Mälardalen University Press Licentiate Theses
No. 116

PERTURBED RENEWAL EQUATIONS WITH NON-POLYNOMIAL PERTURBATIONS

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2010

MÄLARDALEN UNIVERSITY
SWEDEN

School of Education, Culture and Communication

ISSN 1651-9256
ISBN 978-91-86135-58-4
Printed by Mälardalen University, Västerås, Sweden

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Akademisk avhandling

som för avläggande av Filosofie licentiatexamen i Matematik/tillämpad matematik vid Akademin för Utbildning, Kultur & Kommunikation, avdelningen för tillämpad matematik, Mälardalens Högskola, kommer att offentligt försvaras Fredagen, 07 Maj, 2010, 13:15 i Kappa, Hus U, Högskoleplan 1, Mälardalens Högskola.

Granskare: Prof. Yuri Belyaev, Umeå universitet, Sverige

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Abstract

This thesis deals with a model of nonlinearly perturbed continuous-time renewal equation with non-polynomial perturbations. The characteristics, namely the defect and moments, of the distribution function generating the renewal equation are assumed to have expansions with respect to a non-polynomial asymptotic scale: $\{\varphi_{\vec{n}}(\varepsilon) = \varepsilon^{\vec{n}\cdot\vec{\omega}}, \vec{n} \in \mathbf{N}_0^k\}$ as $\varepsilon \to 0$, where $\mathbf{N}_0$ is the set of non-negative integers, $\mathbf{N}_0^k \equiv \mathbf{N}_0 \times \cdots \times \mathbf{N}_0, 1 \leq k < \infty$ with the product being taken k times and $\vec{\omega}$ is a k dimensional parameter vector that satisfies certain properties. For the one-dimensional case, i.e., $k = 1$, this model reduces to the model of nonlinearly perturbed renewal equation with polynomial perturbations which is well studied in the literature. The goal of the present study is to obtain the exponential asymptotics for the solution to the perturbed renewal equation in the form of exponential asymptotic expansions and present possible applications.

The thesis is based on three papers which study successively the model stated above. Paper **A** investigates the two-dimensional case, i.e. where $k = 2$. The corresponding asymptotic exponential expansion for the solution to the perturbed renewal equation is given. The asymptotic results are applied to an example of the perturbed risk process, which leads to diffusion approximation type asymptotics for the ruin probability. Numerical experimental studies on this example of perturbed risk process are conducted in paper **B**, where Monte Carlo simulation are used to study the accuracy and properties of the asymptotic formulas. Paper **C** presents the asymptotic results for the more general case where the dimension k satisfies $1 \leq k < \infty$, which are applied to the asymptotic analysis of the ruin probability in an example of perturbed risk processes with this general type of non-polynomial perturbations. All the proofs of the theorems stated in paper **C** are collected in its supplement: paper **D**.

Acknowledgements

I am most grateful to my supervisor Dmitrii Silvestrov for his patience, encouragement, guidance and support in this sometimes painful, but ultimately rewarding research journey.

I would also like to express my gratitude to Anatoliy Malyarenko, my assistant supervisor, for his helpful and insightful comments.

I am thankful to all my colleagues at the Division of Applied Mathematics at Mälardalen University for being supportive and helpful.

Of course I wish to thank Graduate School of Mathematics and Computing (FMB) for providing the financial support, which enables me to conduct this research work in the first place.

I dedicate my efforts in this work to my fiancé Marcus whose love is a consistent source of encouragement.

Västerås, March 15, 2010
Ying Ni

This work was funded by the Graduate School in Mathematics and Computing.

FMB

List of Papers

The present thesis contains the following papers:

Paper A. Ni,Y., Silvestrov, D., Malyarenko, A. (2008). Exponential asymptotics for nonlinearly perturbed renewal equation with non-polynomial perturbations. *J. Numer. Appl. Math.* **1(96)**, 173-197.

Paper B. Ni,Y. (2009). Exponential asymptotic expansions and Monte Carlo studies for ruin probabilities. Journal of Statistical Planning and Inference, 6th St.Petersburg Workshop on Simulation, St Petersburg, 2009. (submitted)

Paper C. Ni,Y. (2010). Perturbed renewal equations with multivariate non-polynomial perturbations. In: Frenkel, I., Gertsbakh, I., Khvatskin L., Laslo Z. Lisnianski, A.(eds), *Proceedings of the International Symposium on Stochastic Models in Reliability Engineering, Life Science and Operations Management*, Beer Sheva, 2010, 754–763.

Paper D. Ni,Y. (2010). Supplement to paper **C** , which contains the proofs of the theorems presented in Paper **C**.

Parts of the thesis have been presented at the following international conferences:

1. International School Finance, Insurance, Energy Markets – Sustainable Development, Västerås, Sweden, May 5 – 9, 2008.

2. III International Symposium on Semi-Markov Models: Theory and Applications, Cagliari, Italy, June 17 – 19, 2009.

3. International Conference on Mathematical Methods in Reliability, Moscow, Russia, June 22 – 27, 2009.

4. Workshop on Simulation, St-Petersburg, Russia, June 28 – July 4, 2009.

5. International Symposium on Stochastic Models in Reliability Engineering, Life Science and Operations Management, Beer Sheva, Israel, February 8 – 11, 2010.

Parts of the thesis have also been published in the following paper:

- Ni,Y. (2009). Exponential asymptotics for a sepcial type of nonlinearly perturbed renewal equation. In: Kuznetsov, N., Rykov, V. (eds) *MMR 2009-Mathematical Methods in Reliability: Theory. Methods. Applications. VI International Conference: Extended Abstracts*, Moscow, 2009. 27 – 31. (A four-page extended abstract of paper **A**).

