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Establishing Mathematics for Teaching within Classroom Interactions in Teacher Education

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Teacher educators' processes of establishing 'mathematics for teaching' in teacher education programs have been recognized as an important area for further research. In this study we examine how two teacher educators establish and make explicit features of mathematics for teaching within classroom interactions. The study shows how the establishment of mathematics for teaching is dependent on the use of keywords from the mathematics education domain, the introduction of variation, and the use of generic communicative strategies. As such, the study could be seen as a contribution to ongoing research on how mathematics teacher educators interactively deal with mathematics for teaching.

1 Introduction

Mathematics for teaching (MfT; e.g., Ball, Thames, & Phelps, 2008; Davis & Simmt, 2006; Even, 1990; Hill & Ball, 2004) and subsets within it have been introduced to conceptualize specific ways in which teachers need to know mathematics in order to promote successful learning of mathematics for students. As part of this focus researchers engage in studies of how opportunities are created in teacher education programs for prospective and in-service mathematics teachers to develop knowing in teaching mathematics (e.g., Adler & Davis, 2006; Davis, Adler, & Parker, 2007; Huillet, 2009; Rowland, 2008). Adler and Davis (2006) suggest that the dual nature of MfT, involving, as it does, elements from the domains of mathematics and of mathematics education, makes it particularly difficult to teach. In this article we aim to contribute to research into *how* MfT can be established in mathematics teacher education courses by studying the classroom interactions of two Swedish teacher education courses. Our contribution in relation to prior research is that we dig into interactional details of the teacher educators' role in the establishment of MfT and especially how they introduce and use terms and expressions from the mathematics education domain. A guiding research question is: How do these teacher educators establish MfT in the classroom interaction?

2 Theoretical approach

Since the mid-1980s when Shulman (1986, 1987) introduced the concept of pedagogical content knowledge (PCK), an impressive body of research has applied and developed Shulman's ideas, particularly in mathematics education (e.g., Ball et al., 2008; Davis & Simmt, 2006; Even, 1990; Hill & Ball, 2004; Huillet, 2009; Rowland, 2008). Through and within this development, MfT has become established as an important concept. Essential work has been conducted in several areas related to MfT including fine-grained elaborations of subsets of MfT (e.g., Ball et al., 2008), development of assessment and tests for examining MfT (e.g., Hill, Rowan, & Ball, 2005), development of categorizations examining video-recordings of teaching in pre-service education (e.g., Rowland, 2008), and examinations of how MfT is established in mathematics teacher education (e.g., Adler & Davis, 2006, 2011). By drawing on the work of Bernstein, Adler and Davis (2011) develop an analytical approach enabling them to study how teacher educators, in establishing MfT in classrooms, model teaching in different ways ('look at my practice', 'look at your practice', and 'look at mathematics education practice') and how the teacher educators appeal to different domains of knowledge (mathematics, mathematics education, metaphors, experience of teacher educator or students, curriculum, authority of teacher educator). As indicated above, in this article we are also interested in examining how the object of MfT can be established in mathematics teacher education. However, we are slightly more focused on turn-to-turn interactional features of the classroom interactions and how those features affect how MfT is established. To engage in this focus we use a collection of analytical constructs to conceptualize and analyze our data that are to be presented.

2.1 Variation theory

Adler and Davis (2011) use Bernstein to conceptualize the classroom interactions in terms of evaluative events aimed at establishing a specific mathematics/teaching object. In our aim of digging into interactional details of the two classrooms we found variation theory (Marton & Booth, 1997; Marton & Tsui, 2004; Sun, 2011) useful for conceptualizing the establishment of the object of MfT. Variation theory is a theory of learning focused especially on the object of learning. In this study we analyze the *enacted object of learning* which is an interactional phenomenon and part of the classroom interaction. To analyze the enacted object of learning includes revealing the possibilities and limitations for a learner to experience features of the learning object as it is established in the classroom interaction (Runesson, 2005). The interest in what is established in the classroom interaction builds on the assumption that “in order for some content to be learned, it has to be represented as an object available for semiotic mediation in pedagogic interactions between teacher and learner” (Adler & Davis, 2011, p. 143). A feature (e.g., colour) and its values (e.g., blue, red, yellow) are within variation theory denoted a *dimension of variation*. An object of learning typically consists of a number of dimensions of variation. Let us take an example related to our data. Dimensions of variation concerning mathematical problems and critical for prospective teachers to experience in relation to MfT could be, for instance: Mathematical ideas embedded in the problem (statistics, probability); possible mathematical representations useful in teaching and solving a problem (algebraic, graphic); how different mathematical problems are suitable for different students (Year 1, Year 5); ways to appropriately introduce the problem in the classroom (telling a story, drawing a picture).

