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Examples of G-Hom-Associative Algebras

by

Felicia Modin

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DIVISION OF MATHEMATICS AND PHYSICS
MÄLARDALEN UNIVERSITY
SE-721 23 VÄSTERÅS, SWEDEN

Mälardalen University

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Author(s):
Felicia Modin

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Supervisor(s):
Sergei Silvestrov

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Lars Hellström

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Abstract

In this thesis we look at hom-associative algebras (which turn out to be exactly the G_1-hom-associative algebras), by, in two and three dimensions, trying to find the structure constants for which an algebra becomes hom-associative when the homomorphism α is defined as different matrix units. These algebras are also hom-Lie admissible (or G_6-hom-associative, which turn out to be the same thing) with a commutator, so we also find the commutator for each of these hom-Lie admissible algebras. We end up finding every hom-associative and hom-Lie algebra for α defined as each 2×2 matrix unit in two dimensions, each 3×3 matrix unit in three dimensions when the problem is mapped to one dimension, for three 3×3 matrix units in three dimensions when the problem is mapped to two dimensions (but with the commutators not having been calculated), and only a few hom-associative algebras and hom-Lie algebras for one 3×3 matrix unit in the full three dimensions. We also compare the results for the different values of α, and find that in n dimensions it is possible to find the values of the structure constants for all n^2 different α:s simply by finding all of the solutions for n different α:s (chosen in a specific way) and then permutating all of the indices.

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Contents

List of Tables

Chapter 1

Introduction

1.1 Background and Research Questions

In this thesis, we will work with a few different hom-algebras, which are multiplications on a vector space where the hom-algebra structures are twisted by a homomorphism. Hom-algebras were first initiated back in 2005 by Larsson and Silvestrov in [8] and [9], and in 2006 by Hartwig, Larsson and Silvestrov in [7]. The motivation for the hom-algebras was quasi-deformations of Lie algebras of vector fields. In [7], Hartwig, Larsson and Silvestrov also introduced hom-Lie algebras as well as more general quasi-hom-Lie algebras. In 2008, in [10], Makhlouf and Silvestrov looked at various classes of hom-Lie admissible algebras, as well as introduced algebras such as hom-associative algebras and G-hom-associative algebras and showed that they were hom-Lie admissible, meaning that using commutator map as new product these algebras lead to hom-Lie algebras. The algebras we will be looking at in this thesis are the hom-associative algebras, hom-Lie admissible algebras, and G-hom-associative algebras.

While we have the general definitions of these algebras, it is very interesting to look at specific cases of them, to understand them better, which is what we will be doing in this thesis. We will let the homomorphism α be defined as different matrix units, specifically every 2×2 and 3×3 matrix unit – looking at higher dimensions would be interesting, but not possible to accomplish in the scope of this thesis.

Matrix units are denoted by E_{ij} [6], and the matrix E_{ij} is defined as the matrix whose only nonzero entry is a 1 in position (i, j) [2]. It is interesting to let α be defined as specifically matrix units since every other matrix is simply a linear combination of matrix units. Furthermore, matrix units are not invertible, which make them extra interesting to use as our α:s.

Since we always define our α as different matrix units, we are interested in their properties. The matrix units have some useful properties, such as the fact that two n-dimensional matrix units E_{ij} and E_{kl} satisfy the multiplicative relation $E_{ij}E_{kl} = \delta_{jk}E_{il}$ for $1 \le i, j, k, l \le n$, where δ_{jk} is the Kronecker delta [6]. This is interesting if we for example want to look at α^2. If we instead want to multiply a matrix unit with another matrix, we get the following results: Assume that we have an $n\times m$ matrix A and an $m\times p$ matrix unit E_{ij}, then AE_{ij} becomes an $n\times p$ matrix in which all entries are zero except for the j:t column, which is exactly the i:th column from A. Assume that we still have an $n \times m$ matrix A but the matrix unit is instead a $p \times n$ matrix E_{ij}. Then, $E_{ij}A$ becomes a $p \times m$ matrix in which all entries are zero except for the i:th row, which is exactly the j:th row of A. [3]

In this thesis, we will investigate, in two and three dimensions, for which values of the structure constants C_{ij}^k that an algebra becomes G-hom-associative – specifically G_1-hom-associative – when α is defined as a matrix unit? What do the commutator tables for the G_6-hom-associative (hom-Lie admissible) algebras associated to these algebras look like? Is there any connection between the results for α defined as different matrix units? And finally, is it possible to generalise these findings to n dimensions?

1.2 Literature Review

The area of hom-algebras was first introduced by Larsson and Silvestrov in [8, 9], and by Hartwig, Larsson and Silvestrov in [7]. In particular, [7] is important for this project since it was in this article that hom-Lie algebras were first defined. We do not work with hom-Lie algebras directly in this thesis, but we do work with hom-Lie admissible algebras, which are the algebras that lead to hom-Lie algebras using commutator map as new product.

In [10], Makhlouf and Silvestrov introduced the hom-Lie admissible algebras, and they also introduced both hom-associative algebras and G-hom-associative algebras, and showed that they are hom-Lie admissible. Since these are exactly the algebras that we are working with in this thesis – specifically G_1-hom-associative algebras, which turn out to be exactly the hom-associative algebras, and G_6-hom-associative algebras, which turn out to be exactly the hom-Lie admissible algebras – and this article has all of the original definitions, this article is the main source for the thesis.

There has been quite a lot of work done with hom-associative algebras. For example, in [11], Makhlouf and Zahari deform the 2×2 matrix algebra to simple hom-associative algebras, as well as a few other things, in order to study the algebraic varieties and structure of hom-associative algebras. Another example is [5], in which Gohr looks at a hom-associative structure in which he assume that α is surjective, and show that the binary operation turn out to be a twisted version of an associative operation under some additional conditions on the multiplication. We also put some assumptions on α, but different ones.

The aim of this thesis is to let the homomorphism α be defined as the different 2×2 and 3×3 matrix units, and try to find the values of the structure constants for which the algebra becomes hom-associative, in first two and then three dimensions. These hom-associative algebras are also hom-Lie admissible, so we then calculate the commutator for each of these hom-Lie admissible algebras. This is not something that has been done before, so it is interesting to find these examples of hom-associative and hom-Lie admissible algebras.

Since we look at hom-associative and hom-Lie admissible algebras where α is defined as different matrix units, we have looked at matrix units as well. A definition of matrix units along with useful information about their properties have been found in [2], [3] and [6]. Finally, for some other important definitions and theorems that were used throughout the thesis, we used a book on linear algebra, [1], and a book on abstract algebra, [4].

1.3 Overview of the Thesis

In Chapter 2 – the first chapter after this introductory chapter – we begin by defining hom-associative algebras, since those are the basis for the whole thesis. We then move on to defining hom-Lie algebras, looking at the connections between hom-associative algebras and hom-Lie algebras, and defining hom-Lie admissible algebras. The final part of this chapter is spent looking at G-hom-associative algebras, since the focus of the thesis will be on G_1-hom-associative algebras, which turn out to be exactly the hom-associative algebras, and G_6-hom-associative algebras, which turn out to be exactly the hom-Lie admissible algebras.

In Chapter 3 we try to find the values of the structure constants C_{ij}^k under which certain algebras will be hom-associative. We begin in dimension 2, and let α be defined as each of the four different 2×2 matrix units in turn, beginning with E_{11}. When we find the values of the structure constants that give us a hom-associative algebra for α defined as each of the matrix units, it turns out that we get multiple solutions in each case. We then look at each of these solutions, to find the commutator tables for the hom-Lie admissible algebras associated to each of the hom-associative algebras. Finally, we investigate all of the results, and compare the ones for different α:s.

The second part of Chapter 3 is spent doing the exact same thing as in the first part of the chapter, except in dimension 3 and with 3×3 matrix units. However, while working with α defined as E_{11} we find that it takes too much time and work to find all of the different solutions for the structure constants, since the systems of equations we have to solve in three dimensions become too big, so we stop after only finding some of those solutions. For each of these solutions, just like in two dimensions, we find the commutator tables for the hom-Lie admissible algebras associated to the hom-associative algebras. For α defined as each of the remaining matrix units, we simply write out the system of equations without trying to solve them.

Since we still want to find some solutions for α defined as each of the matrix units in three dimensions, even if we cannot find all of them, we spend the final part of Chapter 3 mapping these systems of equations to first one and then two dimensions. When we map to one dimension, we encounter no problem, and we are able to find the solutions for the structure constants for which we get a hom-associative algebra for each different α, and then find the commutator tables for the associated hom-Lie admissible algebras. We discuss the results before then moving on to mapping to two dimensions, and do all of the same things as before for α defined as E_{11}, E_{22} and E_{33}. However, we find that even when mapping the three-dimensional system of equations to two dimensions it takes a lot of time and work to solve it, so we end up choosing not look at the cases when α is defined as the remaining six matrix units. Furthermore, we find a lot of different hom-associative algebras for the three matrix units we did look at, so we choose not to calculate the commutator tables for the associated hom-Lie admissible algebras either.

Finally, in Chapter 4 we sum up all of the results, before going through the comparisons we did in the previous chapter of the results in two dimensions, three dimensions, and when mapping the system of equations from three dimensions to one dimension. We then discuss these results, before finishing the thesis by discussing ideas for future work.

Chapter 2

Introduction to Hom-Algebras

In this chapter, we will look at the hom-algebras that will be used in this thesis; first, hom-associative algebras and hom-Lie algebras, and then G-hom-associative algebras. We include definitions and propositions, and a lemma, that are used either for other definitions/propositions/the lemma or will be important later in the thesis.

2.1 Hom-Associative and Hom-Lie Algebras

Throughout this paper, we will let $\mathbb{F}$ be an algebraically closed field of characteristic 0, and V a linear space over $\mathbb{F}$. Furthermore, we let $\sum_{\circlearrowleft(x,y,z)}$ denote summation over the cyclic permutation on x, y, z. Now, to begin, we need to define hom-associative and hom-Lie algebras.

Definition 1 (Makhlouf, Silvestrov [10])**.** A hom-associative algebra is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying

$$\mu(\alpha(x), \mu(y,z)) = \mu(\mu(x,y), \alpha(z)).$$

Definition 2 (Hartwig, Larsson, Silvestrov [7])**.** A hom-Lie algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \to V$ and a linear space homomorphism $\alpha : V \to V$ satisfying

$$[x, y] = -[y, x] \qquad \text{(skew-symmetry)} \tag{2.1}$$

$$\sum_{\circlearrowleft(x,y,z)} [\alpha(x), [y,z]] = 0 \qquad \text{(hom-Jacobi identity)} \tag{2.2}$$

for all x, y, z from V.

Remark 1. In Definition 2, we define skew-symmetry such that a bilinear map $[\cdot,\cdot] : V \times V \to V$ is skew-symmetric if $[x, y] = -[y, x]$ for all $x, y \in V$. This gives us that, for all $x \in V$, we must have

$$[x, x] = -[x, x] \iff 2 \cdot [x, x] = 0.$$

Now, since we are working over a field $\mathbb{F}$ with characteristic 0, we know that $2 \neq 0$, and therefore we need to have $[x,x] = 0$ for the above equality to hold. Thus, our first definition of skew-symmetry is equivalent to $[x,x] = 0$ for all $x \in V$.

Proposition 1 (Makhlouf, Silvestrov [10])**.** *Every skew-symmetric bilinear map on a 2-dimensional space defines a hom-Lie algebra.*

Proof. To show that every skew-symmetric bilinear map on a 2-dimensional space defines a hom-Lie algebra, we look back at the definition of a hom-Lie algebra, Definition 2. We already have some skew-symmetric bilinear map $[\cdot,\cdot] : V \times V \to V$, where V is a 2-dimensional linear space, but we also need some linear space homomorphism $\alpha : V \to V$. To show that this always defines a hom-Lie algebra, for every skew-symmetric bilinear map and α, we now only need to show that this satisfies the hom-Jacobi identity, $\sum_{\circlearrowleft(x,y,z)} [\alpha(x), [y,z]] = 0$.

We let e_1, e_2 be the basis elements in V. Then every element in this space can be written as a linear combination of these basis elements. We take three elements $x, y, z \in V$, and let them be written as $x = k_1e_1 + k_2e_2$, $y = l_1e_1 + l_2e_2$ and $z = m_1e_1 + m_2e_2$. We can now begin trying to show that the hom-Jacobi identity is satisfied.

$$\begin{aligned}
\sum_{\circlearrowleft(x,y,z)} [\alpha(x), [y,z]] &= [\alpha(x), [y,z]] + [\alpha(y), [z,x]] + [\alpha(z), [x,y]] \\
&= [\alpha(k_1e_1 + k_2e_2), [l_1e_1 + l_2e_2, m_1e_1 + m_2e_2]] \\
&\quad + [\alpha(l_1e_1 + l_2e_2), [m_1e_1 + m_2e_2, k_1e_1 + k_2e_2]] \\
&\quad + [\alpha(m_1e_1 + m_2e_2), [k_1e_1 + k_2e_2, l_1e_1 + l_2e_2]] \\
&= [k_1\alpha(e_1) + k_2\alpha(e_2), l_1m_1[e_1,e_1] + l_1m_2[e_1,e_2] \\
&\quad + l_2m_1[e_2,e_1] + l_2m_2[e_2,e_2]] \\
&\quad + [l_1\alpha(e_1) + l_2\alpha(e_2), m_1k_1[e_1,e_1] + m_1k_2[e_1,e_2] \\
&\quad + m_2k_1[e_2,e_1] + m_2k_2[e_2,e_2]] \\
&\quad + [m_1\alpha(e_1) + m_2\alpha(e_2), k_1l_1[e_1,e_1] + k_1l_2[e_1,e_2] \\
&\quad + k_2l_1[e_2,e_1] + k_2l_2[e_2,e_2]]
\end{aligned}$$

Now, since our map is skew-symmetric, Remark 1 tells us that we must have $[x,x] = 0$ for all $x \in V$. Thus, $[e_1,e_1] = [e_2,e_2] = 0$, and we get

$$\begin{aligned}
\sum_{\circlearrowleft(x,y,z)} [\alpha(x), [y,z]] &= [k_1\alpha(e_1) + k_2\alpha(e_2), l_1m_2[e_1,e_2] + l_2m_1[e_2,e_1]] \\
&\quad + [l_1\alpha(e_1) + l_2\alpha(e_2), m_1k_2[e_1,e_2] + m_2k_1[e_2,e_1]] \\
&\quad + [m_1\alpha(e_1) + m_2\alpha(e_2), k_1l_2[e_1,e_2] + k_2l_1[e_2,e_1]] \\
&= (k_1l_1m_2[\alpha(e_1), [e_1,e_2]] + k_2l_1m_2[\alpha(e_2), [e_1,e_2]] \\
&\quad + k_1l_2m_1[\alpha(e_1), [e_2,e_1]] + k_2l_2m_1[\alpha(e_2), [e_2,e_1]]) \\
&\quad + (k_2l_1m_1[\alpha(e_1), [e_1,e_2]] + k_2l_2m_1[\alpha(e_2), [e_1,e_2]] \\
&\quad + k_1l_1m_2[\alpha(e_1), [e_2,e_1]] + k_1l_2m_2[\alpha(e_2), [e_2,e_1]])
\end{aligned}$$

$$
\begin{aligned}
&+ (k_1 l_2 m_1 [\alpha(e_1), [e_1, e_2]] + k_1 l_2 m_2 [\alpha(e_2), [e_1, e_2]] \\
&+ k_2 l_1 m_1 [\alpha(e_1), [e_2, e_1]] + k_2 l_1 m_2 [\alpha(e_2), [e_2, e_1]]) \\
= &\, k_1 l_1 m_2 ([\alpha(e_1), [e_1, e_2]] + [\alpha(e_1), [e_2, e_1]]) \\
&+ k_1 l_2 m_1 ([\alpha(e_1), [e_1, e_2]] + [\alpha(e_1), [e_2, e_1]]) \\
&+ k_2 l_1 m_1 ([\alpha(e_1), [e_1, e_2]] + [\alpha(e_1), [e_2, e_1]]) \\
&+ k_2 l_2 m_1 ([\alpha(e_2), [e_2, e_1]] + [\alpha(e_2), [e_1, e_2]]) \\
&+ k_2 l_1 m_2 ([\alpha(e_2), [e_2, e_1]] + [\alpha(e_2), [e_1, e_2]]) \\
&+ k_1 l_2 m_2 ([\alpha(e_2), [e_2, e_1]] + [\alpha(e_2), [e_1, e_2]])
\end{aligned}
$$

Now, remember that we want to show that the hom-Jacobi identity is satisfied; that is, that $\sum_{\circlearrowleft(x,y,z)} [\alpha(x), [y, z]] = 0$. We see that the sum in each of the parentheses above can be written as $[\alpha(x), [x, z]] + [\alpha(x), [z, x]]$, where either $x = e_1$ and $z = e_2$, or $x = e_2$ and $z = e_1$. Thus, if we can show that $[\alpha(x), [x, z]] + [\alpha(x), z, x] = 0$ for any x, z, that means that the hom-Jacobi property will be satisfied. We get

$$
\begin{aligned}
[\alpha(x), [x, z]] + [\alpha(x), [z, x]] &= [\alpha(x), [x, z]] + [\alpha(x), -[x, z]] \\
&= [\alpha(x), [x, z]] - [\alpha(x), [x, z]] \\
&= 0,
\end{aligned}
$$

using the fact that we know that the map is skew-symmetric and bilinear. Thus, the hom-Jacobi identity is satisfied, and we have proved that every skew-symmetric bilinear map on a 2-dimensional bilinear map defines a hom-Lie algebra.

□

Proposition 2 (Makhlouf, Silvestrov [10])**.** *To any hom-associative algebra (V, μ, α), one may associate a hom-Lie algebra defined for all $x, y \in V$ by the bracket $[x, y] = \mu(x, y) - \mu(y, x)$.*

Proof. To show that $(V, [\cdot, \cdot], \alpha)$ is a hom-Lie algebra, Definition 2 tells us that we must show that it satisfies the skew-symmetry property (2.1) and the hom-Jacobi identity (2.2). We begin by showing that it is skew-symmetric; that is, that $[x, y] = -[y, x]$.

$$
\begin{aligned}
[x, y] &= \mu(x, y) - \mu(y, x) \\
&= -(\mu(y, x) - \mu(x, y)) \\
&= -[y, x],
\end{aligned}
\tag{2.3}
$$

and thus it is clearly skew-symmetric.

Now, we need to show that the hom-Jacobi identity (2.1) holds as well; that is, that $[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$. Using the way we have defined $[x, y]$ to rewrite the expression, and then using Definition 1, as well as fact that we know that μ is bilinear, we get:

$$\begin{aligned}
&[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]\\
&\quad=[\alpha(x),\mu(y,z)-\mu(z,y)]+[\alpha(y),\mu(z,x)-\mu(x,z)]+[\alpha(z),\mu(x,y)-\mu(y,x)]\\
&\quad=(\mu(\alpha(x),\mu(y,z)-\mu(z,y))-\mu(\mu(y,z)-\mu(z,y),\alpha(x)))\\
&\qquad+(\mu(\alpha(y),\mu(z,x)-\mu(x,z))-\mu(\mu(z,x)-\mu(x,z),\alpha(y)))\\
&\qquad+(\mu(\alpha(z),\mu(x,y)-\mu(y,x))-\mu(\mu(x,y)-\mu(y,x),\alpha(z)))\\
&\quad=((\mu(\alpha(x),\mu(y,z))-\mu(\alpha(x),\mu(z,y)))-(\mu(\mu(y,z),\alpha(x))-(\mu(\mu(z,y),\alpha(x))))\\
&\qquad+((\mu(\alpha(y),\mu(z,x))-\mu(\alpha(y),\mu(x,z)))-(\mu(\mu(z,x),\alpha(y))-\mu(\mu(x,z),\alpha(y))))\\
&\qquad+((\mu(\alpha(z),\mu(x,y))-\mu(\alpha(z),\mu(y,x)))-(\mu(\mu(x,y),\alpha(z))-\mu(\mu(y,x),\alpha(z))))\\
&\quad=\mu(\alpha(x),\mu(y,z))-\mu(\alpha(x),\mu(z,y))-\mu(\mu(y,z),\alpha(x))+(\mu(\mu(z,y),\alpha(x))\\
&\qquad+\mu(\alpha(y),\mu(z,x))-\mu(\alpha(y),\mu(x,z))-\mu(\mu(z,x),\alpha(y))+\mu(\mu(x,z),\alpha(y))\\
&\qquad+\mu(\alpha(z),\mu(x,y))-\mu(\alpha(z),\mu(y,x))-\mu(\mu(x,y),\alpha(z))+\mu(\mu(y,x),\alpha(z))\\
&\quad=(\mu(\alpha(x),\mu(y,z))-\mu(\mu(x,y),\alpha(z)))+(\mu(\mu(x,z),\alpha(y))-\mu(\alpha(x),\mu(z,y)))\\
&\qquad+(\mu(\alpha(y),\mu(z,x))-\mu(\mu(y,z),\alpha(x)))+((\mu(\mu(z,y),\alpha(x))-\mu(\alpha(z),\mu(y,x)))\\
&\qquad+(\mu(\mu(y,x),\alpha(z))-\mu(\alpha(y),\mu(x,z)))+(\mu(\alpha(z),\mu(x,y))-\mu(\mu(z,x),\alpha(y)))\\
&\quad=0+0+0+0+0+0\\
&\quad=0.
\end{aligned}$$

Thus, we have shown that the hom-Jacobi identity holds as well, which concludes our proof. □

Remark 2. From (2.3) we can easily see that every commutator $[x,y]=\mu(x,y)-\mu(y,x)$, where μ is a bilinear map, will be skew-symmetric.

We are also going to look at hom-Lie admissible algebras, so we need to define those as well.

Definition 3 (Makhlouf, Silvestrov [10])**.** Let $\mathcal{A}$ be a hom-algebra structure on V defined by the multiplication μ and a homomorphism α. Then $\mathcal{A}$ is said to be a hom-Lie admissible algebra over V if the bracket defined for all $x,y\in V$ by

$$[x,y]=\mu(x,y)-\mu(y,x) \tag{2.4}$$

satisfies the hom-Jacobi identity $\sum_{\circlearrowleft(x,y,z)}[\alpha(x),[y,z]]=0$ for all $x,y,z\in V$.

A hom-Lie admissible algebra is an algebra that leads to a hom-Lie algebra using commutator map as new product.

Proposition 3 (Makhlouf, Silvestrov [10])**.** *Any hom-Lie algebra* $(V,[\cdot,\cdot],\alpha)$ *is hom-Lie admissible with the same twisting map* α.

Proof. We need to show that (2.4) satisfies the hom-Jacobi identity for $(V,[\cdot,\cdot],\alpha)$. However, we note that in this case $[\cdot,\cdot]$ defines the multiplication μ, so we need to define a new commutator product

$$\langle x,y\rangle=[x,y]-[y,x].$$

From Remark 2 we know that this is skew-symmetric, so we only need to prove that this new commutator product satisfies the hom-Jacobi identity $\sum_{\circlearrowleft(x,y,z)} \langle\alpha(x),\langle y,z\rangle\rangle = 0$ for all $x,y,z \in V$.

We get

$$\begin{aligned}
\sum_{\circlearrowleft(x,y,z)} \langle\alpha(x),\langle y,z\rangle\rangle &= \sum_{\circlearrowleft(x,y,z)} ([\alpha(x),\langle y,z\rangle] - [\langle y,z\rangle,\alpha(x)]) \\
&= \sum_{\circlearrowleft(x,y,z)} ([\alpha(x),[y,z]-[z,y]] - [[y,z]-[z,y],\alpha(x)]) \\
&= \sum_{\circlearrowleft(x,y,z)} ([\alpha(x),[y,z]] - [\alpha(x),[z,y]] - [[y,z],\alpha(x)] + [[z,y],\alpha(x)])
\end{aligned}$$

Since we are working with a hom-Lie algebra, we know that $[\cdot,\cdot]$ is skew-symmetric, and so we get

$$\begin{aligned}
\sum_{\circlearrowleft(x,y,z)} \langle\alpha(x),\langle y,z\rangle\rangle &= \sum_{\circlearrowleft(x,y,z)} ([\alpha(x),[y,z]] - [\alpha(x),[z,y]] + [\alpha(x),[y,z]] - [\alpha(x),[z,y]]) \\
&= \sum_{\circlearrowleft(x,y,z)} (2[\alpha(x),[y,z]] - 2[\alpha(x),[z,y]]) \\
&= \sum_{\circlearrowleft(x,y,z)} (2[\alpha(x),[y,z]] - 2[\alpha(x),-[y,z]]) \\
&= \sum_{\circlearrowleft(x,y,z)} (2[\alpha(x),[y,z]] + 2[\alpha(x),[y,z]]) \\
&= 4\sum_{\circlearrowleft(x,y,z)} [\alpha(x),[y,z]].
\end{aligned}$$

Since we are working with a hom-Lie algebra $(V,[\cdot,\cdot],\alpha)$, Definition 2 tells us that $\sum_{\circlearrowleft(x,y,z)} [\alpha(x),[y,z]] = 0$. Thus, we get

$$\sum_{\circlearrowleft(x,y,z)} \langle\alpha(x),\langle y,z\rangle\rangle = 4\sum_{\circlearrowleft(x,y,z)} [\alpha(x),[y,z]] = 4\cdot 0 = 0.$$

We have thus shown that $\langle\cdot,\cdot\rangle$ satisfied the hom-Jacobi identity, and therefore we have proven that any hom-Lie algebra $(V,[\cdot,\cdot],\alpha)$ is hom-Lie admissible. □

There are other ways to find hom-Lie admissible algebras as well, which is shown with the following definition, lemma and proposition.

Definition 4 (Makhlouf, Silvestrov [10])**.** By α-associator of μ we call a trilinear map $a_{\alpha,\mu}$ over V associated to a product μ and a homomorphism α defined by

$$a_{\alpha,\mu}(x_1,x_2,x_3) = \mu(\mu(x_1,x_2),\alpha(x_3)) - \mu(\alpha(x_1),\mu(x_2,x_3)).$$

Makhlouf and Silvestrov [10] lets $\mathcal{A} = (V, \mu, \alpha)$ be a hom-algebra with $[x, y] = \mu(x, y) - \mu(y,x)$ as its commutator. They then introduce a ternary map defined by $S(x, y, z) := a_{\mu,\alpha}(x, y, z) + a_{\mu,\alpha}(y, z,x) + a_{\mu,\alpha}(z,x, y)$. This has the following properties:

Lemma 1 (Makhlouf, Silvestrov [10])**.**

$$S(x, y, z) = [\mu(x, y),\alpha(z)] + [\mu(y, z),\alpha(x)] + [\mu(z,x),\alpha(y)].$$

Proposition 4 (Makhlouf, Silvestrov [10])**.** *A hom-algebra $\mathcal{A}$ is hom-Lie admissible if and only if it satisfies*

$$S(x, y, z) = S(x, z, y)$$

for any $x, y, z \in V$.

Proof. From Definition 3, we get that we have to show that equation (2.4) satisfies the hom-Jacobi identity if and only if $S(x, y, z) = S(x, z, y)$ for any $x, y, z \in V$. We know, using first Lemma 1, then reordering the terms, then using equation (2.4), and finally using the fact that we know that $[x, y]$ defined as in equation (2.4) is skew-symmetric, that

$$\begin{aligned}
S(x, z, y) - S(x, y, z) &= [\mu(x, z),\alpha(y)] + [\mu(z, y),\alpha(x)] + [\mu(y,x),\alpha(z)] \\
&\quad - ([\mu(x, y),\alpha(z)] + [\mu(y, z),\alpha(x)] + [\mu(z,x),\alpha(y)]) \\
&= [\mu(y,x),\alpha(z)] - [\mu(x, y),\alpha(z)] + [\mu(z, y),\alpha(x)] \\
&\quad - [\mu(y, z),\alpha(x)] + [\mu(x, z),\alpha(y)] - [\mu(z,x),\alpha(y)] \\
&= [\mu(x, y) - [x, y],\alpha(z)] - [\mu(x, y),\alpha(z)] + [\mu(y, z) - [y, z],\alpha(x)] \\
&\quad - [\mu(y, z),\alpha(x)] + [\mu(z,x) - [z,x],\alpha(y)] - [\mu(z,x),\alpha(y)] \\
&= [\mu(x, y),\alpha(z)] - [[x, y],\alpha(z)] - [\mu(x, y),\alpha(z)] \\
&\quad + [\mu(y, z),\alpha(x)] - [[y, z],\alpha(x)] - [\mu(y, z),\alpha(x)] \\
&\quad + [\mu(z,x),\alpha(y)] - [[z,x],\alpha(y)] - [\mu(z,x),\alpha(y)] \\
&= -[[x, y],\alpha(z)] - [[y, z],\alpha(x)] - [[z,x],\alpha(y)] \\
&= \sum_{\circlearrowleft(x,y,z)} -[[x, y],\alpha(z)] \\
&= \sum_{\circlearrowleft(x,y,z)} [\alpha(x),[y, z]].
\end{aligned}$$

Since we have now shown that

$$S(x, z, y) - S(x, y, z) = \sum_{\circlearrowleft(x,y,z)} [\alpha(x),[y, z]],$$

this means that

$$S(x, y, z) = S(x, z, y) \iff \sum_{\circlearrowleft(x,y,z)} [\alpha(x),[y, z]] = 0,$$

which concludes the proof. □

2.2 G-Hom-Associative Algebras

A big part of this thesis will be spent looking at G-hom-associative algebras, so we spend this section defining and looking at them.

Definition 5 (Makhlouf, Silvestrov [10])**.** Let G be a subgroup of the permutation group $\mathcal{S}_3$, a hom-algebra on V defined by the multiplication μ and a homomorphism α is said G-hom-associative if

$$\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)}))-\mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)})))=0 \qquad (2.5)$$

where x_i are in V and $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ.

We note that the condition (2.5) can also be written as

$$\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}a_{\mu,\alpha}\circ\sigma=0,$$

where $\sigma(x_1,x_2,x_3)=(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$.

Proposition 5 (Makhlouf, Silvestrov [10])**.** *Let G be a subgroup of the permutations group $\mathcal{S}_3$. Then any G-hom-associative algebra is a hom-Lie admissible algebra.*

Now, we want to look at the different G-hom-associative algebras. By Definition 5, G is a subgroup of the permutation group $\mathcal{S}_3$, so we need to look at each of the subgroups of $\mathcal{S}_3$. If we let τ_{ij} denote the transposition between i and j, the subgroups of $\mathcal{S}_3$ are the following:

$$\begin{aligned}&G_1=\{Id\},\quad G_2=\{Id,\tau_{12}\},\quad G_3=\{Id,\tau_{23}\},\quad G_4=\{Id,\tau_{13}\},\\&G_5=\{Id,\tau_{13}\tau_{12},\tau_{12}\tau_{13}\},\quad G_6=\mathcal{S}_3=\{Id,\tau_{12},\tau_{23},\tau_{13},\tau_{13}\tau_{12},\tau_{12}\tau_{13}\}.\end{aligned}$$

We now want to find the G-hom-associative algebras for each of the subgroups. We begin with G_1. We want to use Definition 5 to find which condition has to be satisfied for a hom-algebra (V,μ,α) to be G_1-hom-associative. First, we note that the only $\sigma\in G$ is Id, and $Id(1)=1$, $Id(2)=2$, and $Id(3)=3$. Finally, the signature of the identity permutation is even, since it can be written as the composition of an even number of transpositions. Thus, $(-1)^{\epsilon(\sigma)}=1$. This gives us that the condition will be

$$\begin{aligned}0&=\sum_{\sigma\in G_1}(-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)}))-\mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)})))\\&=(-1)^{\varepsilon(Id)}(\mu(\mu(x_{Id(1)},x_{Id(2)}),\alpha(x_{Id(3)}))-\mu(\alpha(x_{Id(1)}),\mu(x_{Id(2)},x_{Id(3)})))\\&=1(\mu(\mu(x_1,x_2),\alpha(x_3))-\mu(\alpha(x_1),\mu(x_2,x_3))),\end{aligned}$$

which implies that

$$\mu(\mu(x_1,x_2),\alpha(x_3))=\mu(\alpha(x_1),\mu(x_2,x_3)).$$

Renaming the variables x_1,x_2,x_3 to x,y,z, we get

$$\mu(\mu(x,y),\alpha(z))=\mu(\alpha(x),\mu(y,z)),$$

which we see is exactly the condition for a triple (V,μ,α) to be a hom-associative algebra. Thus, the G_1-hom-associative algebras are simply the same as the hom-associative algebras.