Contents

1 Introduction

This thesis deals with renewal theory. The intensive development of renewal theory began in 40s of the 20th century and continues at present. Much credit is due to Feller (1941, 1949, 1950, 1961, 1966), Blackwell (1948, 1953), Doob (1948), Smith (1954, 1958), Cox (1962) and others. We refer to the classical books by Feller (1950, 1966, 1971) for a thorough introduction to renewal theory and Smith (1958) for a comprehensive survey of the early works. The later works and results are well presented in Kingman (1972), Sevast'yanov (1974), Silvestrov (1980), Woodroofe (1982), Kovalenko, Kuznetsov and Shurenkov (1983), Shedler (1987), Shurenkov (1989), Shedler (1993), Rolski, Schmidli, Schmidt and Teugels (1999), Thorisson (2000), Grey (2001), and Gyllenberg and Silvestrov (2008).

Briefly speaking renewal theory is concerned with quantities connected to a renewal process or a renewal equation, and the latter is the main object studied in this thesis. By saying renewal equation we mean continuous-time renewal equation on the positive half line unless stated otherwise. It can be shown that many important quantities in various applied probability models satisfy a renewal equation and often the asymptotic behavior of the solution to the renewal equation is of interest. The renewal theorem describes the limit behavior of the solution to the renewal equation at the infinity and plays a fundamental role in renewal theory. For example, it is known that the distribution of a regenerative process at a moment t satisfies a renewal equation, therefore we can apply the renewal theorem to obtain ergodic theorems for regenerative processes. The renewal theorem has its origin in Doob (1948) and has precursors as Blackwell theorem originally developed by Blackwell (1948) and the key renewal theorem formulated originally by Smith (1954, 1958). In Erdös, Feller and Pollard (1949), the renewal theorem for the discrete-time renewal equation was given. The final form of the renewal theorem, given by Feller (1966, 1971) and used in this thesis, proves the existence of the limit for the solution to the renewal equation and gives the expression of this limit under stringent conditions.

The theory of perturbed renewal equations goes back to the work of Silvestrov (1976, 1978, 1979) in the late 1970s. The renewal theorem was generalized for perturbed renewal equations in these works. Shurenkov (1980a, 1980b, 1980c) extended some of these results to the case of perturbed matrix renewal equations. Later Englund and Silvestrov (1997) and Englund (2000, 2001) obtained similar type of asymptotical results for discrete-time perturbed renewal equations. The study of nonlinearly perturbed renewal equations where the characteristics are assumed to have polynomial perturbations, i.e. where the characteristics have expansions in asymptotic power series in terms of the perturbation parameter, originated in Silvestrov (1995). The improved renewal theorems obtained in Silvestrov (1995) provide powerful tools to the study of various perturbed stochastic systems, for instance to the analysis of the so-called quasi-stationary phenomena in nonlinearly perturbed stochastic systems and the derivation of the so-called mixed ergodic and limit/large deviation theorems. Gyllenberg and Silvestrov (1998, 1999b, 2000b) extended further the asymptotic results which were applied to nonlinearly perturbed semi-Markov processes and nonlinearly perturbed regenerative processes. The recent book by Gyllenberg and Silvestrov (2008) has collected the author's earlier results and contains also new results for the theory of perturbed renewal equation. We refer to this book for the general theory and its applications in perturbed regenerative processes, perturbed semi-Markov processes and perturbed risk processes. We also refer to the comprehensive bibliography contained in this book for more works in the area. In particular we would like to mention here some of the works related to asymptotic expansions for Markov type processes, namely, Korolyuk and Turbin (1976, 1978), Courtois and Semal (1984), Latouche (1988), Silvestrov and Abadov (1991, 1993), Kartashov (1996), Avrachenkov and Haviv (2003, 2004) and Koroliuk and Limnios (2005).

In the literature on nonlinearly perturbed renewal equations, models with such polynomial perturbations have attracted most of the attention. This type of model is systematically treated in the book by Gyllenberg and Silvestrov (2008). On the other hand, models of nonlinearly perturbed renewal equations with non-polynomial perturbations is also an object of interest with its own importance. This research area is relatively new and only a few works have been done (Englund and Silvestrov 1997, Englund 1999, 2000, 2001). In these works, the nonlinearly perturbed renewal equation with a special type of non-polynomial perturbations based on polynomial and exponential infinitesimals was considered and the asymptotic behavior of the solution was investigated. As shown in their papers, this type of non-polynomial perturbations appears to be theoretically important and can also come out in applications in particular to analysis of ruin probabilities for nonlinearly perturbed risk processes and perturbed random walks.

In the present thesis, we investigate nonlinearly perturbed renewal equations with a new type of non-polynomial perturbations which can be seen as a generalization of the polynomial case. We highlight a few risk theory examples that are related to the asymptotic analysis of the ruin probability for nonlinearly perturbed classical risk processes.

This introduction reviews the relevant results from the theory of the renewal equation and the perturbed renewal equation, then gives an informal presentation of the main results developed in the thesis.

1 The renewal equation

We begin with introducing the classical renewal equation on the positive half-line:

$$x(t) = q(t) + \int_0^t x(t-s)F(ds), \quad t \geq 0, \tag{1}$$

where $q(t)$ is a measurable function defined on $[0, \infty)$ and bounded on any finite interval, and $F(s)$ is a distribution function on $[0, \infty)$ which can be proper ($F(\infty) = 1$) or improper ($F(\infty) < 1$) but not concentrated on zero ($F(0) < 1$). By convention $q(t)$ and $F(s)$ are called, respectively, the *forcing function* and the *distribution generating the renewal equation.*

Let F and G be any proper or improper distribution functions on $[0, \infty)$, the *convolution* $F * G$ is defined as

$$(F * G)(t) = \begin{cases} 0, & t < 0, \\ \int_0^t F(t-s)G(ds), & t \geq 0, \end{cases}$$

which is again a distribution function. We use notation $F^{(*2)}$ for $F * F$, i.e. the convolution of F with itself. The *r-fold convolution* of F with itself $F^{(*r)}$, where $r \in \mathbf{N}$, is defined as

$$F^{(*r)}(t) = \begin{cases} F(t), & r = 1, \\ \int_0^t F^{*(r-1)}(t-s)F(ds), & r \geq 1. \end{cases}$$

The special case $F^{(*0)}$ is defined as the atomic distribution concentrated at the origin, i.e., such that $F(0) = 1$.