From a variation theory perspective, discerning variation requires a juxtaposition of current and past experience and in approaching our data we consider how dimensions of variations are present in the classroom interaction. Marton, Runesson, & Tsui (2004) delineate three ways in which dimensions of variation could be established that are of relevance for this

study; contrast, generalization and separation. Contrast refers to comparing values (e.g., algebraic representation and graphical representation) while generalization refers to the process of applying the same value (e.g., graphical representation) to different objects (e.g., problems). Separation includes discerning a feature by varying values of that feature while holding other features constant such as when the same problem is solved by means of different representations. These three ways of establishing variation are used as an analytical tool for identifying variation connected with the teaching of mathematical problems in the classroom data.

Further, opportunities to compare and contrast a variety of educational aspects of the mathematical problems and their solutions are facilitated by the guidance of somebody more experienced in helping to introducing explicit distinctions among and within features. Such explicit distinctions are to a large extent dependent upon our way of talking about things, and learning is therefore strongly dependent on language use (Adler & Davis, 2006; Marton & Tsui, 2004). Below we introduce the analytical constructs used to conceptualize language use in this study.

2.2 Word use in knowledge domains

There are numerous ways to conceptualize and analyze interactional data (see, e.g., Ryve, 2011). Adler and Davis (2006) suggest that one essential feature of MfT is that it could be seen as a dual object of learning with close connections to both the mathematics and mathematics education domains of knowledge. One typical characteristic of a knowledge domain is the use of specific keywords or technical terms (Sfard, 2008). In beginning to analyze the data we realized that the use of technical terms related to MfT was of importance for understanding how MfT was established. This seems to be in harmony with, for instance, Mason (1998) who shows how the uses of technical terms connected to certain knowledge

domains are influential in changing the structure of attention and are therefore important in unpacking critical features of the learning object. In analyzing the data we pay attention to how technical terms are used, focusing on three aspects: (1) Are the teacher educators using several different words/terms to refer to the same object or (2) are they using the same word to refer to different objects without being explicit about it. A third aspect (3) is connected to technical terms and the precision of terms. That is, in analyzing how MfT is established we study whether the terms used to talk about mathematics teaching refer to specific aspects of MfT or whether they refer to broader aspects connected to MfT. This is particularly relevant for terms used to establish interactions about teaching and learning of mathematics since one could argue that the word use of the mathematics education domain is not as well established as it is in mathematics (Adler & Davis, 2006). Besides the use of words, expressions, and frameworks from the domain of mathematics education we are alert to how the teacher educators use generic communicative strategies such as formulation of questions, meta-shifts and repetition.

3 Method

On a general level, the study is focused on problem-solving courses for prospective teachers. The specific interest in problem-solving courses relates to the role of problem solving as both an important mathematical competence and as a means to develop other mathematical competences (Kilpatrick, Swaffold, & Findell, 2001; Niss & Jensen, 2002). As described in Ryve (2007), each mathematics teacher education program in Sweden was contacted in a search for courses especially designed for teaching problem-solving. It turned out that rather few teacher education programs in Sweden offered such courses, and this article is based on two of the three courses in which data were collected. We denote these courses Course 1 (C1) and Course 2 (C2) where teacher educator 1 and 2 (TE1, TE2) were teaching. None of the

authors work/worked within the programs in which the courses were held, making it easier for us to “take a skeptical stance towards that work” (cf. Adler, Ball, Krainer, Lin, & Novotna, 2005, p. 372). At the time when the data was collected, prospective teachers in Sweden did not have to choose whether they wanted to become primary or secondary teachers until relatively late in their education, and in the two courses the prospective teachers had not yet decided their specialization¹. There were about 25 prospective teachers and one teacher educator in each course. Both teacher educators are experienced and are generally described as very popular among the students.