We now do the same thing for G_2. The permutations in G_2 are Id and τ_{12}, and since τ_{12} transposes 1 and 2 we get $\tau_{12}(1) = 2$, $\tau_{12}(2) = 1$, $\tau_{12}(3) = 3$ and $(-1)^{\epsilon(\tau_{12})} = -1$. Thus, the condition that G_2-associative algebras need to satisfy is

$$\begin{aligned}
0 &= \sum_{\sigma\in G_2} (-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)})) - \mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)}))) \\
&= (-1)^{\varepsilon(Id)}(\mu(\mu(x_{Id(1)},x_{Id(2)}),\alpha(x_{Id(3)})) - \mu(\alpha(x_{Id(1)}),\mu(x_{Id(2)},x_{Id(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{12})}(\mu(\mu(x_{\tau_{12}(1)},x_{\tau_{12}(2)}),\alpha(x_{\tau_{12}(3)})) - \mu(\alpha(x_{\tau_{12}(1)}),\mu(x_{\tau_{12}(2)},x_{\tau_{12}(3)}))) \\
&= 1(\mu(\mu(x_1,x_2),\alpha(x_3)) - \mu(\alpha(x_1),\mu(x_2,x_3))) \\
&\quad + (-1)(\mu(\mu(x_2,x_1),\alpha(x_3)) - \mu(\alpha(x_2),\mu(x_1,x_3))).
\end{aligned}$$

Again renaming the variables x_1,x_2,x_3 to x,y,z and rewriting the above condition, we get that G_2-hom-associative algebras need to satisfy

$$\mu(\alpha(x),\mu(y,z)) - \mu(\alpha(y),\mu(x,z)) = \mu(\mu(x,y),\alpha(z)) - \mu(\mu(y,x),\alpha(z)).$$

Continuing to the next subgroup, G_3, which we know consists of the permutations Id and τ_{23}. We get $\tau_{23}(1) = 1$, $\tau_{23}(2) = 3$, $\tau_{23}(3) = 2$ and $(-1)^{\epsilon(\tau_{23})} = -1$ Thus, the condition that G_3-associative algebras need to satisfy is

$$\begin{aligned}
0 &= \sum_{\sigma\in G_3} (-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)})) - \mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)}))) \\
&= (-1)^{\varepsilon(Id)}(\mu(\mu(x_{Id(1)},x_{Id(2)}),\alpha(x_{Id(3)})) - \mu(\alpha(x_{Id(1)}),\mu(x_{Id(2)},x_{Id(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{23})}(\mu(\mu(x_{\tau_{23}(1)},x_{\tau_{23}(2)}),\alpha(x_{\tau_{23}(3)})) - \mu(\alpha(x_{\tau_{23}(1)}),\mu(x_{\tau_{23}(2)},x_{\tau_{23}(3)}))) \\
&= 1(\mu(\mu(x_1,x_2),\alpha(x_3)) - \mu(\alpha(x_1),\mu(x_2,x_3))) \\
&\quad + (-1)(\mu(\mu(x_1,x_3),\alpha(x_2)) - \mu(\alpha(x_1),\mu(x_3,x_2))).
\end{aligned}$$

Again renaming the variables x_1,x_2,x_3 to x,y,z and rewriting the above condition, we get that G_3-hom-associative algebras need to satisfy

$$\mu(\alpha(x),\mu(y,z)) - \mu(\alpha(x),\mu(z,y)) = \mu(\mu(x,y),\alpha(z)) - \mu(\mu(x,z),\alpha(y)).$$

We now continue doing the same thing with the next subgroup, G_4. It consists of the permutations Id and τ_{13}, and we get $\tau_{13}(1) = 3$, $\tau_{13}(2) = 2$, $\tau_{13}(3) = 1$ and $(-1)^{\epsilon(\tau_{13})} = -1$. Thus, the condition that G_4-associative algebras need to satisfy is

$$\begin{aligned}
0 &= \sum_{\sigma\in G_4} (-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)})) - \mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)}))) \\
&= (-1)^{\varepsilon(Id)}(\mu(\mu(x_{Id(1)},x_{Id(2)}),\alpha(x_{Id(3)})) - \mu(\alpha(x_{Id(1)}),\mu(x_{Id(2)},x_{Id(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{13})}(\mu(\mu(x_{\tau_{13}(1)},x_{\tau_{13}(2)}),\alpha(x_{\tau_{13}(3)})) - \mu(\alpha(x_{\tau_{13}(1)}),\mu(x_{\tau_{13}(2)},x_{\tau_{13}(3)}))) \\
&= 1(\mu(\mu(x_1,x_2),\alpha(x_3)) - \mu(\alpha(x_1),\mu(x_2,x_3))) \\
&\quad + (-1)(\mu(\mu(x_3,x_2),\alpha(x_1)) - \mu(\alpha(x_3),\mu(x_2,x_1))).
\end{aligned}$$

Again renaming the variables x_1, x_2, x_3 to x, y, z and rewriting the above condition, we get that G_4-hom-associative algebras need to satisfy

$$\mu(\alpha(x), \mu(y,z)) - \mu(\alpha(z), \mu(y,x)) = \mu(\mu(x,y), \alpha(z)) - \mu(\mu(z,y), \alpha(x)).$$

Now, we again do the same thing for the next subgroup, G_5. This subgroup consists of the permutations Id, $\tau_{13}\tau_{12}$ and $\tau_{12}\tau_{13}$. We get

$$\begin{aligned}
\tau_{13}\tau_{12}(1) &= \tau_{13}(2) = 2 \\
\tau_{13}\tau_{12}(2) &= \tau_{13}(1) = 3 \\
\tau_{13}\tau_{12}(3) &= \tau_{13}(3) = 1 \\
\tau_{12}\tau_{13}(1) &= \tau_{12}(3) = 3 \\
\tau_{12}\tau_{13}(2) &= \tau_{12}(2) = 1 \\
\tau_{12}\tau_{13}(3) &= \tau_{12}(1) = 2,
\end{aligned}$$

and $(-1)^{\epsilon(\tau_{13}\tau_{12})} = 1$, $(-1)^{\epsilon(\tau_{12}\tau_{13})} = 1$. This gives us that G_5-hom-associative algebras need to satisfy

$$\begin{aligned}
0 &= \sum_{\sigma\in G_5} (-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)}, x_{\sigma(2)}), \alpha(x_{\sigma(3)})) - \mu(\alpha(x_{\sigma(1)}), \mu(x_{\sigma(2)}, x_{\sigma(3)}))) \\
&= (-1)^{\varepsilon(Id)}(\mu(\mu(x_{Id(1)}, x_{Id(2)}), \alpha(x_{Id(3)})) - \mu(\alpha(x_{Id(1)}), \mu(x_{Id(2)}, x_{Id(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{13}\tau_{12})}(\mu(\mu(x_{\tau_{13}\tau_{12}(1)}, x_{\tau_{13}\tau_{12}(2)}), \alpha(x_{\tau_{13}\tau_{12}(3)})) - \mu(\alpha(x_{\tau_{13}\tau_{12}(1)}), \mu(x_{\tau_{13}\tau_{12}(2)}, x_{\tau_{13}\tau_{12}(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{12}\tau_{13})}(\mu(\mu(x_{\tau_{12}\tau_{13}(1)}, x_{\tau_{12}\tau_{13}(2)}), \alpha(x_{\tau_{12}\tau_{13}(3)})) - \mu(\alpha(x_{\tau_{12}\tau_{13}(1)}), \mu(x_{\tau_{12}\tau_{13}(2)}, x_{\tau_{12}\tau_{13}(3)}))) \\
&= 1(\mu(\mu(x_1, x_2), \alpha(x_3)) - \mu(\alpha(x_1), \mu(x_2, x_3))) \\
&\quad + 1(\mu(\mu(x_2, x_3), \alpha(x_1)) - \mu(\alpha(x_2), \mu(x_3, x_1))) \\
&\quad + 1(\mu(\mu(x_3, x_1), \alpha(x_2)) - \mu(\alpha(x_3), \mu(x_1, x_2))).
\end{aligned}$$

Again renaming the variables x_1, x_2, x_3 to x, y, z and rewriting the above condition, we get that G_5-hom-associative algebras need to satisfy

$$\begin{aligned}
&\mu(\alpha(x), \mu(y,z)) + \mu(\alpha(y), \mu(z,x)) + \mu(\alpha(z), \mu(x,y)) \\
&\quad = \mu(\mu(x,y), \alpha(z)) + \mu(\mu(y,z), \alpha(x)) + \mu(\mu(z,x), \alpha(y)).
\end{aligned}$$

Finally, we look at the subgroup $G_6 = \mathcal{S}_3$. We know that $\mathcal{S}_3 = \{Id, \tau_{12}, \tau_{23}, \tau_{13}, \tau_{13}\tau_{12}, \tau_{12}\tau_{13}\}$. Thus, G_6-hom-associative algebras need to satisfy

$$\begin{aligned}
0 &= \sum_{\sigma\in G_6} (-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)}, x_{\sigma(2)}), \alpha(x_{\sigma(3)})) - \mu(\alpha(x_{\sigma(1)}), \mu(x_{\sigma(2)}, x_{\sigma(3)}))) \\
&= (-1)^{\varepsilon(Id)}(\mu(\mu(x_{Id(1)}, x_{Id(2)}), \alpha(x_{Id(3)})) - \mu(\alpha(x_{Id(1)}), \mu(x_{Id(2)}, x_{Id(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{12})}(\mu(\mu(x_{\tau_{12}(1)}, x_{\tau_{12}(2)}), \alpha(x_{\tau_{12}(3)})) - \mu(\alpha(x_{\tau_{12}(1)}), \mu(x_{\tau_{12}(2)}, x_{\tau_{12}(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{23})}(\mu(\mu(x_{\tau_{23}(1)}, x_{\tau_{23}(2)}), \alpha(x_{\tau_{23}(3)})) - \mu(\alpha(x_{\tau_{23}(1)}), \mu(x_{\tau_{23}(2)}, x_{\tau_{23}(3)}))) \\
&\quad + (-1)^{\varepsilon(\tau_{13})}(\mu(\mu(x_{\tau_{13}(1)}, x_{\tau_{13}(2)}), \alpha(x_{\tau_{13}(3)})) - \mu(\alpha(x_{\tau_{13}(1)}), \mu(x_{\tau_{13}(2)}, x_{\tau_{13}(3)})))
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{\varepsilon(\tau_{13}\tau_{12})}(\mu(\mu(x_{\tau_{13}\tau_{12}(1)},x_{\tau_{13}\tau_{12}(2)}),\alpha(x_{\tau_{13}\tau_{12}(3)}))-\mu(\alpha(x_{\tau_{13}\tau_{12}(1)}),\mu(x_{\tau_{13}\tau_{12}(2)},x_{\tau_{13}\tau_{12}(3)})))\\
&+(-1)^{\varepsilon(\tau_{12}\tau_{13})}(\mu(\mu(x_{\tau_{12}\tau_{13}(1)},x_{\tau_{12}\tau_{13}(2)}),\alpha(x_{\tau_{12}\tau_{13}(3)}))-\mu(\alpha(x_{\tau_{12}\tau_{13}(1)}),\mu(x_{\tau_{12}\tau_{13}(2)},x_{\tau_{12}\tau_{13}(3)})))\\
=\,&1(\mu(\mu(x_1,x_2),\alpha(x_3))-\mu(\alpha(x_1),\mu(x_2,x_3)))\\
&+(-1)(\mu(\mu(x_2,x_1),\alpha(x_3))-\mu(\alpha(x_2),\mu(x_1,x_3)))\\
&+(-1)(\mu(\mu(x_1,x_3),\alpha(x_2))-\mu(\alpha(x_1),\mu(x_3,x_2)))\\
&+(-1)(\mu(\mu(x_3,x_2),\alpha(x_1))-\mu(\alpha(x_3),\mu(x_2,x_1)))\\
&+1(\mu(\mu(x_2,x_3),\alpha(x_1))-\mu(\alpha(x_2),\mu(x_3,x_1)))\\
&+1(\mu(\mu(x_3,x_1),\alpha(x_2))-\mu(\alpha(x_3),\mu(x_1,x_2))).
\end{aligned}
$$

We now, once again rename the variable x_1,x_2,x_3 to x,y,z and rewrite the above condition, as well as multiply each side by -1, and get that G_6-hom-associative algebras need to satisfy

$$
\begin{aligned}
0=\,&\mu(\alpha(x),\mu(y,z))-\mu(\alpha(x),\mu(z,y))-\mu(\mu(y,z),\alpha(x))+\mu(\mu(z,y),\alpha(x))\\
&+\mu(\alpha(y),\mu(z,x))-\mu(\alpha(y),\mu(x,z))-\mu(\mu(z,x),\alpha(y))+\mu(\mu(x,z),\alpha(y))\\
&+\mu(\alpha(z),\mu(x,y))-\mu(\alpha(z),\mu(y,x))-\mu(\mu(x,y),\alpha(z))+\mu(\mu(y,x),\alpha(z))\\
=\,&\mu(\alpha(x),\mu(y,z)-\mu(z,y))-\mu(\mu(y,z)-\mu(z,y),\alpha(x))\\
&+\mu(\alpha(y),\mu(z,x)-\mu(x,z))-\mu(\mu(z,x)-\mu(x,z),\alpha(y))\\
&+\mu(\alpha(z),\mu(x,y)-\mu(y,x))-\mu(\mu(x,y)-\mu(y,x),\alpha(z)).
\end{aligned}
$$

If we define $[x,y]=\mu(x,y)-\mu(y,x)$ for all $x,y\in V$, we can further rewrite the above as

$$
\begin{aligned}
0=\,&\mu(\alpha(x),[y,z])-\mu([y,z],\alpha(x))\\
&+\mu(\alpha(y),[z,x])-\mu([z,x],\alpha(y))\\
&+\mu(\alpha(z),[x,y])-\mu([x,y],\alpha(z))\\
=\,&[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]\\
=\,&\sum_{\circlearrowleft(x,y,z)}[\alpha(x),[y,z]].
\end{aligned}
$$

That is, for a hom-algebra (V,μ,α) to be G_6-hom-associative, $[x,y]=\mu(x,y)-\mu(y,x)$, defined for all $x,y\in V$, has to satisfy the hom-Jacobi property $\sum_{\circlearrowleft(x,y,z)}[\alpha(x),[y,z]]=0$ for all $x,y,z\in V$. We see that this is the exact condition from Definition 3, so the G_6-hom-associative algebras are the same as the hom-Lie admissible algebras. Proposition 5 then also tells us that any G-hom-associative algebra is G_6-hom-Lie admissible.

In this thesis, we will focus on the G_1-hom-associative algebras and the G_6-hom-associative algebras.

Chapter 3

Examples of G-Hom-Associative Algebras

In this chapter, we are going to look at examples of G-hom-associative algebras. More specifically, we are going to look at G_1-hom-associative algebras, which we in the previous section found to be exactly the hom-associative algebras, and G_6-hom-associative algebras, which we found in the previous section to be exactly the hom-Lie admissible algebras. For simplicity, we are therefore henceforth going to simply say that we are looking at hom-associative algebras and hom-Lie admissible algebras.

Our goal is to find the values of the structure constants C_{ij}^k, where i, j, k are all indices, for which (V, μ, α) is a hom-associative algebra – see Definition 1. We also have $\mu(e_i, e_j) = \sum_{k=1}^{n} C_{ij}^k e_k$ and $\alpha(e_i) = \sum_{j=1}^{n} a_{ji} e_j$ [10]. We do this first in two dimension, and then in two dimensions. We also look at the hom-Lie admissible algebras we get from every hom-associative algebra.

3.1 Dimension 2

We begin by looking at hom-associative algebras in dimension 2. We remember from Definition 1 that (V, μ, α) needs to satisfy

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$$

for all $x, y, z \in V$. Since we have $\dim V = 2$, we let $\{e_1, e_2\}$ be a basis for V. Every element in V can be written as a linear combination of the basis elements, and thus, since μ is bilinear, we know that the above equality will hold for all elements in V if it holds for all combinations of the basis elements. That is, we need all of the following equalities to hold:

$$\mu(\alpha(e_1), \mu(e_1, e_1)) = \mu(\mu(e_1, e_1), \alpha(e_1)), \tag{3.1}$$
$$\mu(\alpha(e_1), \mu(e_1, e_2)) = \mu(\mu(e_1, e_1), \alpha(e_2)), \tag{3.2}$$
$$\mu(\alpha(e_1), \mu(e_2, e_1)) = \mu(\mu(e_1, e_2), \alpha(e_1)), \tag{3.3}$$
$$\mu(\alpha(e_2), \mu(e_1, e_1)) = \mu(\mu(e_2, e_1), \alpha(e_1)), \tag{3.4}$$
$$\mu(\alpha(e_1), \mu(e_2, e_2)) = \mu(\mu(e_1, e_2), \alpha(e_2)), \tag{3.5}$$

$$\mu(\alpha(e_2),\mu(e_1,e_2)) = \mu(\mu(e_2,e_1),\alpha(e_2)), \tag{3.6}$$
$$\mu(\alpha(e_2),\mu(e_2,e_1)) = \mu(\mu(e_2,e_2),\alpha(e_1)), \tag{3.7}$$
$$\mu(\alpha(e_2),\mu(e_2,e_2)) = \mu(\mu(e_2,e_2),\alpha(e_2)). \tag{3.8}$$

We have

$$\alpha(e_i) = \sum_{j=1}^{2} a_{ji}e_j = a_{1i}e_1 + a_{2i}e_2,$$

and thus we get

$$\alpha(e_1) = a_{11}e_1 + a_{21}e_2, \tag{3.9}$$
$$\alpha(e_2) = a_{12}e_1 + a_{22}e_2. \tag{3.10}$$

Furthermore, we have

$$\mu(e_i,e_j) = \sum_{k=1}^{2} C_{ij}^k e_k = C_{ij}^1 e_1 + C_{ij}^2 e_2, \tag{3.11}$$

which gives us

$$\mu(e_1,e_1) = C_{11}^1 e_1 + C_{11}^2 e_2,$$
$$\mu(e_1,e_2) = C_{12}^1 e_1 + C_{12}^2 e_2,$$
$$\mu(e_2,e_1) = C_{21}^1 e_1 + C_{21}^2 e_2,$$
$$\mu(e_2,e_2) = C_{22}^1 e_1 + C_{22}^2 e_2.$$

3.1.1 α defined as E_{11}

Now, we will look at what the hom-associative algebras will look like for different α. We begin with

$$[\alpha] = E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives us

$$\alpha(e_1) = 1e_1 + 0e_2 = e_1,$$
$$\alpha(e_2) = 0e_1 + 0e_2 = 0.$$

Using this, we can rewrite all of our equalities as follows:

$$\begin{aligned}
&(3.1): \quad \mu(e_1, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, e_1),\\
&(3.2): \quad \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2) = 0,\\
&(3.3): \quad \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, e_1),\\
&(3.4): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, e_1) \quad \Rightarrow \quad 0 = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, e_1),\\
&(3.5): \quad \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2) = 0,\\
&(3.6): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0,\\
&(3.7): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, e_1) \quad \Rightarrow \quad 0 = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, e_1),\\
&(3.8): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0.
\end{aligned}$$

Since both (3.6) and (3.8) end up giving us only $0 = 0$, we do not have to care about these two equations in our following calculations.

We already know how to calculate $\mu(e_i, e_j)$, but we now need to figure out how to calculate $\mu(e_i, C^1_{jk}e_1 + C^2_{jk}e_2)$ and $\mu(C^1_{jk}e_1 + C^2_{jk}e_2, e_i)$. We get

$$\begin{aligned}
\mu(e_i, C^1_{jk}e_1 + C^2_{jk}e_2) &= \mu(e_i, C^1_{jk}e_1) + \mu(e_i, C^2_{jk}e_2) \\
&= C^1_{jk}\mu(e_i, e_1) + C^2_{jk}\mu(e_i, e_2) \\
&= C^1_{jk}(C^1_{i1}e_1 + C^2_{i1}e_2) + C^2_{jk}(C^1_{i2}e_1 + C^2_{i2}e_2) \\
&= C^1_{jk}C^1_{i1}e_1 + C^1_{jk}C^2_{i1}e_2 + C^2_{jk}C^1_{i2}e_1 + C^2_{jk}C^2_{i2}e_2 \\
&= (C^1_{jk}C^1_{i1} + C^2_{jk}C^1_{i2})e_1 + (C^1_{jk}C^2_{i1} + C^2_{jk}C^2_{i2})e_2
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\mu(C^1_{jk}e_1 + C^2_{jk}e_2, e_i) &= \mu(C^1_{jk}e_1, e_i) + \mu(C^2_{jk}e_2, e_i) \\
&= C^1_{jk}\mu(e_1, e_i) + C^2_{jk}\mu(e_2, e_i) \\
&= C^1_{jk}(C^1_{1i}e_1 + C^2_{1i}e_2) + C^2_{jk}(C^1_{2i}e_1 + C^2_{2i}e_2) \\
&= C^1_{jk}C^1_{1i}e_1 + C^1_{jk}C^2_{1i}e_2 + C^2_{jk}C^1_{2i}e_1 + C^2_{jk}C^2_{2i}e_2 \\
&= (C^1_{jk}C^1_{1i} + C^2_{jk}C^1_{2i})e_1 + (C^1_{jk}C^2_{1i} + C^2_{jk}C^2_{2i})e_2,
\end{aligned} \tag{3.13}$$

where we use the fact that we know that μ is bilinear. Note that in this case, we always have $i = 1$. Using these formulas, our equalities become:

$$\begin{aligned}
(3.1):\quad & (C^1_{11}C^1_{11} + C^2_{11}C^1_{12})e_1 + (C^1_{11}C^2_{11} + C^2_{11}C^2_{12})e_2 = (C^1_{11}C^1_{11} + C^2_{11}C^1_{21})e_1 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (C^1_{11}C^2_{11} + C^2_{11}C^2_{21})e_2 \\
& \Rightarrow\ C^2_{11}C^1_{12}e_1 + C^2_{11}C^2_{12}e_2 = C^2_{11}C^1_{21}e_1 + C^2_{11}C^2_{21}e_2 \\
& \Rightarrow\ (C^2_{11}C^1_{12} - C^2_{11}C^1_{21})e_1 + (C^2_{11}C^2_{12} - C^2_{11}C^2_{21})e_2 = 0, \\
(3.2):\quad & (C^1_{12}C^1_{11} + C^2_{12}C^1_{12})e_1 + (C^1_{12}C^2_{11} + C^2_{12}C^2_{12})e_2 = 0, \\
(3.3):\quad & (C^1_{21}C^1_{11} + C^2_{21}C^1_{12})e_1 + (C^1_{21}C^2_{11} + C^2_{21}C^2_{12})e_2 = (C^1_{12}C^1_{11} + C^2_{12}C^1_{21})e_1 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (C^1_{12}C^2_{11} + C^2_{12}C^2_{21})e_2 \\
& \Rightarrow\ (C^1_{21}C^1_{11} + C^2_{21}C^1_{12} - C^1_{12}C^1_{11} - C^2_{12}C^1_{21})e_1 \\
& \qquad + (C^1_{21}C^2_{11} + C^2_{21}C^2_{12} - C^1_{12}C^2_{11} - C^2_{12}C^2_{21})e_2 = 0, \\
(3.4):\quad & 0 = (C^1_{21}C^1_{11} + C^2_{21}C^1_{21})e_1 + (C^1_{21}C^2_{11} + C^2_{21}C^2_{21})e_2, \\
(3.5):\quad & (C^1_{22}C^1_{11} + C^2_{22}C^1_{12})e_1 + (C^1_{22}C^2_{11} + C^2_{22}C^2_{12})e_2 = 0, \\
(3.7):\quad & 0 = (C^1_{22}C^1_{11} + C^2_{22}C^1_{21})e_1 + (C^1_{22}C^2_{11} + C^2_{22}C^2_{21})e_2.
\end{aligned}$$

Now, we remember that $\{e_1, e_2\}$ is a basis for V. We want to use this fact to find the solutions to the above equation, but to do that we need the following two definitions:

Definition 6 (Anton, Rorres [1]). If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a finite set of vectors in V, then S is called a basis for V if the following two conditions hold:

(a) S is linearly independent.

(b) S spans V.

Definition 7 (Anton, Rorres [1])**.** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a nonempty set of vectors in a vector space V, then the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = 0$$

has at least one solution, namely,

$$k_1 = 0,\ \ k_2 = 0,\ \ \ldots,\ \ k_r = 0.$$

We call this the trivial solution. If this is the only solution, then S is said to be a linearly independent set. If there are solutions in addition to the trivial solution, then S is said to be a linearly dependent set.

Since we know that $\{e_1, e_2\}$ is a basis, Definition 6 tells us that $\{e_1, e_2\}$ is a linearly independent set, and thus, Definition 7 tells us that $k_1e_1 + k_2e_2 = 0$ will only have the trivial solution $k_1 = k_2 = 0$. Using this fact, we get that the solutions to the above equations must be as given in the system of equations below.

$$\begin{aligned}
C^2_{11}C^1_{12} - C^2_{11}C^1_{21} &= 0\\
C^2_{11}C^2_{12} - C^2_{11}C^2_{21} &= 0\\
\underline{C^1_{12}C^1_{11}} + C^2_{12}C^1_{12} &= 0\\
\underline{C^1_{12}C^2_{11}} + C^2_{12}C^2_{12} &= 0\\
\underline{C^1_{21}C^1_{11}} + C^2_{21}C^1_{12} - \underline{C^1_{12}C^1_{11}} - C^2_{12}C^1_{21} &= 0\\
\underline{C^1_{21}C^2_{11}} + C^2_{21}C^2_{12} - \underline{C^1_{12}C^2_{11}} - C^2_{12}C^2_{21} &= 0\\
\underline{C^1_{21}C^1_{11}} + C^2_{21}C^1_{21} &= 0\\
\underline{C^1_{21}C^2_{11}} + C^2_{21}C^2_{21} &= 0\\
\underline{C^1_{22}C^1_{11}} + C^2_{22}C^1_{12} &= 0\\
\underline{C^1_{22}C^2_{11}} + C^2_{22}C^2_{12} &= 0\\
\underline{C^1_{22}C^1_{11}} + C^2_{22}C^1_{21} &= 0\\
\underline{C^1_{22}C^2_{11}} + C^2_{22}C^2_{21} &= 0.
\end{aligned}$$

We notice that the same term is present in multiple equations; for simplicity, each of these terms has been marked with a different colour. To simplify the system of equations, we can eliminate these terms by adding or subtracting the equations containing the terms. We begin by adding the third equation to the fifth equation, adding the fourth equation to the sixth equation, subtracting the eleventh equation from the ninth, and subtracting the twelfth equation from the tenth. This gives us

$$
\begin{aligned}
C^2_{11}C^1_{12} - C^2_{11}C^1_{21} &= 0 \\
C^2_{11}C^2_{12} - C^2_{11}C^2_{21} &= 0 \\
C^1_{12}C^1_{11} + C^2_{12}C^1_{12} &= 0 \\
C^1_{12}C^2_{11} + C^2_{12}C^2_{12} &= 0 \\
C^2_{12}C^1_{12} + \underline{C^1_{21}C^1_{11}} + C^2_{21}C^1_{12} - C^2_{12}C^1_{21} &= 0 \\
C^2_{12}C^2_{12} + \underline{C^1_{21}C^2_{11}} + C^2_{21}C^2_{12} - C^2_{12}C^2_{21} &= 0 \\
\underline{C^1_{21}C^1_{11}} + C^2_{21}C^1_{21} &= 0 \\
\underline{C^1_{21}C^2_{11}} + C^2_{21}C^2_{21} &= 0 \\
C^2_{22}C^1_{12} - C^2_{22}C^1_{21} &= 0 \\
C^2_{22}C^2_{12} - C^2_{22}C^2_{21} &= 0 \\
\underline{C^1_{22}C^1_{11}} + C^2_{22}C^1_{21} &= 0 \\
\underline{C^1_{22}C^2_{11}} + C^2_{22}C^2_{21} &= 0.
\end{aligned}
$$

Now, we see that we also have the same term in the fifth and the seventh equation, and we have the same term in the sixth equation and the eighth equation. To simplify the system of equations we can therefore subtract the seventh equation from the fifth, and subtract the eighth equation from the sixth. Our system of equations then becomes

$$
\begin{aligned}
C^2_{11}C^1_{12} - C^2_{11}C^1_{21} &= 0 \\
C^2_{11}C^2_{12} - C^2_{11}C^2_{21} &= 0 \\
C^1_{12}C^1_{11} + C^2_{12}C^1_{12} &= 0 \\
C^1_{12}C^2_{11} + C^2_{12}C^2_{12} &= 0 \\
C^2_{12}C^1_{12} + C^2_{21}C^1_{12} - C^2_{12}C^1_{21} - C^2_{21}C^1_{21} &= 0 \\
C^2_{12}C^2_{12} + C^2_{21}C^2_{12} - C^2_{12}C^2_{21} - C^2_{21}C^2_{21} &= 0 \\
C^1_{21}C^1_{11} + C^2_{21}C^1_{21} &= 0 \\
C^1_{21}C^2_{11} + C^2_{21}C^2_{21} &= 0 \\
C^2_{22}C^1_{12} - C^2_{22}C^1_{21} &= 0 \\
C^2_{22}C^2_{12} - C^2_{22}C^2_{21} &= 0 \\
C^1_{22}C^1_{11} + C^2_{22}C^1_{21} &= 0 \\
C^1_{22}C^2_{11} + C^2_{22}C^2_{21} &= 0.
\end{aligned}
$$

This can be rewritten as

$$
\begin{aligned}
C^2_{11}(C^1_{12} - C^1_{21}) &= 0 \\
C^2_{11}(C^2_{12} - C^2_{21}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1 &= 0 \\
C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 \\
C_{12}^2 (C_{12}^1 - C_{21}^1) + C_{21}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{12}^2 (C_{12}^2 - C_{21}^2) + C_{21}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{21}^1 C_{11}^1 + C_{21}^2 C_{21}^1 &= 0 \\
C_{21}^1 C_{11}^2 + C_{21}^2 C_{21}^2 &= 0 \\
C_{22}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{22}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^1 C_{11}^2 + C_{22}^2 C_{21}^2 &= 0,
\end{aligned}
$$

since addition is associative, and can further be rewritten as

$$
\begin{aligned}
C_{11}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{11}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1 &= 0 \\
C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 \\
(C_{12}^2 + C_{21}^2)(C_{12}^1 - C_{21}^1) &= 0 \\
(C_{12}^2 + C_{21}^2)(C_{12}^2 - C_{21}^2) &= 0 \\
C_{21}^1 C_{11}^1 + C_{21}^2 C_{21}^1 &= 0 \\
C_{21}^1 C_{11}^2 + C_{21}^2 C_{21}^2 &= 0 \\
C_{22}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{22}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^1 C_{11}^2 + C_{22}^2 C_{21}^2 &= 0.
\end{aligned}
$$

Furthermore, we since the structure constants are constants over a field, multiplication with them is commutative. Thus, we get $C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1 = C_{12}^1 C_{11}^1 + C_{12}^1 C_{12}^2 = C_{12}^1 (C_{11}^1 + C_{12}^2)$, and thus,

$$
\begin{aligned}
C_{11}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{11}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{12}^1 (C_{11}^1 + C_{12}^2) &= 0 \\
C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 \\
(C_{12}^2 + C_{21}^2)(C_{12}^1 - C_{21}^1) &= 0 \\
(C_{12}^2 + C_{21}^2)(C_{12}^2 - C_{21}^2) &= 0 \\
C_{21}^1 C_{11}^1 + C_{21}^2 C_{21}^1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
C_{21}^1 C_{11}^2 + C_{21}^2 C_{21}^2 &= 0 \\
C_{22}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{22}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^1 C_{11}^2 + C_{22}^2 C_{21}^2 &= 0.
\end{aligned}
$$

To try to solve this, we first need the following definitions and theorem:

Definition 8 (Fraleigh, Katz [4]). If a and b are two nonzero elements of a ring R such that $ab = 0$, then a and b are divisors of 0 (or 0 divisors).

Definition 9 (Fraleigh, Katz [4]). An integral domain D is a commutative ring with unity $1 \neq 0$ and containing no divisors of 0.

Theorem 1 (Fraleigh, Katz [4]). *Every field $\mathbb{F}$ is an integral domain.*

We are working over a field, which Theorem 1 tells us is an integral domain. Definition 9 then tells us that the field contains no divisors of 0, and finally Definition 8 tells us that this means that if we have $ab = 0$, then either a or b (or both) has to be equal to zero.

Thus, if we look at the first equation, we see that we need to have either $C_{11}^2 = 0$ or $C_{12}^1 - C_{21}^1 = 0$. We begin by looking at the first case.

Case 1: We let $C_{11}^2 = 0$, and remove the equations that then become $0 = 0$ from the system of equations. We then end up with the following:

$$
\begin{aligned}
C_{12}^1 (C_{11}^1 + C_{12}^2) &= 0 \\
C_{12}^2 C_{12}^2 &= 0 \\
(C_{12}^2 + C_{21}^2)(C_{12}^1 - C_{21}^1) &= 0 \\
(C_{12}^2 + C_{21}^2)(C_{12}^2 - C_{21}^2) &= 0 \\
C_{21}^1 C_{11}^1 + C_{21}^2 C_{21}^1 &= 0 \\
C_{21}^2 C_{21}^2 &= 0 \\
C_{22}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{22}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^2 C_{21}^2 &= 0.
\end{aligned}
$$

We see that the second equation gives us that we must have $C_{12}^2 = 0$. Substituting this into the system of equations, we get

$$\begin{aligned}
C_{12}^1 C_{11}^1 &= 0 \\
C_{21}^2(C_{12}^1 - C_{21}^1) &= 0 \\
-C_{21}^2 C_{21}^2 &= 0 \\
C_{21}^1 C_{11}^1 + C_{21}^2 C_{21}^1 &= 0 \\
C_{21}^2 C_{21}^2 &= 0 \\
C_{22}^2(C_{12}^1 - C_{21}^1) &= 0 \\
-C_{22}^2 C_{21}^2 &= 0 \\
C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^2 C_{21}^2 &= 0.
\end{aligned}$$

Note that we use commutativity to get $C_{21}^2(-C_{21}^2) = -C_{21}^2 C_{21}^2$. Now, the third equation tells us that we must have $C_{21}^2 = 0$. Substituting this too into the system of equations, we now have

$$\begin{aligned}
C_{12}^1 C_{11}^1 &= 0 \\
C_{21}^1 C_{11}^1 &= 0 \\
C_{22}^2(C_{12}^1 - C_{21}^1) &= 0 \\
C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 &= 0.
\end{aligned}$$

Now, we see that we do not immediately get the value of any of the constants. Thus, we look at the first equation, and get that we must have either $C_{12}^1 = 0$ or $C_{11}^1 = 0$.

Case 1.1: We first let $C_{12}^1 = 0$. Then we get

$$\begin{aligned}
C_{21}^1 C_{11}^1 &= 0 \\
-C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 &= 0.
\end{aligned}$$

To simplify the system of equations, we add the second equation to the third equation. Then we end up with

$$\begin{aligned}
C_{21}^1 C_{11}^1 &= 0 \\
-C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^1 C_{11}^1 &= 0.
\end{aligned}$$

Looking at the first equation, we see that the only two possible solutions are $C_{21}^1 = 0$ and $C_{11}^1 = 0$.

Case 1.1.1: If $C^1_{21} = 0$, our system of equations becomes

$$C^1_{22}C^1_{11} = 0.$$

To solve this final equation, we just need to have either $C^1_{22} = 0$ or $C^1_{11} = 0$. The variables that we have not concluded must be zero will be free variables.

Case 1.1.2: If $C^1_{11} = 0$ and $C^1_{21} \neq 0$, our system of equations becomes

$$-C^2_{22}C^1_{21} = 0.$$

The only solution to this equation is $C^2_{22} = 0$. This means we have now solved the entire system of equations, so the remaining variables will be free variables.

Case 1.2: We now look at the case when $C^1_{11} = 0$ and $C^1_{12} \neq 0$. Now, we of course have to go back to the system of equations we got in Case 1. Substituting $C^1_{11} = 0$ into this system of equations, we get

$$\begin{aligned} C^2_{22}(C^1_{12} - C^1_{21}) &= 0 \\ C^2_{22}C^1_{21} &= 0. \end{aligned}$$

To solve the second equation, we see that we must have either $C^2_{22} = 0$ or $C^1_{21} = 0$.

Case 1.2.1: If $C^2_{22} = 0$, we see that we have solved the system of equations.

Case 1.2.2: If $C^1_{21} = 0$ and $C^2_{22} \neq 0$, the system of equations becomes

$$C^2_{22}C^1_{12} = 0.$$

However, we have assumed that both $C^1_{12} \neq 0$ and $C^2_{22} \neq 0$, so there exists no solution to this equation, and this case did not lead to any solution.

Case 2: We now look at the case when we, from the beginning of solving the system of equations, instead assume that $C^2_{11} \neq 0$. Then we must have $C^1_{12} - C^1_{21} = 0$, which clearly gives us $C^1_{12} = C^1_{21}$. If we add this condition to the system of equations, remove the equations that now become redundant, and use commutativity to rewrite one of the equations, we get

$$\begin{aligned} C^2_{11}(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{12}(C^1_{11} + C^2_{12}) &= 0 \\ C^1_{12}C^2_{11} + C^2_{12}C^2_{12} &= 0 \\ (C^2_{12} + C^2_{21})(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{12}(C^1_{11} + C^2_{21}) &= 0 \\ C^1_{12}C^2_{11} + C^2_{21}C^2_{21} &= 0 \\ C^2_{22}(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{22}C^1_{11} + C^2_{22}C^1_{12} &= 0 \\ C^1_{22}C^2_{11} + C^2_{22}C^2_{21} &= 0. \end{aligned}$$

Since we currently have $C_{11}^2 \neq 0$, the second equation gives us that we must have $C_{12}^2 = C_{21}^2$. The system of equations will then look as follows:

$$\begin{aligned} C_{12}^1(C_{11}^1 + C_{12}^2) &= 0 \\ C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 \\ C_{12}^1(C_{11}^1 + C_{12}^2) &= 0 \\ C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 \\ C_{22}^1 C_{11}^1 + C_{22}^2 C_{12}^1 &= 0 \\ C_{22}^1 C_{11}^2 + C_{22}^2 C_{12}^2 &= 0. \end{aligned}$$

Now, we note that some of the equations are the exact same as each other, so we remove them so we only have one copy of each equation. The system of equations then becomes

$$\begin{aligned} C_{12}^1(C_{11}^1 + C_{12}^2) &= 0 \\ C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 \\ C_{22}^1 C_{11}^1 + C_{22}^2 C_{12}^1 &= 0 \\ C_{22}^1 C_{11}^2 + C_{22}^2 C_{12}^2 &= 0. \end{aligned}$$

The solution to the first equation is either $C_{12}^1 = 0$ or $C_{11}^1 + C_{12}^2 = 0$.