Let us also define the so-called *renewal function*

$$U(t) = \sum_{r=0}^{\infty} F^{(*r)}(t), t \geq 0,$$

which, under conditions imposed above on the distribution function $F(t)$, is finite for every $t \geq 0$ and is a right continuous function.

The renewal equation (1) is one of the most important probabilistic equations since such type of equations arise repeatedly in many applied probability models such as queueing systems, reliability models and risk processes. It is known that equation (1) has a unique measurable solution $x(t)$ that is bounded on any finite interval, which can be expressed explicitly in terms of the forcing function $q(t)$ and the distribution function $F(s)$,

$$x(t) = \int_0^t q(t-s)U(ds), t \geq 0,$$

where $U(ds)$ is a so-called renewal measure on $[0, \infty)$ generated by the renewal function as $U((a, b]) = U(b) - U(a)$, $0 \leq a \leq b < \infty$.

1.1 The renewal theorem

In many cases, a knowledge of the asymptotic behavior of the solution $x(t)$ of the renewal equation as $t \to \infty$ can answer most of the questions of interest. The renewal theorem, being one of the most useful results in renewal theory, describes such asymptotic behavior of $x(t)$.

Before we formulate the final form of renewal theorem given by Feller (1966, 1971), we need to introduce notions of non-arithmetic functions and directly Riemann integrable functions. A distribution function $F(t)$ is said to be *non-arithmetic* or *non-lattice* if for any positive number h, we have the inequality:

$$\sum_{r=-\infty}^{\infty} (F(rh) - F(rh-0)) < 1.$$

A measurable function $q(t)$ is said to be directly Riemann integrable on $[0, \infty)$ if it satisfies the following conditions: **(i)** $q(t)$ is continuous almost everywhere on $[0, \infty)$ with respect to the Lebesgue measure; **(ii)** $q(t)$ is bounded on any finite intervals; **(iii)** there exists a positive number h such that

$$\lim_{T\to\infty} h \sum_{r \geq T/h} \sup_{rh \leq t \leq (r+1)h} |q(t)| = 0.$$

The renewal theorem (Feller 1966 or 1971) states that, under conditions: **(i)** $F(t)$ is *non-arithmetic distribution function* with finite mean $m_1 = \int_0^\infty sF(ds)$; **(ii)** $q(t)$ is *directly Riemann integrable* on $[0, \infty)$, the following asymptotic relation takes place,

$$x(t) \to \frac{1}{m_1} \int_0^\infty q(s)ds \quad \text{as } t \to \infty.$$

In the current literature, the renewal theorem stated above is often used under the name of *key renewal theorem.*

The main results of the thesis are essentially new forms of renewal theorems for the models of nonlinearly perturbed renewal equations with new types of non-polynomial perturbations.

1.2 Ruin probability for classical risk process

Applications of renewal theorem are many, in this section we show an example about the ruin probability which is an important subject of study in risk theory. We refer to the book by Asmussen (2000) for a comprehensive discussion on ruin probability.

Let us begin with introducing the following classical risk process which describes the time evolution of the reserves in an insurance company,

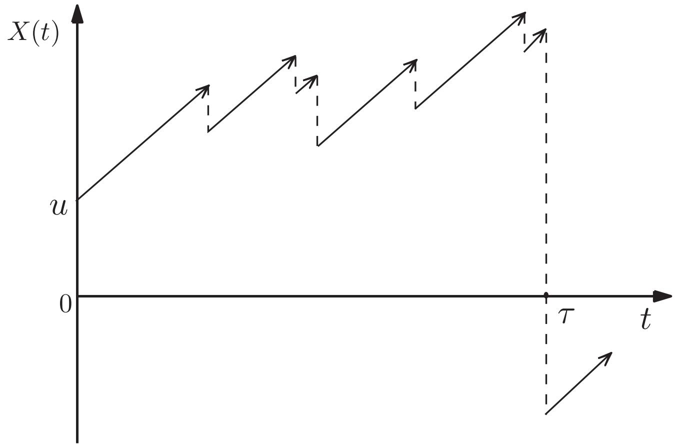
$$X(t) = u + ct - \sum_{k=1}^{N(t)} Z_k, \qquad t \geq 0. \tag{2}$$

In (2), u is a nonnegative constant denoting the initial capital of the insurance company and c is a positive constant referring to the gross risk premium rate. The Poisson claim arrival process $N(t), t \geq 0$ with rate λ counts the number of claims in time interval $[0, t]$, the claim sizes, denoted by $Z_k, k = 1, 2, \ldots$, are i.i.d non-negative random variables that follow a common distribution $G(z)$ with a finite mean β. In addition, the claim sizes $Z_k, k = 1, 2, \ldots$ are assumed to be independent of the process $N(t)$. Figure 1 illustrates a typical sample path of the risk process $X(t), t \geq 0$, where the dashed lines represent the drops in the reserves caused by claim arrivals. The ruin is said to occur if the process $X(t)$ ever falls below zero, in Figure 1, τ is the time of ruin.

The (ultimate) *ruin probability* $\Psi(u)$ refers to the probability of ruin for different values of initial capital u, i.e.

$$\Psi(u) = \mathsf{P}\{\inf_{t \geq 0} X(t) < 0\}, \ u \geq 0.$$

Figure 1: A sample path of a classical risk process



A key parameter for the risk process (2) is the *loading rate of claims* $\alpha = \lambda\beta/c$.

We shall assume hereafter that $\alpha \leq 1$ holds, since for $\alpha > 1$ it is known that $\Psi(u) \equiv 0$ for all u.

The asymptotic behavior of $\Psi(u)$ when the initial capital u takes large values is our object of study. The *Cramér-Lundberg approximation*, which gives the asymptotics of the ruin probability for a fixed $\alpha < 1$ as $u \to \infty$ under the *Cramér-Lundberg condition*, is one of the standard results on ruin probability. The original analytical proofs of Cramér-Lundberg approximation and the closely related *Lundberg inequality* can be found in Lundberg (1926, 1932) and Cramér (1930, 1955).