The overall teaching approach in both courses was very similar. In both courses two types of teaching activities were conducted: teacher educators lecturing and prospective teachers presenting solutions to mathematical tasks at the chalkboard. Our focus is on the latter activity. The prospective teachers solved the tasks in groups before presenting them. The tasks were typically designed for primary or secondary school students. The classroom interaction following the presentations in both courses was focused on the solutions presented at the chalkboard. Even though the focus of the study is not on the selection of tasks we would like to note that in discussions with the teacher educators the first author gained the impression that TE2 was more engaged in considerations of how and why to choose certain tasks than was TE1.

Audio recordings and field notes were used to collect data. The first author of this article acted as a non-participating observer during all data collection. Data were collected at the beginning, in the middle, and at the end of the courses. Approximately ten hours of audio recordings from each classroom interaction were collected and transcribed. In analyzing the data we read the transcripts and listened to the recordings, and in presenting the transcripts we

¹ This was changed in August 2011.

include comments and signs (e.g., pause () and *stress*) to help the reader understand our analytical points.

4 Results of the empirical analysis

On a general level, the design of the classroom activities in which prospective teachers present solutions to primary and secondary school mathematical problems at the chalkboard indicates that the object of learning in both Course 1 (C1) and Course 2 (C2) is connected not only to mathematics but also to the teaching of mathematics.

As Ryve (2007) shows, some parts of the interaction in both courses could be characterized as purely mathematical. We will only briefly mention a few issues about these parts before we delve into an analysis of how other features of MfT are established. Typically, one or two mathematical aspects were introduced by the teacher educator in relation to each problem, and the aspects introduced were of primary or secondary school level (arithmetic and geometric number sequence; the difference between 3 multiplied by $\frac{3}{4}$ and 3 added to $\frac{3}{4}$; discrete versus continuous functions). Therefore, both courses offered opportunities for prospective teachers to consolidate their familiarity with several, often disparate mathematical concepts. In addition, the teacher educators typically introduced these mathematical distinctions using technical terms from the domain of mathematics and spent a relatively short period of time discussing them.

When we analyze other aspects of MfT related to the mathematics teaching we recognize interesting differences between C1 and C2. These differences are presented and analyzed below by means of excerpts from both courses.

4.1 Course 1

A prospective teacher (PT) at the chalkboard is presenting solutions at the beginning of the course to the mathematical task:

Patrik and Pernilla do the pools together. Patrick bets 35 SEK and Pernilla 25 SEK.

They win 8760 SEK. How should they divide the pools win equitably?

The solution presented by PT builds on the idea and calculation that $25/(35+25) \cdot 100 \approx 42\%$ and $35/(35+25) \cdot 100 \approx 58\%$ producing the answers of 3650 SEK and 5110 SEK. The teacher educator in Course 1 (TE1) introduces a discussion of the solution:

21 TE1 – Has anybody else been doing it like this?

22 C – No (quietly).

23 TE1 – Is it, so to say, a natural method?

24 C – No (hesitant).

25 TE1 – Yes, yeah I think it is. Can you follow the way of reasoning, she shows how great a percentage of the total input comes from one of them, then you get the same percentage of the total sum. It's a fairly smooth method, one could say.

TE1 starts by asking 'Has anybody else been doing it like this?'. As becomes a bit clearer in [25], TE1 wants to highlight a feature referring to the idea of splitting the pools win proportionally to the investment. In [22], the prospective teachers in the class (C) state that they have not been doing something similar. When studying the other solutions presented by the prospective teachers to this problem during this lesson, all solutions build on the same idea of splitting the pools win in proportion to the investment. How could we then understand the prospective teachers' ways of answering in [22]? Let us continue before we return to this