Case 2.1: If $C_{12}^1 = 0$, the system of equations becomes

$$\begin{aligned} C_{12}^2 C_{12}^2 &= 0 \\ C_{22}^1 C_{11}^1 &= 0 \\ C_{22}^1 C_{11}^2 + C_{22}^2 C_{12}^2 &= 0. \end{aligned}$$

The first equation gives us that we must have $C_{12}^2 = 0$. Then the system of equations becomes

$$\begin{aligned} C_{22}^1 C_{11}^1 &= 0 \\ C_{22}^1 C_{11}^2 &= 0. \end{aligned}$$

Since we have assumed that $C_{11}^2 \neq 0$, the second equation gives us that we must have $C_{22}^1 = 0$. It is easy to see that we have then solved the system of equations.

Case 2.2: If $C_{12}^1 \neq 0$ $C_{11}^1 + C_{12}^2 = 0$, or $C_{11}^1 = -C_{12}^2$, the system of equations becomes

$$\begin{aligned} C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 \\ -C_{22}^1 C_{12}^2 + C_{22}^2 C_{12}^1 &= 0 \\ C_{22}^1 C_{11}^2 + C_{22}^2 C_{12}^2 &= 0. \end{aligned}$$

Since we have previously handled the case where $C_{12}^1 = 0$, we have assumed that $C_{12}1 \neq 0$ here. Thus, we can write the solution to the second equation as $C_{11}^2 = -C_{12}^2 C_{12}^2 / C_{12}^1$. Substituting this into the system of equations, we get

$$\begin{aligned} -C_{22}^1 C_{12}^2 + C_{22}^2 C_{12}^1 &= 0 \\ -C_{22}^1 C_{12}^2 C_{12}^2 / C_{12}^1 + C_{22}^2 C_{12}^2 &= 0. \end{aligned}$$

We can now write the solution to the first equation as $C_{22}^2 = C_{22}^1 C_{12}^2 / C_{12}^1$. Substituting this into the system of equations, the only remaining equation becomes

$$-C_{22}^1 C_{12}^2 C_{12}^2 / C_{12}^1 + C_{22}^1 C_{12}^2 C_{12}^2 / C_{12}^1 = 0.$$

Thus, we have now solved the system of equations.

Now, if we go through each of these cases and write down what the variables must be in each of the solutions (which we found in each of the cases) – marked with "free" if it is a free variable – the results can be seen in Table 3.1. Note that each column represents one solution.

	Soln 1	Soln 2	Soln 3	Soln 4	Soln 5	Soln 6
C_{11}^1	free	0	0	0	free	$-C_{12}^2$
C_{11}^2	0	0	0	0	free	$C_{11}^2 \neq 0$
C_{12}^1	0	0	0	free	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{12}^2	0	0	0	0	0	free
C_{21}^1	0	0	free	free	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{21}^2	0	0	0	0	0	C_{12}^2
C_{22}^1	0	free	free	free	0	$-C_{22}^2 C_{12}^2 / C_{11}^2$
C_{22}^2	free	free	0	0	free	free

Table 3.1: The values of the structure constants that give hom-associative algebras when α is defined as E_{11} in two dimensions

Looking at the solutions, we see that some of the solutions are actually special cases of another of the solutions. For example, solution 1 is a special case of solution 5, since the only difference between them is that C_{11}^2 is free in solution 6, while in solution 2 we have $C_{11}^2 = 0$. In the same way, solution 3 is a special cases of solution 4. Since these special cases have only appeared due to the way we have solved the system of equations – if we for example had chosen to begin with another equation, we might have gotten other special cases – we can remove them. We are then left with the solutions shown in Table 3.2

Now, we want to look at the hom-Lie admissible algebras associated to the hom-associative algebras we have found above. Using Proposition 2, we can associate a hom-Lie algebra defined by $[x, y] = \mu(x, y) - \mu(y, x)$ for all $x, y \in V$ to our hom-associative algebra (V, μ, α). This gives us the hom-Lie algebra $(V, [\cdot, \cdot], \alpha)$, which Proposition 3 tells us is hom-Lie admissible with

	Soln 2	Soln 4	Soln 5	Soln 6
C_{11}^1	0	0	free	$-C_{12}^2$
C_{11}^2	0	0	free	$C_{11}^2 \neq 0$
C_{12}^1	0	free	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{12}^2	0	0	0	free
C_{21}^1	0	free	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{21}^2	0	0	0	C_{12}^2
C_{22}^1	free	free	0	$-C_{22}^2 C_{12}^2 / C_{11}^2$
C_{22}^2	free	0	free	free

Table 3.2: The values of the structure constants that give hom-associative algebras when α is defined as E_{11} in two dimensions, with the special cases removed

the same twisting map α. However, remembering that our hom-associative algebra is in fact G_1-hom-associative, we could just use Proposition 5 which tells us that any G-hom-associative algebra is a hom-Lie admissible algebra.

Thus, each of the hom-associative algebras we have found is hom-Lie admissible. We now want to calculate the commutator $[\cdot,\cdot]$ for each of these hom-Lie admissible algebras. That is, we want to calculate $[e_i, e_j]$ – where $i, j = \{1,2\}$ since we are currently working in two dimensions – for each of the hom-Lie admissible algebras. By definition $[e_1, e_1] = 0$ and $[e_2, e_2] = 0$, so all we need to calculate is $[e_1, e_2]$ and $[e_2, e_1]$. We begin by looking at solution 2 in the table above. Clearly, the first thing we need to do is calculate $\mu(e_1, e_2)$ and $\mu(e_2, e_1)$. Using Formula (3.11),

$$\mu(e_i, e_j) = \sum_{k=1}^{2} C_{ij}^k e_k = C_{ij}^1 e_1 + C_{ij}^2 e_2,$$

we get

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = 0e_1 + 0e_2 = 0,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = 0e_1 + 0e_2 = 0.$$

This gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = 0 - 0 = 0,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = 0 - 0 = 0.$$

Moving on to solution 4, we get

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = C_{12}^1 e_1 + 0e_2 = C_{12}^1 e_1,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = C_{21}^1 e_1 + 0e_2 = C_{21}^1 e_1,$$

which gives us

$$[e_1,e_2] = \mu(e_1,e_2) - \mu(e_2,e_1) = C_{12}^1 e_1 - C_{21}^1 e_1 = (C_{12}^1 - C_{21}^1)e_1,$$
$$[e_2,e_1] = \mu(e_2,e_1) - \mu(e_1,e_2) = C_{21}^1 e_1 - C_{12}^1 e_1 = (C_{21}^1 - C_{12}^1)e_1.$$

Solution 5 instead gives us

$$\mu(e_1,e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = 0e_1 + 0e_2 = 0,$$
$$\mu(e_2,e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = 0e_1 + 0e_2 = 0.$$

Thus, we get

$$[e_1,e_2] = \mu(e_1,e_2) - \mu(e_2,e_1) = 0 - 0 = 0,$$
$$[e_2,e_1] = \mu(e_2,e_1) - \mu(e_1,e_2) = 0 - 0 = 0,$$

just as for solution 1. Finally, solution 6 gives us

$$\mu(e_1,e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = -(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2,$$
$$\mu(e_2,e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = -(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2,$$

where $C_{11}^2 \neq 0$, and we end up with

$$[e_1,e_2] = \mu(e_1,e_2) - \mu(e_2,e_1) = -(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2 - (-(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2) = 0,$$
$$[e_2,e_1] = \mu(e_2,e_1) - \mu(e_1,e_2) = -(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2 - (-(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2) = 0.$$

This means that from our four hom-associative algebras (V, μ, E_{11}), we were able to get four different hom-Lie admissible algebras $(V, [\cdot,\cdot], E_{11})$, where $[\cdot,\cdot]$ is defined as in Table 3.3 for the algebras with structure constants as in solutions 2, 5 or 6 above, or Table 3.4 for the algebra with structure constants as in solution 4 above.

$[\cdot,\cdot]$	e_1	e_2
e_1	0	0
e_2	0	0

Table 3.3: The commutator table for the hom-Lie admissible algebras with α defined as E_{11} and the structure constants defined as in either solution 2, solution 5 or solution 6 in Table 3.2

The hom-Lie admissible algebras with commutators as in Table 3.3 are abelian.

$[\cdot,\cdot]$	e_1	e_2
e_1	0	$(C_{12}^1 - C_{21}^1)e_1$
e_2	$(C_{21}^1 - C_{12}^1)e_1$	0

Table 3.4: The commutator table for the hom-Lie admissible algebras with α defined as E_{11} and the structure constants defined as in solution 4 in Table 3.2

3.1.2 α defined as E_{12}

We will now instead investigate what the hom-associative algebras will look like for

$$[\alpha] = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Using formulas (3.9) and (3.10), we get

$$\begin{aligned} \alpha(e_1) &= 0e_1 + 0e_2 = 0, \\ \alpha(e_2) &= 1e_1 + 0e_2 = e_1. \end{aligned}$$

Using this, we can rewrite all of our equalities as follows:

$$\begin{aligned}
&(3.1): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0, \\
&(3.2): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, e_1) \quad \Rightarrow \quad 0 = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, e_1), \\
&(3.3): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0, \\
&(3.4): \quad \mu(e_1, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_1, C_{11}^1 e_1 + C_{11}^2 e_2) = 0, \\
&(3.5): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, e_1) \quad \Rightarrow \quad 0 = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, e_1), \\
&(3.6): \quad \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, e_1), \\
&(3.7): \quad \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2) = 0, \\
&(3.8): \quad \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, e_1).
\end{aligned}$$

We now use formulas (3.12) and (3.13) again, and we note that we again always have $i = 1$. Using these formulas, our equalities become:

$$\begin{aligned}
&(3.2): \quad 0 = (C_{11}^1 C_{11}^1 + C_{11}^2 C_{21}^1)e_1 + (C_{11}^1 C_{11}^2 + C_{11}^2 C_{21}^2)e_2, \\
&(3.4): \quad (C_{11}^1 C_{11}^1 + C_{11}^2 C_{12}^1)e_1 + (C_{11}^1 C_{11}^2 + C_{11}^2 C_{12}^2)e_2 = 0, \\
&(3.5): \quad 0 = (C_{12}^1 C_{11}^1 + C_{12}^2 C_{21}^1)e_1 + (C_{12}^1 C_{11}^2 + C_{12}^2 C_{21}^2)e_2, \\
&(3.6): \quad (C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1)e_1 + (C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2)e_2 = (C_{21}^1 C_{11}^1 + C_{21}^2 C_{21}^1)e_1 \\
&\qquad\qquad + (C_{21}^1 C_{11}^2 + C_{21}^2 C_{21}^2)e_2 \\
&\qquad \Rightarrow \quad (C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1 - C_{21}^1 C_{11}^1 - C_{21}^2 C_{21}^1)e_1 \\
&\qquad\qquad + (C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 - C_{21}^1 C_{11}^2 - C_{21}^2 C_{21}^2)e_2 = 0, \\
&(3.7): \quad (C_{21}^1 C_{11}^1 + C_{21}^2 C_{12}^1)e_1 + (C_{21}^1 C_{11}^2 + C_{21}^2 C_{12}^2)e_2 = 0,
\end{aligned}$$

$$
\begin{aligned}
(3.8): \quad & (C^1_{22}C^1_{11}+C^2_{22}C^1_{12})e_1+(C^1_{22}C^2_{11}+C^2_{22}C^2_{12})e_2=(C^1_{22}C^1_{11}+C^2_{22}C^1_{21})e_1 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad +(C^1_{22}C^2_{11}+C^2_{22}C^2_{21})e_2 \\
\Rightarrow \quad & C^2_{22}C^1_{12}e_1+C^2_{22}C^2_{12}e_2=C^2_{22}C^1_{21}e_1+C^2_{22}C^2_{21}e_2 \\
\Rightarrow \quad & (C^2_{22}C^1_{12}-C^2_{22}C^1_{21})e_1+(C^2_{22}C^2_{12}-C^2_{22}C^2_{21})e_2=0.
\end{aligned}
$$

Remembering that $k_1e_1+k_1e_2=0$ will only have the trivial solution $k_1=k_2=0$, this gives us the system of equations

$$
\begin{aligned}
\underline{C^1_{11}C^1_{11}}+C^2_{11}C^1_{21}&=0\\
\underline{C^1_{11}C^2_{11}}+C^2_{11}C^2_{21}&=0\\
\underline{C^1_{11}C^1_{11}}+C^2_{11}C^1_{12}&=0\\
\underline{C^1_{11}C^2_{11}}+C^2_{11}C^2_{12}&=0\\
\underline{C^1_{12}C^1_{11}}+C^2_{12}C^1_{21}&=0\\
\underline{C^1_{12}C^2_{11}}+C^2_{12}C^2_{21}&=0\\
\underline{C^1_{12}C^1_{11}}+C^2_{12}C^1_{12}-\underline{C^1_{21}C^1_{11}}-C^2_{21}C^1_{21}&=0\\
\underline{C^1_{12}C^2_{11}}+C^2_{12}C^2_{12}-\underline{C^1_{21}C^2_{11}}-C^2_{21}C^2_{21}&=0\\
\underline{C^1_{21}C^1_{11}}+C^2_{21}C^1_{12}&=0\\
\underline{C^1_{21}C^2_{11}}+C^2_{21}C^2_{12}&=0\\
C^2_{22}C^1_{12}-C^2_{22}C^1_{21}&=0\\
C^2_{22}C^2_{12}-C^2_{22}C^2_{21}&=0.
\end{aligned}
$$

To eliminate these terms, we subtract the first equation from the third, subtract the second equation from the fourth, add the ninth and seventh equations, and add the tenth and eighth equations. This gives us

$$
\begin{aligned}
C^1_{11}C^1_{11}+C^2_{11}C^1_{21}&=0\\
C^2_{11}C^1_{12}-C^2_{11}C^1_{21}&=0\\
C^1_{11}C^1_{11}+C^2_{11}C^1_{12}&=0\\
C^2_{11}C^2_{12}-C^2_{11}C^2_{21}&=0\\
\underline{C^1_{12}C^1_{11}}+C^2_{12}C^1_{21}&=0\\
\underline{C^1_{12}C^2_{11}}+C^2_{12}C^2_{21}&=0\\
C^2_{21}C^1_{12}+\underline{C^1_{12}C^1_{11}}+C^2_{12}C^1_{12}-C^2_{21}C^1_{21}&=0\\
C^2_{21}C^2_{12}+\underline{C^1_{12}C^2_{11}}+C^2_{12}C^2_{12}-C^2_{21}C^2_{21}&=0\\
C^1_{21}C^1_{11}+C^2_{21}C^1_{12}&=0\\
C^1_{21}C^2_{11}+C^2_{21}C^2_{12}&=0\\
C^2_{22}C^1_{12}-C^2_{22}C^1_{21}&=0\\
C^2_{22}C^2_{12}-C^2_{22}C^2_{21}&=0.
\end{aligned}
$$

Now subtracting the third equation from the fifth, and subtracting the fourth equation from the sixth, we get

$$\begin{aligned}
C_{11}^1 C_{11}^1 + C_{11}^2 C_{21}^1 &= 0 \\
C_{11}^2 C_{12}^1 - C_{11}^2 C_{21}^1 &= 0 \\
C_{11}^1 C_{11}^1 + C_{11}^2 C_{12}^1 &= 0 \\
C_{11}^2 C_{12}^2 - C_{11}^2 C_{21}^2 &= 0 \\
C_{12}^1 C_{11}^1 + C_{12}^2 C_{21}^1 &= 0 \\
C_{12}^1 C_{11}^2 + C_{12}^2 C_{21}^2 &= 0 \\
C_{21}^2 C_{12}^1 + C_{12}^2 C_{12}^1 - C_{21}^2 C_{21}^1 - C_{12}^2 C_{21}^1 &= 0 \\
C_{21}^2 C_{12}^2 + C_{12}^2 C_{12}^2 - C_{21}^2 C_{21}^2 - C_{12}^2 C_{21}^2 &= 0 \\
C_{21}^1 C_{11}^1 + C_{21}^2 C_{12}^1 &= 0 \\
C_{21}^1 C_{11}^2 + C_{21}^2 C_{12}^2 &= 0 \\
C_{22}^2 C_{12}^1 - C_{22}^2 C_{21}^1 &= 0 \\
C_{22}^2 C_{12}^2 - C_{22}^2 C_{21}^2 &= 0.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
C_{11}^1 C_{11}^1 + C_{11}^2 C_{21}^1 &= 0 \\
C_{11}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{11}^1 C_{11}^1 + C_{11}^2 C_{12}^1 &= 0 \\
C_{11}^2 (C_{12}^2 - C_{21}^2) &= 0 \\
C_{12}^1 C_{11}^1 + C_{12}^2 C_{21}^1 &= 0 \\
C_{12}^1 C_{11}^2 + C_{12}^2 C_{21}^2 &= 0 \\
(C_{21}^2 + C_{12}^2)(C_{12}^1 - C_{21}^1) &= 0 \\
(C_{21}^2 + C_{12}^2)(C_{12}^2 - C_{21}^2) &= 0 \\
C_{21}^1 C_{11}^1 + C_{21}^2 C_{12}^1 &= 0 \\
C_{21}^1 C_{11}^2 + C_{21}^2 C_{12}^2 &= 0 \\
C_{22}^2 (C_{12}^1 - C_{21}^1) &= 0 \\
C_{22}^2 (C_{12}^2 - C_{21}^2) &= 0.
\end{aligned}$$

Remember that multiplication with our constants C_{ij}^k is commutative. Thus, subtracting the first equation from the third, subtracting the tenth equation from the sixth, and rewriting, gives us

$$\begin{aligned}
C_{11}^1 C_{11}^1 + C_{11}^2 C_{21}^1 &= 0 \\
C_{11}^2 (C_{12}^1 - C_{21}^1) &= 0
\end{aligned}$$

$$
\begin{aligned}
C^2_{11}(C^1_{12} - C^1_{21}) &= 0\\
C^2_{11}(C^2_{12} - C^2_{21}) &= 0\\
C^1_{12}C^1_{11} + C^2_{12}C^1_{21} &= 0\\
C^2_{11}(C^1_{12} - C^1_{21}) &= 0\\
(C^2_{21} + C^2_{12})(C^1_{12} - C^1_{21}) &= 0\\
(C^2_{21} + C^2_{12})(C^2_{12} - C^2_{21}) &= 0\\
C^1_{21}C^1_{11} + C^2_{21}C^1_{12} &= 0\\
C^1_{21}C^2_{11} + C^2_{21}C^2_{12} &= 0\\
C^2_{22}(C^1_{12} - C^1_{21}) &= 0\\
C^2_{22}(C^2_{12} - C^2_{21}) &= 0.
\end{aligned}
$$

We see that some of the equalities are now identical, so we can remove equations so we only have one of each. We then have

$$
\begin{aligned}
C^1_{11}C^1_{11} + C^2_{11}C^1_{21} &= 0\\
C^2_{11}(C^1_{12} - C^1_{21}) &= 0\\
C^2_{11}(C^2_{12} - C^2_{21}) &= 0\\
C^1_{12}C^1_{11} + C^2_{12}C^1_{21} &= 0\\
(C^2_{21} + C^2_{12})(C^1_{12} - C^1_{21}) &= 0\\
(C^2_{21} + C^2_{12})(C^2_{12} - C^2_{21}) &= 0\\
C^1_{21}C^1_{11} + C^2_{21}C^1_{12} &= 0\\
C^1_{21}C^2_{11} + C^2_{21}C^2_{12} &= 0\\
C^2_{22}(C^1_{12} - C^1_{21}) &= 0\\
C^2_{22}(C^2_{12} - C^2_{21}) &= 0.
\end{aligned}
$$

We now try to solve this system of equations by looking at different cases. We begin to look at the second equation. The solution to this has to be either $C^2_{11} = 0$ or $C^1_{12} - C^1_{21} = 0$, or both. We begin by looking at the first case, where $C^2_{11} = 0$, before moving on to the second case, where $C^2_{11} \neq 0$ and $C^1_{12} - C^1_{21} = 0$.

Case 1: If $C^2_{11} = 0$ our system of equations will look as follows:

$$
\begin{aligned}
C^1_{11}C^1_{11} &= 0\\
C^1_{12}C^1_{11} + C^2_{12}C^1_{21} &= 0\\
(C^2_{21} + C^2_{12})(C^1_{12} - C^1_{21}) &= 0\\
(C^2_{21} + C^2_{12})(C^2_{12} - C^2_{21}) &= 0\\
C^1_{21}C^1_{11} + C^2_{21}C^1_{12} &= 0\\
C^2_{21}C^2_{12} &= 0
\end{aligned}
$$

$$C^2_{22}(C^1_{12} - C^1_{21}) = 0$$
$$C^2_{22}(C^2_{12} - C^2_{21}) = 0.$$

The only solution to the first equation is $C^1_{11} = 0$. Our system of equations thus becomes

$$\begin{aligned}
C^2_{12}C^1_{21} &= 0\\
(C^2_{21} + C^2_{12})(C^1_{12} - C^1_{21}) &= 0\\
(C^2_{21} + C^2_{12})(C^2_{12} - C^2_{21}) &= 0\\
C^2_{21}C^1_{12} &= 0\\
C^2_{21}C^2_{12} &= 0\\
C^2_{22}(C^1_{12} - C^1_{21}) &= 0\\
C^2_{22}(C^2_{12} - C^2_{21}) &= 0.
\end{aligned}$$

The solutions to the first equation are $C^2_{12} = 0$ and $C^1_{21} = 0$. We look at both cases.

Case 1.1: If $C^2_{12} = 0$ our system of equations becomes

$$\begin{aligned}
C^2_{21}(C^1_{12} - C^1_{21}) &= 0\\
C^2_{21}(-C^2_{21}) &= 0\\
C^2_{21}C^1_{12} &= 0\\
C^2_{22}(C^1_{12} - C^1_{21}) &= 0\\
C^2_{22}(-C^2_{21}) &= 0.
\end{aligned}$$

Clearly, the only solution to the second equation is $C^2_{21} = 0$. This gives us

$$C^2_{22}(C^1_{12} - C^1_{21}) = 0.$$

The two solutions to this equation are $C^2_{22} = 0$ and $C^1_{12} - C^1_{21} = 0$. We have thus solved the system of equations.

Case 1.2: If $C^1_{21} = 0$ and $C^2_{12} \neq 0$, our system of equations becomes

$$\begin{aligned}
(C^2_{21} + C^2_{12})C^1_{12} &= 0\\
(C^2_{21} + C^2_{12})(C^2_{12} - C^2_{21}) &= 0\\
C^2_{21}C^1_{12} &= 0\\
C^2_{21}C^2_{12} &= 0\\
C^2_{22}C^1_{12} &= 0\\
C^2_{22}(C^2_{12} - C^2_{21}) &= 0.
\end{aligned}$$

Since $C^2_{12} \neq 0$, the only solution to equation four is $C^2_{21} = 0$. This gives us

$$\begin{aligned} C^2_{12}C^1_{12} &= 0 \\ C^2_{12}C^2_{12} &= 0 \\ C^2_{22}C^1_{12} &= 0 \\ C^2_{22}C^2_{12} &= 0. \end{aligned}$$

Now, the only solution to the second equation is $C^2_{12} = 0$, but we have assumed that $C^2_{12} \neq 0$. Thus, we do not get any solution to the system of equations in this case.

Case 2: If instead $C^1_{12} - C^1_{21} = 0$ and $C^2_{11} \neq 0$, we get

$$\begin{aligned} C^1_{11}C^1_{11} + C^2_{11}C^1_{12} &= 0 \\ C^2_{11}(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{12}C^1_{11} + C^2_{12}C^1_{12} &= 0 \\ (C^2_{21} + C^2_{12})(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{12}C^1_{11} + C^2_{21}C^1_{12} &= 0 \\ C^1_{12}C^2_{11} + C^2_{21}C^2_{12} &= 0 \\ C^2_{22}(C^2_{12} - C^2_{21}) &= 0. \end{aligned}$$

Using commutativity, this can be rewritten as

$$\begin{aligned} C^1_{11}C^1_{11} + C^2_{11}C^1_{12} &= 0 \\ C^2_{11}(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{12}(C^1_{11} + C^2_{12}) &= 0 \\ (C^2_{21} + C^2_{12})(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{12}(C^1_{11} + C^2_{21}) &= 0 \\ C^1_{12}C^2_{11} + C^2_{21}C^2_{12} &= 0 \\ C^2_{22}(C^2_{12} - C^2_{21}) &= 0. \end{aligned}$$

Since $C^2_{11} \neq 0$, the second equation gives us that we must have $C^2_{12} - C^2_{21} = 0$. The system of equations then become

$$\begin{aligned} C^1_{11}C^1_{11} + C^2_{11}C^1_{12} &= 0 \\ C^1_{12}(C^1_{11} + C^2_{12}) &= 0 \\ C^1_{12}(C^1_{11} + C^2_{12}) &= 0 \\ C^1_{12}C^2_{11} + C^2_{12}C^2_{12} &= 0, \end{aligned}$$

which is the same as

$$C_{11}^1 C_{11}^1 + C_{11}^2 C_{12}^1 = 0$$
$$C_{12}^1 (C_{11}^1 + C_{12}^2) = 0$$
$$C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 = 0.$$

The solutions to the second equation are $C_{12}^1 = 0$ and $C_{11}^1 + C_{12}^2 = 0$. We look at the case where $C_{12}^1 = 0$ first, and then the case where $C_{12}^1 \neq 0$ and $C_{11}^1 + C_{12}^2 = 0$.

Case 2.1: If $C_{12}^1 = 0$, the system of equations becomes

$$C_{11}^1 C_{11}^1 = 0$$
$$C_{12}^2 C_{12}^2 = 0.$$

The solution to the first equation is $C_{11}^1 = 0$ and the solution to the second equation is $C_{12}^2 = 0$. We have then solved the system of equations.

Case 2.2: If instead $C_{12}^1 \neq 0$ and $C_{11}^1 + C_{12}^2 = 0$, the system of equations becomes

$$(-C_{12}^2)(-C_{12}^2) + C_{11}^2 C_{12}^1 = 0$$
$$C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 = 0.$$

Due to commutativity these are the same equation, so all that is left to do is to solve $C_{12}^2 C_{12}^2 + C_{11}^2 C_{12}^1 = 0$. Since we have assumed that $C_{11}^2 \neq 0$, the solution to this is $C_{12}^1 = -C_{12}^2 C_{12}^2 / C_{11}^2$. We have now again solved the system of equations.

We sum up all of the results in Table 3.5.

	Soln 1	Soln 2	Soln 3	Soln 4
C_{11}^1	0	0	0	$-C_{12}^2$
C_{11}^2	0	0	free	$C_{11}^2 \neq 0$
C_{12}^1	free	free	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{12}^2	0	0	0	free
C_{21}^1	free	C_{12}^1	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{21}^2	0	0	0	C_{12}^2
C_{22}^1	free	free	free	free
C_{22}^2	0	free	free	free

Table 3.5: The values of the structure constants that give hom-associative algebras when α is defined as E_{12} in two dimensions

Now, we see that if, in solution 4, we let $C_{12}^2 = 0$, we get exactly solution 3. This means that solution 3 is actually just a special case of solution 4, and we can remove it. Our solutions then look as seen in Table 3.6.

	Soln 1	Soln 2	Soln 4
C_{11}^1	0	0	$-C_{12}^2$
C_{11}^2	0	0	$C_{11}^2 \neq 0$
C_{12}^1	free	free	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{12}^2	0	0	free
C_{21}^1	free	C_{12}^1	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{21}^2	0	0	C_{12}^2
C_{22}^1	free	free	free
C_{22}^2	0	free	free

Table 3.6: The values of the structure constants that give hom-associative algebras when α is defined as E_{12} in two dimensions, with the special cases removed

Now, just as in the previous case, we want to find the commutator for the hom-Lie admissible algebras we get from the hom-associative algebras found above. Like we did before, we must calculate $[e_i, e_j]$, where $i, j = \{1, 2\}$, for each of the solutions in Table 3.6, and since $[e_1, e_1] = [e_2, e_2] = 0$, we only have to calculate $[e_1, e_2]$ and $[e_2, e_1]$. Beginning with solution 1, we get

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = C_{12}^1 e_1 + 0 e_2 = C_{12}^1 e_1,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = C_{21}^1 e_1 + 0 e_2 = C_{21}^1 e_1,$$

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^1 e_1 - C_{21}^1 e_1 = (C_{12}^1 - C_{21}^1) e_1,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = C_{21}^1 e_1 - C_{12}^1 e_1 = (C_{21}^1 - C_{12}^1) e_1.$$

Moving on to solution 2, we get

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = C_{12}^1 e_1 + 0 e_2 = C_{12}^1 e_1,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = C_{12}^1 e_1 + 0 e_2 = C_{12}^1 e_1,$$

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^1 e_1 - C_{12}^1 e_1 = 0,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = C_{12}^1 e_1 - C_{12}^1 e_1 = 0.$$

Finally, looking at solution 4, we get

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = -(C_{12}^2 C_{12}^2 / C_{11}^2) e_1 + C_{12}^2 e_2,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = -(C_{12}^2 C_{12}^2 / C_{11}^2) e_1 + C_{12}^2 e_2,$$

which gives us

$$[e_1,e_2] = \mu(e_1,e_2) - \mu(e_2,e_1) = -(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2 - (-(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2) = 0,$$
$$[e_2,e_1] = \mu(e_2,e_1) - \mu(e_1,e_2) = -(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2 - (-(C_{12}^2 C_{12}^2 / C_{11}^2)e_1 + C_{12}^2 e_2) = 0.$$

This means that from our three hom-associative algebras (V, μ, E_{12}), we were able to get three different hom-Lie admissible algebras $(V, [\cdot,\cdot], E_{12})$, where $[\cdot,\cdot]$ is defined as in Table 3.7 for the structure constants given in solution 1 above, or Table 3.8 for the structure constants given in solution 2 and 4 above.

$[\cdot,\cdot]$	e_1	e_2
e_1	0	$(C_{12}^1 - C_{21}^1)e_1$
e_2	$(C_{21}^1 - C_{12}^1)e_1$	0

Table 3.7: The commutator table for the hom-Lie admissible algebras with α defined as E_{12} and the structure constants defined as in solution 1 in Table 3.6

$[\cdot,\cdot]$	e_1	e_2
e_1	0	0
e_2	0	0

Table 3.8: The commutator table for the hom-Lie admissible algebras with α defined as E_{12} and the structure constants defined as in either solution 2 or solution 4 in Table 3.6

We note that these commutator tables are the exact same as the ones for α defined as E_{11}, given in Table 3.3 and Table 3.4.

3.1.3 α defined as E_{21}

We now look at

$$[\alpha] = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Recalling formulas (3.9) and (3.10), we get

$$\alpha(e_1) = 0e_1 + 1e_2 = e_2,$$
$$\alpha(e_2) = 0e_1 + 0e_2 = 0.$$

Using this, we can rewrite all of our equalities as follows:

$$
\begin{aligned}
&(3.1): \quad \mu(e_2, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, e_2),\\
&(3.2): \quad \mu(e_2, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_2, C_{12}^1 e_1 + C_{12}^2 e_2) = 0,\\
&(3.3): \quad \mu(e_2, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, e_2),\\
&(3.4): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, e_2) \quad \Rightarrow \quad 0 = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, e_2),\\
&(3.5): \quad \mu(e_2, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_2, C_{22}^1 e_1 + C_{22}^2 e_2) = 0,\\
&(3.6): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0,\\
&(3.7): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, e_2) \quad \Rightarrow \quad 0 = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, e_2),\\
&(3.8): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0.
\end{aligned}
$$

We now use formulas (3.12) and (3.13) again, but this time we note that we always have $i = 2$. Using these formulas, our equalities become:

$$
\begin{aligned}
(3.1): \quad & (C_{11}^1 C_{21}^1 + C_{11}^2 C_{22}^1)e_1 + (C_{11}^1 C_{21}^2 + C_{11}^2 C_{22}^2)e_2 = (C_{11}^1 C_{12}^1 + C_{11}^2 C_{22}^1)e_1\\
& \qquad + (C_{11}^1 C_{12}^2 + C_{11}^2 C_{22}^2)e_2\\
& \Rightarrow \quad C_{11}^1 C_{21}^1 e_1 + C_{11}^1 C_{21}^2 e_2 = C_{11}^1 C_{12}^1 e_1 + C_{11}^1 C_{12}^2 e_2\\
& \Rightarrow \quad (C_{11}^1 C_{21}^1 - C_{11}^1 C_{12}^1)e_1 + (C_{11}^1 C_{21}^2 - C_{11}^1 C_{12}^2)e_2 = 0,\\
(3.2): \quad & (C_{12}^1 C_{21}^1 + C_{12}^2 C_{22}^1)e_1 + (C_{12}^1 C_{21}^2 + C_{12}^2 C_{22}^2)e_2 = 0,\\
(3.3): \quad & (C_{21}^1 C_{21}^1 + C_{21}^2 C_{22}^1)e_1 + (C_{21}^1 C_{21}^2 + C_{21}^2 C_{22}^2)e_2 = (C_{12}^1 C_{12}^1 + C_{12}^2 C_{22}^1)e_1\\
& \qquad + (C_{12}^1 C_{12}^2 + C_{12}^2 C_{22}^2)e_2\\
& \Rightarrow \quad (C_{21}^1 C_{21}^1 + C_{21}^2 C_{22}^1 - C_{12}^1 C_{12}^1 - C_{12}^2 C_{22}^1)e_1\\
& \qquad + (C_{21}^1 C_{21}^2 + C_{21}^2 C_{22}^2 - C_{12}^1 C_{12}^2 - C_{12}^2 C_{22}^2)e_2 = 0,\\
(3.4): \quad & 0 = (C_{21}^1 C_{12}^1 + C_{21}^2 C_{22}^1)e_1 + (C_{21}^1 C_{12}^2 + C_{21}^2 C_{22}^2)e_2,\\
(3.5): \quad & (C_{22}^1 C_{21}^1 + C_{22}^2 C_{22}^1)e_1 + (C_{22}^1 C_{21}^2 + C_{22}^2 C_{22}^2)e_2 = 0,\\
(3.7): \quad & 0 = (C_{22}^1 C_{12}^1 + C_{22}^2 C_{22}^1)e_1 + (C_{22}^1 C_{12}^2 + C_{22}^2 C_{22}^2)e_2.
\end{aligned}
$$

Again remembering that $k_1 e_1 + k_1 e_2 = 0$ will only have the trivial solution $k_1 = k_2 = 0$, this gives us the system of equations

$$
\begin{aligned}
C_{11}^1 C_{21}^1 - C_{11}^1 C_{12}^1 &= 0\\
C_{11}^1 C_{21}^2 - C_{11}^1 C_{12}^2 &= 0\\
\underline{C_{12}^1 C_{21}^1} + \underline{C_{12}^2 C_{22}^1} &= 0\\
C_{12}^1 C_{21}^2 + \underline{C_{12}^2 C_{22}^2} &= 0\\
C_{21}^1 C_{21}^1 + \underline{C_{21}^2 C_{22}^1} - C_{12}^1 C_{12}^1 - \underline{C_{12}^2 C_{22}^1} &= 0\\
C_{21}^1 C_{21}^2 + \underline{C_{21}^2 C_{22}^2} - C_{12}^1 C_{12}^2 - \underline{C_{12}^2 C_{22}^2} &= 0\\
\underline{C_{21}^1 C_{12}^1} + \underline{C_{21}^2 C_{22}^1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
C^1_{21}C^2_{12} + C^2_{21}C^2_{22} &= 0\\
C^1_{22}C^1_{21} + C^2_{22}C^1_{22} &= 0\\
C^1_{22}C^2_{21} + C^2_{22}C^2_{22} &= 0\\
C^1_{22}C^1_{12} + C^2_{22}C^1_{22} &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

In the system of equations, we have again marked the terms that are present in multiple equations with one colour for each of the terms. Remember that multiplication with the constants is commutative. We want to use these terms that appear in multiple equations to simplify the system, by adding or subtracting certain equations from each other. This way, we will hopefully be able to rewrite the equations as products, which will make the system easier to solve.