However the proof of Cramér-Lundberg approximation can be alternatively done using the technique of renewal equations and the renewal theorem. The corresponding method is presented in Feller (1966). This method uses the fact that the ruin probability $\Psi(u)$ satisfies the following *improper (defective) renewal equation*,

$$\Psi(u) = \alpha(1 - \widetilde{G}(u)) + \alpha \int_0^u \Psi(u-s)\widetilde{G}(ds), \qquad u \geq 0, \tag{3}$$

where $\widetilde{G}(u) = 1/\beta \int_0^u (1 - G(s))ds$ is the *integrated tail distribution*, or the *equilibrium distribution*, of G.

Under some Cramér type condition that guarantees the existence of *Lundberg exponent* ρ_0 which is the root of the characteristic equation

$$\alpha \int_0^\infty e^{\rho s} \widetilde{G}(ds) = 1,$$

one can transform (3) into a proper renewal equation after multiplying both sides by $e^{\rho_0 u}$. Applying next the renewal theorem to the transformed renewal equation, the Cramér-Lundberg asymptotics can be obtained which has the following form:

$$\Psi(u)e^{\rho_0 u} \to \frac{\int_0^\infty e^{\rho_0 s}(1 - \widetilde{G}(s))ds}{\int_0^\infty s e^{\rho_0 s} \widetilde{G}(ds)}, \text{ as } u \to \infty.$$

Another classical result on ruin probability is the diffusion approximation which describes the asymptotics of the ruin probability in situations where $u \to \infty$ and $\alpha \to 1$ simultaneously. Here some balancing conditions are imposed on the speeds at which $u \to \infty$ and $\alpha \to 1$, namely, $(1 - \alpha)u \to \lambda_1$. Also it is assumed that the second moment for the claim size distribution G, denote it by γ, is finite. Under these conditions, the diffusion approximation asymptotics takes the form

$$\Psi(u) \to e^{-\lambda_1/a_1} \text{ as } u \to \infty,$$

where the constant $a_1 = \gamma/2\beta$. Recall that β is the first moment of claim size distribution G.

Traditionally the diffusion type asymptotics was obtained by using a Wiener process with drift to approximate the risk processes, see Grandell (1977) or the more recent presentation in Grandell (1991). However, this result can also be proved in an alternative way, by applying the theory of perturbed renewal equations developed by Silvestrov (1976, 1978, 1979). For the details of using this method to obtain the diffusion approximation asymptotics, we refer to Gyllenberg and Silvestrov (1999a, 2000a).

The main results obtained from the applications in the present thesis are related to the improvement of diffusion type approximation to the more advanced form of exponential asymptotic expansions.

2 The perturbed renewal equation

The theory of perturbed renewal equations was initiated by Silvestrov (1976, 1978, 1979) and has nowadays developed into a comprehensive and active subject. The model of perturbed renewal equation refers to the following family of equations:

$$x_\varepsilon(t) = q_\varepsilon(t) + \int_0^t x_\varepsilon(t-s)F_\varepsilon(ds), \quad t \geq 0, \tag{4}$$

where $\varepsilon \geq 0$ is a perturbation parameter on which the force function $q_\varepsilon(t)$ and distribution function $F_\varepsilon(s)$ depend. It is usually assumed that, as $\varepsilon \to 0$, $q_\varepsilon(t)$ and $F_\varepsilon(s)$ converge in some natural sense to a limiting function $q_0(t)$ and a limiting distribution $F_0(s)$. Under such continuity conditions for $q_\varepsilon(t)$ and $F_\varepsilon(s)$ at the point $\varepsilon = 0$, equation (4) can be seen as a perturbed version of the classical renewal equation (1), and it reduces to the classical renewal equation when $\varepsilon = 0$. The renewal theorem was generalized to such models of perturbed renewal equations by Silvestrov (1976, 1978, 1979).

In this thesis we consider the case of an improper perturbed renewal equation, i.e. where $F_\varepsilon(s)$ can be improper i.e. $F_\varepsilon(\infty) < 1$, for $\varepsilon > 0$, but the limiting distribution $F_0(s)$ is proper, i.e. $F_0(\infty) = 1$. In this case the *defect* $f_\varepsilon = 1 - F_\varepsilon(\infty) > 0$ for $\varepsilon > 0$ but $f_\varepsilon \to 0$ as $\varepsilon \to 0$.

Let us introduce the following Cramér type moment condition for F_ε.

Cramér type condition: There exists $\delta > 0$ such that

$$\varlimsup_{0 \leq \varepsilon \to 0} \int_0^\infty e^{\delta s} F_\varepsilon(ds) < \infty,$$

where notation $\varlimsup$ is equivalent to $\limsup$.

The renewal theorem was generalized by Silvestrov (1976, 1978, 1979) for this case under the above Cramér type condition and conditions on $F_\varepsilon(s)$ and $q_\varepsilon(t)$ that are generalized from the corresponding conditions of the classical renewal theorem, see condition **A** and **C** in paper **A** (Ni, Silvestrov and Malyarenko 2008). It states that (see Section 2 in paper **A**), under these conditions, we have the following asymptotic relation for the solution to perturbed renewal equation (4),

$$e^{\rho_\varepsilon t_\varepsilon} x_\varepsilon(t_\varepsilon) \to x_0(\infty) \qquad \text{as } \varepsilon \to 0, \tag{5}$$

where ρ_ε is implicitly given as the root of the *characteristic equation*

$$\phi_\varepsilon(\rho) \equiv \int_0^\infty e^{\rho s} F_\varepsilon(ds) = 1. \tag{6}$$

Note that we use symbol t_ε in (5) to mean that t is changed together with ε.

It should be noted that, in the works of Silvestrov (1976, 1978, 1979), the asymptotic relation analogous to (5) were also proved under a minimal set of moment conditions where the Cramér type condition is not required, moreover, similar asymptotical results were obtained also for the more general case where $F_0(\infty) \leq 1$.