question. In [23] TE1 specifies the expression in [21] by using the term of ‘natural method’ and at the same time poses it as a question about qualities about the solution method rather than, as in [21], a question about the relation between the solution at the chalkboard and the prospective teachers’ own solutions. The prospective teachers answer this question with a ‘No’ ([24]) that seems strange in relation to the fact that they a few moments later present solutions building on the same idea of splitting the pools win in proportion to the investment. However, from another perspective their way of answering seems rational; the prospective teachers in the audience do not exactly understand what TE1 is referring to in [21] and [23]. This, in turn, we argue is related to the expressions and terms used to denote aspects of the solution to the problem. That is, the expressions of ‘doing it like this’ and ‘natural method’ do not seem to direct students’ attention towards the idea of dividing the pools win proportionally to the bet. In [25] TE1 suggests that the solution at the chalkboard is a ‘natural method’ and further specifies what he refers to by introducing two new expressions (‘ways of reasoning’ and ‘fairly smooth method’) and uses them to denote the idea of splitting the pools win proportionally to the investment. From a variation theory perspective it becomes relevant to note that this is the first solution presented to the problem, and no variation of, for instance, strategies, ideas, or mathematical representations for splitting the pools win has been established in the enacted object of learning. However, by the end of the interaction related to this problem three different solutions have been presented, making it possible to highlight variation. TE1 summarizes the discussion in [71] by making a final comment on features and the variation among them:

71 TE1 – It is very interesting since I suspect that you feel pretty differently, as noted, so perhaps that [the first solution] feels like a very strange percentage calculation for many of you. It becomes a little bit abstract, undoubtedly; Why should one include 100?

Even that one [the second solution] is a pretty sophisticated way of approaching the problem, the one to the right over there, but if one is used to it, it's of course possible. That one [the third solution] has certain advantages; it illustrates things graphically.

A number of features are highlighted by TE1 when he sums up and establishes variation by comparing the solutions presented at the chalkboard. The expressions and terms used to denote aspects of the solutions to the task are 'very strange percentage calculation', 'a little bit abstract', 'more natural', 'pretty sophisticated', and 'illustrates things graphically'. Earlier, the expressions and terms 'doing it like this', 'natural method', 'type of reasoning', 'fairly smooth method' were used to comment upon the solution. What could be said about these expressions and terms used for denoting aspects of the solutions to the problems? First, the expressions used by TE1 consist of a combination of terms from the everyday domain (e.g. natural, smooth, very strange) and from the mathematics domain (percentage calculation, graphically). In the cases presented thus far, while TE1 uses expressions and terms from the mathematics domain to discuss purely mathematical aspects he does not use terms and expressions from the mathematics education research domain to discuss the solutions presented by the prospective teachers. Second, and related to the former aspect, the use of terms from the everyday domain make it, at least for us and arguably also for the prospective teachers, somewhat complicated to figure out exactly which aspects of the solutions TE1 is referring to. We must, however, consider that this is the first lesson during the course and it is possible that ways to talk about solutions to mathematical tasks are established throughout the course. Third, TE1 uses several expressions and terms to refer to the same object as, for instance, 'natural method' seems to be used interchangeable with 'ways of reasoning' and 'fairly smooth method'. This is worth noting from a variation theory perspective and gives us a first opportunity to consider the relation between the establishment of variation in the enacted

object of learning and the use of terms and expressions. TE1 enacts dimensions of variation of the object of MfT by comparing the solutions at the chalkboard to each other and does so by using a variety of terms and expressions from the everyday and mathematics domain, often using many expressions referring to the same object.

Let us move on to a discussion from one of the last lessons of C1. Below we present the whole interaction related to the following problem discussed in C1:

A rabbit eats $\frac{3}{4}$ of a carrot per day. How many carrots could three rabbits eat in four days?

[1] PT – A rabbit eats three quarters (TE1 – per day), yeah and three rabbits then eat three times three quarters and that is nine quarters and then in four days it becomes $\frac{9}{4}$ times four which is nine carrots.

[2] TE1 – Do we agree on this fact, now let us see, thus in such a problem it's often the way one structures things that is most important (PT – Mm) because if one does it the right way everything becomes crystal clear and you naturally start, one could of course start with the problem but now you have taken another step and written one rabbit eats $\frac{3}{4}$ per day and then it is 3 (PT – but it's per day) yeah that's right since the days vary later in the problem that's important, a way of denoting here that is connected to the way you write $3\frac{3}{4}$, it's a little bit ambiguous because sometimes it means, when you write 3 and then nothing and then $\frac{3}{4}$ it could mean two things, here it is absolutely clear since you know that 3 and $\frac{3}{4}$ could mean three plus $\frac{3}{4}$, which is a little bit funny, in everyday situations it's like that and that's something one should be very aware of when one teaches these things since it's difficult to separate these things and it's important to do it the right way, ok and multiply by the number of days, so it's a simple problem when it

comes to calculations with fractions but dividing it into natural steps is so important in this kind of case so that everybody can follow, is there anything anybody would like to say, alternatives or comments? No? You can erase it.