We see that the third equation has a term equal to a term in the fifth equation, so we add the third equation to the fifth equation to eliminate the term in the fifth equation. In the same way, we add the fourth equation to the sixth. We also subtract the eleventh equation from the ninth, and subtract the twelfth equation from the eighth. This gives us

$$
\begin{aligned}
C^1_{11}C^1_{21} - C^1_{11}C^1_{12} &= 0\\
C^1_{11}C^2_{21} - C^1_{11}C^2_{12} &= 0\\
C^1_{12}C^1_{21} + C^2_{12}C^1_{22} &= 0\\
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0\\
C^1_{21}C^1_{21} + C^2_{21}C^1_{22} - C^1_{12}C^1_{12} + C^1_{12}C^1_{21} &= 0\\
C^1_{21}C^2_{21} + C^2_{21}C^2_{22} - C^1_{12}C^2_{12} + C^1_{12}C^2_{21} &= 0\\
C^1_{21}C^1_{12} + C^2_{21}C^1_{22} &= 0\\
C^1_{21}C^2_{12} + C^2_{21}C^2_{22} &= 0\\
C^1_{22}C^1_{21} - C^1_{22}C^1_{12} &= 0\\
C^1_{22}C^2_{21} - C^1_{22}C^2_{12} &= 0\\
C^1_{22}C^1_{12} + C^2_{22}C^1_{22} &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

We see that it now makes sense to subtract the seventh equation from the fifth, and in the same way we also subtract the eight equation from the sixth. We then – if we also remove the colour of the terms that now only exist in one place – end up with:

$$
\begin{aligned}
C^1_{11}C^1_{21} - C^1_{11}C^1_{12} &= 0\\
C^1_{11}C^2_{21} - C^1_{11}C^2_{12} &= 0\\
C^1_{12}C^1_{21} + C^2_{12}C^1_{22} &= 0\\
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0
\end{aligned}
$$

$$
\begin{aligned}
C^1_{21}C^1_{21} - C^1_{12}C^1_{12} + \underline{C^1_{12}C^1_{21}} - \underline{C^1_{21}C^1_{12}} &= 0 \\
C^1_{21}C^2_{21} - C^1_{12}C^2_{12} + C^1_{12}C^2_{21} - C^1_{21}C^2_{12} &= 0 \\
\underline{C^1_{21}C^1_{12}} + \underline{C^2_{21}C^1_{22}} &= 0 \\
C^1_{21}C^2_{12} + C^2_{21}C^2_{22} &= 0 \\
C^1_{22}C^1_{21} - C^1_{22}C^1_{12} &= 0 \\
\underline{C^1_{22}C^2_{21}} - \underline{C^1_{22}C^2_{12}} &= 0 \\
C^1_{22}C^1_{12} + C^2_{22}C^1_{22} &= 0 \\
\underline{C^1_{22}C^2_{12}} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

Note that we kept $\underline{C^1_{12}C^1_{21}} - \underline{C^1_{21}C^1_{12}}$ in the third equation. This is because if we choose to let them cancel each other out, we will end up with $C^1_{21}C^1_{21} - C^1_{12}C^1_{12} = 0$, while if we keep them instead we get $C^1_{21}C^1_{21} - C^1_{12}C^1_{12} + \underline{C^1_{12}C^1_{21}} - \underline{C^1_{21}C^1_{12}} = C^1_{21}C^1_{21} - \underline{C^1_{21}C^1_{12}} + \underline{C^1_{12}C^1_{21}} - C^1_{12}C^1_{12} = C^1_{21}(C^1_{21} - C^1_{12}) + C^1_{12}(C^1_{21} - C^1_{12}) = (C^1_{21} + C^1_{12})(C^1_{21} - C^1_{12}) = 0$, which will turn out to fit better with the other equations (as we can guess from α was defined as E_{11} and E_{12}). In the same way, we can rewrite the sixth equation as $C^1_{21}C^2_{21} - C^1_{12}C^2_{12} + C^1_{12}C^2_{21} - C^1_{21}C^2_{12} = C^1_{21}C^2_{21} - C^1_{21}C^2_{12} + C^1_{12}C^2_{21} - C^1_{12}C^2_{12} = C^1_{21}(C^2_{21} - C^2_{12}) + C^1_{12}(C^2_{21} - C^2_{12}) = (C^1_{21} + C^1_{12})(C^2_{21} - C^2_{12}) = 0$. We also rewrite the other equations that can be written as a product, to see which equations we do not have to simplify further. This gives us

$$
\begin{aligned}
C^1_{11}(C^1_{21} - C^1_{12}) &= 0 \\
C^1_{11}(C^2_{21} - C^2_{12}) &= 0 \\
\underline{C^1_{12}C^1_{21}} + \underline{C^2_{12}C^1_{22}} &= 0 \\
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0 \\
(C^1_{21} + C^1_{12})(C^1_{21} - C^1_{12}) &= 0 \\
(C^1_{21} + C^1_{12})(C^2_{21} - C^2_{12}) &= 0 \\
\underline{C^1_{21}C^1_{12}} + \underline{C^2_{21}C^1_{22}} &= 0 \\
C^1_{21}C^2_{12} + C^2_{21}C^2_{22} &= 0 \\
C^1_{22}(C^1_{21} - C^1_{12}) &= 0 \\
C^1_{22}(C^2_{21} - C^2_{12}) &= 0 \\
C^1_{22}(C^1_{12} + C^2_{22}) &= 0 \\
\underline{C^1_{22}C^2_{12}} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

Looking at the equations that cannot be rewritten as products, we see that the only fairly obvious thing we can do is add the tenth equation to the third. The third equation would then become $\underline{C^1_{12}C^1_{21}} + \underline{C^2_{12}C^1_{22}} + \underline{C^1_{22}C^2_{21}} - \underline{C^1_{22}C^2_{12}} = \underline{C^1_{12}C^1_{21}} + \underline{C^1_{22}C^2_{21}} = 0$. We see that this is exactly the seventh equation, and since we only need one copy of the same equation we can remove the third equation completely. Since we cannot see any other simply way to simplify the system of equations, the final system becomes:

$$
\begin{aligned}
C^1_{11}(C^1_{21} - C^1_{12}) &= 0\\
C^1_{11}(C^2_{21} - C^2_{12}) &= 0\\
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0\\
(C^1_{21} + C^1_{12})(C^1_{21} - C^1_{12}) &= 0\\
(C^1_{21} + C^1_{12})(C^2_{21} - C^2_{12}) &= 0\\
C^1_{21}C^1_{12} + C^2_{21}C^1_{22} &= 0\\
C^1_{21}C^2_{12} + C^2_{21}C^2_{22} &= 0\\
C^1_{22}(C^1_{21} - C^1_{12}) &= 0\\
C^1_{22}(C^2_{21} - C^2_{12}) &= 0\\
C^1_{22}(C^1_{12} + C^2_{22}) &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

We can now try to solve the system of equations. As we have done earlier, we have to look at cases. We begin by looking at the first equation, and seeing that the solution has to be either $C^1_{11} = 0$ or $C^1_{21} - C^1_{12} = 0$ – or both.

Case 1: If $C^1_{11} = 0$ the system of equations becomes

$$
\begin{aligned}
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0\\
(C^1_{21} + C^1_{12})(C^1_{21} - C^1_{12}) &= 0\\
(C^1_{21} + C^1_{12})(C^2_{21} - C^2_{12}) &= 0\\
C^1_{21}C^1_{12} + C^2_{21}C^1_{22} &= 0\\
C^1_{21}C^2_{12} + C^2_{21}C^2_{22} &= 0\\
C^1_{22}(C^1_{21} - C^1_{12}) &= 0\\
C^1_{22}(C^2_{21} - C^2_{12}) &= 0\\
C^1_{22}(C^1_{12} + C^2_{22}) &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

We now look at the sixth equation, and see that the solutions to this are $C^1_{22} = 0$ and $C^1_{21} - C^1_{12} = 0$.

Case 1.1: If $C^1_{22} = 0$, the system of equations now becomes

$$
\begin{aligned}
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0\\
(C^1_{21} + C^1_{12})(C^1_{21} - C^1_{12}) &= 0\\
(C^1_{21} + C^1_{12})(C^2_{21} - C^2_{12}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
C^1_{21}C^1_{12} &= 0\\
C^1_{21}C^2_{12} + C^2_{21}C^2_{22} &= 0\\
C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

From the last equation we get that we must have $C^2_{22} = 0$. This gives us

$$
\begin{aligned}
C^1_{12}C^2_{21} &= 0\\
(C^1_{21} + C^1_{12})(C^1_{21} - C^1_{12}) &= 0\\
(C^1_{21} + C^1_{12})(C^2_{21} - C^2_{12}) &= 0\\
C^1_{21}C^1_{12} &= 0\\
C^1_{21}C^2_{12} &= 0.
\end{aligned}
$$

From the second equation we get that we must have either $C^1_{21} = -C^1_{12}$ or $C^1_{21} = C^1_{12}$. In the first case the fourth equation becomes $-C^1_{12}C^1_{12} = 0$ and in the second case it becomes $C^1_{12}C^1_{12} = 0$. Clearly, the solution to both is $C^1_{12} = 0$, which gives us that we must also have $C^1_{21} = 0$. This solves all of the remaining equations in our system of equations.

Case 1.2: If we instead assume that $C^1_{22} \neq 0$ and $C^1_{21} - C^1_{12} = 0$, our system of equations becomes

$$
\begin{aligned}
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0\\
(C^1_{12} + C^1_{12})(C^2_{21} - C^2_{12}) &= 0\\
C^1_{12}C^1_{12} + C^2_{21}C^1_{22} &= 0\\
C^1_{12}C^2_{12} + C^2_{21}C^2_{22} &= 0\\
C^1_{22}(C^2_{21} - C^2_{12}) &= 0\\
C^1_{22}(C^1_{12} + C^2_{22}) &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

Since we assume that $C^1_{22} \neq 0$, the only solution to the fifth equation is $C^2_{21} = C^2_{12}$. This gives us

$$
\begin{aligned}
C^1_{12}C^2_{12} + C^2_{12}C^2_{22} &= 0\\
C^1_{12}C^1_{12} + C^2_{12}C^1_{22} &= 0\\
C^1_{12}C^2_{12} + C^2_{12}C^2_{22} &= 0\\
C^1_{22}(C^1_{12} + C^2_{22}) &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

For the same reason, the only solution to the fourth equation is $C^2_{22} = -C^1_{12}$. This gives us

$$
\begin{aligned}
C^1_{12}C^2_{12} - C^2_{12}C^1_{12} &= 0\\
C^1_{12}C^1_{12} + C^2_{12}C^1_{22} &= 0\\
C^1_{12}C^2_{12} - C^2_{12}C^1_{12} &= 0\\
C^1_{22}C^2_{12} + (-C^1_{12})(-C^1_{12}) &= 0.
\end{aligned}
$$

We see that the terms in the first and third equation (which are equal to each other) both cancel, so any value of the variables will solve the equations. We also see that the second and fourth equations are now equal, so we can remove one of them, and thus we have ended up with only one equation left to solve, $C^1_{12}C^1_{12} + C^2_{12}C^1_{22} = 0$. Since we have already assumed that $C^1_{22} \neq 0$, the solution to this equation is $C^2_{12} = -C^1_{12}C^1_{12}/C^1_{22}$, and we have now solved the system of equations.

Case 2: If we instead, from the beginning, have $C^1_{11} \neq 0$ and $C^1_{21} - C^1_{12} = 0$, the system of equations becomes

$$
\begin{aligned}
C^1_{11}(C^2_{21} - C^2_{12}) &= 0\\
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} &= 0\\
(C^1_{12} + C^1_{12})(C^2_{21} - C^2_{12}) &= 0\\
C^1_{12}C^1_{12} + C^2_{21}C^1_{22} &= 0\\
C^1_{12}C^2_{12} + C^2_{21}C^2_{22} &= 0\\
C^1_{22}(C^2_{21} - C^2_{12}) &= 0\\
C^1_{22}(C^1_{12} + C^2_{22}) &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

Since we have assumed that $C^1_{11} \neq 0$, we see that the only solution to the first equation is $C^2_{21} - C^2_{12} = 0$. This gives us

$$
\begin{aligned}
C^1_{12}C^2_{12} + C^2_{12}C^2_{22} &= 0\\
C^1_{12}C^1_{12} + C^2_{12}C^1_{22} &= 0\\
C^1_{12}C^2_{12} + C^2_{12}C^2_{22} &= 0\\
C^1_{22}(C^2_{12} - C^2_{12}) &= 0\\
C^1_{22}(C^1_{12} + C^2_{22}) &= 0\\
C^1_{22}C^2_{12} + C^2_{22}C^2_{22} &= 0.
\end{aligned}
$$

We see that any value of the variables will solve the fourth equation, so we can remove that equation. Moving on, we see that the solutions to the fifth equation are $C^1_{22} = 0$ and $C^1_{12} + C^2_{22} = 0$, so we will look at both of those cases.

Case 2.1: If $C_{22}^1 = 0$, the system of equations becomes

$$\begin{aligned} C_{12}^1 C_{12}^2 + C_{12}^2 C_{22}^2 &= 0 \\ C_{12}^1 C_{12}^1 &= 0 \\ C_{12}^1 C_{12}^2 + C_{12}^2 C_{22}^2 &= 0 \\ C_{22}^2 C_{22}^2 &= 0. \end{aligned}$$

We see immediately that the only solutions to the second and fourth equations are $C_{12}^1 = 0$ and $C_{22}^2 = 0$, respectively. This also solves the remaining two equations, which means we have now solved the system of equations.

Case 2.2: If we instead have $C_{22}^1 \neq 0$ and $C_{12}^1 + C_{22}^2 = 0$, the system of equations becomes

$$\begin{aligned} C_{12}^1 C_{12}^2 - C_{12}^2 C_{12}^1 &= 0 \\ C_{12}^1 C_{12}^1 + C_{12}^2 C_{22}^1 &= 0 \\ C_{12}^1 C_{12}^2 - C_{12}^2 C_{12}^1 &= 0 \\ C_{22}^1 C_{12}^2 + (-C_{12}^1)(-C_{12}^1) &= 0. \end{aligned}$$

The first and third equation are now clearly identical, and due to commutativity the second and fourth equation are as well. Thus, we can remove the third and fourth equation. However, due to commutativity the first (and the third) equation doesn't give us anything since any value of the variables will solve the equation, so we can remove that as well. Thus, only one equation remains to solve:

$$C_{12}^1 C_{12}^1 + C_{12}^2 C_{22}^1 = 0.$$

Since we have $C_{22}^1 \neq 0$ the solution to this equation will be $C_{12}^2 = -C_{12}^1 C_{12}^1 / C_{22}^1$. We have now solved the system of equation.

We have now found all possible solutions to the system of equations. We sum them up in Table 3.9.

Looking at the solutions, we see that if we set the free variable C_{11}^1 in solution 4 to 0, we get exactly solution 2. This means that the assumption that $C_{11}^1 = 0$ was unnecessary, and solution 2 is actually just a special case of solution 4, and can thus be removed. Our remaining solutions are then shown in Table 3.10.

Again, these hom-associative algebras are hom-Lie admissible with commutator $[\cdot,\cdot]$, so for each of these hom-Lie admissible algebras we must now find these commutators. Like we did before, we calculate $[e_i, e_j]$, where $i, j = \{1, 2\}$, for each of the solutions in Table 3.6, and since $[e_1, e_1] = [e_2, e_2] = 0$, we only have to calculate $[e_1, e_2]$ and $[e_2, e_1]$. We begin by looking at solution 1.

$$\begin{aligned} \mu(e_1, e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 = 0 e_1 + C_{12}^2 e_2 = C_{12}^2 e_2, \\ \mu(e_2, e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 = 0 e_1 + C_{21}^2 e_2 = C_{21}^2 e_2, \end{aligned}$$

	Soln 1	Soln 2	Soln 3	Soln 4
C_{11}^1	0	0	free	free
C_{11}^2	free	free	free	free
C_{12}^1	0	free	0	free
C_{12}^2	free	$-C_{12}^1C_{12}^1/C_{22}^1$	free	$-C_{12}^1C_{12}^1/C_{22}^1$
C_{21}^1	0	C_{12}^1	0	C_{12}^1
C_{21}^2	free	$-C_{12}^1C_{12}^1/C_{22}^1$	C_{12}^2	$-C_{12}^1C_{12}^1/C_{22}^1$
C_{22}^1	0	$C_{22}^1 \neq 0$	0	$C_{22}^1 \neq 0$
C_{22}^2	0	$-C_{12}^1$	0	$-C_{12}^1$

Table 3.9: The values of the structure constants that give hom-associative algebras when α is defined as E_{21} in two dimensions

	Soln 1	Soln 3	Soln 4
C_{11}^1	0	free	free
C_{11}^2	free	free	free
C_{12}^1	0	0	free
C_{12}^2	free	free	$-C_{12}^1C_{12}^1/C_{22}^1$
C_{21}^1	0	0	C_{12}^1
C_{21}^2	free	C_{12}^2	$-C_{12}^1C_{12}^1/C_{22}^1$
C_{22}^1	0	0	$C_{22}^1 \neq 0$
C_{22}^2	0	0	$-C_{12}^1$

Table 3.10: The values of the structure constants that give hom-associative algebras when α is defined as E_{21} in two dimensions, with the special cases removed

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^2 e_2 - C_{21}^2 e_2 = (C_{12}^2 - C_{21}^2)e_2,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = C_{21}^2 e_2 - C_{12}^2 e_2 = (C_{21}^2 - C_{12}^2)e_2.$$

We can then move on to solution 3.

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = 0e_1 + C_{12}^2 e_2 = C_{12}^2 e_2,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = 0e_1 + C_{21}^2 e_2 = C_{12}^2 e_2,$$

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^2 e_2 - C_{12}^2 e_2 = 0,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = C_{12}^2 e_2 - C_{12}^2 e_2 = 0.$$

Finally, we look at solution 4:

$$\mu(e_1,e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = C_{12}^1 e_1 - (C_{12}^2 C_{12}^2 / C_{22}^1) e_2,$$
$$\mu(e_2,e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = C_{12}^1 e_1 - (C_{12}^2 C_{12}^2 / C_{22}^1) e_2,$$

which gives us

$$[e_1,e_2] = \mu(e_1,e_2) - \mu(e_2,e_1) = C_{12}^1 e_1 - (C_{12}^2 C_{12}^2 / C_{22}^1) e_2 - (C_{12}^1 e_1 - (C_{12}^2 C_{12}^2 / C_{22}^1) e_2) = 0,$$
$$[e_2,e_1] = \mu(e_2,e_1) - \mu(e_1,e_2) = C_{12}^1 e_1 - (C_{12}^2 C_{12}^2 / C_{22}^1) e_2 - (C_{12}^1 e_1 - (C_{12}^2 C_{12}^2 / C_{22}^1) e_2) = 0.$$

This means that from our three hom-associative algebras (V, μ, E_{21}), we were able to get three different hom-Lie admissible algebras $(V, [\cdot,\cdot], E_{21})$, where $[\cdot,\cdot]$ is defined as in Table 3.11 for the structure constants given in solution 1 above, or Table 3.12 for the structure constants given in solution 3 or 4 above.

$[\cdot,\cdot]$	e_1	e_2
e_1	0	$(C_{12}^2 - C_{21}^2)e_2$
e_2	$(C_{21}^2 - C_{12}^2)e_2$	0

Table 3.11: The commutator table for the hom-Lie admissible algebras with α defined as E_{21} and the structure constants defined as in solution 1 in Table 3.10

$[\cdot,\cdot]$	e_1	e_2
e_1	0	0
e_2	0	0

Table 3.12: The commutator table for the hom-Lie admissible algebras with α defined as E_{21} and the structure constants defined as in either solution 3 or solution 4 in Table 3.10

3.1.4 α defined as E_{22}

We now let

$$[\alpha] = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Again using formulas (3.9) and (3.10), we get

$$\alpha(e_1) = 0e_1 + 0e_2 = 0,$$
$$\alpha(e_2) = 0e_1 + 1e_2 = e_2.$$

Using this, we can rewrite all of our equalities as follows:

$$
\begin{aligned}
&(3.1): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0,\\
&(3.2): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, e_2) \quad \Rightarrow \quad 0 = \mu(C_{11}^1 e_1 + C_{11}^2 e_2, e_2),\\
&(3.3): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, 0) \quad \Rightarrow \quad 0 = 0,\\
&(3.4): \quad \mu(e_2, C_{11}^1 e_1 + C_{11}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_2, C_{11}^1 e_1 + C_{11}^2 e_2) = 0,\\
&(3.5): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, e_2) \quad \Rightarrow \quad 0 = \mu(C_{12}^1 e_1 + C_{12}^2 e_2, e_2),\\
&(3.6): \quad \mu(e_2, C_{12}^1 e_1 + C_{12}^2 e_2) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2, e_2),\\
&(3.7): \quad \mu(e_2, C_{21}^1 e_1 + C_{21}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, 0) \quad \Rightarrow \quad \mu(e_2, C_{21}^1 e_1 + C_{21}^2 e_2) = 0,\\
&(3.8): \quad \mu(e_2, C_{22}^1 e_1 + C_{22}^2 e_2) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2, e_2).
\end{aligned}
$$

We now use formulas (3.12) and (3.13) again, but this time we note that we again always have $i = 2$. Using these formulas, our equalities become:

$$
\begin{aligned}
&(3.2): \quad 0 = (C_{11}^1 C_{12}^1 + C_{11}^2 C_{22}^1) e_1 + (C_{11}^1 C_{12}^2 + C_{11}^2 C_{22}^2) e_2,\\
&(3.4): \quad (C_{11}^1 C_{21}^1 + C_{11}^2 C_{22}^1) e_1 + (C_{11}^1 C_{21}^2 + C_{11}^2 C_{22}^2) e_2 = 0,\\
&(3.5): \quad 0 = (C_{12}^1 C_{12}^1 + C_{12}^2 C_{22}^1) e_1 + (C_{12}^1 C_{12}^2 + C_{12}^2 C_{22}^2) e_2,\\
&(3.6): \quad (C_{12}^1 C_{21}^1 + C_{12}^2 C_{22}^1) e_1 + (C_{12}^1 C_{21}^2 + C_{12}^2 C_{22}^2) e_2 = (C_{21}^1 C_{12}^1 + C_{21}^2 C_{22}^1) e_1\\
&\qquad\qquad + (C_{21}^1 C_{12}^2 + C_{21}^2 C_{22}^2) e_2\\
&\qquad \Rightarrow \quad (C_{12}^1 C_{21}^1 + C_{12}^2 C_{22}^1 - C_{21}^1 C_{12}^1 - C_{21}^2 C_{22}^1) e_1\\
&\qquad\qquad + (C_{12}^1 C_{21}^2 + C_{12}^2 C_{22}^2 - C_{21}^1 C_{12}^2 - C_{21}^2 C_{22}^2) e_2 = 0,\\
&(3.7): \quad (C_{21}^1 C_{21}^1 + C_{21}^2 C_{22}^1) e_1 + (C_{21}^1 C_{21}^2 + C_{21}^2 C_{22}^2) e_2 = 0,\\
&(3.8): \quad (C_{22}^1 C_{21}^1 + C_{22}^2 C_{22}^1) e_1 + (C_{22}^1 C_{21}^2 + C_{22}^2 C_{22}^2) e_2 = (C_{22}^1 C_{12}^1 + C_{22}^2 C_{22}^1) e_1\\
&\qquad\qquad + (C_{22}^1 C_{12}^2 + C_{22}^2 C_{22}^2) e_2\\
&\qquad \Rightarrow \quad C_{22}^1 C_{21}^1 e_1 + C_{22}^1 C_{21}^2 e_2 = C_{22}^1 C_{12}^1 e_1 + C_{22}^1 C_{12}^2 e_2\\
&\qquad \Rightarrow \quad (C_{22}^1 C_{21}^1 - C_{22}^1 C_{12}^1) e_1 + (C_{22}^1 C_{21}^2 - C_{22}^1 C_{12}^2) e_2 = 0.
\end{aligned}
$$

Once again remembering that $k_1 e_1 + k_1 e_2 = 0$ will only have the trivial solution $k_1 = k_2 = 0$, this gives us the system of equations

$$
\begin{aligned}
C_{11}^1 C_{12}^1 + \underline{C_{11}^2 C_{22}^1} &= 0\\
C_{11}^1 C_{12}^2 + \underline{C_{11}^2 C_{22}^2} &= 0\\
C_{11}^1 C_{21}^1 + \underline{C_{11}^2 C_{22}^1} &= 0\\
C_{11}^1 C_{21}^2 + \underline{C_{11}^2 C_{22}^2} &= 0\\
C_{12}^1 C_{12}^1 + \underline{C_{12}^2 C_{22}^1} &= 0\\
C_{12}^1 C_{12}^2 + \underline{C_{12}^2 C_{22}^2} &= 0\\
\underline{C_{12}^1 C_{21}^1} + \underline{C_{12}^2 C_{22}^1} - \underline{C_{21}^1 C_{12}^1} - \underline{C_{21}^2 C_{22}^1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} - C^1_{21}C^2_{12} - C^2_{21}C^2_{22} &= 0 \\
C^1_{21}C^1_{21} + C^2_{21}C^1_{22} &= 0 \\
C^1_{21}C^2_{21} + C^2_{21}C^2_{22} &= 0 \\
C^1_{22}C^1_{21} - C^1_{22}C^1_{12} &= 0 \\
C^1_{22}C^2_{21} - C^1_{22}C^2_{12} &= 0.
\end{aligned}
$$

We now want to simplify the system of equations, which we do by using the terms that appear in more than one of the equations, marked with different colours. To begin, we subtract the third equation from the first equation, and the fourth equation from the second. We also see that we can simplify the seventh equation due to commutativity. This gives us

$$
\begin{aligned}
C^1_{11}C^1_{12} - C^1_{11}C^1_{21} &= 0 \\
C^1_{11}C^2_{12} - C^1_{11}C^2_{21} &= 0 \\
C^1_{11}C^1_{21} + C^2_{11}C^1_{22} &= 0 \\
C^1_{11}C^2_{21} + C^2_{11}C^2_{22} &= 0 \\
C^1_{12}C^1_{12} + C^2_{12}C^1_{22} &= 0 \\
C^1_{12}C^2_{12} + C^2_{12}C^2_{22} &= 0 \\
C^2_{12}C^1_{22} - C^2_{21}C^1_{22} &= 0 \\
C^1_{12}C^2_{21} + C^2_{12}C^2_{22} - C^1_{21}C^2_{12} - C^2_{21}C^2_{22} &= 0 \\
C^1_{21}C^1_{21} + C^2_{21}C^1_{22} &= 0 \\
C^1_{21}C^2_{21} + C^2_{21}C^2_{22} &= 0 \\
C^1_{22}C^1_{21} - C^1_{22}C^1_{12} &= 0 \\
C^1_{22}C^2_{21} - C^1_{22}C^2_{12} &= 0.
\end{aligned}
$$

We see that the seventh equation is now identical to the negative of the last equation, which means we can rewrite it to be exactly the last equation, and thus we can remove it since we only need one of the same equation. Then, we subtract the sixth equation from and add the twelfth equation to the eighth equation. After this, we end up with

$$
\begin{aligned}
C^1_{11}C^1_{12} - C^1_{11}C^1_{21} &= 0 \\
C^1_{11}C^2_{12} - C^1_{11}C^2_{21} &= 0 \\
C^1_{11}C^1_{21} + C^2_{11}C^1_{22} &= 0 \\
C^1_{11}C^2_{21} + C^2_{11}C^2_{22} &= 0 \\
C^1_{12}C^1_{12} + C^2_{12}C^1_{22} &= 0 \\
C^1_{12}C^2_{12} + C^2_{12}C^2_{22} &= 0 \\
C^1_{12}C^2_{21} - C^1_{21}C^2_{12} - C^1_{12}C^2_{12} + C^1_{21}C^2_{21} &= 0 \\
C^1_{21}C^1_{21} + C^2_{21}C^1_{22} &= 0
\end{aligned}
$$

$$C^1_{21}C^2_{21} + C^2_{21}C^2_{22} = 0$$
$$C^1_{22}C^1_{21} - C^1_{22}C^1_{12} = 0$$
$$C^1_{22}C^2_{21} - C^1_{22}C^2_{12} = 0.$$

If we look at the equations that still have common terms, there doesn't seem to be an obvious way to simplify the system further, so we remove the colours and rewrite as many equations as possible into products. Our system of equations the ends up being as follows:

$$C^1_{11}(C^1_{12} - C^1_{21}) = 0$$
$$C^1_{11}(C^2_{12} - C^2_{21}) = 0$$
$$C^1_{11}C^1_{21} + C^2_{11}C^1_{22} = 0$$
$$C^1_{11}C^2_{21} + C^2_{11}C^2_{22} = 0$$
$$C^1_{12}C^1_{12} + C^2_{12}C^1_{22} = 0$$
$$C^1_{12}C^2_{12} + C^2_{12}C^2_{22} = 0$$
$$(C^1_{12} + C^1_{21})(C^2_{21} - C^2_{12}) = 0$$
$$C^1_{21}C^1_{21} + C^2_{21}C^1_{22} = 0$$
$$C^2_{21}(C^1_{21} + C^2_{22}) = 0$$
$$C^1_{22}(C^1_{21} - C^1_{12}) = 0$$
$$C^1_{22}(C^2_{21} - C^2_{12}) = 0.$$

We can now try to solve the system of equations. First, we see that the solutions to the last equation are $C^1_{22} = 0$ and $C^2_{21} - C^2_{12} = 0$. We look at each of those cases.

Case 1: If $C^1_{22} = 0$, the system of equations becomes

$$C^1_{11}(C^1_{12} - C^1_{21}) = 0$$
$$C^1_{11}(C^2_{12} - C^2_{21}) = 0$$
$$C^1_{11}C^1_{21} = 0$$
$$C^1_{11}C^2_{21} + C^2_{11}C^2_{22} = 0$$
$$C^1_{12}C^1_{12} = 0$$
$$C^1_{12}C^2_{12} + C^2_{12}C^2_{22} = 0$$
$$(C^1_{12} + C^1_{21})(C^2_{21} - C^2_{12}) = 0$$
$$C^1_{21}C^1_{21} = 0$$
$$C^2_{21}(C^1_{21} + C^2_{22}) = 0.$$

Clearly the only possible solution to the fifth equation is $C^1_{12} = 0$, and the only solution to the eighth equation is $C^1_{21} = 0$. Inserting these solutions into the system of equations gives us:

$$\begin{aligned} C^1_{11}(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{11}C^2_{21} + C^2_{11}C^2_{22} &= 0 \\ C^2_{12}C^2_{22} &= 0 \\ C^2_{21}(C^2_{22}) &= 0. \end{aligned}$$

To solve the third and fourth equation, we must have either $C^2_{22} = 0$ or both $C^2_{12} = 0$ and $C^2_{21} = 0$.

Case 1.1: If $C^2_{22} = 0$, the system of equations becomes

$$\begin{aligned} C^1_{11}(C^2_{12} - C^2_{21}) &= 0 \\ C^1_{11}C^2_{21} &= 0. \end{aligned}$$

We see that one possible solution to this is $C^1_{11} = 0$. The only other possible solution is both $C^2_{12} - C^2_{21} = 0$ and $C^2_{21} = 0$ – which of course means that $C^2_{12} = C^2_{21} = 0$. Thus, we have now solved the entire system of equations.

Case 1.2: If we instead have $C^2_{22} \neq 0$, $C^2_{12} = 0$ and $C^2_{21} = 0$, the system of equations becomes

$$C^2_{11}C^2_{22} = 0.$$

Since we have assumed that $C^2_{22} \neq 0$, the only possible solution to this equation it $C^2_{11} = 0$, and we have now solved the entire system of equations.

Case 2: If instead $C^1_{22} \neq 0$ and $C^2_{21} - C^2_{12} = 0$, the system of equations becomes

$$\begin{aligned} C^1_{11}(C^1_{12} - C^1_{21}) &= 0 \\ C^1_{11}C^1_{21} + C^2_{11}C^1_{22} &= 0 \\ C^1_{11}C^2_{12} + C^2_{11}C^2_{22} &= 0 \\ C^1_{12}C^1_{12} + C^2_{12}C^1_{22} &= 0 \\ C^1_{12}C^2_{12} + C^2_{12}C^2_{22} &= 0 \\ C^1_{21}C^1_{21} + C^2_{12}C^1_{22} &= 0 \\ C^2_{12}(C^1_{21} + C^2_{22}) &= 0 \\ C^1_{22}(C^1_{21} - C^1_{12}) &= 0. \end{aligned}$$

Now, since $C^1_{22} \neq 0$, the only possible solution to the last equations is $C^1_{21} - C^1_{12} = 0$.

This gives us

$$\begin{aligned}
C^1_{11}C^1_{12}+C^2_{11}C^1_{22} &= 0\\
C^1_{11}C^2_{12}+C^2_{11}C^2_{22} &= 0\\
C^1_{12}C^1_{12}+C^2_{12}C^1_{22} &= 0\\
C^1_{12}C^2_{12}+C^2_{12}C^2_{22} &= 0\\
C^1_{12}C^1_{12}+C^2_{12}C^1_{22} &= 0\\
C^2_{12}(C^1_{12}+C^2_{22}) &= 0.
\end{aligned}$$

Looking at the last equation, we see that the two possible solutions are $C^2_{12} = 0$ and $C^1_{12}+C^2_{22} = 0$.

Case 2.1: If $C^2_{12} = 0$, the system of equations becomes

$$\begin{aligned}
C^1_{11}C^1_{12}+C^2_{11}C^1_{22} &= 0\\
C^2_{11}C^2_{22} &= 0\\
C^1_{12}C^1_{12} &= 0\\
C^1_{12}C^1_{12} &= 0.
\end{aligned}$$

The only solution to the last two (identical) equations is clearly $C^1_{12} = 0$. This gives us

$$\begin{aligned}
C^2_{11}C^1_{22} &= 0\\
C^2_{11}C^2_{22} &= 0.
\end{aligned}$$

Now, since we have assumed that $C^1_{22} \neq 0$, the only solution to the first equations is $C^2_{11} = 0$. This is also a solution to the second equation, so we have now solved the entire system of equations.

Case 2.2: If instead $C^2_{12} \neq 0$ and $C^1_{12}+C^2_{22} = 0$, the system of equations becomes

$$\begin{aligned}
C^1_{11}C^1_{12}+C^2_{11}C^1_{22} &= 0\\
C^1_{11}C^2_{12}-C^2_{11}C^1_{12} &= 0\\
C^1_{12}C^1_{12}+C^2_{12}C^1_{22} &= 0\\
C^1_{12}C^2_{12}-C^2_{12}C^1_{12} &= 0\\
C^1_{12}C^1_{12}+C^2_{12}C^1_{22} &= 0.
\end{aligned}$$

First of all, we see that we can now remove the fifth equation since it is identical to the third equation. Second of all, the fourth equation can be removed as well since the left hand side will be equal to zero regardless of the values of the structure constants, due to commutativity. Furthermore, we use the fact that we have assumed that $C^1_{22} \neq 0$ to write the solutions of equations 1 and 3 as $C_{11}^2 = -C_{11}^1 C_{12}^1 / C_{22}^1$ and $C_{12}^2 = -C_{12}^1 C_{12}^1 / C_{22}^1$, respectively. Plugging these into the second equation, we get $C_{11}^1 C_{12}^2 - C_{11}^2 C_{12}^1 = -C_{11}^1 C_{12}^1 C_{12}^1 / C_{22}^1 + C_{11}^1 C_{12}^1 C_{12}^1 / C_{22}^1 = 0$. Clearly, the solutions to the first and third equations were a solution to the second equation as well. Thus, we have now solved the entire system of equations.

We have now found all possible solutions to the system of equations, and we write all of these solutions down in Table 3.13 for an easy overview.