The results of perturbed renewal equations from these works have stimulated further research in the area. To name a few examples, Shurenkov (1980a, 1980b, 1980c) extended some of these asymptotic results to the case of perturbed matrix renewal equations, via the approach of embedding the matrix model to the scalar model. Englund and Silvestrov (1997) and Englund (2000, 2001) obtained similar type of asymptotical results for discrete-time perturbed renewal equation.

3 The nonlinearly perturbed renewal equation

In the following discussions we will assume that the Cramér type condition holds. Under this condition, the exponent ρ_ε in (5) is given implicitly as the root of characteristic equation (6). Therefore, to obtain ρ_ε, one needs to solve a nonlinear equation for every $\varepsilon \geq 0$. This is not very convenient. An approach to overcome the inconvenience was introduced by Silvestrov (1995), where the asymptotic relation (5) was improved by giving an explicit asymptotic expansion for ρ_ε. His method was based on the assumption that the characteristics of $F_\varepsilon(s)$, namely the defect f_ε and the moments $m_{\varepsilon r} = \int_0^\infty s^r F_\varepsilon(ds), r = 1, \ldots, k$ have asymptotic expansions up to and including order k with respect to the standard polynomial *asymptotic scale:* $\{\varphi_n(\varepsilon) = \varepsilon^n, n = 0, 1, 2, \ldots\}$ as $\varepsilon \to 0$. Let us call this assumption as condition $\mathbf{P}^k$,

$\mathbf{P}^k$: **(a)** $1 - f_\varepsilon = 1 + b_{1,0}\varepsilon + b_{2,0}\varepsilon^2 + \cdots + b_{k,0}\varepsilon^k + o(\varepsilon^k)$ as $\varepsilon \to 0$, where all coefficients are finite numbers;

(b) $m_{\varepsilon r} = m_{0r} + b_{1,r}\varepsilon + \cdots + b_{k-r,r}\varepsilon^{k-r} + o(\varepsilon^{k-r})$ as $\varepsilon \to 0$ for $r = 1, \ldots, k$, where all coefficients are finite numbers.

The term "nonlinearly" perturbed renewal equation originally comes from the fact that the above additional perturbation conditions are polynomial hence nonlinear expansions.

The corresponding asymptotic expansion for ρ_ε was given in Silvestrov (1995), i.e.,

$$\rho_\varepsilon = a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_k\varepsilon^k + o(\varepsilon^k), \text{ as } \varepsilon \to 0, \tag{7}$$

with explicit effective algorithm for computing the coefficients $a_1, a_2, \ldots, a_k$ in terms of the coefficients in the perturbation condition $\mathbf{P}^k$.

As a consequence, an improved version of asymptotic relation (5) was obtained by imposing some balancing condition which describes how the perturbation parameter ε goes to zero and t goes to infinity simultaneously. That is, for $1 \leq r \leq k$, under balancing condition

$$\varepsilon^r t_\varepsilon \to \lambda_r < \infty, \text{ as } \varepsilon \to 0, \tag{8}$$

the following asymptotic exponential expansion holds:

$$\frac{x_\varepsilon(t_\varepsilon)}{\exp\{-(a_1\varepsilon + \cdots + a_{r-1}\varepsilon^{r-1})t_\varepsilon\}} \to e^{-a_r\lambda_r} x_0(\infty), \text{ as } \varepsilon \to 0. \tag{9}$$

Note that the results above were obtained for the case of asymptotically proper renewal equation which allows the distribution function $F_\varepsilon(s)$ to be improper, i.e., $F_\varepsilon(\infty) \leq 1$ for $\varepsilon > 0$ but requires the limiting distribution function to be proper i.e. $F_0(\infty) = 1$. In the works of Gyllenberg and Silvestrov (1998, 1999b, 2000b), the general case, where the limiting renewal equation can be either proper or improper i.e. $F_0(\infty) \leq 1$, was investigated and the corresponding asymptotical results were obtained and applied to the analysis of nonlinearly perturbed semi-Markov processes and nonlinearly perturbed regenerative processes. In addition, asymptotical expansions for the renewal limit were also obtained in these papers.

The discrete time renewal equations under perturbation conditions similar to $\mathbf{P}^k$ were considered in the papers of Englund and Silvestrov (1997) and Englund (2000, 2001), and the asymptotical results analogous to (7) and (9) were proved and then applied to the study of regenerative processes, random walk, queueing systems, and risk processes.

As mentioned in the beginning of this introduction, the book by Gyllenberg and Silvestrov (2008) contains a thorough discussion on nonlinearly perturbed renewal equations and its applications. We would like to note that all expansions in this book are based on the polynomial asymptotic scale $\{\varphi_n(\varepsilon) = \varepsilon^n, n = 0, 1, 2, \ldots\}$, as in expansions in $\mathbf{P}^k$.

4 Nonlinearly perturbed renewal equation with non-polynomial perturbations

Englund and Silvestrov (1997), Englund (1999) and Englund (2000, 2001) have dealt with nonlinearly perturbed renewal equations with a type of non-polynomial perturbations, namely, perturbations of the polynomial and mixed polynomial-exponential type. In the present thesis, we investigate nonlinearly perturbed renewal equations with another type of non-polynomial perturbations that can be seen as a generalization of the polynomial counterpart.

4.1 Problem formulation of the thesis

The study conducted in thesis follows the line of research described in Sections 2 and 3. It deals with nonlinearly perturbed renewal equations with a new type of non-polynomial perturbations.

Denoting the dot product of vector $\vec{n}$ and $\vec{\omega}$ by $\vec{n} \cdot \vec{\omega}$ and letting ε be some small perturbation parameter, we begin with introducing the following asymptotic scale,

$$\{\varphi_{\vec{n}}(\varepsilon) = \varepsilon^{\vec{n}\cdot\vec{\omega}}, \vec{n} \in \mathbf{N}_0^k\}, \text{ as } \varepsilon \to 0, \tag{10}$$

where $\mathbf{N}_0$ is the set of non-negative integers; $\mathbf{N}_0^k \equiv \mathbf{N}_0 \times \cdots \times \mathbf{N}_0, 1 \leq k < \infty$ with the product being taken k times; $\vec{\omega}$ is a parameter vector of dimension k. By the definition of asymptotic scale, the *gauge functions* $\varphi_{\vec{n}}(\varepsilon)$ are ordered by index $\vec{n}$ in such way that the next function in the sequence is always o-function of the previous one.

