ϕ_τ it is clear that if $\ker \tau = V$ then $\phi_\tau(x_1, \dots, x_{n+1}) = 0$ for all $x_1, \dots, x_{n+1} \in V$. Now, assume that $\ker \tau = \{0\}$. Since τ is a ϕ -trace one has that $\tau(\phi(x_1, \dots, x_n)) = 0$ for all $x_1, \dots, x_n \in V$. Hence, $\phi(x_1, \dots, x_n)$ is in $\ker \tau$ which implies that $\phi(x_1, \dots, x_n) = 0$. From this it immediately follows that $\phi_\tau(x_1, \dots, x_{n+1}) = 0$ for all $x_1, \dots, x_{n+1} \in V$. \square

5. TWISTING OF n -ARY HOM-NAMBU-LIE ALGEBRAS

In [47] the general property that an n -Lie algebra induces a $(n-k)$ -Lie algebra by fixing k elements in the bracket, was extended to Hom-Nambu-Lie algebras.

Definition 5.1. Let $\phi : V^n \rightarrow V$ be a linear map and let a_1, \dots, a_k (with $k < n$) be elements of V . By $\pi_{a_1 \dots a_k} \phi$ we denote the map $V^{n-k} \rightarrow V$ defined by

$$(5.1) \quad (\pi_{a_1 \dots a_k} \phi)(x_1, \dots, x_{n-k}) = \phi(x_1, \dots, x_{n-k}, a_1, \dots, a_k).$$

The result in [47] can now be stated as

Proposition 5.2. *Let $(V, \phi, \alpha_1, \dots, \alpha_{n-1})$ be an n -Hom-Nambu-Lie algebra and let $a_1, \dots, a_k \in V$ (with $k < n$) be elements such that $\alpha_{n-k-1+i}(a_i) = a_i$ for $i = 1, \dots, k$. Then $(V, \pi_{a_1 \dots a_k} \phi, \alpha_1, \dots, \alpha_{n-k-1})$ is a $(n-k)$ -Hom-Nambu-Lie algebra.*

Hence, given an n -Hom-Nambu-Lie algebra one can create a ‘‘twisted’’ n -Hom-Nambu-Lie algebra by applying Proposition 5.2 and Theorem 3.4.

Proposition 5.3. *Let $(V, \phi, \alpha_1, \dots, \alpha_{n-1})$ be an n -Hom-Nambu-Lie algebra, τ a ϕ -trace and $\alpha_n : V \rightarrow V$ a linear map such that equations (3.2) and (3.3) are fulfilled. If there exists an element $a \in V$ such that $\alpha_n(a) = a$ then $(V, \pi_a \phi_\tau, \alpha_1, \dots, \alpha_{n-1})$ is an n -Hom-Nambu-Lie algebra with*

$$\begin{aligned} (\pi_a \phi_\tau)(x_1, \dots, x_n) &= \sum_{k=1}^n (-1)^k \tau(x_k) \phi(x_1, \dots, \hat{x}_k, \dots, x_n, a) \\ &\quad + (-1)^{n+1} \tau(a) \phi(x_1, \dots, x_n). \end{aligned}$$

One can also go the other way around: first applying π and then inducing a higher order algebra.

Proposition 5.4. *Let $(V, \phi, \alpha_1, \dots, \alpha_{n-1})$ be an n -Hom-Nambu-Lie algebra and τ a $\pi_a \phi$ -trace such that equations (3.2) and (3.3) are fulfilled. If there exists an element $a \in V$ such that $\alpha_{n-1}(a) = a$ then it holds that $(V, (\pi_a \phi)_\tau, \alpha_1, \dots, \alpha_{n-1})$ is a n -Hom-Nambu-Lie algebra with*

$$(\pi_a \phi)_\tau(x_1, \dots, x_n) = \sum_{k=1}^n (-1)^k \tau(x_k) \phi(x_1, \dots, \hat{x}_k, \dots, x_n, a).$$

One notes that the two types of twistings are in general not equivalent. In fact, assuming the two procedures in Propositions 5.3 and 5.4 are well defined for some element $a \in V$ and denoting $\phi_\tau \equiv i_\tau \phi$, one can write

$$(5.2) \quad ([i_\tau, \pi_a] \phi)(x_1, \dots, x_n) = (-1)^n \tau(a) \phi(x_1, \dots, x_n).$$

Thus, unless $a \in \ker \tau$, one recovers the ‘‘untwisted’’ n -Hom-Nambu-Lie algebra as the commutator of the maps i_τ and π_a . If $a \in \ker \tau$ then the two procedures yield the same result.