	Soln 1	Soln 2	Soln 3	Soln 4	Soln 5
C_{11}^1	0	free	free	free	free
C_{11}^2	free	free	0	0	$-C_{11}^1 C_{12}^1 / C_{22}^1$
C_{12}^1	0	0	0	0	free
C_{12}^2	free	0	0	0	$-C_{12}^1 C_{12}^1 / C_{22}^1$
C_{21}^1	0	0	0	0	C_{12}^1
C_{21}^2	free	0	0	0	$-C_{12}^1 C_{12}^1 / C_{22}^1$
C_{22}^1	0	0	0	free	$C_{22}^1 \neq 0$
C_{22}^2	0	0	free	free	$-C_{12}^1$

Table 3.13: The values of the structure constants that give hom-associative algebras when α is defined as E_{22} in two dimensions

Looking at the solutions, we see that if we set C_{22}^1 to 0 in solution 4 we get solution 3, which means that solution 3 is just a special case of solution 4, and we can remove it. This leaves us with the solutions shown in Table 3.14.

	Soln 1	Soln 2	Soln 4	Soln 5
C_{11}^1	0	free	free	free
C_{11}^2	free	free	0	$-C_{11}^1 C_{12}^1 / C_{22}^1$
C_{12}^1	0	0	0	free
C_{12}^2	free	0	0	$-C_{12}^1 C_{12}^1 / C_{22}^1$
C_{21}^1	0	0	0	C_{12}^1
C_{21}^2	free	0	0	$-C_{12}^1 C_{12}^1 / C_{22}^1$
C_{22}^1	0	0	free	$C_{22}^1 \neq 0$
C_{22}^2	0	0	free	$-C_{12}^1$

Table 3.14: The values of the structure constants that give hom-associative algebras when α is defined as E_{22} in two dimensions, with the special cases removed

Now, we want to find the commutators for the hom-Lie admissible algebras that we get from the hom-associative algebras found above, just like we did before. Again, we must calculate $[e_i, e_j]$, where $i, j = \{1,2\}$, for each of the solutions in Table 3.6, and since $[e_1, e_1] = [e_2, e_2] = 0$, we only have to calculate $[e_1, e_2]$ and $[e_2, e_1]$. To begin, we look at solution 1:

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = 0e_1 + C_{12}^2 e_2 = C_{12}^2 e_2,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = 0e_1 + C_{21}^2 e_2 = C_{21}^2 e_2,$$

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^2 e_2 - C_{21}^2 e_2 = (C_{12}^2 - C_{21}^2) e_2,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = C_{21}^2 e_2 - C_{12}^2 e_2 = (C_{21}^2 - C_{12}^2) e_2.$$

We can then look at solution 2.

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = 0e_1 + 0e_2 = 0,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = 0e_1 + 0e_2 = 0,$$

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = 0 - 0 = 0,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = 0 - 0 = 0.$$

Moving on, we look at solution 4.

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = 0e_1 + 0e_2 = 0,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = 0e_1 + 0e_2 = 0,$$

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = 0 - 0 = 0,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = 0 - 0 = 0.$$

Finally, we look at solution 5.

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = C_{12}^1 e_1 - (C_{12}^1 C_{12}^1 / C_{22}^1) e_2,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = C_{12}^1 e_1 - (C_{12}^1 C_{12}^1 / C_{22}^1) e_2,$$

which gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^1 e_1 + C_{12}^2 e_2 - (C_{12}^1 e_1 + C_{12}^2 e_2) = 0,$$
$$[e_2, e_1] = \mu(e_2, e_1) - \mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 - (C_{12}^1 e_1 + C_{12}^2 e_2) = 0.$$

This means that from our four hom-associative algebras (V, μ, E_{22}), we were able to get four different hom-Lie admissible algebras $(V, [\cdot,\cdot], E_{22})$, where $[\cdot,\cdot]$ is defined as in Table 3.15 for the structure constants given in solution1 above, or Table 3.16 for the structure constants given in solutions 2, 4 and 5 above.

We note that these commutator tables are actually exactly the same as the ones for α defined as E_{21}, given in Table 3.11 and Table 3.12.

$[\cdot,\cdot]$	e_1	e_2
e_1	0	$(C^2_{12}-C^2_{21})e_2$
e_2	$(C^2_{21}-C^2_{12})e_2$	0

Table 3.15: The commutator table for the hom-Lie admissible algebras with α defined as E_{22} and the structure constants defined as in solution 1 in Table 3.14

$[\cdot,\cdot]$	e_1	e_2
e_1	0	0
e_2	0	0

Table 3.16: The commutator table for the hom-Lie admissible algebras with α defined as E_{22} and the structure constants defined as in either solution 2, solution 4, or solution 5 in Table 3.14

3.1.5 Comparison

We see that for the hom-associative algebras, we get the same results for E_{11} and E_{22} if you permutate all of the indices of the structure constants by the permutation which in cyclic notation can be written as $\sigma_2 = (1,2)$. Looking back at the systems of equations we solved for each α, we see that this is actually because if you permutate the system of equations for either E_{11} or E_{22} by the same permutation σ_2, you get the other system of equations exactly. Looking instead at E_{12} and E_{21}, we see that their systems of equations – and hence also their results – actually have the same relationship, with the same permutation σ_2. From this we hypothesise that this is because E_{11} and E_{22} are the α:s with a one on the main diagonal, and E_{12} and E_{21} the ones with a one on the anti-diagonal. If this is the case, we should see this relationship between the α:s with ones on the same diagonals in higher dimensions as well, so we will investigate this in the next section, when we look at three dimensions.

Moving on to the hom-Lie admissible algebras, we see that we get the same commutator tables for $[\alpha] = E_{11}$ (Table 3.3 and Table 3.4) and $[\alpha] = E_{12}$ (Table 3.7 and Table 3.8), and we also get the same commutator tables for $[\alpha] = E_{21}$ (Table 3.11 and Table 3.12) and $[\alpha] = E_{22}$ (Table 3.15 and Table 3.16). One table they all have in common is the one with only zeroes; that is, the commutator belonging to the abelian hom-Lie admissible algebras. This is because to get this table, all we need is to have $[e_1,e_2] = \mu(e_1,e_2) - \mu(e_2,e_1) = (C^1_{12}e_1 + C^2_{12}e_2) - (C^1_{21}e_1 + C^2_{21}e_2) = (C^1_{12} - C^1_{21})e_1 + (C^2_{12} - C^2_{21})e_2 = 0$ (since $[e_1,e_1] = 0$ and $[e_2,e_2] = 0$ by definition, and $[e_2,e_1] = -[e_1,e_2]$). From α defined as each of our two-dimensional matrix units it turns out that we get at least one hom-Lie admissible algebra for which $C^1_{12} = C^1_{21}$ and $C^2_{12} = C^2_{21}$, which is why the zero commutator table appears in all four cases. In fact, for $[\alpha] = E_{11}$ and $[\alpha] = E_{22}$ three out of four of their respective hom-Lie admissible algebras have a zero commutator table, while for $[\alpha] = E_{12}$ and $[\alpha] = E_{21}$ two of our their three respective hom-Lie admissible algebras have a zero commutator table.

Now, what about the commutator tables that are not equal to zero? We see that for $[\alpha] = E_{11}$, the one hom-Lie admissible algebra that has a commutator table that is not all zero actually has the exact same values of all structure constants as the one hom-Lie admissible algebra for $[\alpha] = E_{12}$ that has a commutator table that is not all zero. This, of course, explains why the one commutator table that is not all zero is identical for these two cases. The exact same thing is the case for $[\alpha] = E_{21}$ and $[\alpha] = E_{22}$. However, even though all of the structure constants have the same values for both different hom-Lie admissible algebras in this case, it is not necessary in order to get the same commutator table; only C_{12}^1, C_{12}^2, C_{21}^1 and C_{21}^2 actually affect the commutator tables, so as long as two different hom-Lie admissible algebras (in two dimensions) have the same value of C_{12}^1 as each other, the same value of C_{12}^2 as each other, the same value of C_{21}^1 as each other, and the same value of C_{21}^2 as each other, they will end up with the exact same commutator tables.

3.2 Dimension 3

We now move on to looking at hom-associative algebras in dimension 3. We again remember from Definition 1 that (V, μ, α) needs to satisfy

$$\mu(\alpha(x), \mu(y,z)) = \mu(\mu(x,y), \alpha(z))$$

for all $x, y, z \in V$. Since we now have $\dim V = 3$, we let $\{e_1, e_2, e_3\}$ be a basis for V, in the same way as we did in dimension 2. Every element in V can be written as a linear combination of the basis elements, and thus, since μ is bilinear, we know that the above equality will hold for all elements in V if it holds for all combinations of the basis elements. That is, we need all of the following equalities to hold:

$$\mu(\alpha(e_1), \mu(e_1,e_1)) = \mu(\mu(e_1,e_1), \alpha(e_1)), \tag{3.14}$$
$$\mu(\alpha(e_1), \mu(e_1,e_2)) = \mu(\mu(e_1,e_1), \alpha(e_2)), \tag{3.15}$$
$$\mu(\alpha(e_1), \mu(e_1,e_3)) = \mu(\mu(e_1,e_1), \alpha(e_3)), \tag{3.16}$$
$$\mu(\alpha(e_1), \mu(e_2,e_1)) = \mu(\mu(e_1,e_2), \alpha(e_1)), \tag{3.17}$$
$$\mu(\alpha(e_1), \mu(e_3,e_1)) = \mu(\mu(e_1,e_3), \alpha(e_1)), \tag{3.18}$$
$$\mu(\alpha(e_1), \mu(e_2,e_2)) = \mu(\mu(e_1,e_2), \alpha(e_2)), \tag{3.19}$$
$$\mu(\alpha(e_1), \mu(e_2,e_3)) = \mu(\mu(e_1,e_2), \alpha(e_3)), \tag{3.20}$$
$$\mu(\alpha(e_1), \mu(e_3,e_2)) = \mu(\mu(e_1,e_3), \alpha(e_2)), \tag{3.21}$$
$$\mu(\alpha(e_1), \mu(e_3,e_3)) = \mu(\mu(e_1,e_3), \alpha(e_3)), \tag{3.22}$$
$$\mu(\alpha(e_2), \mu(e_1,e_1)) = \mu(\mu(e_2,e_1), \alpha(e_1)), \tag{3.23}$$
$$\mu(\alpha(e_2), \mu(e_1,e_2)) = \mu(\mu(e_2,e_1), \alpha(e_2)), \tag{3.24}$$
$$\mu(\alpha(e_2), \mu(e_1,e_3)) = \mu(\mu(e_2,e_1), \alpha(e_3)), \tag{3.25}$$
$$\mu(\alpha(e_2), \mu(e_2,e_1)) = \mu(\mu(e_2,e_2), \alpha(e_1)), \tag{3.26}$$
$$\mu(\alpha(e_2), \mu(e_3,e_1)) = \mu(\mu(e_2,e_3), \alpha(e_1)), \tag{3.27}$$
$$\mu(\alpha(e_2), \mu(e_2,e_2)) = \mu(\mu(e_2,e_2), \alpha(e_2)), \tag{3.28}$$
$$\mu(\alpha(e_2), \mu(e_2,e_3)) = \mu(\mu(e_2,e_2), \alpha(e_3)), \tag{3.29}$$
$$\mu(\alpha(e_2), \mu(e_3,e_2)) = \mu(\mu(e_2,e_3), \alpha(e_2)), \tag{3.30}$$

$$\mu(\alpha(e_2),\mu(e_3,e_3)) = \mu(\mu(e_2,e_3),\alpha(e_3)), \tag{3.31}$$
$$\mu(\alpha(e_3),\mu(e_1,e_1)) = \mu(\mu(e_3,e_1),\alpha(e_1)), \tag{3.32}$$
$$\mu(\alpha(e_3),\mu(e_1,e_2)) = \mu(\mu(e_3,e_1),\alpha(e_2)), \tag{3.33}$$
$$\mu(\alpha(e_3),\mu(e_1,e_3)) = \mu(\mu(e_3,e_1),\alpha(e_3)), \tag{3.34}$$
$$\mu(\alpha(e_3),\mu(e_2,e_1)) = \mu(\mu(e_3,e_2),\alpha(e_1)), \tag{3.35}$$
$$\mu(\alpha(e_3),\mu(e_3,e_1)) = \mu(\mu(e_3,e_3),\alpha(e_1)), \tag{3.36}$$
$$\mu(\alpha(e_3),\mu(e_2,e_2)) = \mu(\mu(e_3,e_2),\alpha(e_2)), \tag{3.37}$$
$$\mu(\alpha(e_3),\mu(e_2,e_3)) = \mu(\mu(e_3,e_2),\alpha(e_3)), \tag{3.38}$$
$$\mu(\alpha(e_3),\mu(e_3,e_2)) = \mu(\mu(e_3,e_3),\alpha(e_2)), \tag{3.39}$$
$$\mu(\alpha(e_3),\mu(e_3,e_3)) = \mu(\mu(e_3,e_3),\alpha(e_3)). \tag{3.40}$$

We have

$$\alpha(e_i) = \sum_{j=1}^{3} a_{ji}e_j = a_{1i}e_1 + a_{2i}e_2 + a_{3i}e_3,$$

and thus we get

$$\alpha(e_1) = a_{11}e_1 + a_{21}e_2 + a_{31}e_3, \tag{3.41}$$
$$\alpha(e_2) = a_{12}e_1 + a_{22}e_2 + a_{32}e_3, \tag{3.42}$$
$$\alpha(e_3) = a_{13}e_1 + a_{23}e_2 + a_{33}e_3. \tag{3.43}$$

Furthermore, we have

$$\mu(e_i,e_j) = \sum_{k=1}^{3} C_{ij}^k e_k = C_{ij}^1 e_1 + C_{ij}^2 e_2 + C_{ij}^3 e_3,$$

which gives us

$$\begin{aligned}
\mu(e_1,e_1) &= C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3,\\
\mu(e_1,e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3,\\
\mu(e_1,e_3) &= C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3,\\
\mu(e_2,e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3,\\
\mu(e_2,e_2) &= C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3,\\
\mu(e_2,e_3) &= C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3,\\
\mu(e_3,e_1) &= C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3,\\
\mu(e_3,e_2) &= C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3,\\
\mu(e_3,e_3) &= C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3.
\end{aligned}$$

3.2.1 α defined as E_{11}

Just as we did in the two-dimensional case, we want to investigate what the hom-associative algebras will look like for different α. We begin with

$$[\alpha] = E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using formulas (3.41), (3.42) and (3.43), we get

$$\alpha(e_1) = 1e_1 + 0e_2 + 0e_3 = e_1, \tag{3.44}$$
$$\alpha(e_2) = 0e_1 + 0e_2 + 0e_3 = 0, \tag{3.45}$$
$$\alpha(e_3) = 0e_1 + 0e_2 + 0e_3 = 0. \tag{3.46}$$

We can thus rewrite our equalities as follows:

$$\begin{aligned}
(3.14):&\quad \mu(e_1, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, e_1),\\
(3.15):&\quad \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0)\\
&\quad \Rightarrow\ \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = 0,\\
(3.16):&\quad \mu(e_1, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0)\\
&\quad \Rightarrow\ \mu(e_1, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = 0,\\
(3.17):&\quad \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_1),\\
(3.18):&\quad \mu(e_1, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_1),\\
(3.19):&\quad \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0)\\
&\quad \Rightarrow\ \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = 0,\\
(3.20):&\quad \mu(e_1, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0)\\
&\quad \Rightarrow\ \mu(e_1, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = 0,\\
(3.21):&\quad \mu(e_1, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0)\\
&\quad \Rightarrow\ \mu(e_1, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = 0,\\
(3.22):&\quad \mu(e_1, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0)\\
&\quad \Rightarrow\ \mu(e_1, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = 0,\\
(3.23):&\quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_1)\\
&\quad \Rightarrow\ 0 = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_1),\\
(3.24):&\quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0)\ \Rightarrow\ 0 = 0,\\
(3.25):&\quad \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0)\ \Rightarrow\ 0 = 0,\\
(3.26):&\quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_1)\\
&\quad \Rightarrow\ 0 = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_1),\\
(3.27):&\quad \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_1)
\end{aligned}$$

$$\Rightarrow \quad 0 = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_1),$$

$$(3.28): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.29): \quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.30): \quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.31): \quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.32): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_1)$$

$$\Rightarrow \quad 0 = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_1),$$

$$(3.33): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.34): \quad \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.35): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_1)$$

$$\Rightarrow \quad 0 = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_1),$$

$$(3.36): \quad \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_1)$$

$$\Rightarrow \quad 0 = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_1),$$

$$(3.37): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.38): \quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.39): \quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0,$$

$$(3.40): \quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \quad \Rightarrow \quad 0 = 0.$$

Note that we keep the original names of the equations, from the beginning of Section 3.2, as we rewrite them. Now, we cannot use the formulas we calculated for dimension 2 anymore, since we are now in dimension 3, which means we have to use formula (3.11) to figure out how to calculate $\mu(e_i, C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3)$:

$$\begin{aligned}
\mu(e_i, C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3) &= \mu(e_i, C_{jk}^1 e_1) + \mu(e_i, C_{jk}^2 e_2) + \mu(e_i, C_{jk}^3 e_3) \\
&= C_{jk}^1 \mu(e_i, e_1) + C_{jk}^2 \mu(e_i, e_2) + C_{jk}^3 \mu(e_i, e_3) \\
&= C_{jk}^1 (C_{i1}^1 e_1 + C_{i1}^2 e_2 + C_{i1}^3 e_3) + C_{jk}^2 (C_{i2}^1 e_1 + C_{i2}^2 e_2 + C_{i2}^3 e_3) \\
&\quad + C_{jk}^3 (C_{i3}^1 e_1 + C_{i3}^2 e_2 + C_{i3}^3 e_3) \\
&= (C_{jk}^1 C_{i1}^1 + C_{jk}^2 C_{i2}^1 + C_{jk}^3 C_{i3}^1) e_1 + (C_{jk}^1 C_{i1}^2 + C_{jk}^2 C_{i2}^2 + C_{jk}^3 C_{i3}^2) e_2 \\
&\quad + (C_{jk}^1 C_{i1}^3 + C_{jk}^2 C_{i2}^3 + C_{jk}^3 C_{i3}^3) e_3.
\end{aligned}$$

We also use the same formula to figure out how to calculate $\mu(C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3, e_i)$:

$$\begin{aligned}
\mu(C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3, e_i) &= \mu(C_{jk}^1 e_1, e_i) + \mu(C_{jk}^2 e_2, e_i) + \mu(C_{jk}^3 e_3, e_i) \\
&= C_{jk}^1 \mu(e_1, e_i) + C_{jk}^2 \mu(e_2, e_i) + C_{jk}^3 \mu(e_3, e_i) \\
&= C_{jk}^1 (C_{1i}^1 e_1 + C_{1i}^2 e_2 + C_{1i}^3 e_3) + C_{jk}^2 (C_{2i}^1 e_1 + C_{2i}^2 e_2 + C_{2i}^3 e_3) \\
&\quad + C_{jk}^3 (C_{3i}^1 e_1 + C_{3i}^2 e_2 + C_{3i}^3 e_3)
\end{aligned}$$

$$= (C^1_{jk}C^1_{1i} + C^2_{jk}C^1_{2i} + C^3_{jk}C^1_{3i})e_1 + (C^1_{jk}C^2_{1i} + C^2_{jk}C^2_{2i} + C^3_{jk}C^2_{3i})e_2$$
$$+ (C^1_{jk}C^3_{1i} + C^2_{jk}C^3_{2i} + C^3_{jk}C^3_{3i})e_3.$$

We can now use these formulas to rewrite our equations.

$$\begin{aligned}
(3.14):\quad & (C^1_{11}C^1_{11} + C^2_{11}C^1_{12} + C^3_{11}C^1_{13})e_1 + (C^1_{11}C^2_{11} + C^2_{11}C^2_{12} + C^3_{11}C^2_{13})e_2 \\
& + (C^1_{11}C^3_{11} + C^2_{11}C^3_{12} + C^3_{11}C^3_{13})e_3 \\
& = (C^1_{11}C^1_{11} + C^2_{11}C^1_{21} + C^3_{11}C^1_{31})e_1 + (C^1_{11}C^2_{11} + C^2_{11}C^2_{21} + C^3_{11}C^2_{31})e_2 \\
& \quad + (C^1_{11}C^3_{11} + C^2_{11}C^3_{21} + C^3_{11}C^3_{31})e_3 \\
& \Rightarrow\ (C^2_{11}C^1_{12} + C^3_{11}C^1_{13} - C^2_{11}C^1_{21} - C^3_{11}C^1_{31})e_1 + (C^2_{11}C^2_{12} + C^3_{11}C^2_{13} - C^2_{11}C^2_{21} - C^3_{11}C^2_{31})e_2 \\
& \quad + (C^2_{11}C^3_{12} + C^3_{11}C^3_{13} - C^2_{11}C^3_{21} - C^3_{11}C^3_{31})e_3 = 0, \\
(3.15):\quad & (C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13})e_1 + (C^1_{12}C^2_{11} + C^2_{12}C^2_{12} + C^3_{12}C^2_{13})e_2 \\
& + (C^1_{12}C^3_{11} + C^2_{12}C^3_{12} + C^3_{12}C^3_{13})e_3 = 0, \\
(3.16):\quad & (C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13})e_1 + (C^1_{13}C^2_{11} + C^2_{13}C^2_{12} + C^3_{13}C^2_{13})e_2 \\
& + (C^1_{13}C^3_{11} + C^2_{13}C^3_{12} + C^3_{13}C^3_{13})e_3 = 0, \\
(3.17):\quad & (C^1_{21}C^1_{11} + C^2_{21}C^1_{12} + C^3_{21}C^1_{13})e_1 + (C^1_{21}C^2_{11} + C^2_{21}C^2_{12} + C^3_{21}C^2_{13})e_2 \\
& + (C^1_{21}C^3_{11} + C^2_{21}C^3_{12} + C^3_{21}C^3_{13})e_3 \\
& = (C^1_{12}C^1_{11} + C^2_{12}C^1_{21} + C^3_{12}C^1_{31})e_1 + (C^1_{12}C^2_{11} + C^2_{12}C^2_{21} + C^3_{12}C^2_{31})e_2 \\
& \quad + (C^1_{12}C^3_{11} + C^2_{12}C^3_{21} + C^3_{12}C^3_{31})e_3 \\
& \Rightarrow\ (C^1_{21}C^1_{11} + C^2_{21}C^1_{12} + C^3_{21}C^1_{13} - C^1_{12}C^1_{11} - C^2_{12}C^1_{21} - C^3_{12}C^1_{31})e_1 \\
& \quad + (C^1_{21}C^2_{11} + C^2_{21}C^2_{12} + C^3_{21}C^2_{13} - C^1_{12}C^2_{11} - C^2_{12}C^2_{21} - C^3_{12}C^2_{31})e_2 \\
& \quad + (C^1_{21}C^3_{11} + C^2_{21}C^3_{12} + C^3_{21}C^3_{13} - C^1_{12}C^3_{11} - C^2_{12}C^3_{21} - C^3_{12}C^3_{31})e_3 = 0, \\
(3.18):\quad & (C^1_{31}C^1_{11} + C^2_{31}C^1_{12} + C^3_{31}C^1_{13})e_1 + (C^1_{31}C^2_{11} + C^2_{31}C^2_{12} + C^3_{31}C^2_{13})e_2 \\
& + (C^1_{31}C^3_{11} + C^2_{31}C^3_{12} + C^3_{31}C^3_{13})e_3 \\
& = (C^1_{13}C^1_{11} + C^2_{13}C^1_{21} + C^3_{13}C^1_{31})e_1 + (C^1_{13}C^2_{11} + C^2_{13}C^2_{21} + C^3_{13}C^2_{31})e_2 \\
& \quad + (C^1_{13}C^3_{11} + C^2_{13}C^3_{21} + C^3_{13}C^3_{31})e_3 \\
& \Rightarrow\ (C^1_{31}C^1_{11} + C^2_{31}C^1_{12} + C^3_{31}C^1_{13} - C^1_{13}C^1_{11} - C^2_{13}C^1_{21} - C^3_{13}C^1_{31})e_1 \\
& \quad + (C^1_{31}C^2_{11} + C^2_{31}C^2_{12} + C^3_{31}C^2_{13} - C^1_{13}C^2_{11} - C^2_{13}C^2_{21} - C^3_{13}C^2_{31})e_2 \\
& \quad + (C^1_{31}C^3_{11} + C^2_{31}C^3_{12} + C^3_{31}C^3_{13} - C^1_{13}C^3_{11} - C^2_{13}C^3_{21} - C^3_{13}C^3_{31})e_3 = 0, \\
(3.19):\quad & (C^1_{22}C^1_{11} + C^2_{22}C^1_{12} + C^3_{22}C^1_{13})e_1 + (C^1_{22}C^2_{11} + C^2_{22}C^2_{12} + C^3_{22}C^2_{13})e_2 \\
& + (C^1_{22}C^3_{11} + C^2_{22}C^3_{12} + C^3_{22}C^3_{13})e_3 = 0, \\
(3.20):\quad & (C^1_{23}C^1_{11} + C^2_{23}C^1_{12} + C^3_{23}C^1_{13})e_1 + (C^1_{23}C^2_{11} + C^2_{23}C^2_{12} + C^3_{23}C^2_{13})e_2 \\
& + (C^1_{23}C^3_{11} + C^2_{23}C^3_{12} + C^3_{23}C^3_{13})e_3 = 0, \\
(3.21):\quad & (C^1_{32}C^1_{11} + C^2_{32}C^1_{12} + C^3_{32}C^1_{13})e_1 + (C^1_{32}C^2_{11} + C^2_{32}C^2_{12} + C^3_{32}C^2_{13})e_2 \\
& + (C^1_{32}C^3_{11} + C^2_{32}C^3_{12} + C^3_{32}C^3_{13})e_3 = 0,
\end{aligned}$$

$$(3.22):\quad (C^1_{33}C^1_{11}+C^2_{33}C^1_{12}+C^3_{33}C^1_{13})e_1+(C^1_{33}C^2_{11}+C^2_{33}C^2_{12}+C^3_{33}C^2_{13})e_2 + (C^1_{33}C^3_{11}+C^2_{33}C^3_{12}+C^3_{33}C^3_{13})e_3=0,$$

$$(3.23):\quad 0=(C^1_{21}C^1_{11}+C^2_{21}C^1_{21}+C^3_{21}C^1_{31})e_1+(C^1_{21}C^2_{11}+C^2_{21}C^2_{21}+C^3_{21}C^2_{31})e_2 + (C^1_{21}C^3_{11}+C^2_{21}C^3_{21}+C^3_{21}C^3_{31})e_3,$$

$$(3.26):\quad 0=(C^1_{22}C^1_{11}+C^2_{22}C^1_{21}+C^3_{22}C^1_{31})e_1+(C^1_{22}C^2_{11}+C^2_{22}C^2_{21}+C^3_{22}C^2_{31})e_2 + (C^1_{22}C^3_{11}+C^2_{22}C^3_{21}+C^3_{22}C^3_{31})e_3,$$

$$(3.27):\quad 0=(C^1_{23}C^1_{11}+C^2_{23}C^1_{21}+C^3_{23}C^1_{31})e_1+(C^1_{23}C^2_{11}+C^2_{23}C^2_{21}+C^3_{23}C^2_{31})e_2 + (C^1_{23}C^3_{11}+C^2_{23}C^3_{21}+C^3_{23}C^3_{31})e_3,$$

$$(3.32):\quad 0=(C^1_{31}C^1_{11}+C^2_{31}C^1_{21}+C^3_{31}C^1_{31})e_1+(C^1_{31}C^2_{11}+C^2_{31}C^2_{21}+C^3_{31}C^2_{31})e_2 + (C^1_{31}C^3_{11}+C^2_{31}C^3_{21}+C^3_{31}C^3_{31})e_3,$$

$$(3.35):\quad 0=(C^1_{32}C^1_{11}+C^2_{32}C^1_{21}+C^3_{32}C^1_{31})e_1+(C^1_{32}C^2_{11}+C^2_{32}C^2_{21}+C^3_{32}C^2_{31})e_2 + (C^1_{32}C^3_{11}+C^2_{32}C^3_{21}+C^3_{32}C^3_{31})e_3,$$

$$(3.36):\quad 0=(C^1_{33}C^1_{11}+C^2_{33}C^1_{21}+C^3_{33}C^1_{31})e_1+(C^1_{33}C^2_{11}+C^2_{33}C^2_{21}+C^3_{33}C^2_{31})e_2 + (C^1_{33}C^3_{11}+C^2_{33}C^3_{21}+C^3_{33}C^3_{31})e_3.$$

Just as we found in Section 3.1.1 for two dimensions, since $\{e_1, e_2, e_3\}$ is a basis it is a linearly independent set, and $k_1e_1 + k_2e_2 + k_3e_3 = 0$ will thus only have the trivial solution $k_1 = k_2 = k_3 = 0$. Thus, each of our original equations will give us three new equations. To easily see where each equation comes from, we give the three equations we get from equation (3.14) the names (R3.14.1), (R3.14.2) and (R3.14.3), where the first one was originally the coefficient for e_1, the second one the coefficient for e_2, and the third one the coefficient for e_3. Rewriting and naming all of the other equations in the same way, we get the following system of equations:

$$C^2_{11}C^1_{12}+C^3_{11}C^1_{13}-C^2_{11}C^1_{21}-C^3_{11}C^1_{31}=0 \tag{R3.14.1}$$

$$C^2_{11}C^2_{12}+C^3_{11}C^2_{13}-C^2_{11}C^2_{21}-C^3_{11}C^2_{31}=0 \tag{R3.14.2}$$

$$C^2_{11}C^3_{12}+C^3_{11}C^3_{13}-C^2_{11}C^3_{21}-C^3_{11}C^3_{31}=0 \tag{R3.14.3}$$

$$C^1_{12}C^1_{11}+C^2_{12}C^1_{12}+C^3_{12}C^1_{13}=0 \tag{R3.15.1}$$

$$C^1_{12}C^2_{11}+C^2_{12}C^2_{12}+C^3_{12}C^2_{13}=0 \tag{R3.15.2}$$

$$C^1_{12}C^3_{11}+C^2_{12}C^3_{12}+C^3_{12}C^3_{13}=0 \tag{R3.15.3}$$

$$C^1_{13}C^1_{11}+C^2_{13}C^1_{12}+C^3_{13}C^1_{13}=0 \tag{R3.16.1}$$

$$C^1_{13}C^2_{11}+C^2_{13}C^2_{12}+C^3_{13}C^2_{13}=0 \tag{R3.16.2}$$

$$C^1_{13}C^3_{11}+C^2_{13}C^3_{12}+C^3_{13}C^3_{13}=0 \tag{R3.16.3}$$

$$C^1_{21}C^1_{11}+C^2_{21}C^1_{12}+C^3_{21}C^1_{13}-C^1_{12}C^1_{11}-C^2_{12}C^1_{21}-C^3_{12}C^1_{31}=0 \tag{R3.17.1}$$

$$C^1_{21}C^2_{11}+C^2_{21}C^2_{12}+C^3_{21}C^2_{13}-C^1_{12}C^2_{11}-C^2_{12}C^2_{21}-C^3_{12}C^2_{31}=0 \tag{R3.17.2}$$

$$C^1_{21}C^3_{11}+C^2_{21}C^3_{12}+C^3_{21}C^3_{13}-C^1_{12}C^3_{11}-C^2_{12}C^3_{21}-C^3_{12}C^3_{31}=0 \tag{R3.17.3}$$

$$C^1_{31}C^1_{11}+C^2_{31}C^1_{12}+C^3_{31}C^1_{13}-C^1_{13}C^1_{11}-C^2_{13}C^1_{21}-C^3_{13}C^1_{31}=0 \tag{R3.18.1}$$

$$C^1_{31}C^2_{11}+C^2_{31}C^2_{12}+C^3_{31}C^2_{13}-C^1_{13}C^2_{11}-C^2_{13}C^2_{21}-C^3_{13}C^2_{31}=0 \quad \text{(R3.18.2)}$$
$$C^1_{31}C^3_{11}+C^2_{31}C^3_{12}+C^3_{31}C^3_{13}-C^1_{13}C^3_{11}-C^2_{13}C^3_{21}-C^3_{13}C^3_{31}=0 \quad \text{(R3.18.3)}$$
$$C^1_{22}C^1_{11}+C^2_{22}C^1_{12}+C^3_{22}C^1_{13}=0 \quad \text{(R3.19.1)}$$
$$C^1_{22}C^2_{11}+C^2_{22}C^2_{12}+C^3_{22}C^2_{13}=0 \quad \text{(R3.19.2)}$$
$$C^1_{22}C^3_{11}+C^2_{22}C^3_{12}+C^3_{22}C^3_{13}=0 \quad \text{(R3.19.3)}$$
$$C^1_{23}C^1_{11}+C^2_{23}C^1_{12}+C^3_{23}C^1_{13}=0 \quad \text{(R3.20.1)}$$
$$C^1_{23}C^2_{11}+C^2_{23}C^2_{12}+C^3_{23}C^2_{13}=0 \quad \text{(R3.20.2)}$$
$$C^1_{23}C^3_{11}+C^2_{23}C^3_{12}+C^3_{23}C^3_{13}=0 \quad \text{(R3.20.3)}$$
$$C^1_{32}C^1_{11}+C^2_{32}C^1_{12}+C^3_{32}C^1_{13}=0 \quad \text{(R3.21.1)}$$
$$C^1_{32}C^2_{11}+C^2_{32}C^2_{12}+C^3_{32}C^2_{13}=0 \quad \text{(R3.21.2)}$$
$$C^1_{32}C^3_{11}+C^2_{32}C^3_{12}+C^3_{32}C^3_{13}=0 \quad \text{(R3.21.3)}$$
$$C^1_{33}C^1_{11}+C^2_{33}C^1_{12}+C^3_{33}C^1_{13}=0 \quad \text{(R3.22.1)}$$
$$C^1_{33}C^2_{11}+C^2_{33}C^2_{12}+C^3_{33}C^2_{13}=0 \quad \text{(R3.22.2)}$$
$$C^1_{33}C^3_{11}+C^2_{33}C^3_{12}+C^3_{33}C^3_{13}=0 \quad \text{(R3.22.3)}$$
$$C^1_{21}C^1_{11}+C^2_{21}C^1_{21}+C^3_{21}C^1_{31}=0 \quad \text{(R3.23.1)}$$
$$C^1_{21}C^2_{11}+C^2_{21}C^2_{21}+C^3_{21}C^2_{31}=0 \quad \text{(R3.23.2)}$$
$$C^1_{21}C^3_{11}+C^2_{21}C^3_{21}+C^3_{21}C^3_{31}=0 \quad \text{(R3.23.3)}$$
$$C^1_{22}C^1_{11}+C^2_{22}C^1_{21}+C^3_{22}C^1_{31}=0 \quad \text{(R3.26.1)}$$
$$C^1_{22}C^2_{11}+C^2_{22}C^2_{21}+C^3_{22}C^2_{31}=0 \quad \text{(R3.26.2)}$$
$$C^1_{22}C^3_{11}+C^2_{22}C^3_{21}+C^3_{22}C^3_{31}=0 \quad \text{(R3.26.3)}$$
$$C^1_{23}C^1_{11}+C^2_{23}C^1_{21}+C^3_{23}C^1_{31}=0 \quad \text{(R3.27.1)}$$
$$C^1_{23}C^2_{11}+C^2_{23}C^2_{21}+C^3_{23}C^2_{31}=0 \quad \text{(R3.27.2)}$$
$$C^1_{23}C^3_{11}+C^2_{23}C^3_{21}+C^3_{23}C^3_{31}=0 \quad \text{(R3.27.3)}$$
$$C^1_{31}C^1_{11}+C^2_{31}C^1_{21}+C^3_{31}C^1_{31}=0 \quad \text{(R3.32.1)}$$
$$C^1_{31}C^2_{11}+C^2_{31}C^2_{21}+C^3_{31}C^2_{31}=0 \quad \text{(R3.32.2)}$$
$$C^1_{31}C^3_{11}+C^2_{31}C^3_{21}+C^3_{31}C^3_{31}=0 \quad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11}+C^2_{32}C^1_{21}+C^3_{32}C^1_{31}=0 \quad \text{(R3.35.1)}$$
$$C^1_{32}C^2_{11}+C^2_{32}C^2_{21}+C^3_{32}C^2_{31}=0 \quad \text{(R3.35.2)}$$
$$C^1_{32}C^3_{11}+C^2_{32}C^3_{21}+C^3_{32}C^3_{31}=0 \quad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11}+C^2_{33}C^1_{21}+C^3_{33}C^1_{31}=0 \quad \text{(R3.36.1)}$$
$$C^1_{33}C^2_{11}+C^2_{33}C^2_{21}+C^3_{33}C^2_{31}=0 \quad \text{(R3.36.2)}$$
$$C^1_{33}C^3_{11}+C^2_{33}C^3_{21}+C^3_{33}C^3_{31}=0. \quad \text{(R3.36.3)}$$