We impose the following assumptions for the parameter vector $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_k)$: **(i)** $1 = \omega_1 < \omega_2 < \ldots < \omega_k$; **(ii)** the components are linearly independent over the field $\mathbb{Q}$ of rational numbers, i.e., ω_i/ω_j is an irrational number for any $i \neq j, i, j = 1, \ldots, k$. Note that assumptions **(i)** and **(ii)** imply that $\omega_2, \ldots, \omega_k$ are irrational numbers.

We study nonlinearly perturbed renewal equations where the defect f_ε and moments $m_{\varepsilon r}$ can be expanded in perturbation parameter ε with respect to the above non-polynomial asymptotic scale up to some order $\alpha \geq 1$, i.e., the following perturbation condition holds:

$\mathbf{P}_{\vec{\omega}}^{(\alpha)}$: **(a)** $1 - f_\varepsilon = 1 + \sum_{1 \leq \vec{n}\cdot\vec{\omega} \leq \alpha} b_{\vec{n},0}\varepsilon^{\vec{n}\cdot\vec{\omega}} + o(\varepsilon^{[\alpha]_{\vec{\omega}}})$ as $\varepsilon \to 0$

where all coefficients are finite numbers;

(b) $m_{\varepsilon r} = m_{0r} + \sum_{1 \le \vec{n}\cdot\vec{\omega} \le \alpha - r} b_{\vec{n},r}\varepsilon^{\vec{n}\cdot\vec{\omega}} + o(\varepsilon^{[\alpha-r]_{\vec{\omega}}})$ as $\varepsilon \to 0$
for $r = 1, \ldots, [\alpha]$, where all coefficients are finite numbers.

The notation $[\alpha]_{\vec{\omega}}$ is defined as $[\alpha]_{\vec{\omega}} \equiv \max(\vec{n}\cdot\vec{\omega} :\ \vec{n}\cdot\vec{\omega} \le \alpha)$.

It should be noted that asymptotic expansion of the type $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ is more "dense" than its polynomial counterpart when both are expanded up to the same order. For example, a possible expansion up to order three with respect to the standard polynomial asymptotic scale can take the form:

$$f_\varepsilon = b_{1,0}\varepsilon + b_{2,0}\varepsilon^2 + b_{3,0}\varepsilon^3 + o(\varepsilon^3),\ \text{as}\ \varepsilon \to 0. \tag{11}$$

An example of expansion with respect to scale (10) with $\vec{\omega} = (1, \sqrt{2})$ i.e. for $k = 2$ can have the form:

$$\begin{aligned} f_\varepsilon = {} & b_{(1,0),0}\varepsilon + b_{(0,1),0}\varepsilon^{\sqrt{2}} + b_{(2,0),0}\varepsilon^2 + b_{(1,1),0}\varepsilon^{1+\sqrt{2}} \\ & + b_{(0,2),0}\varepsilon^{2\sqrt{2}} + b_{(3,0),0}\varepsilon^3 + o(\varepsilon^3),\ \text{as}\ \varepsilon \to 0, \end{aligned} \tag{12}$$

whereas if with respect to scale (10) with $\vec{\omega} = (1, \sqrt{2}, \sqrt{3})$ for $k = 3$,

$$\begin{aligned} f_\varepsilon = {} & b_{(1,0,0),0}\varepsilon + b_{(0,1,0),0}\varepsilon^{\sqrt{2}} + b_{(0,0,1),0}\varepsilon^{\sqrt{3}} + b_{(2,0,0),0}\varepsilon^2 \\ & + b_{(1,1,0),0}\varepsilon^{1+\sqrt{2}} + b_{(1,0,1),0}\varepsilon^{1+\sqrt{3}} + b_{(0,2,0),0}\varepsilon^{2\sqrt{2}} \\ & + b_{(3,0,0),0}\varepsilon^3 + o(\varepsilon^3)\ \text{as}\ \varepsilon \to 0. \end{aligned} \tag{13}$$

As illustrated above, all three expansions are expanded up to order three, however asymptotic expansions (13) contains more terms than (12) and the latter consists of more term than the polynomial case (11). A graphical illustration of such phenomenon can be found in Figure 1 in paper **A**. Note also that when the value of k gets greater, the corresponding expansion gets more "dense".

The objective of our study is to obtain the asymptotic behavior of the solution to nonlinearly perturbed renewal equation with this type of perturbation conditions and illustrate the theoretical results by applications.

As regards the motivation of the study, note that if we set $k = 1$ in (10) so that $\vec{\omega} = 1$, the asymptotic scale (10) reduces to the standard polynomial asymptotic scale $\{\varepsilon^0, \varepsilon, \varepsilon^2, \ldots\}$ as $\varepsilon \to 0$, which is the case for models with polynomial perturbation conditions $\mathbf{P}^k$. Therefore the result obtained in this thesis can be considered as a generalization to the corresponding result for models with polynomial perturbations, i.e. the asymptotic relations (7) and (9). Moreover, such non-polynomial perturbations can appear in some models of the perturbed stochastic processes when the perturbation depends not only on ε but also on $\varepsilon^{\omega_1}, \varepsilon^{\omega_2}$ etc. A concrete example of perturbed risk process is given in Section 4.3.

Finally we would like to mention again the works of Englund and Silvestrov (1997), Englund (1999, 2000, 2001) which deal with nonlinearly perturbed renewal equations with another type of non-polynomial perturbations, more specifically, perturbations of the polynomial and mixed polynomial-exponential type.