Recalling the possible cases for the relation between $\alpha_1, \dots, \alpha_n$ and the kernel of τ , one notes that in case (C2) any fixed point of α_i is necessarily in the kernel of τ , which implies that $[i_\tau, \pi_a] = 0$. In case (C1) there might be fixed points in U .

6. HIGHER ORDER CONSTRUCTIONS

A natural extension of the current framework would be to construct a $(n+p)$ -Lie algebra from an n -Lie algebra and a p -form by using the wedge product. Let us illustrate how this can be done and investigate the connection to a closely related kind of algebras, satisfying the so called *generalized Jacobi identity*.

Definition 6.1. Let $(V, \phi, \alpha_1, \dots, \alpha_{n-1})$ be an n -ary Hom-Nambu-Lie algebra and let $\tau \in \wedge^p V$ be a p -form. Define $\tau \wedge \phi : V^{n+p} \rightarrow V$ by

$$\begin{aligned} (\tau \wedge \phi)(x_1, \dots, x_{n+p}) &= \frac{1}{n!p!} \sum_{\sigma \in S_{n+p}} \text{sgn}(\sigma) \tau(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \phi(x_{\sigma(p+1)}, \dots, x_{\sigma(n+p)}) \\ &\equiv \phi_\tau(x_1, \dots, x_{n+p}) \end{aligned}$$

for all $x_1, \dots, x_{n+p} \in V$.

There is a natural extension of the concept of ϕ -traces to p -forms.

Definition 6.2. For $\phi : V^n \rightarrow V$ we call $\tau \in \wedge^p V$ a ϕ -compatible p -form if $\tau(\phi(x_1, \dots, x_n), y_1, \dots, y_{p-1}) = 0$ for all $x_1, \dots, x_n, y_1, \dots, y_{p-1} \in V$.

The fundamental identity of n -Lie algebras is not a complete symmetrization of an iterated bracket. The generalized Jacobi identity is a more symmetric extension of the standard Jacobi identity.

Definition 6.3. A vector valued form $\phi \in \wedge^n(V, V)$ is said to satisfy the *generalized Jacobi identity* if

$$(6.1) \quad \sum_{\sigma \in S_{2n-1}} \operatorname{sgn}(\sigma) \phi(\phi(x_{\sigma(1)}, \dots, x_{\sigma(n)}), x_{\sigma(n+1)}, \dots, x_{\sigma(2n-1)}) = 0$$

for all $x_1, \dots, x_{2n-1} \in V$.

In the following we study the question when $\tau \wedge \phi$ satisfies the generalized Jacobi identity, or the fundamental identity of n -Lie algebras.

For $\phi \in \wedge^k(V, V)$ and $\psi \in \wedge^{l+1}(V)$ (or $\wedge^{l+1}(V, V)$) one defines the *interior product* $i_\phi \psi \in \wedge^{k+l}(V)$ (resp. $\wedge^{l+1}(V, V)$) as

$$(6.2) \quad i_\phi \psi(x_1, \dots, x_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \psi(\phi(x_{\sigma(1)}, \dots, x_{\sigma(k)}), x_{\sigma(k+1)}, \dots, x_{\sigma(k+l)}).$$

The generalized Jacobi identity for ϕ can then be expressed as $i_\phi \phi = 0$. The interior product satisfies the following properties with respect to the wedge product (see e.g. [41])

$$(6.3) \quad i_{\tau \wedge \phi} \psi = \tau \wedge (i_\phi \psi)$$

$$(6.4) \quad i_\phi(\tau \wedge \psi) = (i_\phi \tau) \wedge \psi + (-1)^{(k-1)p} \tau \wedge (i_\phi \psi),$$

where $\tau \in \wedge^p(V)$. With these set of relations at hand, one can now easily prove the following statement.