Just as we did in two dimensions, we look at the terms that exist in multiple equations, and add or subtract certain equations. We marked the equations so we could easily show how we added or subtracted equations, as seen below.

$$C^2_{11}C^1_{12}+C^3_{11}C^1_{13}-C^2_{11}C^1_{21}-C^3_{11}C^1_{31}=0 \quad \text{(R3.14.1)}$$
$$C^2_{11}C^2_{12}+C^3_{11}C^2_{13}-C^2_{11}C^2_{21}-C^3_{11}C^2_{31}=0 \quad \text{(R3.14.2)}$$
$$C^2_{11}C^3_{12}+C^3_{11}C^3_{13}-C^2_{11}C^3_{21}-C^3_{11}C^3_{31}=0 \quad \text{(R3.14.3)}$$
$$C^1_{12}C^1_{11}+C^2_{12}C^1_{12}+C^3_{12}C^1_{13}=0 \quad \text{(R3.15.1)}$$
$$C^1_{12}C^2_{11}+C^2_{12}C^2_{12}+C^3_{12}C^2_{13}=0 \quad \text{(R3.15.2)}$$
$$C^1_{12}C^3_{11}+C^2_{12}C^3_{12}+C^3_{12}C^3_{13}=0 \quad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11}+C^2_{13}C^1_{12}+C^3_{13}C^1_{13}=0 \quad \text{(R3.16.1)}$$
$$C^1_{13}C^2_{11}+C^2_{13}C^2_{12}+C^3_{13}C^2_{13}=0 \quad \text{(R3.16.2)}$$
$$C^1_{13}C^3_{11}+C^2_{13}C^3_{12}+C^3_{13}C^3_{13}=0 \quad \text{(R3.16.3)}$$
$$C^2_{21}C^1_{12}+C^3_{21}C^1_{13}-C^2_{12}C^1_{21}-C^3_{12}C^1_{31}+C^2_{12}C^1_{12}+C^3_{12}C^1_{13}-C^2_{21}C^1_{21}-C^3_{21}C^1_{31}=0 \quad \text{(R3.17.1)+(R3.15.1)−(R3.23.1)}$$
$$C^2_{21}C^2_{12}+C^3_{21}C^2_{13}-C^2_{12}C^2_{21}-C^3_{12}C^2_{31}+C^2_{12}C^2_{12}+C^3_{12}C^2_{13}-C^2_{21}C^2_{21}-C^3_{21}C^2_{31}=0 \quad \text{(R3.17.2)+(R3.15.2)−(R3.23.2)}$$
$$C^2_{21}C^3_{12}+C^3_{21}C^3_{13}-C^2_{12}C^3_{21}-C^3_{12}C^3_{31}+C^2_{12}C^3_{12}+C^3_{12}C^3_{13}-C^2_{21}C^3_{21}-C^3_{21}C^3_{31}=0 \quad \text{(R3.17.3)+(R3.15.3)−(R3.23.3)}$$
$$C^2_{31}C^1_{12}+C^3_{31}C^1_{13}-C^2_{13}C^1_{21}-C^3_{13}C^1_{31}+C^2_{13}C^1_{12}+C^3_{13}C^1_{13}-C^2_{31}C^1_{21}-C^3_{31}C^1_{31}=0 \quad \text{(R3.18.1)+(R3.16.1)−(R3.32.1)}$$
$$C^2_{31}C^2_{12}+C^3_{31}C^2_{13}-C^2_{13}C^2_{21}-C^3_{13}C^2_{31}+C^2_{13}C^2_{12}+C^3_{13}C^2_{13}-C^2_{31}C^2_{21}-C^3_{31}C^2_{31}=0 \quad \text{(R3.18.2)+(R3.16.2)−(R3.32.2)}$$
$$C^2_{31}C^3_{12}+C^3_{31}C^3_{13}-C^2_{13}C^3_{21}-C^3_{13}C^3_{31}+C^2_{13}C^3_{12}+C^3_{13}C^3_{13}-C^2_{31}C^3_{21}-C^3_{31}C^3_{31}=0 \quad \text{(R3.18.3)+(R3.16.3)−(R3.32.3)}$$
$$C^2_{22}C^1_{12}+C^3_{22}C^1_{13}-C^2_{22}C^1_{21}-C^3_{22}C^1_{31}=0 \quad \text{(R3.19.1)−(R3.26.1)}$$
$$C^2_{22}C^2_{12}+C^3_{22}C^2_{13}-C^2_{22}C^2_{21}-C^3_{22}C^2_{31}=0 \quad \text{(R3.19.2)−(R3.26.2)}$$
$$C^2_{22}C^3_{12}+C^3_{22}C^3_{13}-C^2_{22}C^3_{21}-C^3_{22}C^3_{31}=0 \quad \text{(R3.19.3)−(R3.26.3)}$$
$$C^2_{23}C^1_{12}+C^3_{23}C^1_{13}-C^2_{23}C^1_{21}-C^3_{23}C^1_{31}=0 \quad \text{(R3.20.1)−(R3.27.1)}$$
$$C^2_{23}C^2_{12}+C^3_{23}C^2_{13}-C^2_{23}C^2_{21}-C^3_{23}C^2_{31}=0 \quad \text{(R3.20.2)−(R3.27.2)}$$
$$C^2_{23}C^3_{12}+C^3_{23}C^3_{13}-C^2_{23}C^3_{21}-C^3_{23}C^3_{31}=0 \quad \text{(R3.20.3)−(R3.27.3)}$$
$$C^2_{32}C^1_{12}+C^3_{32}C^1_{13}-C^2_{32}C^1_{21}-C^3_{32}C^1_{31}=0 \quad \text{(R3.21.1)−(R3.35.1)}$$
$$C^2_{32}C^2_{12}+C^3_{32}C^2_{13}-C^2_{32}C^2_{21}-C^3_{32}C^2_{31}=0 \quad \text{(R3.21.2)−(R3.35.2)}$$
$$C^2_{32}C^3_{12}+C^3_{32}C^3_{13}-C^2_{32}C^3_{21}-C^3_{32}C^3_{31}=0 \quad \text{(R3.21.3)−(R3.35.3)}$$
$$C^2_{33}C^1_{12}+C^3_{33}C^1_{13}-C^2_{33}C^1_{21}-C^3_{33}C^1_{31}=0 \quad \text{(R3.22.1)−(R3.36.1)}$$
$$C^2_{33}C^2_{12}+C^3_{33}C^2_{13}-C^2_{33}C^2_{21}-C^3_{33}C^2_{31}=0 \quad \text{(R3.22.2)−(R3.36.2)}$$
$$C^2_{33}C^3_{12}+C^3_{33}C^3_{13}-C^2_{33}C^3_{21}-C^3_{33}C^3_{31}=0 \quad \text{(R3.22.3)−(R3.36.3)}$$
$$C^1_{21}C^1_{11}+C^2_{21}C^1_{21}+C^3_{21}C^1_{31}=0 \quad \text{(R3.23.1)}$$
$$C^1_{21}C^2_{11}+C^2_{21}C^2_{21}+C^3_{21}C^2_{31}=0 \quad \text{(R3.23.2)}$$
$$C^1_{21}C^3_{11}+C^2_{21}C^3_{21}+C^3_{21}C^3_{31}=0 \quad \text{(R3.23.3)}$$
$$C^1_{22}C^1_{11}+C^2_{22}C^1_{21}+C^3_{22}C^1_{31}=0 \quad \text{(R3.26.1)}$$
$$C^1_{22}C^2_{11}+C^2_{22}C^2_{21}+C^3_{22}C^2_{31}=0 \quad \text{(R3.26.2)}$$
$$C^1_{22}C^3_{11}+C^2_{22}C^3_{21}+C^3_{22}C^3_{31}=0 \quad \text{(R3.26.3)}$$
$$C^1_{23}C^1_{11}+C^2_{23}C^1_{21}+C^3_{23}C^1_{31}=0 \quad \text{(R3.27.1)}$$
$$C^1_{23}C^2_{11}+C^2_{23}C^2_{21}+C^3_{23}C^2_{31}=0 \quad \text{(R3.27.2)}$$

$$C_{23}^1C_{11}^3+C_{23}^2C_{21}^3+C_{23}^3C_{31}^3=0 \quad \text{(R3.27.3)}$$
$$C_{31}^1C_{11}^1+C_{31}^2C_{21}^1+C_{31}^3C_{31}^1=0 \quad \text{(R3.32.1)}$$
$$C_{31}^1C_{11}^2+C_{31}^2C_{21}^2+C_{31}^3C_{31}^2=0 \quad \text{(R3.32.2)}$$
$$C_{31}^1C_{11}^3+C_{31}^2C_{21}^3+C_{31}^3C_{31}^3=0 \quad \text{(R3.32.3)}$$
$$C_{32}^1C_{11}^1+C_{32}^2C_{21}^1+C_{32}^3C_{31}^1=0 \quad \text{(R3.35.1)}$$
$$C_{32}^1C_{11}^2+C_{32}^2C_{21}^2+C_{32}^3C_{31}^2=0 \quad \text{(R3.35.2)}$$
$$C_{32}^1C_{11}^3+C_{32}^2C_{21}^3+C_{32}^3C_{31}^3=0 \quad \text{(R3.35.3)}$$
$$C_{33}^1C_{11}^1+C_{33}^2C_{21}^1+C_{33}^3C_{31}^1=0 \quad \text{(R3.36.1)}$$
$$C_{33}^1C_{11}^2+C_{33}^2C_{21}^2+C_{33}^3C_{31}^2=0 \quad \text{(R3.36.2)}$$
$$C_{33}^1C_{11}^3+C_{33}^2C_{21}^3+C_{33}^3C_{31}^3=0 \quad \text{(R3.36.3).}$$

Before we continue, we rename the equations so that equation (R3.17.1) + (R3.15.1) − (R3.23.1) instead becomes (R2.17.1.2), and then follow the same way of renaming the rest of the equations as well, so that for example equation (R3.20.2) − (R3.27.2) becomes renamed to (R2.20.2.2), and so on. This ends up looking as follows:

$$C_{11}^2(C_{12}^1-C_{21}^1)+C_{11}^3(C_{13}^1-C_{31}^1)=0 \qquad \text{(R3.14.1)}$$
$$C_{11}^2(C_{12}^2-C_{21}^2)+C_{11}^3(C_{13}^2-C_{31}^2)=0 \qquad \text{(R3.14.2)}$$
$$C_{11}^2(C_{12}^3-C_{21}^3)+C_{11}^3(C_{13}^3-C_{31}^3)=0 \qquad \text{(R3.14.3)}$$
$$C_{12}^1C_{11}^1+C_{12}^2C_{12}^1+C_{12}^3C_{13}^1=0 \qquad \text{(R3.15.1)}$$
$$C_{12}^1C_{11}^2+C_{12}^2C_{12}^2+C_{12}^3C_{13}^2=0 \qquad \text{(R3.15.2)}$$
$$C_{12}^1C_{11}^3+C_{12}^2C_{12}^3+C_{12}^3C_{13}^3=0 \qquad \text{(R3.15.3)}$$
$$C_{13}^1C_{11}^1+C_{13}^2C_{12}^1+C_{13}^3C_{13}^1=0 \qquad \text{(R3.16.1)}$$
$$C_{13}^1C_{11}^2+C_{13}^2C_{12}^2+C_{13}^3C_{13}^2=0 \qquad \text{(R3.16.2)}$$
$$C_{13}^1C_{11}^3+C_{13}^2C_{12}^3+C_{13}^3C_{13}^3=0 \qquad \text{(R3.16.3)}$$
$$(C_{12}^2+C_{21}^2)(C_{12}^1-C_{21}^1)+(C_{12}^3+C_{21}^3)(C_{13}^1-C_{31}^1)=0 \qquad \text{(R3.17.1.2)}$$
$$(C_{12}^2+C_{21}^2)(C_{12}^2-C_{21}^2)+(C_{12}^3+C_{21}^3)(C_{13}^2-C_{31}^2)=0 \qquad \text{(R3.17.2.2)}$$
$$(C_{12}^2+C_{21}^2)(C_{12}^3-C_{21}^3)+(C_{12}^3+C_{21}^3)(C_{13}^3-C_{31}^3)=0 \qquad \text{(R3.17.3.2)}$$
$$(C_{13}^2+C_{31}^2)(C_{12}^1-C_{21}^1)+(C_{13}^3+C_{31}^3)(C_{13}^1-C_{31}^1)=0 \qquad \text{(R3.18.1.2)}$$
$$(C_{13}^2+C_{31}^2)(C_{12}^2-C_{21}^2)+(C_{13}^3+C_{31}^3)(C_{13}^2-C_{31}^2)=0 \qquad \text{(R3.18.2.2)}$$
$$(C_{13}^2+C_{31}^2)(C_{12}^3-C_{21}^3)+(C_{13}^3+C_{31}^3)(C_{13}^3-C_{31}^3)=0 \qquad \text{(R3.18.3.2)}$$
$$C_{22}^2(C_{12}^1-C_{21}^1)+C_{22}^3(C_{13}^1-C_{31}^1)=0 \qquad \text{(R3.19.1.2)}$$
$$C_{22}^2(C_{12}^2-C_{21}^2)+C_{22}^3(C_{13}^2-C_{31}^2)=0 \qquad \text{(R3.19.2.2)}$$
$$C_{22}^2(C_{12}^3-C_{21}^3)+C_{22}^3(C_{13}^3-C_{31}^3)=0 \qquad \text{(R3.19.3.2)}$$
$$C_{23}^2(C_{12}^1-C_{21}^1)+C_{23}^3(C_{13}^1-C_{31}^1)=0 \qquad \text{(R3.20.1.2)}$$
$$C_{23}^2(C_{12}^2-C_{21}^2)+C_{23}^3(C_{13}^2-C_{31}^2)=0 \qquad \text{(R3.20.2.2)}$$
$$C_{23}^2(C_{12}^3-C_{21}^3)+C_{23}^3(C_{13}^3-C_{31}^3)=0 \qquad \text{(R3.20.3.2)}$$

$$C_{32}^2(C_{12}^1 - C_{21}^1) + C_{32}^3(C_{13}^1 - C_{31}^1) = 0 \quad \text{(R3.21.1.2)}$$
$$C_{32}^2(C_{12}^2 - C_{21}^2) + C_{32}^3(C_{13}^2 - C_{31}^2) = 0 \quad \text{(R3.21.2.2)}$$
$$C_{32}^2(C_{12}^3 - C_{21}^3) + C_{32}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.21.3.2)}$$
$$C_{33}^2(C_{12}^1 - C_{21}^1) + C_{33}^3(C_{13}^1 - C_{31}^1) = 0 \quad \text{(R3.22.1.2)}$$
$$C_{33}^2(C_{12}^2 - C_{21}^2) + C_{33}^3(C_{13}^2 - C_{31}^2) = 0 \quad \text{(R3.22.2.2)}$$
$$C_{33}^2(C_{12}^3 - C_{21}^3) + C_{33}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.22.3.2)}$$
$$C_{21}^1C_{11}^1 + C_{21}^2C_{21}^1 + C_{21}^3C_{31}^1 = 0 \quad \text{(R3.23.1)}$$
$$C_{21}^1C_{11}^2 + C_{21}^2C_{21}^2 + C_{21}^3C_{31}^2 = 0 \quad \text{(R3.23.2)}$$
$$C_{21}^1C_{11}^3 + C_{21}^2C_{21}^3 + C_{21}^3C_{31}^3 = 0 \quad \text{(R3.23.3)}$$
$$C_{22}^1C_{11}^1 + C_{22}^2C_{21}^1 + C_{22}^3C_{31}^1 = 0 \quad \text{(R3.26.1)}$$
$$C_{22}^1C_{11}^2 + C_{22}^2C_{21}^2 + C_{22}^3C_{31}^2 = 0 \quad \text{(R3.26.2)}$$
$$C_{22}^1C_{11}^3 + C_{22}^2C_{21}^3 + C_{22}^3C_{31}^3 = 0 \quad \text{(R3.26.3)}$$
$$C_{23}^1C_{11}^1 + C_{23}^2C_{21}^1 + C_{23}^3C_{31}^1 = 0 \quad \text{(R3.27.1)}$$
$$C_{23}^1C_{11}^2 + C_{23}^2C_{21}^2 + C_{23}^3C_{31}^2 = 0 \quad \text{(R3.27.2)}$$
$$C_{23}^1C_{11}^3 + C_{23}^2C_{21}^3 + C_{23}^3C_{31}^3 = 0 \quad \text{(R3.27.3)}$$
$$C_{31}^1C_{11}^1 + C_{31}^2C_{21}^1 + C_{31}^3C_{31}^1 = 0 \quad \text{(R3.32.1)}$$
$$C_{31}^1C_{11}^2 + C_{31}^2C_{21}^2 + C_{31}^3C_{31}^2 = 0 \quad \text{(R3.32.2)}$$
$$C_{31}^1C_{11}^3 + C_{31}^2C_{21}^3 + C_{31}^3C_{31}^3 = 0 \quad \text{(R3.32.3)}$$
$$C_{32}^1C_{11}^1 + C_{32}^2C_{21}^1 + C_{32}^3C_{31}^1 = 0 \quad \text{(R3.35.1)}$$
$$C_{32}^1C_{11}^2 + C_{32}^2C_{21}^2 + C_{32}^3C_{31}^2 = 0 \quad \text{(R3.35.2)}$$
$$C_{32}^1C_{11}^3 + C_{32}^2C_{21}^3 + C_{32}^3C_{31}^3 = 0 \quad \text{(R3.35.3)}$$
$$C_{33}^1C_{11}^1 + C_{33}^2C_{21}^1 + C_{33}^3C_{31}^1 = 0 \quad \text{(R3.36.1)}$$
$$C_{33}^1C_{11}^2 + C_{33}^2C_{21}^2 + C_{33}^3C_{31}^2 = 0 \quad \text{(R3.36.2)}$$
$$C_{33}^1C_{11}^3 + C_{33}^2C_{21}^3 + C_{33}^3C_{31}^3 = 0. \quad \text{(R3.36.3)}$$

We now have to try to solve this system of equations. Looking at it, we see that in all of the equations containing products, the terms $C_{12}^i - C_{21}^i$ and $C_{13}^i - C_{31}^i$, where $i = 1, 2, 3$, are present, and they follow a clear pattern, where the three equations that originated from the same equation are identical except for the terms $C_{12}^i - C_{21}^i$ and $C_{13}^i - C_{31}^i$, where $i = 1$ for the equation that was the coefficient for e_1, $i = 2$ for the equation that was the coefficient for e_2, and $i = 3$ for the equation that was the coefficient for e_3. We try to use this pattern to solve the system of equations. We begin by looking at the case where $C_{13}^1 - C_{31}^1 = 0$ before looking at the case where $C_{13}^1 - C_{31}^1 \neq 0$.

Case 1: If $C_{13}^1 - C_{31}^1 = 0$ the system of equations becomes

$$C_{11}^2(C_{12}^1 - C_{21}^1) = 0 \quad \text{(R3.14.1)}$$

$$C_{11}^{2}(C_{12}^{2}-C_{21}^{2})+C_{11}^{3}(C_{13}^{2}-C_{31}^{2})=0 \qquad \text{(R3.14.2)}$$
$$C_{11}^{2}(C_{12}^{3}-C_{21}^{3})+C_{11}^{3}(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.14.3)}$$
$$C_{12}^{1}C_{11}^{1}+C_{12}^{2}C_{12}^{1}+C_{12}^{3}C_{13}^{1}=0 \qquad \text{(R3.15.1)}$$
$$C_{12}^{1}C_{11}^{2}+C_{12}^{2}C_{12}^{2}+C_{12}^{3}C_{13}^{2}=0 \qquad \text{(R3.15.2)}$$
$$C_{12}^{1}C_{11}^{3}+C_{12}^{2}C_{12}^{3}+C_{12}^{3}C_{13}^{3}=0 \qquad \text{(R3.15.3)}$$
$$C_{13}^{1}C_{11}^{1}+C_{13}^{2}C_{12}^{1}+C_{13}^{3}C_{13}^{1}=0 \qquad \text{(R3.16.1)}$$
$$C_{13}^{1}C_{11}^{2}+C_{13}^{2}C_{12}^{2}+C_{13}^{3}C_{13}^{2}=0 \qquad \text{(R3.16.2)}$$
$$C_{13}^{1}C_{11}^{3}+C_{13}^{2}C_{12}^{3}+C_{13}^{3}C_{13}^{3}=0 \qquad \text{(R3.16.3)}$$
$$(C_{12}^{2}+C_{21}^{2})(C_{12}^{1}-C_{21}^{1})=0 \qquad \text{(R3.17.1.2)}$$
$$(C_{12}^{2}+C_{21}^{2})(C_{12}^{2}-C_{21}^{2})+(C_{12}^{3}+C_{21}^{3})(C_{13}^{2}-C_{31}^{2})=0 \qquad \text{(R3.17.2.2)}$$
$$(C_{12}^{2}+C_{21}^{2})(C_{12}^{3}-C_{21}^{3})+(C_{12}^{3}+C_{21}^{3})(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.17.3.2)}$$
$$(C_{13}^{2}+C_{31}^{2})(C_{12}^{1}-C_{21}^{1})=0 \qquad \text{(R3.18.1.2)}$$
$$(C_{13}^{2}+C_{31}^{2})(C_{12}^{2}-C_{21}^{2})+(C_{13}^{3}+C_{31}^{3})(C_{13}^{2}-C_{31}^{2})=0 \qquad \text{(R3.18.2.2)}$$
$$(C_{13}^{2}+C_{31}^{2})(C_{12}^{3}-C_{21}^{3})+(C_{13}^{3}+C_{31}^{3})(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.18.3.2)}$$
$$C_{22}^{2}(C_{12}^{1}-C_{21}^{1})=0 \qquad \text{(R3.19.1.2)}$$
$$C_{22}^{2}(C_{12}^{2}-C_{21}^{2})+C_{22}^{3}(C_{13}^{2}-C_{31}^{2})=0 \qquad \text{(R3.19.2.2)}$$
$$C_{22}^{2}(C_{12}^{3}-C_{21}^{3})+C_{22}^{3}(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.19.3.2)}$$
$$C_{23}^{2}(C_{12}^{1}-C_{21}^{1})=0 \qquad \text{(R3.20.1.2)}$$
$$C_{23}^{2}(C_{12}^{2}-C_{21}^{2})+C_{23}^{3}(C_{13}^{2}-C_{31}^{2})=0 \qquad \text{(R3.20.2.2)}$$
$$C_{23}^{2}(C_{12}^{3}-C_{21}^{3})+C_{23}^{3}(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.20.3.2)}$$
$$C_{32}^{2}(C_{12}^{1}-C_{21}^{1})=0 \qquad \text{(R3.21.1.2)}$$
$$C_{32}^{2}(C_{12}^{2}-C_{21}^{2})+C_{32}^{3}(C_{13}^{2}-C_{31}^{2})=0 \qquad \text{(R3.21.2.2)}$$
$$C_{32}^{2}(C_{12}^{3}-C_{21}^{3})+C_{32}^{3}(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.21.3.2)}$$
$$C_{33}^{2}(C_{12}^{1}-C_{21}^{1})=0 \qquad \text{(R3.22.1.2)}$$
$$C_{33}^{2}(C_{12}^{2}-C_{21}^{2})+C_{33}^{3}(C_{13}^{2}-C_{31}^{2})=0 \qquad \text{(R3.22.2.2)}$$
$$C_{33}^{2}(C_{12}^{3}-C_{21}^{3})+C_{33}^{3}(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.22.3.2)}$$
$$C_{21}^{1}C_{11}^{1}+C_{21}^{2}C_{21}^{1}+C_{21}^{3}C_{13}^{1}=0 \qquad \text{(R3.23.1)}$$
$$C_{21}^{1}C_{11}^{2}+C_{21}^{2}C_{21}^{2}+C_{21}^{3}C_{31}^{2}=0 \qquad \text{(R3.23.2)}$$
$$C_{21}^{1}C_{11}^{3}+C_{21}^{2}C_{21}^{3}+C_{21}^{3}C_{31}^{3}=0 \qquad \text{(R3.23.3)}$$
$$C_{22}^{1}C_{11}^{1}+C_{22}^{2}C_{21}^{1}+C_{22}^{3}C_{13}^{1}=0 \qquad \text{(R3.26.1)}$$
$$C_{22}^{1}C_{11}^{2}+C_{22}^{2}C_{21}^{2}+C_{22}^{3}C_{31}^{2}=0 \qquad \text{(R3.26.2)}$$
$$C_{22}^{1}C_{11}^{3}+C_{22}^{2}C_{21}^{3}+C_{22}^{3}C_{31}^{3}=0 \qquad \text{(R3.26.3)}$$
$$C_{23}^{1}C_{11}^{1}+C_{23}^{2}C_{21}^{1}+C_{23}^{3}C_{13}^{1}=0 \qquad \text{(R3.27.1)}$$

$$C^1_{23}C^2_{11}+C^2_{23}C^2_{21}+C^3_{23}C^2_{31}=0 \qquad \text{(R3.27.2)}$$
$$C^1_{23}C^3_{11}+C^2_{23}C^3_{21}+C^3_{23}C^3_{31}=0 \qquad \text{(R3.27.3)}$$
$$C^1_{13}C^1_{11}+C^2_{31}C^1_{21}+C^3_{31}C^1_{13}=0 \qquad \text{(R3.32.1)}$$
$$C^1_{13}C^2_{11}+C^2_{31}C^2_{21}+C^3_{31}C^2_{31}=0 \qquad \text{(R3.32.2)}$$
$$C^1_{13}C^3_{11}+C^2_{31}C^3_{21}+C^3_{31}C^3_{31}=0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11}+C^2_{32}C^1_{21}+C^3_{32}C^1_{13}=0 \qquad \text{(R3.35.1)}$$
$$C^1_{32}C^2_{11}+C^2_{32}C^2_{21}+C^3_{32}C^2_{31}=0 \qquad \text{(R3.35.2)}$$
$$C^1_{32}C^3_{11}+C^2_{32}C^3_{21}+C^3_{32}C^3_{31}=0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11}+C^2_{33}C^1_{21}+C^3_{33}C^1_{13}=0 \qquad \text{(R3.36.1)}$$
$$C^1_{33}C^2_{11}+C^2_{33}C^2_{21}+C^3_{33}C^2_{31}=0 \qquad \text{(R3.36.2)}$$
$$C^1_{33}C^3_{11}+C^2_{33}C^3_{21}+C^3_{33}C^3_{31}=0. \qquad \text{(R3.36.3)}$$

Looking at equations (R3.14.1), (R3.17.1.2), (R3.18.1.2), (R3.19.1.2), (R3.20.1.2), (R3.21.1.2) and (R3.22.1.2), we see that one solution to all of them is $C^1_{12}-C^1_{21}=0$, but if $C^1_{12}-C^1_{21}\neq 0$, then the only solutions to all of those equations are $C^2_{11}=0$, $C^2_{12}+C^2_{21}=0$, $C^2_{13}+C^2_{31}=0$, $C^2_{22}=0$, $C^2_{23}=0$, $C^2_{32}=0$, and $C^2_{33}=0$, respectively. We look at each of these two cases: either $C^1_{12}-C^1_{21}=0$, or $C^1_{12}-C^1_{21}\neq 0$ and $C^2_{11}=C^2_{12}+C^2_{21}=C^2_{13}+C^2_{31}=C^2_{22}=C^2_{23}=C^2_{32}=C^2_{33}=0$.

Case 1.1: If $C^1_{12}-C^1_{21}=0$ our system of equations becomes

$$C^2_{11}(C^2_{12}-C^2_{21})+C^3_{11}(C^2_{13}-C^2_{31})=0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}(C^3_{12}-C^3_{21})+C^3_{11}(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.14.3)}$$
$$C^1_{12}C^1_{11}+C^2_{12}C^1_{12}+C^3_{12}C^1_{13}=0 \qquad \text{(R3.15.1)}$$
$$C^1_{12}C^2_{11}+C^2_{12}C^2_{12}+C^3_{12}C^2_{13}=0 \qquad \text{(R3.15.2)}$$
$$C^1_{12}C^3_{11}+C^2_{12}C^3_{12}+C^3_{12}C^3_{13}=0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11}+C^2_{13}C^1_{12}+C^3_{13}C^1_{13}=0 \qquad \text{(R3.16.1)}$$
$$C^1_{13}C^2_{11}+C^2_{13}C^2_{12}+C^3_{13}C^2_{13}=0 \qquad \text{(R3.16.2)}$$
$$C^1_{13}C^3_{11}+C^2_{13}C^3_{12}+C^3_{13}C^3_{13}=0 \qquad \text{(R3.16.3)}$$
$$(C^2_{12}+C^2_{21})(C^2_{12}-C^2_{21})+(C^3_{12}+C^3_{21})(C^2_{13}-C^2_{31})=0 \qquad \text{(R3.17.2.2)}$$
$$(C^2_{12}+C^2_{21})(C^3_{12}-C^3_{21})+(C^3_{12}+C^3_{21})(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.17.3.2)}$$
$$(C^2_{13}+C^2_{31})(C^2_{12}-C^2_{21})+(C^3_{13}+C^3_{31})(C^2_{13}-C^2_{31})=0 \qquad \text{(R3.18.2.2)}$$
$$(C^2_{13}+C^2_{31})(C^3_{12}-C^3_{21})+(C^3_{13}+C^3_{31})(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{22}(C^2_{12}-C^2_{21})+C^3_{22}(C^2_{13}-C^2_{31})=0 \qquad \text{(R3.19.2.2)}$$
$$C^2_{22}(C^3_{12}-C^3_{21})+C^3_{22}(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.19.3.2)}$$
$$C^2_{23}(C^2_{12}-C^2_{21})+C^3_{23}(C^2_{13}-C^2_{31})=0 \qquad \text{(R3.20.2.2)}$$

$$C_{23}^2(C_{12}^3 - C_{21}^3) + C_{23}^3(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.20.3.2)}$$
$$C_{32}^2(C_{12}^2 - C_{21}^2) + C_{32}^3(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.21.2.2)}$$
$$C_{32}^2(C_{12}^3 - C_{21}^3) + C_{32}^3(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.21.3.2)}$$
$$C_{33}^2(C_{12}^2 - C_{21}^2) + C_{33}^3(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.22.2.2)}$$
$$C_{33}^2(C_{12}^3 - C_{21}^3) + C_{33}^3(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.22.3.2)}$$
$$C_{12}^1C_{11}^1 + C_{21}^2C_{12}^1 + C_{21}^3C_{13}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{12}^1C_{11}^2 + C_{21}^2C_{21}^2 + C_{21}^3C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{12}^1C_{11}^3 + C_{21}^2C_{21}^3 + C_{21}^3C_{31}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^1C_{11}^1 + C_{22}^2C_{12}^1 + C_{22}^3C_{13}^1 = 0 \qquad \text{(R3.26.1)}$$
$$C_{22}^1C_{11}^2 + C_{22}^2C_{21}^2 + C_{22}^3C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^1C_{11}^3 + C_{22}^2C_{21}^3 + C_{22}^3C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^1C_{11}^1 + C_{23}^2C_{12}^1 + C_{23}^3C_{13}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1C_{11}^2 + C_{23}^2C_{21}^2 + C_{23}^3C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^1C_{11}^3 + C_{23}^2C_{21}^3 + C_{23}^3C_{31}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{13}^1C_{11}^1 + C_{31}^2C_{12}^1 + C_{31}^3C_{13}^1 = 0 \qquad \text{(R3.32.1)}$$
$$C_{13}^1C_{11}^2 + C_{31}^2C_{21}^2 + C_{31}^3C_{31}^2 = 0 \qquad \text{(R3.32.2)}$$
$$C_{13}^1C_{11}^3 + C_{31}^2C_{21}^3 + C_{31}^3C_{31}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^1C_{11}^1 + C_{32}^2C_{12}^1 + C_{32}^3C_{13}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1C_{11}^2 + C_{32}^2C_{21}^2 + C_{32}^3C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^1C_{11}^3 + C_{32}^2C_{21}^3 + C_{32}^3C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1C_{11}^1 + C_{33}^2C_{12}^1 + C_{33}^3C_{13}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1C_{11}^2 + C_{33}^2C_{21}^2 + C_{33}^3C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^1C_{11}^3 + C_{33}^2C_{21}^3 + C_{33}^3C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

Case 1.2: If we instead have $C_{12}^1 - C_{21}^1 \neq 0$ and $C_{11}^2 = C_{12}^2 + C_{21}^2 = C_{13}^2 + C_{31}^2 = C_{22}^2 = C_{23}^2 = C_{32}^2 = C_{33}^2 = 0$, our system of equations becomes

$$C_{11}^3(C_{13}^2 + C_{13}^2) = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^3(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.14.3)}$$
$$C_{12}^1C_{11}^1 + C_{12}^2C_{12}^1 + C_{12}^3C_{13}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{12}^2C_{12}^2 + C_{12}^3C_{13}^2 = 0 \qquad \text{(R3.15.2)}$$
$$C_{12}^1C_{11}^3 + C_{12}^2C_{12}^3 + C_{12}^3C_{13}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^1C_{11}^1 + C_{13}^2C_{12}^1 + C_{13}^3C_{13}^1 = 0 \qquad \text{(R3.16.1)}$$
$$C_{13}^2C_{12}^2 + C_{13}^3C_{13}^2 = 0 \qquad \text{(R3.16.2)}$$
$$C_{13}^1C_{11}^3 + C_{13}^2C_{12}^3 + C_{13}^3C_{13}^3 = 0 \qquad \text{(R3.16.3)}$$

$$(C^3_{12}+C^3_{21})(C^2_{13}+C^2_{13})=0 \tag{R3.17.2.2}$$
$$(C^3_{12}+C^3_{21})(C^3_{13}-C^3_{31})=0 \tag{R3.17.3.2}$$
$$(C^3_{13}+C^3_{31})(C^2_{13}+C^2_{13})=0 \tag{R3.18.2.2}$$
$$(C^3_{13}+C^3_{31})(C^3_{13}-C^3_{31})=0 \tag{R3.18.3.2}$$
$$C^3_{22}(C^2_{13}+C^2_{13})=0 \tag{R3.19.2.2}$$
$$C^3_{22}(C^3_{13}-C^3_{31})=0 \tag{R3.19.3.2}$$
$$C^3_{23}(C^2_{13}+C^2_{13})=0 \tag{R3.20.2.2}$$
$$C^3_{23}(C^3_{13}-C^3_{31})=0 \tag{R3.20.3.2}$$
$$C^3_{32}(C^2_{13}+C^2_{13})=0 \tag{R3.21.2.2}$$
$$C^3_{32}(C^3_{13}-C^3_{31})=0 \tag{R3.21.3.2}$$
$$C^3_{33}(C^2_{13}+C^2_{13})=0 \tag{R3.22.2.2}$$
$$C^3_{33}(C^3_{13}-C^3_{31})=0 \tag{R3.22.3.2}$$
$$C^1_{21}C^1_{11}-C^2_{12}C^1_{21}+C^3_{21}C^1_{13}=0 \tag{R3.23.1}$$
$$C^2_{12}C^2_{12}-C^3_{21}C^2_{13}=0 \tag{R3.23.2}$$
$$C^1_{21}C^3_{11}-C^2_{12}C^3_{21}+C^3_{21}C^3_{31}=0 \tag{R3.23.3}$$
$$C^1_{22}C^1_{11}+C^3_{22}C^1_{13}=0 \tag{R3.26.1}$$
$$-C^3_{22}C^2_{13}=0 \tag{R3.26.2}$$
$$C^1_{22}C^3_{11}+C^3_{22}C^3_{31}=0 \tag{R3.26.3}$$
$$C^1_{23}C^1_{11}+C^3_{23}C^1_{13}=0 \tag{R3.27.1}$$
$$-C^3_{23}C^2_{13}=0 \tag{R3.27.2}$$
$$C^1_{23}C^3_{11}+C^3_{23}C^3_{31}=0 \tag{R3.27.3}$$
$$C^1_{13}C^1_{11}-C^2_{13}C^1_{21}+C^3_{31}C^1_{13}=0 \tag{R3.32.1}$$
$$C^2_{13}C^2_{12}-C^3_{31}C^2_{13}=0 \tag{R3.32.2}$$
$$C^1_{13}C^3_{11}-C^2_{13}C^3_{21}+C^3_{31}C^3_{31}=0 \tag{R3.32.3}$$
$$C^1_{32}C^1_{11}+C^3_{32}C^1_{13}=0 \tag{R3.35.1}$$
$$-C^3_{32}C^2_{13}=0 \tag{R3.35.2}$$
$$C^1_{32}C^3_{11}+C^3_{32}C^3_{31}=0 \tag{R3.35.3}$$
$$C^1_{33}C^1_{11}+C^3_{33}C^1_{13}=0 \tag{R3.36.1}$$
$$-C^3_{33}C^2_{13}=0 \tag{R3.36.2}$$
$$C^1_{33}C^3_{11}+C^3_{33}C^3_{31}=0. \tag{R3.36.3}$$

We see that equation (R3.26.2), $-C^3_{22}C^2_{13}=0$ has the two solutions $C^3_{22}=0$ and $C^2_{13}=0$, so we look at each of those cases.