4.2 Theoretical results of the thesis

Paper **A** deals with perturbed renewal equation (4) where the defects and moments of $F_\varepsilon(s)$ can be expanded in ε up to order α with respect to asymptotic scale:

$$\{\varphi_{n,m}(\varepsilon) = \varepsilon^{n+m\omega}, (n,m) \in \mathbf{N_0} \times \mathbf{N_0}\}, \text{ as } \varepsilon \to 0, \tag{14}$$

where parameter $\omega > 1$ is an irrational number, $\mathbf{N_0}$ refer to the set of nonnegative integers. Note that asymptotical scale (14) is a particular case of (10) with $k = 2$, therefore the expansions of defect and moments take the forms of $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ for the special case $\vec{\omega} = (1, \omega)$.

Under the Cramér type condition and some convergence condition for $F_\varepsilon(s)$, the following asymptotic expansion is obtained for ρ_ε:

$$\rho_\varepsilon = \sum_{1 \le n+m\omega \le \alpha} a_{n,m}\varepsilon^{n+m\omega} + o(\varepsilon^{[\alpha]_\omega}), \tag{15}$$

which is provided by the explicit effective recurrence algorithm for calculating the coefficients $a_{n,m}, 1 \le n + m\omega \le \alpha$.

Further, under some convergence conditions for the forcing function $q_\varepsilon(t)$ and the balancing condition $\varepsilon^{[\beta]_\omega} t_\varepsilon \to \lambda_\beta \in [0, \infty)$ for some $1 \le \beta \le \alpha$ as $0 \le t_\varepsilon \to \infty$, the following asymptotic relation is obtained for the solution to the perturbed renewal equation,

$$\begin{aligned} &\exp\{(\sum_{1 \le n+m\omega < [\beta]_\omega} a_{n,m}\varepsilon^{n+m\omega})t_\varepsilon\}x_\varepsilon(t_\varepsilon) \\ &\quad \to e^{-\lambda_\beta a^{(1)}} x_0(\infty) \text{ as } \varepsilon \to 0, \end{aligned} \tag{16}$$

where $a^{(1)}$ refers to the coefficient of the term which is of order $O(\varepsilon^{[\beta]_\omega})$ in expansion (15).

Paper **C** deals with the general case where the defect and moments have asymptotic expansions with respect to (10) for an arbitrary $k < \infty$, i.e. perturbation condition $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$. Under this perturbation condition, the following asymptotic expansion of ρ_ε is given,

$$\rho_\varepsilon = \sum_{1 \le \vec{n}\cdot\vec{\omega} \le \alpha} a_{\vec{n}} \varepsilon^{\vec{n}\cdot\vec{\omega}} + o(\varepsilon^{[\alpha]_{\vec{\omega}}}), \tag{17}$$

provided by the algorithm for determining the coefficients $a_{\vec{n}}, 1 \le \vec{n}\cdot\vec{\omega} \le \alpha$.

Under balancing conditions of the form,

$$\varepsilon^{[\beta]_{\vec{\omega}}} t_\varepsilon \to \lambda_\beta \in [0, \infty) \text{ as } \varepsilon \to 0,$$

where $\beta \in [1, \alpha]$, the following asymptotic exponential expansion for the solution of the perturbed renewal equation is given,

$$\exp\{(\sum_{1 \le \vec{n}\cdot\vec{\omega} < [\beta]_{\vec{\omega}}} a_{\vec{n}} \varepsilon^{\vec{n}\cdot\vec{\omega}}) t_\varepsilon\} x_\varepsilon(t_\varepsilon) \to e^{-\lambda_\beta a^{(1)}} x_0(\infty) \text{ as } \varepsilon \to 0, \tag{18}$$

where, as in (16), $a^{(1)}$ refers to the coefficient of the term that is of order $O(\varepsilon^{[\beta]_\omega})$ in the expansion of ρ_ε.

Recall that k refers to the dimension of parameter vector $\vec{\omega}$. Asymptotic formula (18) reduces to the formula obtained in Paper **A** if $k = 2$, and if $k = 1$ it reduces to the formula for the case with polynomial perturbation discussed in Section 3, i.e. (9).

4.3 Applications of the theoretical results

Let us consider the risk process (2) with the following claim size distribution

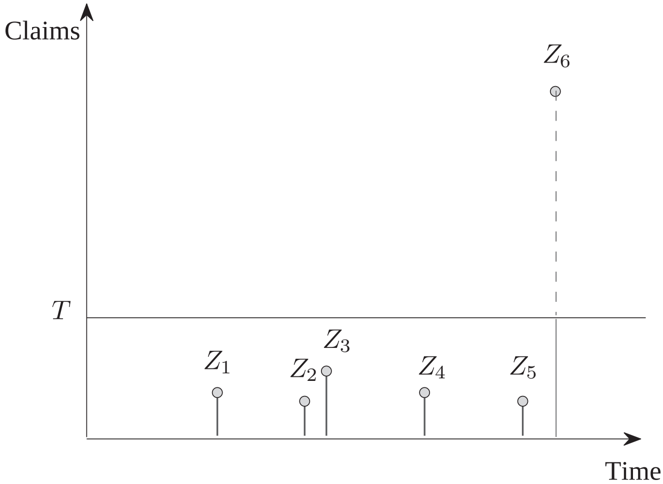
$$G_0(z) = \begin{cases} 1 - \frac{(T_0 - z)^\omega}{T_0^\omega}, & 0 \le z \le T_0, \\ 1, & z > T_0, \end{cases} \tag{19}$$

where T_0 is a constant parameter and parameter $\omega > 1$ is some irrational number.

Suppose now the process is perturbed by an excess-of-loss reinsurance with retention level T, as illustrated in Figure 2. The vertical lines in Figure 2 represent the claim arrivals, with the lengthes being the sizes of each claim. The effect of reinsurance with retention level T is illustrated by the horizontal line that cuts the vertical lines, i.e. the claims, at length of T.