[3] C – Are the carrots of the same size?

[4] TE1 – Yeah, perhaps they are.

[5] C – They assume they are (TE1 – yes) and they usually aren't.

[6] TE1 – No, that's my experience too.

A first aspect worth noting is that the whole interaction in relation to the solution is rather short. TE1 highlights several aspects of the solution in [2] but, for some reason, just one aspect of the solution is introduced by the prospective teachers in the class. That is, in [3] and [5] a prospective teacher is trying to introduce a relevant feature in the interaction (the size of the carrot), but this attempt is not built upon by TE1 and the discussion ends after turn [6]. In addition, the question ending turn [2] is formulated using expressions and terms from the everyday domain “is there anything anybody would like to say, alternatives or comments?”. Hence, TE1's ways of formulating questions and using everyday expressions open up many possible responses (cf. closed questions). Further, in establishing aspects of MfT connected to the solution at the chalkboard we find a similar pattern. That is, the terms and expressions used are taken from the everyday domain; “the way one structures things”, “right way”, and “dividing it into natural steps”. Further, when it comes to aspects typically characterized as common content knowledge (cf. Ball et al., 2008) TE1 uses technical terms from the mathematics domain to make variation explicit. In this case, the expression $3\frac{3}{4}$ is explicitly introduced using a meta-shift: “it could mean two things”, one presented at the chalkboard and contrasted to the other explained by TE1 “three and $\frac{3}{4}$ could mean three plus $\frac{3}{4}$ ”.

4.2 Course 2

We turn now to C2 by introducing excerpts from the data corpus. The prospective teachers have been working with and presenting solutions at the chalkboard to the problem:

Five students have a number of cards. The one that has the most has 40, the mean is 22, the median is 20 and the mode is 20. a) How many cards does each person have? b) Construct a similar problem and solve it.

The prospective teachers at the chalkboard present three different solutions to the problem using the representations of (1) concrete cards, (2) logical reasoning, and (3) algebraic symbols. They thus create variation through separation (Marton et al., 2004). That is, the problem and features such as ‘the right answer’ are held constant while the form of mathematical representations is varied. We enter the interaction just after TE2 asks the prospective teachers in the class (C) about which mathematical representations the prospective teachers at the chalkboard have used. TE2 continues:

[24] TE2 – What kinds of mathematical ideas are they working with? Which headings, chapters are you browsing in your textbooks? I heard somebody whisper statistics, sounds good, could you whisper a little bit louder, statistics – what more within statistics?

[25] C – Mean, median, mode.

[26] TE2 – You consolidate these definitions, you get to know what they imply, what is a mean, what is a median, what is a mode and you perform a number of operations, have you done multiplication, yes, have you done addition, yes, have you done division, yes, have you done subtraction, yes. () Thus you’ve worked with all four operations, you’ve

worked with mathematical ideas connected to the area of statistics, you've worked with representations.

[27] PT – We also have a graphical solution.

In [24] TE2 initiates a discussion about mathematical ideas used to solve the problem and specifies her question by referring to headings in textbooks. The prospective teachers suggest statistics and TE2 further asks the prospective teachers to exemplify more precisely within statistics. Worth noting here is that TE2 uses the technical expression of 'mathematical idea' and we will see below that she sticks to exactly this expression throughout the interaction. Further, the formulations of "which mathematical ideas" and "what more within statistics" do not produce elaborations or discussions among the prospective teachers but rather provoke them to denote aspects of the solution. That is, both TE2's questions in [23] and her own elaboration in [26] give a first indication that it is TE2 who strongly directs the interaction.

If we take a variation theory perspective, the excerpt includes three dimensions of variation of teaching this particular problem. The feature of mathematical ideas used to solve the problem and statistics together with concepts within it such as mean, median and mode is established. As will be shown below, TE2 and the prospective teachers further use this feature to construct and analyze 'similar problems' adjusted to students of different age. Further, TE2 establishes the feature of mathematical operations used in the solution as something worth noting for the prospective teachers by introducing the terms addition, subtraction, multiplication and division. In [27], one prospective teacher at the chalkboard uses the label of a graphic representation to highlight what he/she is about to present. That is, the solution at the chalkboard is contrasted to a graphical solution and therefore establishes a dimension of variation.