Case 1.2.1: If $C_{22}^3 = 0$, our system of equations becomes

$$C_{11}^3(C_{13}^2 + C_{13}^2) = 0 \quad \text{(R3.14.2)}$$
$$C_{11}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.14.3)}$$
$$C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1 + C_{12}^3 C_{13}^1 = 0 \quad \text{(R3.15.1)}$$
$$C_{12}^2 C_{12}^2 + C_{12}^3 C_{13}^2 = 0 \quad \text{(R3.15.2)}$$
$$C_{12}^1 C_{11}^3 + C_{12}^2 C_{12}^3 + C_{12}^3 C_{13}^3 = 0 \quad \text{(R3.15.3)}$$
$$C_{13}^1 C_{11}^1 + C_{13}^2 C_{12}^1 + C_{13}^3 C_{13}^1 = 0 \quad \text{(R3.16.1)}$$
$$C_{13}^2 C_{12}^2 + C_{13}^3 C_{13}^2 = 0 \quad \text{(R3.16.2)}$$
$$C_{13}^1 C_{11}^3 + C_{13}^2 C_{12}^3 + C_{13}^3 C_{13}^3 = 0 \quad \text{(R3.16.3)}$$
$$(C_{12}^3 + C_{21}^3)(C_{13}^2 + C_{13}^2) = 0 \quad \text{(R3.17.2.2)}$$
$$(C_{12}^3 + C_{21}^3)(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.17.3.2)}$$
$$(C_{13}^3 + C_{31}^3)(C_{13}^2 + C_{13}^2) = 0 \quad \text{(R3.18.2.2)}$$
$$(C_{13}^3 + C_{31}^3)(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.18.3.2)}$$
$$C_{23}^3(C_{13}^2 + C_{13}^2) = 0 \quad \text{(R3.20.2.2)}$$
$$C_{23}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.20.3.2)}$$
$$C_{32}^3(C_{13}^2 + C_{13}^2) = 0 \quad \text{(R3.21.2.2)}$$
$$C_{32}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.21.3.2)}$$
$$C_{33}^3(C_{13}^2 + C_{13}^2) = 0 \quad \text{(R3.22.2.2)}$$
$$C_{33}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.22.3.2)}$$
$$C_{21}^1 C_{11}^1 - C_{12}^2 C_{21}^1 + C_{21}^3 C_{13}^1 = 0 \quad \text{(R3.23.1)}$$
$$C_{12}^2 C_{12}^2 - C_{21}^3 C_{13}^2 = 0 \quad \text{(R3.23.2)}$$
$$C_{21}^1 C_{11}^3 - C_{12}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \quad \text{(R3.23.3)}$$
$$C_{22}^1 C_{11}^1 = 0 \quad \text{(R3.26.1)}$$
$$C_{22}^1 C_{11}^3 = 0 \quad \text{(R3.26.3)}$$
$$C_{23}^1 C_{11}^1 + C_{23}^3 C_{13}^1 = 0 \quad \text{(R3.27.1)}$$
$$-C_{23}^3 C_{13}^2 = 0 \quad \text{(R3.27.2)}$$
$$C_{23}^1 C_{11}^3 + C_{23}^3 C_{31}^3 = 0 \quad \text{(R3.27.3)}$$
$$C_{13}^1 C_{11}^1 - C_{13}^2 C_{21}^1 + C_{31}^3 C_{13}^1 = 0 \quad \text{(R3.32.1)}$$
$$C_{13}^2 C_{12}^2 - C_{31}^3 C_{13}^2 = 0 \quad \text{(R3.32.2)}$$
$$C_{13}^1 C_{11}^3 - C_{13}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \quad \text{(R3.32.3)}$$
$$C_{32}^1 C_{11}^1 + C_{32}^3 C_{13}^1 = 0 \quad \text{(R3.35.1)}$$
$$-C_{32}^3 C_{13}^2 = 0 \quad \text{(R3.35.2)}$$
$$C_{32}^1 C_{11}^3 + C_{32}^3 C_{31}^3 = 0 \quad \text{(R3.35.3)}$$

$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$-C^3_{33}C^2_{13} = 0 \qquad \text{(R3.36.2)}$$
$$C^1_{33}C^3_{11} + C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

Now, we see that equations (R3.26.1) and (R3.26.3), $C^1_{22}C^1_{11} = 0$ and $C^1_{22}C^3_{11} = 0$ both have one solution, $C^1_{22} = 0$, and if $C^1_{22} \neq 0$ the only solutions are $C^1_{11} = 0$ and $C^3_{11} = 0$, so we look at each of these cases.

Case 1.2.1.1: If $C^1_{22} = 0$, the system of equations becomes

$$C^3_{11}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.14.2)}$$
$$C^3_{11}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.14.3)}$$
$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^1_{12}C^3_{11} + C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = 0 \qquad \text{(R3.16.2)}$$
$$C^1_{13}C^3_{11} + C^2_{13}C^3_{12} + C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{13} + C^3_{31})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{23}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.20.2.2)}$$
$$C^3_{23}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.20.3.2)}$$
$$C^3_{32}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.21.2.2)}$$
$$C^3_{32}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.21.3.2)}$$
$$C^3_{33}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.22.2.2)}$$
$$C^3_{33}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} - C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$C^1_{21}C^3_{11} - C^2_{12}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{23}C^1_{11} + C^3_{23}C^1_{13} = 0 \qquad \text{(R3.27.1)}$$
$$-C^3_{23}C^2_{13} = 0 \qquad \text{(R3.27.2)}$$
$$C^1_{23}C^3_{11} + C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^1_{13}C^1_{11} - C^2_{13}C^1_{21} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} - C^3_{31}C^2_{13} = 0 \qquad \text{(R3.32.2)}$$

$$C^1_{13}C^3_{11} - C^2_{13}C^3_{21} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} + C^3_{32}C^1_{13} = 0 \qquad \text{(R3.35.1)}$$
$$-C^3_{32}C^2_{13} = 0 \qquad \text{(R3.35.2)}$$
$$C^1_{32}C^3_{11} + C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$-C^3_{33}C^2_{13} = 0 \qquad \text{(R3.36.2)}$$
$$C^1_{33}C^3_{11} + C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

We now see that the solutions to equations (R3.27.2) and (R3.35.2) are either $C^2_{13} = 0$ or if $C^2_{13} \neq 0$, then $C^3_{23} = 0$ and $C^3_{32} = 0$.

Case 1.2.1.1.1: If $C^2_{13} = 0$, our system of equations becomes

$$C^3_{11}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.14.3)}$$
$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} = 0 \qquad \text{(R3.15.2)}$$
$$C^1_{12}C^3_{11} + C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^1_{13}C^3_{11} + C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{23}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.20.3.2)}$$
$$C^3_{32}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.21.3.2)}$$
$$C^3_{33}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} = 0 \qquad \text{(R3.23.2)}$$
$$C^1_{21}C^3_{11} - C^2_{12}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{23}C^1_{11} + C^3_{23}C^1_{13} = 0 \qquad \text{(R3.27.1)}$$
$$C^1_{23}C^3_{11} + C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^1_{13}C^1_{11} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^1_{13}C^3_{11} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} + C^3_{32}C^1_{13} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{32}C^3_{11} + C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$C^1_{33}C^3_{11} + C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

The only solution to equation (R3.15.2) is $C^2_{12} = 0$, which gives us

$$C^3_{11}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.14.3)}$$
$$C^1_{12}C^1_{11} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^1_{12}C^3_{11} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^1_{13}C^3_{11} + C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{23}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.20.3.2)}$$
$$C^3_{32}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.21.3.2)}$$
$$C^3_{33}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^1_{21}C^3_{11} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{23}C^1_{11} + C^3_{23}C^1_{13} = 0 \qquad \text{(R3.27.1)}$$
$$C^1_{23}C^3_{11} + C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^1_{13}C^1_{11} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^1_{13}C^3_{11} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} + C^3_{32}C^1_{13} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{32}C^3_{11} + C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$C^1_{33}C^3_{11} + C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

Since we no longer have any equation that is just a product of two variables, we look at the first equation, which is the product of a variable and the difference between two variables. It is easy to see that its two solutions are $C^3_{11} = 0$ and $C^3_{13} - C^3_{31} = 0$, so we will look at each of those cases.

Case 1.2.1.1.1.1: If $C^3_{11} = 0$, the system of equations becomes

$$C^1_{12}C^1_{11} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{23}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.20.3.2)}$$

$$C^3_{32}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.21.3.2)}$$
$$C^3_{33}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{23}C^1_{11} + C^3_{23}C^1_{13} = 0 \qquad \text{(R3.27.1)}$$
$$C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^1_{13}C^1_{11} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} + C^3_{32}C^1_{13} = 0 \qquad \text{(R3.35.1)}$$
$$C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

The only solution to equation (R3.16.3) is $C^3_{13} = 0$, which gives us:

$$C^1_{12}C^1_{11} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^1_{13}C^1_{11} = 0 \qquad \text{(R3.16.1)}$$
$$(C^3_{12} + C^3_{21})(-C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{31})(-C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{23}(-C^3_{31}) = 0 \qquad \text{(R3.20.3.2)}$$
$$C^3_{32}(-C^3_{31}) = 0 \qquad \text{(R3.21.3.2)}$$
$$C^3_{33}(-C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{23}C^1_{11} + C^3_{23}C^1_{13} = 0 \qquad \text{(R3.27.1)}$$
$$C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^1_{13}C^1_{11} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} + C^3_{32}C^1_{13} = 0 \qquad \text{(R3.35.1)}$$
$$C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

The only solution to equations (R3.18.3.2) and (R3.32.3) is clearly

$C^3_{31} = 0$. Inserting this into our system of equations gives us

$$
\begin{aligned}
C^1_{12}C^1_{11} + C^3_{12}C^1_{13} &= 0 && \text{(R3.15.1)}\\
C^1_{13}C^1_{11} &= 0 && \text{(R3.16.1)}\\
C^1_{21}C^1_{11} + C^3_{21}C^1_{13} &= 0 && \text{(R3.23.1)}\\
C^1_{23}C^1_{11} + C^3_{23}C^1_{13} &= 0 && \text{(R3.27.1)}\\
C^1_{13}C^1_{11} &= 0 && \text{(R3.32.1)}\\
C^1_{32}C^1_{11} + C^3_{32}C^1_{13} &= 0 && \text{(R3.35.1)}\\
C^1_{33}C^1_{11} + C^3_{33}C^1_{13} &= 0. && \text{(R3.36.1).}
\end{aligned}
$$

Now, first, we can remove equation (R3.32.1) since it is identical to equation (R3.16.1). Then, we look at equation (R3.16.1) and see that its solutions are $C^1_{13} = 0$ and $C^1_{11} = 0$.
Case 1.2.1.1.1.1.1: If $C^1_{13} = 0$, the system of equations becomes

$$
\begin{aligned}
C^1_{12}C^1_{11} &= 0 && \text{(R3.15.1)}\\
C^1_{21}C^1_{11} &= 0 && \text{(R3.23.1)}\\
C^1_{23}C^1_{11} &= 0 && \text{(R3.27.1)}\\
C^1_{32}C^1_{11} &= 0 && \text{(R3.35.1)}\\
C^1_{33}C^1_{11} &= 0. && \text{(R3.36.1)}
\end{aligned}
$$

One solution to this whole system of equations is clearly $C^1_{11} = 0$, but if $C^1_{11} \neq 0$ we get our second solution, which is that $C^1_{12} = C^1_{21} = C^1_{23} = C^1_{32} = C^1_{33} = 0$. However, we have assumed that $C^1_{12} \neq C^1_{21}$, so this solution is not possible. Thus, the only possible solution is $C^1_{11} = 0$, and we have now solved the entire system of equations.
Case 1.2.1.1.1.1.2: If we instead look at the possible solution $C^1_{13} \neq 0$ and $C^1_{11} = 0$, the system of equations becomes:

$$
\begin{aligned}
C^3_{21}C^1_{13} &= 0 && \text{(R3.23.1)}\\
C^3_{23}C^1_{13} &= 0 && \text{(R3.27.1)}\\
C^3_{32}C^1_{13} &= 0 && \text{(R3.35.1)}\\
C^3_{33}C^1_{13} &= 0. && \text{(R3.36.1)}
\end{aligned}
$$

Since we have assumed that $C^1_{13} \neq 0$, we see that the only solution to this system of equations is $C^3_{21} = C^3_{23} = C^3_{32} = C^3_{33} = 0$. We have now solved the entire system of equations.
Case 1.2.1.1.1.2: If we now instead have $C^3_{11} \neq 0$ and $C^3_{13} - C^3_{31} = 0$,

the system of equations becomes

$$C_{12}^1 C_{11}^1 + C_{12}^3 C_{13}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 + C_{12}^3 C_{13}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^1 C_{11}^1 + C_{13}^3 C_{13}^1 = 0 \qquad \text{(R3.16.1)}$$
$$C_{13}^1 C_{11}^3 + C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.16.3)}$$
$$C_{21}^1 C_{11}^1 + C_{21}^3 C_{13}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 + C_{21}^3 C_{13}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{23}^1 C_{11}^1 + C_{23}^3 C_{13}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 + C_{23}^3 C_{13}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{13}^1 C_{11}^1 + C_{13}^3 C_{13}^1 = 0 \qquad \text{(R3.32.1)}$$
$$C_{13}^1 C_{11}^3 + C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^1 C_{11}^1 + C_{32}^3 C_{13}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 + C_{32}^3 C_{13}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 + C_{33}^3 C_{13}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 + C_{33}^3 C_{13}^3 = 0. \qquad \text{(R3.36.3)}$$

We can remove equations (R3.32.1) and (R3.32.3) since they are identical to equations (R3.16.1) and (R3.16.3). Then, due to commutativity, we see that we can rewrite equation (R3.16.1) as $C_{13}^1 C_{11}^1 + C_{13}^3 C_{13}^1 = C_{13}^1 C_{11}^1 + C_{13}^1 C_{13}^3 = C_{13}^1 (C_{11}^1 + C_{13}^3) = 0$. The system of equations will then look as follows:

$$C_{12}^1 C_{11}^1 + C_{12}^3 C_{13}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 + C_{12}^3 C_{13}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^1 (C_{11}^1 + C_{13}^3) = 0 \qquad \text{(R3.16.1)}$$
$$C_{13}^1 C_{11}^3 + C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.16.3)}$$
$$C_{21}^1 C_{11}^1 + C_{21}^3 C_{13}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 + C_{21}^3 C_{13}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{23}^1 C_{11}^1 + C_{23}^3 C_{13}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 + C_{23}^3 C_{13}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{32}^1 C_{11}^1 + C_{32}^3 C_{13}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 + C_{32}^3 C_{13}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 + C_{33}^3 C_{13}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 + C_{33}^3 C_{13}^3 = 0. \qquad \text{(R3.36.3)}$$

The solutions to (R3.16.1) are $C_{13}^1 = 0$ and $C_{11}^1 + C_{13}^3 = 0$.

Case 1.2.1.1.1.2.1: If $C_{13}^1 = 0$, our system of equations becomes

$$C_{12}^1 C_{11}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 + C_{12}^3 C_{13}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.16.3)}$$
$$C_{21}^1 C_{11}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 + C_{21}^3 C_{13}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{23}^1 C_{11}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 + C_{23}^3 C_{13}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{32}^1 C_{11}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 + C_{32}^3 C_{13}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 + C_{33}^3 C_{13}^3 = 0. \qquad \text{(R3.36.3)}$$

The only solution to equation (R3.16.3) is $C_{13}^3 = 0$. Thus, our system of equations becomes

$$C_{12}^1 C_{11}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{21}^1 C_{11}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{23}^1 C_{11}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{32}^1 C_{11}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 = 0. \qquad \text{(R3.36.3)}$$

Now, since we have assumed that $C_{11}^3 \neq 0$, the only solution to equation (R3.15.3) is $C_{12}^1 = 0$, the only solution to equation (R3.23.3) is $C_{21}^1 = 0$, the only solution to equation (R3.27.3) is $C_{23}^1 = 0$, the only solution to equation (R3.35.3) is $C_{32}^1 = 0$, and the only solution to equation (R3.36.3) is $C_{33}^1 = 0$. However, we have assumed that $C_{12}^1 \neq C_{21}^1$, which means that our solution $C_{12}^1 = C_{21}^1 = 0$ is not possible. Thus, this case did not lead to a solution of the system of equations.
Case 1.2.1.1.1.2.2: If we instead have $C_{13}^1 \neq 0$ and $C_{11}^1 + C_{13}^3 = 0$,

the system of equations becomes

$$C_{12}^1 C_{11}^1 + C_{12}^3 C_{13}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 - C_{12}^3 C_{11}^1 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^1 C_{11}^3 + C_{11}^1 C_{11}^1 = 0 \qquad \text{(R3.16.3)}$$
$$C_{21}^1 C_{11}^1 + C_{21}^3 C_{13}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 - C_{21}^3 C_{11}^1 = 0 \qquad \text{(R3.23.3)}$$
$$C_{23}^1 C_{11}^1 + C_{23}^3 C_{13}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 - C_{23}^3 C_{11}^1 = 0 \qquad \text{(R3.27.3)}$$
$$C_{32}^1 C_{11}^1 + C_{32}^3 C_{13}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 - C_{32}^3 C_{11}^1 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 + C_{33}^3 C_{13}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 - C_{33}^3 C_{11}^1 = 0. \qquad \text{(R3.36.3)}$$

Now, looking at the equations we see that they follow some kind of pattern, where the first two equations are a pair, and the remaining equations pair up in order as well – except for equation (R3.16.3), which is on its own.
We begin by looking at equation (R3.15.3) Since we have already assumed that $C_{11}^3 \neq 0$, the solution to this equation is $C_{12}^1 = C_{12}^3 C_{11}^1 / C_{11}^3$. We now substitute this solution into equation (R3.15.1) and get $C_{12}^1 C_{11}^1 + C_{12}^3 C_{13}^1 = C_{12}^3 C_{11}^1 C_{11}^1 / C_{11}^3 + C_{12}^3 C_{13}^1 = 0$. Now, we look at equation (R3.16.3), and see that $C_{13}^1 = -C_{11}^1 C_{11}^1 / C_{11}^3$ solves the equation. Since C_{13}^1 is a variable in equation (R3.15.1) we plug this new formula into that equation, and get $C_{12}^3 C_{11}^1 C_{11}^1 / C_{11}^3 + C_{12}^3 C_{13}^1 = C_{12}^3 C_{11}^1 C_{11}^1 / C_{11}^3 - C_{12}^3 C_{11}^1 C_{11}^1 / C_{11}^3 = 0$. Clearly the solutions to equations (R3.15.3) and (R3.16.3) are a solution to equation (R3.15.1) as well. We now look at the rest of the equations in the system of equations in the same way:

$$\begin{aligned}
\text{(R3.23.3)}: &\quad & C_{21}^1 &= C_{21}^3 C_{11}^1 / C_{11}^3, \\
\text{(R3.23.1)}: &\quad C_{21}^1 C_{11}^1 + C_{21}^3 C_{13}^1 & &= C_{21}^3 C_{11}^1 C_{11}^1 / C_{11}^3 + C_{21}^3 C_{13}^1 \\
& & &= C_{21}^3 C_{11}^1 C_{11}^1 / C_{11}^3 - C_{21}^3 C_{11}^1 C_{11}^1 / C_{11}^3 \\
& & &= 0, \\
\text{(R3.27.3)}: &\quad & C_{23}^1 &= C_{23}^3 C_{11}^1 / C_{11}^3, \\
\text{(R3.27.1)}: &\quad C_{23}^1 C_{11}^1 + C_{23}^3 C_{13}^1 & &= C_{23}^3 C_{11}^1 C_{11}^1 / C_{11}^3 + C_{23}^3 C_{13}^1 \\
& & &= C_{23}^3 C_{11}^1 C_{11}^1 / C_{11}^3 - C_{23}^3 C_{11}^1 C_{11}^1 / C_{11}^3 \\
& & &= 0, \\
\text{(R3.35.3)}: &\quad & C_{32}^1 &= C_{32}^3 C_{11}^1 / C_{11}^3,
\end{aligned}$$

$$\begin{aligned}
\text{(R3.35.1)}: \quad C^1_{32}C^1_{11} + C^3_{32}C^1_{13} &= C^3_{32}C^1_{11}C^1_{11}/C^3_{11} + C^3_{32}C^1_{13} \\
&= C^3_{32}C^1_{11}C^1_{11}/C^3_{11} - C^3_{32}C^1_{11}C^1_{11}/C^3_{11} \\
&= 0, \\
\text{(R3.36.3)}: \quad C^1_{33} &= C^3_{33}C^1_{11}/C^3_{11}, \\
\text{(R3.36.1)}: \quad C^1_{33}C^1_{11} + C^3_{33}C^1_{13} &= C^3_{33}C^1_{11}C^1_{11}/C^3_{11} + C^3_{33}C^1_{13} \\
&= C^3_{33}C^1_{11}C^1_{11}/C^3_{11} - C^3_{33}C^1_{11}C^1_{11}/C^3_{11} \\
&= 0.
\end{aligned}$$

Thus, we have shown that the following values of our variables solve the remainder of the system of equations:

$$\begin{aligned}
C^1_{12} &= C^3_{12}C^1_{11}/C^3_{11}, \\
C^1_{13} &= -C^1_{11}C^1_{11}/C^3_{11}, \\
C^1_{21} &= C^3_{21}C^1_{11}/C^3_{11}, \\
C^1_{23} &= C^3_{23}C^1_{11}/C^3_{11}, \\
C^1_{32} &= C^3_{32}C^1_{11}/C^3_{11}, \\
C^1_{33} &= C^3_{33}C^1_{11}/C^3_{11}.
\end{aligned}$$

We can also note that $C^1_{11} = -C^3_{13}$, so $C^1_{13} = -C^1_{11}C^1_{11}/C^3_{11} = C^3_{13}C^1_{11}/C^3_{11}$, which means it follows the same pattern as the other variables.

Case 1.2.1.1.2: If we have $C^2_{13} \neq 0$, $C^3_{23} = 0$ and $C^3_{32} = 0$, our system of equations becomes

$$\begin{aligned}
C^3_{11}(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.14.2)} \\
C^3_{11}(C^3_{13} - C^3_{31}) &= 0 && \text{(R3.14.3)} \\
C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} &= 0 && \text{(R3.15.1)} \\
C^2_{12}C^2_{12} + C^3_{12}C^2_{13} &= 0 && \text{(R3.15.2)} \\
C^1_{12}C^3_{11} + C^2_{12}C^3_{12} + C^3_{12}C^3_{13} &= 0 && \text{(R3.15.3)} \\
C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} &= 0 && \text{(R3.16.1)} \\
C^2_{13}C^2_{12} + C^3_{13}C^2_{13} &= 0 && \text{(R3.16.2)} \\
C^1_{13}C^3_{11} + C^2_{13}C^3_{12} + C^3_{13}C^3_{13} &= 0 && \text{(R3.16.3)} \\
(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.17.2.2)} \\
(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) &= 0 && \text{(R3.17.3.2)} \\
(C^3_{13} + C^3_{31})(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.18.2.2)} \\
(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) &= 0 && \text{(R3.18.3.2)} \\
C^3_{33}(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.22.2.2)} \\
C^3_{33}(C^3_{13} - C^3_{31}) &= 0 && \text{(R3.22.3.2)}
\end{aligned}$$

$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} - C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$C^1_{21}C^3_{11} - C^2_{12}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{23}C^1_{11} = 0 \qquad \text{(R3.27.1)}$$
$$C^1_{23}C^3_{11} = 0 \qquad \text{(R3.27.3)}$$
$$C^1_{13}C^1_{11} - C^2_{13}C^1_{21} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} - C^3_{31}C^2_{13} = 0 \qquad \text{(R3.32.2)}$$
$$C^1_{13}C^3_{11} - C^2_{13}C^3_{21} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{32}C^3_{11} = 0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$-C^3_{33}C^2_{13} = 0 \qquad \text{(R3.36.2)}$$
$$C^1_{33}C^3_{11} + C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

If we look at equations (R3.27.3) and (R3.35.3), we see that one solution to both equations is $C^3_{11} = 0$, but if we instead assume that $C^3_{11} \neq 0$, we get another solution $C^1_{23} = C^1_{32} = 0$. We look at both cases.

Case 1.2.1.1.2.1: If $C^3_{11} = 0$ our system of equations becomes

$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = 0 \qquad \text{(R3.16.2)}$$
$$C^2_{13}C^3_{12} + C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{13} + C^3_{31})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{33}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.22.2.2)}$$
$$C^3_{33}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} - C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$-C^2_{12}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{23}C^1_{11} = 0 \qquad \text{(R3.27.1)}$$

$$C^1_{13}C^1_{11} - C^2_{13}C^1_{21} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} - C^3_{31}C^2_{13} = 0 \qquad \text{(R3.32.2)}$$
$$-C^2_{13}C^3_{21} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$-C^3_{33}C^2_{13} = 0 \qquad \text{(R3.36.2)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

Looking at equations (R3.27.1) and (R3.35.1), we see that one solution to both equations is $C^1_{11} = 0$, while if we assume $C^1_{11} \neq 0$ we need to have $C^1_{23} = C^1_{32} = 0$, which is another solution. We look at both of these solutions.

Case 1.2.1.1.2.1.1: If $C^1_{11} = 0$, the system of equations becomes

$$C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^2_{13}C^1_{12} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = 0 \qquad \text{(R3.16.2)}$$
$$C^2_{13}C^3_{12} + C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{13} + C^3_{31})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{33}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.22.2.2)}$$
$$C^3_{33}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$-C^2_{12}C^1_{21} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} - C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$-C^2_{12}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$-C^2_{13}C^1_{21} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} - C^3_{31}C^2_{13} = 0 \qquad \text{(R3.32.2)}$$
$$-C^2_{13}C^3_{21} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$-C^3_{33}C^2_{13} = 0 \qquad \text{(R3.36.2)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

If we look at equations (R3.36.1) and (R3.36.3) we see that the only two possible solutions are $C^3_{33} = 0$, or, if $C^3_{33} \neq 0$, $C^1_{13} = C^3_{31} = 0$.

Case 1.2.1.1.2.1.1.1: If $C^3_{33} = 0$, the system of equations becomes

$$
\begin{aligned}
C^2_{12}C^1_{12} + C^3_{12}C^1_{13} &= 0 && \text{(R3.15.1)}\\
C^2_{12}C^2_{12} + C^3_{12}C^2_{13} &= 0 && \text{(R3.15.2)}\\
C^2_{12}C^3_{12} + C^3_{12}C^3_{13} &= 0 && \text{(R3.15.3)}\\
C^2_{13}C^1_{12} + C^3_{13}C^1_{13} &= 0 && \text{(R3.16.1)}\\
C^2_{13}C^2_{12} + C^3_{13}C^2_{13} &= 0 && \text{(R3.16.2)}\\
C^2_{13}C^3_{12} + C^3_{13}C^3_{13} &= 0 && \text{(R3.16.3)}\\
(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.17.2.2)}\\
(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) &= 0 && \text{(R3.17.3.2)}\\
(C^3_{13} + C^3_{31})(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.18.2.2)}\\
(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) &= 0 && \text{(R3.18.3.2)}\\
-C^2_{12}C^1_{21} + C^3_{21}C^1_{13} &= 0 && \text{(R3.23.1)}\\
C^2_{12}C^2_{12} - C^3_{21}C^2_{13} &= 0 && \text{(R3.23.2)}\\
-C^2_{12}C^3_{21} + C^3_{21}C^3_{31} &= 0 && \text{(R3.23.3)}\\
-C^2_{13}C^1_{21} + C^3_{31}C^1_{13} &= 0 && \text{(R3.32.1)}\\
C^2_{13}C^2_{12} - C^3_{31}C^2_{13} &= 0 && \text{(R3.32.2)}\\
-C^2_{13}C^3_{21} + C^3_{31}C^3_{31} &= 0. && \text{(R3.32.3)}
\end{aligned}
$$

Looking at (R3.16.2), we see that due to commutativity, we can rewrite it as $C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = C^2_{13}C^2_{12} + C^2_{13}C^3_{13} = C^2_{13}(C^2_{12} + C^3_{13}) = 0$, and since we have assumed that $C^2_{13} \neq 0$, the only solution to this equations is $C^2_{12} + C^3_{13} = 0$. This changes our system of equations to

$$
\begin{aligned}
C^2_{12}C^1_{12} + C^3_{12}C^1_{13} &= 0 && \text{(R3.15.1)}\\
C^2_{12}C^2_{12} + C^3_{12}C^2_{13} &= 0 && \text{(R3.15.2)}\\
C^2_{13}C^1_{12} - C^2_{12}C^1_{13} &= 0 && \text{(R3.16.1)}\\
C^2_{13}C^3_{12} + C^2_{12}C^2_{12} &= 0 && \text{(R3.16.3)}\\
(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.17.2.2)}\\
(C^3_{12} + C^3_{21})(-C^2_{12} - C^3_{31}) &= 0 && \text{(R3.17.3.2)}\\
(-C^2_{12} + C^3_{31})(C^2_{13} + C^2_{13}) &= 0 && \text{(R3.18.2.2)}\\
(-C^2_{12} + C^3_{31})(-C^2_{12} - C^3_{31}) &= 0 && \text{(R3.18.3.2)}\\
-C^2_{12}C^1_{21} + C^3_{21}C^1_{13} &= 0 && \text{(R3.23.1)}\\
C^2_{12}C^2_{12} - C^3_{21}C^2_{13} &= 0 && \text{(R3.23.2)}
\end{aligned}
$$

$$-C^2_{12}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$-C^2_{13}C^1_{21} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} - C^3_{31}C^2_{13} = 0 \qquad \text{(R3.32.2)}$$
$$-C^2_{13}C^3_{21} + C^3_{31}C^3_{31} = 0. \qquad \text{(R3.32.3)}$$

We can remove equation (R3.16.3) since we see that, due to commutativity and associativity, it is identical to (R3.15.2). Now, $C^2_{13} + C^2_{13} = 2 \cdot C^2_{13} \neq 0$ since $C^2_{13} \neq 0$, which means that equation (R3.18.2.2) only has the solution $-C^2_{12} + C^3_{31} = 0$. Substituting this into our system of equations gives us

$$C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{13}C^1_{12} - C^2_{12}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^2_{13}C^3_{12} + C^2_{12}C^2_{12} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^3_{12} + C^3_{21})(-C^2_{12} - C^2_{12}) = 0 \qquad \text{(R3.17.3.2)}$$
$$-C^2_{12}C^1_{21} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} + -C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$-C^2_{12}C^3_{21} + C^3_{21}C^2_{12} = 0 \qquad \text{(R3.23.3)}$$
$$-C^2_{13}C^1_{21} + C^2_{12}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$-C^2_{13}C^3_{21} + C^2_{12}C^2_{12} = 0. \qquad \text{(R3.32.3)}$$

Now, adding equation (R3.32.1) to equation (R3.16.1), we get

$$\begin{aligned} & C^2_{13}C^1_{12} - C^2_{12}C^1_{13} - C^2_{13}C^1_{21} + C^2_{12}C^1_{13} \\ &= C^2_{13}C^1_{12} - C^2_{13}C^1_{21} \\ &= C^2_{13}(C^1_{12} - C^1_{21}) \\ &= 0. \end{aligned}$$

Since we have already assumed both $C^2_{13} \neq 0$ and $C^1_{12} - C^1_{21} \neq 0$, this equation does not have a solution, which means that this case did not lead to a solution of the system of equations.

Case 1.2.1.1.2.1.1.2: If we instead have $C^3_{33} \neq 0$ and $C^1_{13} = C^3_{31} = 0$, the system of equations becomes

$$C^2_{12}C^1_{12} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$

$$C^2_{13}C^1_{12} = 0 \qquad \text{(R3.16.1)}$$
$$C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = 0 \qquad \text{(R3.16.2)}$$
$$C^2_{13}C^3_{12} + C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^3_{12} + C^3_{21})C^3_{13} = 0 \qquad \text{(R3.17.3.2)}$$
$$C^3_{13}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.18.2.2)}$$
$$C^3_{13}C^3_{13} = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{33}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.22.2.2)}$$
$$C^3_{33}C^3_{13} = 0 \qquad \text{(R3.22.3.2)}$$
$$-C^2_{12}C^1_{21} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} - C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$-C^2_{12}C^3_{21} = 0 \qquad \text{(R3.23.3)}$$
$$-C^2_{13}C^1_{21} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} = 0 \qquad \text{(R3.32.2)}$$
$$-C^2_{13}C^3_{21} = 0 \qquad \text{(R3.32.3)}$$
$$-C^3_{33}C^2_{13} = 0. \qquad \text{(R3.36.2)}$$

We see that the only solution to equation (R3.18.3.2) is $C^3_{13} = 0$, and inserting this into the system of equations we get

$$C^2_{12}C^1_{12} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12} = 0 \qquad \text{(R3.15.3)}$$
$$C^2_{13}C^1_{12} = 0 \qquad \text{(R3.16.1)}$$
$$C^2_{13}C^2_{12} = 0 \qquad \text{(R3.16.2)}$$
$$C^2_{13}C^3_{12} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.17.2.2)}$$
$$C^3_{33}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.22.2.2)}$$
$$-C^2_{12}C^1_{21} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} - C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$-C^2_{12}C^3_{21} = 0 \qquad \text{(R3.23.3)}$$
$$-C^2_{13}C^1_{21} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} = 0 \qquad \text{(R3.32.2)}$$
$$-C^2_{13}C^3_{21} = 0 \qquad \text{(R3.32.3)}$$
$$-C^3_{33}C^2_{13} = 0. \qquad \text{(R3.36.2)}$$

Now, since we have assumed that $C^3_{33} \neq 0$, the only solution to equation (R3.36.2) is $C^2_{13} = 0$. However, we have also assumed that $C^2_{13} \neq 0$, which is a contradiction. Thus, this case did not lead to any solution of the system of equations.