Let us use $\varepsilon = T_0 - T$ as the small perturbation parameter. It can be shown that, under the condition that $\alpha_0 = \lambda\beta_0/c = 1$ where λ is the Poisson

Figure 2: Effect of excess-of-loss reinsurance



arrival rate and β_0 the mean of $G_0(z)$, the defects and moments of the distribution function generating the perturbed version of renewal equation (3) take a special non-polynomial form:

$$f_\varepsilon = \frac{1}{T_0^{1+\omega}}\varepsilon^{1+\omega}, \tag{20}$$

and for $r = 1, 2, \ldots$,

$$m_{\varepsilon r} = \frac{(r!)\cdot T_0^r}{\prod_{i=2}^{r+1}(\omega+i)} + \sum_{k=0}^{r}(-1)^{k+1}\binom{r}{k}\frac{\omega+1}{\omega+k+1}T_0^{r-k-\omega-1}\varepsilon^{k+1+\omega}. \tag{21}$$

The above perturbation conditions can be easily rewritten in terms of asymptotic expansions with respect to asymptotic scale (14). A direct application of the theoretical results developed in the first part of paper **A** leads to the diffusion approximation type asymptotics for the ruin probability. The corresponding results are given in the second part of paper **A**.

Numerical studies on the same example of risk process are carried out in Paper **B**. Monte Carlo simulation experiments are performed for different values of ε and the initial capital u to obtained estimates of the ruin probability which are compared to the approximations by our asymptotic formulas. It is shown that the asymptotical formulas can provide good approximations when the values of ε are relatively small. Moreover, the results of these experiment suggest that the asymptotical formula which involves only one non-zero term from the expansion of ρ_ε works less satisfactory than those that involve more terms. The quality of our approximations is comparable to the classical diffusion approximation method for ruin probabilities.

An example of perturbed risk process with a more complex type of perturbations is treated in the application part of paper **C**. The diffusion approximation type asymptotic relation for the ruin probability is presented, numerical studies are carried out to give insight into the accuracy and other properties of these asymptotic formulas. The accuracy of the formulas appears to be good for small ε. It is shown that involving more terms from the expansions of ρ_ε into the asymptotic formulas is oftentimes desirable for a good approximation.

It should be mentioned that in Englund (2001), a related example of nonlinearly perturbed risk processes with non-polynomial perturbations is studied. This example assumes that the risk process (2) is imposed by an excess-of-loss reinsurance with retention level T and the claim size distribution is exponential distribution or a mixture of exponential distributions. Using $\varepsilon = 1/T$ as perturbation parameter, it turns out that, the defects and moments of the distribution function generating the resulting perturbed renewal equation have a mixture of polynomial and exponential form in terms of ε. The corresponding diffusion approximation asymptotics for ruin probability are given in this work.

5 Summaries of the papers

The thesis includes four papers, namely, paper **A**:"Exponential asymptotics for nonlinearly perturbed renewal equation with non-polynomial perturbations"; paper **B**: "Exponential asymptotic expansions and Monte Carlo studies for ruin probabilities"; paper **C**: "Perturbed renewal equations with multivariate non-polynomial perturbations"; and paper **D**: "Supplement to paper **C**".

5.1 Paper A

We investigate in this paper the model of perturbed renewal equation:

$$x_\varepsilon(t) = q_\varepsilon(t) + \int_0^t x_\varepsilon(t-s)F_\varepsilon(ds), \quad t \geq 0,$$

with the non-polynomial perturbations $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ for the two-dimensional case $\vec{\omega} = (1, \omega)$. We also impose some natural conditions describing the convergence of distribution function $F_\varepsilon(s)$ and forcing function $q_\varepsilon(t)$ to the corresponding limits $F_0(s)$ and $q_0(t)$. In addition, the Cramér type condition is assumed. The main result is a theorem that describes the asymptotic behavior of the solution to the perturbed renewal equation, via the asymptotic expansion of the characteristic root ρ_ε, i.e. the root of characteristic equation (6).

The result is then applied to the example of perturbed risk processes with such type of non-polynomial perturbations, as described in Section 4.3. We assume also that the loading rate of claims α_ε tends to 1 in the limit as $\varepsilon \to 0$, as typical in a diffusion approximation. Under balancing conditions that describes how the initial capital $u \to \infty$ and $\varepsilon \to 0$ simultaneously, we obtain exponential asymptotic expansions for the corresponding ruin probabilities.

5.2 Paper B

This paper is a natural continuation of paper **A**. We review and extend the asymptotical results obtained for the example of perturbed risk process introduced in paper **A**. For the purpose of gaining insight into the asymptotical results, we carry out conditional Monte Carlo simulation experiments to estimate ruin probability for various combinations of values of ε and initial capital u. The estimated ruin probability is then treated as the "reference probability" and compared to the approximations by our asymptotic formulas.

5.3 Paper C

In this paper, we study the model of perturbed renewal equations with more general non-polynomial perturbations than in paper **A**. These perturbations are of type $\mathbf{P}_{\vec{\omega}}^{(\alpha)}$ for the general case $\vec{\omega} = (\omega_1 = 1, \omega_2, \ldots, \omega_k)$ for $1 \leq k < \infty$, in other words, we have a multivariate type of non-polynomial perturbations. The asymptotical expansions are expanded with respect to asymptotic scale

(10) which generalizes the standard polynomial scale $\{\varepsilon^0, \varepsilon, \varepsilon^2, \ldots\}$ as $\varepsilon \to 0$ and also generalizes the asymptotic scale treated in paper **A**.

The asymptotic exponential expansion for the solution for the perturbed renewal equation is then provided in the paper **C**. As expected, if we take $k = 1$ in (10), the asymptotic result reduces to the corresponding result for the model with polynomial perturbations, i.e. to asymptotic exponential expansion (9) as obtained in Silvestrov (1995); and if $k = 2$, the asymptotic result reduces to its counterpart in paper **A**.

We also study a numerical example of perturbed risk process with this multivariate type of non-polynomial perturbations. The claim size distribution is assumed to be a mixture of exponential distributions. The exact formulas for ruin probability exist, which are used to compare to the approximations by our asymptotic formulas for different combinations of values for ε and u.

5.4 Paper D

Paper **D** is a supplement to paper **C**. It presents the proofs to the results formulated in Paper **C**.

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