In addition, a few other things are worth noting. First, TE2 uses technical terms (mathematical ideas, operations, representations) to label features and these are used consistently throughout the interaction. Second, the communicative strategy of introducing meta-shifts with the marking device of pause “Thus you’ve...” is used to withdraw from the action, summarize the discussion and stress all three features again. Third, the prospective teachers are starting to contribute to interaction by using technical terms from the domains of mathematics and mathematics education.

The excerpt from C2 gives a first picture of how TE2 uses technical terms from the domain of mathematics education (mathematical ideas, representations) to establish the enacted object of MfT and how those technical terms, together with specific questions posed by TE2, influence both the content and type of interaction. Let us delineate a few more excerpts from C2 to give a glimpse of which types, and in which ways, features of MfT are introduced by TE2. Since our focus in this article is on how teacher educators establish aspects of MfT, we have chosen excerpts in which TE2 produces the majority of turns. TE2 is active and produces many turns in the whole data material but we would like to note that, in comparison to C1, prospective teachers in C2 are more active in the interaction.

The prospective teachers have produced a number of their own problems related to the “cards problem” above, and are presenting them at the chalkboard:

[39] PT – Now we really want to get comments on this. Level 1 is for early primary school, level 2 for year 5 and then we’re thinking about using the original problem for year 9, secondary school.

[40] TE2 – You could start already in year 7 with the original problem, just make sure they’ve encountered the concepts. What’s the most important aspect when you use these problems in practice?

[41] C – When you use them in practice?

[42] TE2 - Yes, when somebody's about to start working with the problem, what's the most important aspect you have to consider? Really, it's just *one thing* () the pupils must understand the problem. You could tell them a story, make drawings, sing if you like, but the most important aspect to consider is that they understand what they should do otherwise it's not a problem. () That is, the one facing it should want to or need to solve it, they should not possess a ready strategy and some effort is needed, these are the criteria of Lester, one should want to or need to, one should not possess a ready strategy and some kind of effort should be required, and it should absolutely *not* be the case that the students don't understand the problem.

The prospective teacher at the chalkboard poses a rather general request, seeking input on three presented versions of the 'Card Problem' adjusted to primary, year 5 and year 9 respectively. TE2 immediately takes the lead and comments upon one aspect of the problem and then poses a question without technical terms (cf. [40]). The prospective teachers produce a *repair* (Schegloff, 2007), a request to TE2 to rephrase the question, in [41]. TE2 specifies the question and takes the opportunity to answer it herself. TE2 makes explicit certain aspects of teaching problem solving by stressing features of what a mathematical problem is and the importance of helping the pupils, if necessary, to understand the problem. The meta-shift of "That is" in the middle of [42] is followed by the establishment of a variety of features of the dimension of variation of distinguishing a problem from a mathematical exercise. Notice also that TE2 repeats the criteria, yet another communicative strategy she often uses. In repeating the criteria it is also worth noting that TE2 uses the same formulations that have been used before. Hence, TE2 is careful in her use of key words belonging to the mathematics education domain. The excerpts strengthen the picture of how TE2 establishes MfT; she uses technical

terms to establish dimensions of variations of MfT and to a large extent, she directs the interaction.

The discussion of the prospective teachers' designed problems adjusted to different kinds of pupils continues, and the feature of mathematical ideas is applied within another setting which creates variation through generalization (Marton et al., 2004).

[57] TE2 – is it, first we ask, is it the same mathematical idea in these problems as in the original one, what do you say?

[58] C – Perhaps not the first one but the second.

[59] TE2 – The second one has got it.

[60] C – The first one, I believe, is not really the same.

[61] C – One could write; how many cards could Erik and Magnus have, that is, you are not telling how many they have got when they have got the same number.

[62] PT – We thought it might be a little bit easier for the smaller children just to keep x and not to pair them, but, in fact one could...

[63] TE2 - What kinds of mathematical ideas are used in level 1?

[64] C – Really, it's just addition and subtraction.

[65] TE2 - It's just addition and subtraction () it's an operational problem, it's about reading the problem and understanding that one should perform addition and subtraction, it has nothing to do with statistics, no mean, no mode.