Proposition 6.4. *If $\phi \in \wedge^n(V, V)$ satisfies the generalized Jacobi identity and τ is a ϕ -compatible p -form, then $\tau \wedge \phi$ satisfies the generalized Jacobi identity.*

Proof. The generalized Jacobi identity can be expressed as $i_{\tau \wedge \phi}(\tau \wedge \phi) = 0$. One computes

$$\begin{aligned} i_{\tau \wedge \phi}(\tau \wedge \phi) &= \tau \wedge (i_\phi(\tau \wedge \phi)) = \tau \wedge ((i_\phi \tau) \wedge \phi + (-1)^{(n-1)p} \tau \wedge (i_\phi \phi)) \\ &= \tau \wedge (i_\phi \tau) \wedge \phi = 0, \end{aligned}$$

since τ is ϕ -compatible (which implies $i_\phi \tau = 0$) and $i_\phi \phi = 0$ by assumption. \square

It is known that the bracket of any n -Lie algebra satisfies the generalized Jacobi identity (see e.g. [12]), which can be shown by symmetrizing the fundamental identity of the n -Lie algebra. Starting from an n -Lie algebra, Proposition 6.4 does not guarantee that $\tau \wedge \phi$ defines a Nambu-Lie algebra. However, if we assume that

$$\left((i_{x_1 \dots x_{p-1}} \tau) \wedge \tau \right) (y_1, \dots, y_{p+1}) = (\tau(x_1, \dots, x_{p-1}, \cdot) \wedge \tau)(y_1, \dots, y_{p+1}) = 0$$

for all $x_1, \dots, x_{p-1}, y_1, \dots, y_{p+1} \in V$, then the desired result follows.

Proposition 6.5. *Let (V, ϕ) be an n -Nambu-Lie algebra and assume that τ is a ϕ -compatible p -form such that $(i_{x_1 \dots x_{p-1}} \tau) \wedge \tau = 0$ for all $x_1, \dots, x_{p-1} \in V$. Then $(V, \tau \wedge \phi)$ is a $(n+p)$ -Nambu-Lie algebra.*

Proof. Let us start by introducing some notation. Let $\mathcal{X} = (x_1, \dots, x_{m-1})$ and $\tilde{\mathcal{X}} = (x_1, \dots, x_m)$ be ordered sets, and let X be an ordered subset of $\tilde{\mathcal{X}}$ such that $X = (x_{i_1}, \dots, x_{i_p})$ with $i_k < i_l$ if $k < l$. By $\tilde{\mathcal{X}} \setminus X$ (or $\mathcal{X} \setminus X$) we mean the ordered set obtained from $\tilde{\mathcal{X}}$ (or \mathcal{X}) by removing the elements in X . Similarly, we set $\mathcal{Y} = (y_1, \dots, y_m)$ and let Y be an ordered subset of \mathcal{Y} .

Let $x_1, \dots, x_{m-1}, y_1, \dots, y_m$ be arbitrary elements of V and set

$$x_m = (\tau \wedge \phi)(y_1, \dots, y_m).$$

The fundamental identity can then be written as

$$(6.5) \quad \text{FI} = \sum_{k=1}^m (\tau \wedge \phi)(y_1, \dots, y_{k-1}, (\tau \wedge \phi)(x_1, \dots, x_{m-1}, y_k), y_{k+1}, \dots, y_m) \\ - (\tau \wedge \phi)(x_1, \dots, x_m) = 0.$$

Since τ is compatible with ϕ , all of the terms with x_m inside τ vanish, and one can rewrite the second term in (6.5) as

$$(6.6) \quad (\tau \wedge \phi)(x_1, \dots, x_m) = \sum_{X \subset \mathcal{X}, Y \subset \mathcal{Y}} \text{sgn}(X) \text{sgn}(Y) \tau(X) \tau(Y) \phi(\mathcal{X} \setminus X, \phi(\mathcal{Y} \setminus Y)),$$

where $\text{sgn}(X)$ denotes the sign of the permutation $\sigma \in S_m$ such that

$$X = (x_{\sigma(1)}, \dots, x_{\sigma(p)}) \\ (\mathcal{X} \setminus X, x_m) = (x_{\sigma(p+1)}, \dots, x_{\sigma(m)}),$$

and $\text{sgn}(Y)$ denotes the sign of the permutation $\rho \in S_m$ such that

$$Y = (y_{\rho(1)}, \dots, y_{\rho(p)}) \\ \mathcal{Y} \setminus Y = (y_{\rho(p+1)}, \dots, y_{\rho(m)}).$$

Let us now turn to the first term in (6.5), which will generate two types of terms. The first kind, A , will include a τ acting on y_k , and the second kind, B , include no such τ . One can now rewrite B in a fashion similar to (6.6)

$$B = \sum_{X \subset \mathcal{X}, Y \subset \mathcal{Y}} \text{sgn}(X) \text{sgn}(Y) \tau(X) \tau(Y) \times \\ \sum_{y_{i_k} \in \mathcal{Y} \setminus Y} \phi(y_{i_1}, \dots, y_{i_{k-1}}, \phi(\mathcal{X} \setminus X, y_{i_k}), y_{i_{k+1}}, \dots, y_{i_n}),$$

where $(y_{i_1}, \dots, y_{i_n}) = \mathcal{Y} \setminus Y$. Now, one notes that subtracting B from (6.6) gives zero due to the fact that ϕ satisfies the fundamental identity. Thus, there will only be terms of type A left in (6.5). To rewrite these terms we introduce yet some notation.