The excerpt shows how TE2 formulates a closed question ([57]) using the technical expression 'mathematical idea'. It seems that the prospective teachers understand what TE2 is

referring to and engage in an interaction about both whether or not the different problems include the same mathematical idea ([58], [60]) and how to reformulate the first problem ([61], [62]). We find it interesting that TE2 does not continue the line of reasoning in [61] and [62] but instead poses a closed question using technical terms. Prospective teachers in the class shortly answer the question in [64] and TE2 summarizes ([65]) by referring back, at least implicitly, to the discussion of mathematical ideas used to solve the problem (e.g., statistics, mean, mode) and by contrasting the problem formulated by the students with the original problem. A few further aspects are worth noting and consistent with what were found above. First, TE2 seems to be careful to use technical terms to elicit and denote aspects of the problems and solutions, and the data analysis suggests that the prospective teachers understand what TE2 is referring to when using those terms and often manage to produce adequate answers to her questions. Second, on the other hand, the data analysis suggests that TE2's uses of specific closed questions diminish the prospective teachers' possibilities to engage in deep discussions of the problems.

As indicated above, the prospective teachers at the chalkboard typically produce several solutions to the same problem using different mathematical representations. In the interaction below, a dimension of variation connected to the *use* of mathematical representations is introduced:

[88] TE2 – Nice. The last two solutions, the first one he calls graphical, what would you call b?

[89] A- Algebraic.

[90] TE2 - Yeah, he's taken support from the picture () there exists a connection between, exactly like in the second solution, between this graphical solution and the algebraic one, one has not solved the problem with the picture but one has understood

the problem using the picture then you've worked with an algebraic solution using the picture as a support and I think that's pretty important to be clear about that the picture also is helpful for understanding both the problem and the solution.

This excerpt follows the same interactional pattern as shown above. TE2 uses technical terms to pose a question ([88]) with one correct answer and that answer, in turn, is a technical term. In [90], TE2 continues to use technical terms from the mathematics education domain to enact a dimension of variation of mathematical representations and how such representations could be *used* to understand the problem and the solution as well as to solve the problem. Hence, the feature of mathematical representation is generalized in terms of being used in a new context and a new dimension of variation is introduced including values such as 'to understand the problem', 'to understand the solution', and 'to solve the problem'. We can further see how different values within this dimension of variation are explicitly related to and contrasted with each other, both using technical terms and referring to the solutions at the chalkboard.

4.3 Summing up the results from Course 1 and Course 2

The analysis shows that the establishment of mathematical aspects in C1 and C2 is characterized by the use of precise terms from the domain of mathematics. When it comes to the establishment of MfT the analysis shows clear differences between C1 and C2. TE1 uses terms and expressions from the everyday domain to establish interaction about the problems and the solutions. TE1 use very few established words, terms, frameworks and expressions from the mathematics education domain (cf. Adler & Davis, 2006; Davis et al., 2007). The analysis indicates that students' engagement in the discussions of the solutions of the problems is relatively low.

The analysis of C2 shows that TE2's establishment of MfT is characterized by variation, the use of technical terms from mathematics and mathematics education, and the use of communicative strategies such as meta-shifts and repetition. Variation of mathematical ideas and representations of solutions to mathematical problems is established both by drawing attention to different solutions to the same problem but also through the consistent use of words, expressions, and frameworks from the mathematics education domain. Besides the consistent use of terms, expressions, and frameworks TE2 uses generic communicative strategies such as meta-shifts, repetition and pauses to stress dimensions of variation. The interaction is to a large extent directed by TE2 and students' contributions are of a responsive nature. That is, their contributions to the discussions of the problems often consist of short answers using the technical terms introduced by TE2.

5 Conclusions

This study is concerned with how the establishment of MfT is accomplished in two courses in Swedish pre-service mathematics teacher education. Several studies within mathematics education have engaged in the analysis of how mathematics is established in classrooms through variation (e.g., Goldenberg & Mason, 2008; Huang, Mok, & Leung, 2006; Park & Leung, 2006; Watson & Mason, 2005) and how MfT is established in mathematics teacher education (e.g., Adler & Davis, 2006; Huillet, 2009). In relation to these studies our study contributes to three interrelated research topics, described below.

First, this study is a contribution to research on how MfT (rather than mathematics) is established through variation. The study shows that TE2 uses several ways to introduce variation in the classroom. First, TE2 and the prospective teachers use separation by holding the problem constant and using a variety of solutions to solve it. Particular solutions could

therefore be seen as values within a dimension of variation of solutions to one mathematics problem. Second, throughout the course, several problems are analyzed using the same technical terms (mathematical ideas, strategies, mathematics-reality connection) and frameworks (mathematical representations, problem-task) creating generalizations and opportunities for the prospective teachers to use and discern features in relation to each term and framework. That is, the terms and frameworks are held constant and the problems and solutions vary. Third, the framework of mathematical representation could in itself be seen as a dimension of variation. One might therefore argue that the introduction and use of explicit frameworks could be seen as a particularly explicit way of contrasting and introducing dimensions of variation. Fourth, single values within the dimension of variation of mathematical representations are related to a variety of stages in solving a single problem (graphic representation related to understanding the problem, solving the problem, and/or understanding the solution). Hence, values within one dimension of variation are used as features within another hierarchical level, connecting different dimensions of variation to each other. Fifth, single concepts do not include variation, but different problems and solutions (as mentioned above) as well as prospective teachers' pre-knowledge create dimensions of variation. For instance, the dimension of variation of mathematical ideas is exemplified by the value of statistics, while other values such as geometry, probability or calculus could reasonably be assumed to belong to prospective teachers' pre-knowledge. In addition, TE1 uses a variety of terms and expressions to refer to the same object. We hypothesize that this way of enacting variation opens up ambiguity rather than establishing an object of learning but further research is required to substantiate this claim.

The second area of research to which this study adds is the understanding of the teacher educators' role in using the words, expressions and frameworks from the mathematics education domain in the establishment of MfT (Adler & Davis, 2006). The data analysis

suggests that TE1 and TE2 act in different ways. This study shows how TE2, through the use of technical terms and communicative strategies, makes variation, and therefore the establishment of MfT, explicit (cf. Hulliet, 2009). Five aspects of TE2's use of technical terms and expression are important. First, technical and specific terms from the domain of mathematics education are used to make explicit critical features of MfT. Second, TE2 is careful not to mix different terms referring to the same thing. Third, communicative strategies such as meta-shifts and repetition are used to consolidate features of MfT. Fourth, the frameworks of technical terms used by TE2 are connected to the prospective teachers' solutions, giving them ample opportunity to connect the technical terms to the visual and concrete solutions they themselves have produced. Finally, the use of technical terms and the ways TE2 asks questions seem to affect the prospective teachers' engagement in the interaction. That is, while the prospective teachers often seem to understand the closed questions posed by TE2, especially when the questions include technical terms, their engagement in the interaction is often of a responsive nature in which they fill in the missing, correct, technical term that TE2 is asking for. The relation between the use of technical terms to make explicit features and how this affects prospective students' engagement in interactions about MfT seems to require further research. This leads us to the final contribution.

The third contribution of this study is connected to the two others, and relates to developments in analytical approaches in mathematics education that stress both mathematical (or MfT) and interactional features of classroom talk (e.g., Adler & Davis, 2011; Franke, Kazemi, & Battey, 2007; Marton & Tsui, 2004; Nilsson & Ryve, 2010; Ryve, 2006, 2011; Ryve, Larsson, & Nilsson, in press). Marton (2009) suggests that developments to take account of interactional phenomena in the establishment of dimensions of variation are important for strengthening variation theory as an analytical approach. By analyzing the data

using variation theory, technical terms and communicative strategies, we are able to identify both features of MfT that are established and how this is accomplished. As such, we capture both content aspects of what is established and how this is interactionally accomplished (cf. Adler & Davis, 2006), and therefore also indicate how variation theory could be complemented to take account of *how* variation is established in classrooms interactions.

To conclude, the study suggests that the teacher educators' use of variation, technical terms from the mathematics education domain and communicative strategies are important in establishing a classroom presentations and interactions about MfT. Further studies are needed to examine the establishment of MfT in mathematics teacher education programs (Adler & Davis, 2011). It would be interesting to understand more about the teacher educators' mathematics-education knowledge for teaching (MEfT) and how that knowledge influences the constitution of MfT. We would also benefit from future research that examines the prospective teachers' contributions in the establishment of MfT and how this is connected to teacher educators' introduction of words, expressions, and frameworks from the mathematics education domain.

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