Let $\tilde{Y} = (y_{i_1}, \dots, y_{i_{n-1}})$ be an ordered subset of Y and let \tilde{X} be an ordered subset of \mathcal{X} with $|\tilde{X}| = n$. By $\text{sgn}(\tilde{Y}_k)$ we denote the sign of the permutation σ such that $\sigma(m) = k$ and

$$\tilde{Y} = (y_{\sigma(p+1)}, \dots, y_{\sigma(m-1)}) \\ \mathcal{Y} \setminus \tilde{Y} = (y_{\sigma(1)}, \dots, y_{\sigma(p)}),$$

and by $\text{sgn}(\bar{X})$ we denote the sign of the permutation ρ such that $\rho(p) = m$ and

$$\begin{aligned}\bar{X} &= (x_{\rho(p+1)}, \dots, x_{\rho(m)}) \\ \mathcal{X} \setminus \bar{X} &= (x_{\rho(1)}, \dots, x_{\rho(p-1)})\end{aligned}$$

with the definition $x_m = y_k$. In this notation, the terms of type A can be written as

$$\begin{aligned}A &= \sum_{\tilde{Y} \subset \mathcal{Y}, \bar{X} \subset \mathcal{X}} \text{sgn}(\bar{X}) \phi(\tilde{Y}, \phi(\bar{X})) \sum_{y_k \in \mathcal{Y} \setminus \tilde{Y}} \text{sgn}(\tilde{Y}_k) \tau(\mathcal{X} \setminus \bar{X}, y_k) \tau(\mathcal{Y} \setminus \tilde{Y} \setminus \{y_k\}) \\ &\propto \sum_{\tilde{Y} \subset \mathcal{Y}, \bar{X} \subset \mathcal{X}} \text{sgn}(\bar{X}) \phi(\tilde{Y}, \phi(\bar{X})) ((i_{x_{\rho(1)} \dots x_{\rho(p-1)}} \tau) \wedge \tau) (\mathcal{Y} \setminus \tilde{Y}) = 0,\end{aligned}$$

since $(i_{x_1 \dots x_{p-1}} \tau) \wedge \tau = 0$ for all $x_1, \dots, x_{p-1} \in V$. \square

The following example provides a generic construction in the case when ϕ maps V^n onto a proper subspace of V .

Example 6.6. *Let (V, ϕ) be an n -Lie algebra, with $\dim(V) = m$, such that $\phi : V^n \rightarrow U$, where U is a $m - p$ dimensional subspace of V ($p \geq 1$). Given a basis u_1, \dots, u_{m-p} of U , we define $\tau \in \bigwedge^p V$ as*

$$\tau(v_1, \dots, v_p) = \det(v_1, \dots, v_p, u_1, \dots, u_{m-p}).$$

Then τ is a ϕ -compatible p -form on V . Let us now show that $(i_{v_1 \dots v_{p-1}} \tau) \wedge \tau = 0$. From the definition of τ one obtains

$$\begin{aligned}& ((i_{v_1 \dots v_{p-1}} \tau) \wedge \tau)(w_1, \dots, w_{p+1}) \\ &= \sum_{\sigma \in S_{p+1}} \text{sgn}(\sigma) \det(v_1, \dots, v_{p-1}, w_{\sigma(1)}, u_1, \dots, u_{m-p}) \\ &\quad \times \det(w_{\sigma(2)}, \dots, w_{\sigma(p+1)}, u_1, \dots, u_{m-p}).\end{aligned}$$

For this to be non-zero, one needs first of all that $v_1, \dots, v_{p-1}, u_1, \dots, u_{m-p}$ and w_1, \dots, w_{p+1} are two sets of linearly independent vectors. Even though this is the case every term in the sum is zero since for a non-zero result one needs that w_1, \dots, w_{p+1} are linearly independent of u_1, \dots, u_{m-p} , which is impossible due to the fact that w_1, \dots, w_{p+1} are linearly independent and $\dim(V) = m$. Hence, by Proposition 6.5, $(V, \tau \wedge \phi)$ is a $(n + p)$ -Nambu-Lie algebra.

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