Case 1.2.1.1.2.1.2: If we now instead have $C^1_{11} \neq 0$ and $C^1_{23} = C^1_{32} = 0$, the system of equations becomes

$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = 0 \qquad \text{(R3.16.2)}$$
$$C^2_{13}C^3_{12} + C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$
$$(C^3_{12} + C^3_{21})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^3_{13} + C^3_{31})(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{33}(C^2_{13} + C^2_{13}) = 0 \qquad \text{(R3.22.2.2)}$$
$$C^3_{33}(C^3_{13} - C^3_{31}) = 0 \qquad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} + C^3_{21}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} - C^3_{21}C^2_{13} = 0 \qquad \text{(R3.23.2)}$$
$$-C^2_{12}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{13}C^1_{11} - C^2_{13}C^1_{21} + C^3_{31}C^1_{13} = 0 \qquad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} - C^3_{31}C^2_{13} = 0 \qquad \text{(R3.32.2)}$$
$$-C^2_{13}C^3_{21} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{33}C^1_{11} + C^3_{33}C^1_{13} = 0 \qquad \text{(R3.36.1)}$$
$$-C^3_{33}C^2_{13} = 0 \qquad \text{(R3.36.2)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

One solution to equations (R3.36.2) and (R3.36.3) is $C^3_{33} = 0$, while if $C^3_{33} \neq$ the only solution is $C^2_{13} = C^3_{31} = 0$.

Case 1.2.1.1.2.1.2.1: If $C^3_{33} = 0$, the system of equations becomes

$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$

$$C_{13}^{2}C_{12}^{2}+C_{13}^{3}C_{13}^{2}=0 \qquad \text{(R3.16.2)}$$
$$C_{13}^{2}C_{12}^{3}+C_{13}^{3}C_{13}^{3}=0 \qquad \text{(R3.16.3)}$$
$$(C_{12}^{3}+C_{21}^{3})(C_{13}^{2}+C_{13}^{2})=0 \qquad \text{(R3.17.2.2)}$$
$$(C_{12}^{3}+C_{21}^{3})(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.17.3.2)}$$
$$(C_{13}^{3}+C_{31}^{3})(C_{13}^{2}+C_{13}^{2})=0 \qquad \text{(R3.18.2.2)}$$
$$(C_{13}^{3}+C_{31}^{3})(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.18.3.2)}$$
$$C_{21}^{1}C_{11}^{1}-C_{12}^{2}C_{21}^{1}+C_{21}^{3}C_{13}^{1}=0 \qquad \text{(R3.23.1)}$$
$$C_{12}^{2}C_{12}^{2}-C_{21}^{3}C_{13}^{2}=0 \qquad \text{(R3.23.2)}$$
$$-C_{12}^{2}C_{21}^{3}+C_{21}^{3}C_{31}^{3}=0 \qquad \text{(R3.23.3)}$$
$$C_{13}^{1}C_{11}^{1}-C_{13}^{2}C_{21}^{1}+C_{31}^{3}C_{13}^{1}=0 \qquad \text{(R3.32.1)}$$
$$C_{13}^{2}C_{12}^{2}-C_{31}^{3}C_{13}^{2}=0 \qquad \text{(R3.32.2)}$$
$$-C_{13}^{2}C_{21}^{3}+C_{31}^{3}C_{31}^{3}=0 \qquad \text{(R3.32.3)}$$
$$C_{33}^{1}C_{11}^{1}=0. \qquad \text{(R3.36.1)}$$

Since we have assumed that $C_{11}^{1}\neq 0$, the only solution to equation (R3.36.1) is $C_{33}^{1}=0$. This gives us

$$C_{12}^{1}C_{11}^{1}+C_{12}^{2}C_{12}^{1}+C_{12}^{3}C_{13}^{1}=0 \qquad \text{(R3.15.1)}$$
$$C_{12}^{2}C_{12}^{2}+C_{12}^{3}C_{13}^{2}=0 \qquad \text{(R3.15.2)}$$
$$C_{12}^{2}C_{12}^{3}+C_{12}^{3}C_{13}^{3}=0 \qquad \text{(R3.15.3)}$$
$$C_{13}^{1}C_{11}^{1}+C_{13}^{2}C_{12}^{1}+C_{13}^{3}C_{13}^{1}=0 \qquad \text{(R3.16.1)}$$
$$C_{13}^{2}C_{12}^{2}+C_{13}^{3}C_{13}^{2}=0 \qquad \text{(R3.16.2)}$$
$$C_{13}^{2}C_{12}^{3}+C_{13}^{3}C_{13}^{3}=0 \qquad \text{(R3.16.3)}$$
$$(C_{12}^{3}+C_{21}^{3})(C_{13}^{2}+C_{13}^{2})=0 \qquad \text{(R3.17.2.2)}$$
$$(C_{12}^{3}+C_{21}^{3})(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.17.3.2)}$$
$$(C_{13}^{3}+C_{31}^{3})(C_{13}^{2}+C_{13}^{2})=0 \qquad \text{(R3.18.2.2)}$$
$$(C_{13}^{3}+C_{31}^{3})(C_{13}^{3}-C_{31}^{3})=0 \qquad \text{(R3.18.3.2)}$$
$$C_{21}^{1}C_{11}^{1}-C_{12}^{2}C_{21}^{1}+C_{21}^{3}C_{13}^{1}=0 \qquad \text{(R3.23.1)}$$
$$C_{12}^{2}C_{12}^{2}-C_{21}^{3}C_{13}^{2}=0 \qquad \text{(R3.23.2)}$$
$$-C_{12}^{2}C_{21}^{3}+C_{21}^{3}C_{31}^{3}=0 \qquad \text{(R3.23.3)}$$
$$C_{13}^{1}C_{11}^{1}-C_{13}^{2}C_{21}^{1}+C_{31}^{3}C_{13}^{1}=0 \qquad \text{(R3.32.1)}$$
$$C_{13}^{2}C_{12}^{2}-C_{31}^{3}C_{13}^{2}=0 \qquad \text{(R3.32.2)}$$
$$-C_{13}^{2}C_{21}^{3}+C_{31}^{3}C_{31}^{3}=0. \qquad \text{(R3.32.3)}$$

Now, looking at equations (R3.17.2.2) and (R3.18.2.2), since $C_{13}^{2}\neq 0$ and thus $C_{13}^{2}+C_{13}^{2}\neq 0$, the only solutions to these two equations are $C^3_{12} + C^3_{21} = 0$ and $C^3_{13} + C^3_{31} = 0$. Substituting both of these solutions into the system of equations gives us

$$
\begin{aligned}
C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} &= 0 && \text{(R3.15.1)}\\
C^2_{12}C^2_{12} + C^3_{12}C^2_{13} &= 0 && \text{(R3.15.2)}\\
C^2_{12}C^3_{12} + C^3_{12}C^3_{13} &= 0 && \text{(R3.15.3)}\\
C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} &= 0 && \text{(R3.16.1)}\\
C^2_{13}C^2_{12} + C^3_{13}C^2_{13} &= 0 && \text{(R3.16.2)}\\
C^2_{13}C^3_{12} + C^3_{13}C^3_{13} &= 0 && \text{(R3.16.3)}\\
C^1_{21}C^1_{11} - C^2_{12}C^1_{21} - C^3_{12}C^1_{13} &= 0 && \text{(R3.23.1)}\\
C^2_{12}C^2_{12} + C^3_{12}C^2_{13} &= 0 && \text{(R3.23.2)}\\
C^2_{12}C^3_{12} + C^3_{12}C^3_{13} &= 0 && \text{(R3.23.3)}\\
C^1_{13}C^1_{11} - C^2_{13}C^1_{21} - C^3_{13}C^1_{13} &= 0 && \text{(R3.32.1)}\\
C^2_{13}C^2_{12} + C^3_{13}C^2_{13} &= 0 && \text{(R3.32.2)}\\
C^2_{13}C^3_{12} + C^3_{13}C^3_{13} &= 0. && \text{(R3.32.3)}
\end{aligned}
$$

We see that multiple equations are now identical to one of the other equations, so we remove one of them since having multiple copies of the same equation is clearly unnecessary. This gives us

$$
\begin{aligned}
C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} &= 0 && \text{(R3.15.1)}\\
C^2_{12}C^2_{12} + C^3_{12}C^2_{13} &= 0 && \text{(R3.15.2)}\\
C^2_{12}C^3_{12} + C^3_{12}C^3_{13} &= 0 && \text{(R3.15.3)}\\
C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} &= 0 && \text{(R3.16.1)}\\
C^2_{13}C^2_{12} + C^3_{13}C^2_{13} &= 0 && \text{(R3.16.2)}\\
C^2_{13}C^3_{12} + C^3_{13}C^3_{13} &= 0 && \text{(R3.16.3)}\\
C^1_{21}C^1_{11} - C^2_{12}C^1_{21} - C^3_{12}C^1_{13} &= 0 && \text{(R3.23.1)}\\
C^1_{13}C^1_{11} - C^2_{13}C^1_{21} - C^3_{13}C^1_{13} &= 0. && \text{(R3.32.1)}
\end{aligned}
$$

We now see that we can rewrite equation (R3.16.2) as $C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = C^2_{13}(C^2_{12} + C^3_{13}) = 0$. Since we have assumed that $C^2_{13} \neq 0$, the only solution to this equation is $C^2_{12} + C^3_{13} = 0$, which makes our system of equations look as follows:

$$
\begin{aligned}
C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} &= 0 && \text{(R3.15.1)}\\
C^2_{12}C^2_{12} + C^3_{12}C^2_{13} &= 0 && \text{(R3.15.2)}\\
C^1_{13}C^1_{11} + C^2_{13}C^1_{12} - C^2_{12}C^1_{13} &= 0 && \text{(R3.16.1)}\\
C^2_{13}C^3_{12} + C^2_{12}C^2_{12} &= 0 && \text{(R3.16.3)}
\end{aligned}
$$

$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} - C^3_{12}C^1_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^1_{13}C^1_{11} - C^2_{13}C^1_{21} + C^2_{12}C^1_{13} = 0. \qquad \text{(R3.32.1)}$$

Now, we see that equation (R3.15.2) and equation (R3.16.3) are identical, and since we have assumed that $C^2_{13} \neq 0$ we can write the solution to this equation as $C^3_{12} = -C^2_{12}C^2_{12}/C^2_{13}$. Inserting this into the system of equations gives us

$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} - C^2_{12}C^2_{12}C^1_{13}/C^2_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} - C^2_{12}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} + C^2_{12}C^2_{12}C^1_{13}/C^2_{13} = 0 \qquad \text{(R3.23.1)}$$
$$C^1_{13}C^1_{11} - C^2_{13}C^1_{21} + C^2_{12}C^1_{13} = 0. \qquad \text{(R3.32.1)}$$

Remembering again that we have assumed that $C^2_{13} \neq 0$, we can write the solutions to equation (R3.16.1) and equation (R3.32.1) as $C^1_{12} = C^1_{13}(C^2_{12} - C^1_{11})/C^2_{13}$ and $C^1_{21} = C^1_{13}(C^2_{12} + C^1_{11})/C^2_{13}$, respectively. Now, inserting this into equation (R3.15.1), we get

$$\begin{aligned}
&C^1_{12}C^1_{11} + C^2_{12}C^1_{12} - C^2_{12}C^2_{12}C^1_{13}/C^2_{13} \\
&\quad = C^1_{13}(C^2_{12} - C^1_{11})C^1_{11}/C^2_{13} + C^2_{12}C^1_{13}(C^2_{12} - C^1_{11})/C^2_{13} - C^2_{12}C^2_{12}C^1_{13}/C^2_{13} \\
&\quad = C^1_{13}\left((C^2_{12} - C^1_{11})C^1_{11} + C^2_{12}(C^2_{12} - C^1_{11}) - C^2_{12}C^2_{12}\right)/C^2_{13} \\
&\quad = C^1_{13}\left(C^2_{12}C^1_{11} - C^1_{11}C^1_{11} + C^2_{12}C^2_{12} - C^2_{12}C^1_{11} - C^2_{12}C^2_{12}\right)/C^2_{13} \\
&\quad = C^1_{13}\left(-C^1_{11}C^1_{11}\right)/C^2_{13} \\
&= 0.
\end{aligned}$$

The only possible solutions to this are $C^1_{11} = 0$ and $C^1_{13} = 0$. However, if we have $C^1_{11} = 0$ we get $C^1_{12} = C^1_{13}C^2_{12}/C^2_{13}$ and $C^1_{21} = C^1_{13}C^2_{12}/C^2_{13}$, and if we have $C^1_{13} = 0$ we get $C^1_{12} = 0$ and $C^1_{21} = 0$, so in both cases we get $C^1_{12} = C^1_{21}$. However, we have assumed that $C^1_{12} \neq C^1_{21}$, so this is not possible. Thus, this case did not lead to a solution.

Case 1.2.1.1.2.1.2.2: If we instead have $C^3_{33} \neq 0$ and $C^2_{13} = C^3_{31} = 0$, our system of equations will be

$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11} + C^3_{13}C^1_{13} = 0 \qquad \text{(R3.16.1)}$$
$$C^3_{13}C^3_{13} = 0 \qquad \text{(R3.16.3)}$$

$$(C_{12}^3 + C_{21}^3)C_{13}^3 = 0 \qquad \text{(R3.17.3.2)}$$
$$C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{33}^3 C_{13}^3 = 0 \qquad \text{(R3.22.3.2)}$$
$$C_{21}^1 C_{11}^1 - C_{12}^2 C_{21}^1 + C_{21}^3 C_{13}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{12}^2 C_{12}^2 - C_{21}^3 C_{13}^2 = 0 \qquad \text{(R3.23.2)}$$
$$-C_{12}^2 C_{21}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{13}^1 C_{11}^1 = 0 \qquad \text{(R3.32.1)}$$
$$-C_{13}^2 C_{21}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{33}^1 C_{11}^1 + C_{33}^3 C_{13}^1 = 0. \qquad \text{(R3.36.1)}$$

The only solution to equation (R3.15.2) is $C_{12}^2 = 0$, and the only solution to equation (R3.16.3) (and equation (R3.18.3.2), which is identical) is $C_{13}^3 = 0$. Substituting this solution into our system of equations gives us

$$C_{12}^1 C_{11}^1 + C_{12}^3 C_{13}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{13}^1 C_{11}^1 = 0 \qquad \text{(R3.16.1)}$$
$$C_{21}^1 C_{11}^1 - C_{21}^3 C_{13}^1 = 0 \qquad \text{(R3.23.1)}$$
$$-C_{21}^3 C_{13}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{13}^1 C_{11}^1 = 0 \qquad \text{(R3.32.1)}$$
$$-C_{13}^2 C_{21}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{33}^1 C_{11}^1 + C_{33}^3 C_{13}^1 = 0. \qquad \text{(R3.36.1)}$$

We can now remove equation (R3.32.1) and (R3.32.3) since they are identical to equation (R3.16.1) and (R3.23.2). Then, since we have assumed that $C_{11}^1 \neq 0$, the only solution to equation (R3.16.1) is $C_{13}^1 = 0$, which means our system of equations becomes

$$C_{12}^1 C_{11}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{21}^1 C_{11}^1 = 0 \qquad \text{(R3.23.1)}$$
$$-C_{21}^3 C_{13}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{33}^1 C_{11}^1 = 0. \qquad \text{(R3.36.1)}$$

Again since $C_{11}^1 \neq 0$, we see that they only solution to equations (R3.15.1) and (R3.23.1) is $C_{12}^1 = C_{21}^1 = 0$. However, we have assumed that $C_{12}^1 \neq C_{21}^1$, which is a contradiction, so this case did not lead to a solution.

Case 1.2.1.1.2.2: If we instead have $C_{11}^3 \neq 0$ and $C_{23}^1 = C_{32}^1 = 0$,

the system of equations becomes

$$C^3_{11}(C^2_{13}+C^2_{13})=0 \qquad \text{(R3.14.2)}$$

$$C^3_{11}(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.14.3)}$$

$$C^1_{12}C^1_{11}+C^2_{12}C^1_{12}+C^3_{12}C^1_{13}=0 \qquad \text{(R3.15.1)}$$

$$C^2_{12}C^2_{12}+C^3_{12}C^2_{13}=0 \qquad \text{(R3.15.2)}$$

$$C^1_{12}C^3_{11}+C^2_{12}C^3_{12}+C^3_{12}C^3_{13}=0 \qquad \text{(R3.15.3)}$$

$$C^1_{13}C^1_{11}+C^2_{13}C^1_{12}+C^3_{13}C^1_{13}=0 \qquad \text{(R3.16.1)}$$

$$C^2_{13}C^2_{12}+C^3_{13}C^2_{13}=0 \qquad \text{(R3.16.2)}$$

$$C^1_{13}C^3_{11}+C^2_{13}C^3_{12}+C^3_{13}C^3_{13}=0 \qquad \text{(R3.16.3)}$$

$$(C^3_{12}+C^3_{21})(C^2_{13}+C^2_{13})=0 \qquad \text{(R3.17.2.2)}$$

$$(C^3_{12}+C^3_{21})(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.17.3.2)}$$

$$(C^3_{13}+C^3_{31})(C^2_{13}+C^2_{13})=0 \qquad \text{(R3.18.2.2)}$$

$$(C^3_{13}+C^3_{31})(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.18.3.2)}$$

$$C^3_{33}(C^2_{13}+C^2_{13})=0 \qquad \text{(R3.22.2.2)}$$

$$C^3_{33}(C^3_{13}-C^3_{31})=0 \qquad \text{(R3.22.3.2)}$$

$$C^1_{21}C^1_{11}-C^2_{12}C^1_{21}+C^3_{21}C^1_{13}=0 \qquad \text{(R3.23.1)}$$

$$C^2_{12}C^2_{12}-C^3_{21}C^2_{13}=0 \qquad \text{(R3.23.2)}$$

$$C^1_{21}C^3_{11}-C^2_{12}C^3_{21}+C^3_{21}C^3_{31}=0 \qquad \text{(R3.23.3)}$$

$$C^1_{13}C^1_{11}-C^2_{13}C^1_{21}+C^3_{31}C^1_{13}=0 \qquad \text{(R3.32.1)}$$

$$C^2_{13}C^2_{12}+C^3_{31}C^2_{31}=0 \qquad \text{(R3.32.2)}$$

$$C^1_{13}C^3_{11}-C^2_{13}C^3_{21}+C^3_{31}C^3_{31}=0 \qquad \text{(R3.32.3)}$$

$$C^1_{33}C^1_{11}+C^3_{33}C^1_{13}=0 \qquad \text{(R3.36.1)}$$

$$-C^3_{33}C^2_{13}=0 \qquad \text{(R3.36.2)}$$

$$C^1_{33}C^3_{11}+C^3_{33}C^3_{31}=0. \qquad \text{(R3.36.3)}$$

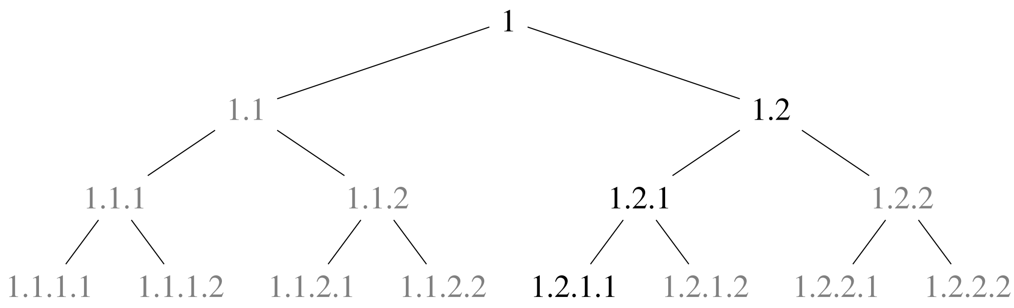
Now, since $C^3_{11}\neq 0$, the only solution to equation (R3.14.2) is $C^2_{13}+C^2_{13}=0$, which here implies that $C^2_{13}=0$. However, we have already assumed that $C^2_{13}\neq 0$, so this is a contradiction, which means that this case does not lead to a solution.

Case 1.2.1.2: $C^1_{22}\neq 0$, $C^1_{11}=0$ and $C^3_{11}=0$

Case 1.2.2: $C^3_{22}\neq 0$ and $C^2_{13}=0$

Case 2: $C^1_{13}-C^1_{31}\neq 0$

We see that we have yet only found a fraction of the possible solutions to the system of equations, with cases 1.2.1.2, 1.2.2, and 2, and all of the cases each of these will split into, remaining. We also remember that we skipped case 1.1 in the beginning, to instead focus on case 1.2. Now, 1.2.1.1 is the "highest" case we have looked at completely (meaning all other cases are branches from this case), and this is one of two cases we get from case 1.2.1, which in turn is one of two branches we get from case 1.2, which in turn is one of two branches we get from case 1, which in turn is one of the two original branches – case 1 and case 2. We write out the cases we get from case 1 in a tree below, down to the level where 1.2.1.1 is, with the cases we have not looked at at all written in grey. Note that we of course cannot know if there will be for example a case 1.1.1.1 or if it will stop at say case 1.1.1, but we assume that there will be.



Now, of course, there would be a tree of identical size for case 2, except all in grey since we have not looked at any of those cases. Thus, there are a total of sixteen different cases on the fourth level, including case 1.2.1.1, the only one we have looked at so far. This means that if each case has approximately as many possible solutions as each other, the ones we have found so far might only be approximately one sixteenth of all possible solutions. Thus, we realise that trying to find all possible solutions will take too long to be possible to include it in this thesis, and we decide to simply stop here instead.

Since we are still interested in the results we *have* found, we write down all possible solutions – in this case only three, since every other solution we found violated some of our assumptions – in Table 3.17.

Now, each of these hom-associative algebras is hom-Lie admissible with commutator $[\cdot,\cdot]$, so just as we did in the two-dimensional case in Section 3.1.1 we need to find these commutators. Since we are now in three dimensions, to find the commutators for each hom-Lie admissible algebra we must calculate $[e_i, e_j]$ for $i, j = \{1,2,3\}$ for α defined as E_{11}. By definition, $[e_1, e_1] = [e_2, e_2] = [e_3, e_3] = 0$, so we only have to calculate the other combinations of i and j. Remembering that

$$\mu(e_i, e_j) = \sum_{k=1}^{3} C_{ij}^k e_k = C_{ij}^1 e_1 + C_{ij}^2 e_2 + C_{ij}^3 e_3,$$

and $[e_i, e_j] = \mu(e_i, e_j) - \mu(e_j, e_i)$, we can now move on to calculating the commutator for the

	Soln 1	Soln 2	Soln 3
C_{11}^1	0	0	free
C_{11}^2	0	0	0
C_{11}^3	0	0	$C_{11}^3 \neq 0$
C_{12}^1	free	free	$C_{12}^3 C_{11}^1 / C_{11}^3$
C_{12}^2	0	0	0
C_{12}^3	free	free	free
C_{13}^1	0	free	$-C_{11}^1 C_{11}^1 / C_{11}^3$
C_{13}^2	0	0	0
C_{13}^3	free	free	$-C_{11}^1$
C_{21}^1	free	free	$C_{21}^3 C_{11}^1 / C_{11}^3$
C_{21}^2	0	0	0
C_{21}^3	free	0	free
C_{22}^1	0	0	0
C_{22}^2	0	0	0
C_{22}^3	0	0	0
C_{23}^1	free	free	$C_{23}^3 C_{11}^1 / C_{11}^3$
C_{23}^2	0	0	0
C_{23}^3	free	0	free
C_{31}^1	0	C_{13}^1	$-C_{11}^1 C_{11}^1 / C_{11}^3$
C_{31}^2	0	0	0
C_{31}^3	0	0	$-C_{11}^1$
C_{32}^1	free	free	$C_{32}^3 C_{11}^1 / C_{11}^3$
C_{32}^2	0	0	0
C_{32}^3	free	0	free
C_{33}^1	free	free	$C_{33}^3 C_{11}^1 / C_{11}^3$
C_{33}^2	0	0	0
C_{33}^3	free	0	free

Table 3.17: The values of the structure constants that give hom-associative algebras when α is defined as E_{11} in three dimensions

hom-Lie admissible algebra we get from each solution. We begin with solution 1.

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = C_{12}^1 e_1 + 0 e_2 + C_{12}^3 e_3 = C_{12}^1 e_1 + C_{12}^3 e_3,$$

$$\mu(e_1,e_3)=C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3=0e_1+0e_2+C^3_{13}e_3=C^3_{13}e_3,$$
$$\mu(e_2,e_1)=C^1_{21}e_1+C^2_{21}e_2+C^3_{21}e_3=C^1_{21}e_1+0e_2+C^3_{21}e_3=C^1_{21}e_1+C^3_{21}e_3,$$
$$\mu(e_2,e_3)=C^1_{23}e_1+C^2_{23}e_2+C^3_{23}e_3=C^1_{23}e_1+0e_2+C^3_{23}e_3=C^1_{23}e_1+C^3_{23}e_3,$$
$$\mu(e_3,e_1)=C^1_{31}e_1+C^2_{31}e_2+C^3_{31}e_3=0e_1+0e_2+0e_3=0,$$
$$\mu(e_3,e_2)=C^1_{32}e_1+C^2_{32}e_2+C^3_{32}e_3=C^1_{32}e_1+0e_2+C^3_{32}e_3=C^1_{32}e_1+C^3_{32}e_3.$$

This gives us

$$[e_1,e_2]=\mu(e_1,e_2)-\mu(e_2,e_1)=C^1_{12}e_1+C^3_{12}e_3-(C^1_{21}e_1+C^3_{21}e_3)=(C^1_{12}-C^1_{21})e_1+(C^3_{12}-C^3_{21})e_3,$$
$$[e_1,e_3]=\mu(e_1,e_3)-\mu(e_3,e_1)=C^3_{13}e_3-0=C^3_{13}e_3,$$
$$[e_2,e_1]=-[e_1,e_2]=(C^1_{21}-C^1_{12})e_1+(C^3_{21}-C^3_{12})e_3,$$
$$[e_2,e_3]=\mu(e_2,e_3)-\mu(e_3,e_2)=C^1_{23}e_1+C^3_{23}e_3-(C^1_{32}e_1+C^3_{32}e_3)=(C^1_{23}-C^1_{32})e_1+(C^3_{23}-C^3_{32})e_3,$$
$$[e_3,e_1]=-[e_1,e_3]=-C^3_{13}e_3,$$
$$[e_3,e_2]=-[e_2,e_3]=(C^1_{32}-C^1_{23})e_1+(C^3_{32}-C^3_{23})e_3.$$

Moving on to solution 2, we get

$$\mu(e_1,e_2)=C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3=C^1_{12}e_1+0e_2+C^3_{12}e_3=C^1_{12}e_1+C^3_{12}e_3,$$
$$\mu(e_1,e_3)=C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3=C^1_{13}e_1+0e_2+C^3_{13}e_3=C^1_{13}e_1+C^3_{13}e_3,$$
$$\mu(e_2,e_1)=C^1_{21}e_1+C^2_{21}e_2+C^3_{21}e_3=C^1_{21}e_1+0e_2+0e_3=C^1_{21}e_1,$$
$$\mu(e_2,e_3)=C^1_{23}e_1+C^2_{23}e_2+C^3_{23}e_3=C^1_{23}e_1+0e_2+0e_3=C^1_{23}e_1,$$
$$\mu(e_3,e_1)=C^1_{31}e_1+C^2_{31}e_2+C^3_{31}e_3=C^1_{13}e_1+0e_2+0e_3=C^1_{13}e_1,$$
$$\mu(e_3,e_2)=C^1_{32}e_1+C^2_{32}e_2+C^3_{32}e_3=C^1_{32}e_1+0e_2+0e_3=C^1_{32}e_1.$$

This gives us

$$[e_1,e_2]=\mu(e_1,e_2)-\mu(e_2,e_1)=C^1_{12}e_1+C^3_{12}e_3-C^1_{21}e_1=(C^1_{12}-C^1_{21})e_1+C^3_{12}e_3,$$
$$[e_1,e_3]=\mu(e_1,e_3)-\mu(e_3,e_1)=C^1_{13}e_1+C^3_{13}e_3-C^1_{13}e_1=C^3_{13}e_3,$$
$$[e_2,e_1]=-[e_1,e_2]=(C^1_{21}-C^1_{12})e_1-C^3_{12}e_3,$$
$$[e_2,e_3]=\mu(e_2,e_3)-\mu(e_3,e_2)=C^1_{23}e_1-C^1_{32}e_1=(C^1_{23}-C^1_{32})e_1,$$
$$[e_3,e_1]=-[e_1,e_3]=-C^3_{13}e_3,$$
$$[e_3,e_2]=-[e_2,e_3]=(C^1_{32}-C^1_{23})e_1.$$

Finally, looking at solution 3, we get

$$\mu(e_1,e_2)=C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3=(C^3_{12}C^1_{11}/C^3_{11})e_1+0e_2+C^3_{12}e_3=(C^3_{12}C^1_{11}/C^3_{11})e_1+C^3_{12}e_3,$$
$$\mu(e_1,e_3)=C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3=-(C^1_{11}C^1_{11}/C^3_{11})e_1+0e_2-C^1_{11}e_3=-(C^1_{11}C^1_{11}/C^3_{11})e_1-C^1_{11}e_3,$$
$$\mu(e_2,e_1)=C^1_{21}e_1+C^2_{21}e_2+C^3_{21}e_3=(C^3_{21}C^1_{11}/C^3_{11})e_1+0e_2+C^3_{21}e_3=(C^3_{21}C^1_{11}/C^3_{11})e_1+C^3_{21}e_3,$$
$$\mu(e_2,e_3)=C^1_{23}e_1+C^2_{23}e_2+C^3_{23}e_3=(C^3_{23}C^1_{11}/C^3_{11})e_1+0e_2+C^3_{23}e_3=(C^3_{23}C^1_{11}/C^3_{11})e_1+C^3_{23}e_3,$$
$$\mu(e_3,e_1)=C^1_{31}e_1+C^2_{31}e_2+C^3_{31}e_3=-(C^1_{11}C^1_{11}/C^3_{11})e_1+0e_2-C^1_{11}e_3=-(C^1_{11}C^1_{11}/C^3_{11})e_1-C^1_{11}e_3,$$
$$\mu(e_3,e_2)=C^1_{32}e_1+C^2_{32}e_2+C^3_{32}e_3=(C^3_{32}C^1_{11}/C^3_{11})e_1+0e_2+C^3_{32}e_3=(C^3_{32}C^1_{11}/C^3_{11})e_1+C^3_{32}e_3.$$

This gives us

$$
\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2)-\mu(e_2,e_1) = (C_{12}^3C_{11}^1/C_{11}^3)e_1 + C_{12}^3e_3 - ((C_{21}^3C_{11}^1/C_{11}^3)e_1 + C_{21}^3e_3)\\
&= (C_{11}^1(C_{12}^3 - C_{21}^3)/C_{11}^3)e_1 + (C_{12}^3 - C_{21}^3)e_3,\\
[e_1,e_3] &= \mu(e_1,e_3)-\mu(e_3,e_1) = -(C_{11}^1C_{11}^1/C_{11}^3)e_1 - C_{11}^1e_3 - (-(C_{11}^1C_{11}^1/C_{11}^3)e_1 - C_{11}^1e_3) = 0,\\
[e_2,e_1] &= -[e_1,e_2] = (C_{11}^1(C_{21}^3 - C_{12}^3)/C_{11}^3)e_1 + (C_{21}^3 - C_{12}^3)e_3,\\
[e_2,e_3] &= \mu(e_2,e_3)-\mu(e_3,e_2) = (C_{23}^3C_{11}^1/C_{11}^3)e_1 + C_{23}^3e_3 - ((C_{32}^3C_{11}^1/C_{11}^3)e_1 + C_{32}^3e_3)\\
&= (C_{11}^1(C_{23}^3 - C_{32}^3)/C_{11}^3)e_1 + (C_{23}^3 - C_{32}^3)e_3,\\
[e_3,e_1] &= -[e_1,e_3] = 0,\\
[e_3,e_2] &= -[e_2,e_3] = (C_{11}^1(C_{32}^3 - C_{23}^3)/C_{11}^3)e_1 + (C_{32}^3 - C_{23}^3)e_3.
\end{aligned}
$$

Thus, we managed to find three different hom-Lie admissible algebras $(V,[\cdot,\cdot],E_{11})$ in three dimensions, where $[\cdot,\cdot]$ is defined as in Table 3.18 for the hom-Lie admissible algebra with the structure constants given in solution 1 above, Table 3.19 for the one with the structure constants given in solution 2 above, and Table 3.20 for the one with the structure constants given in solution 3 above.

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^1 - C_{21}^1)e_1 + (C_{12}^3 - C_{21}^3)e_3$	$C_{13}^3e_3$
e_2	$(C_{21}^1 - C_{12}^1)e_1 + (C_{21}^3 - C_{12}^3)e_3$	0	$(C_{23}^1 - C_{32}^1)e_1 + (C_{23}^3 - C_{32}^3)e_3$
e_3	$-C_{13}^3e_3$	$(C_{32}^1 - C_{23}^1)e_1 + (C_{32}^3 - C_{23}^3)e_3$	0

Table 3.18: The commutator table for the hom-Lie admissible algebras with α defined as E_{11} and the structure constants defined as in solution 1 in Table 3.17

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^1 - C_{21}^1)e_1 + C_{12}^3e_3$	$C_{13}^3e_3$
e_2	$(C_{21}^1 - C_{12}^1)e_1 + C_{12}^3e_3$	0	$(C_{23}^1 - C_{32}^1)e_1$
e_3	$-C_{13}^3e_3$	$(C_{32}^1 - C_{23}^1)e_1$	0

Table 3.19: The commutator table for the hom-Lie admissible algebras with α defined as E_{11} and the structure constants defined as in solution 2 in Table 3.17

Now, since we concluded that we would not be able to find all solutions to the values of the structure constants that give us a hom-associative algebra when α is defined as E_{11} in the scope of this thesis, we assume that the same will be true for α defined as any of the other tree-dimensional matrix units. Thus, for α defined as the remaining matrix units, we will still find and write out the system of equations that would have to be solved, but we will not actually solve them.

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$\frac{C^1_{11}(C^3_{12}-C^3_{21})}{C^3_{11}}e_1+(C^3_{12}-C^3_{21})e_3$	0
e_2	$\frac{C^1_{11}(C^3_{21}-C^3_{12})}{C^3_{11}}e_1+(C^3_{21}-C^3_{12})e_3$	0	$\frac{C^1_{11}(C^3_{23}-C^3_{32})}{C^3_{11}}e_1+(C^3_{23}-C^3_{32})e_3$
e_3	0	$\frac{C^1_{11}(C^3_{32}-C^3_{23})}{C^3_{11}}e_1+(C^3_{32}-C^3_{23})e_3$	0

Table 3.20: The commutator table for the hom-Lie admissible algebras with α defined as E_{11} and the structure constants defined as in solution 3 in Table 3.17

3.2.2 α defined as E_{12}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha] = E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

However, due to our results in the previous section, we will only write out the system of equations, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\begin{aligned} \alpha(e_1) &= 0e_1+0e_2+0e_3 = 0, \\ \alpha(e_2) &= 1e_1+0e_2+0e_3 = e_1, \\ \alpha(e_3) &= 0e_1+0e_2+0e_3 = 0. \end{aligned}$$

We can thus rewrite our equations as follows:

$$\begin{aligned}
(3.14): \quad & \mu(0, C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3) = \mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3, 0) \;\Rightarrow\; 0=0, \\
(3.15): \quad & \mu(0, C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3) = \mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3, e_1) \\
& \Rightarrow\; 0 = \mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3, e_1), \\
(3.16): \quad & \mu(0, C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3) = \mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3, 0) \;\Rightarrow\; 0=0, \\
(3.17): \quad & \mu(0, C^1_{21}e_1+C^2_{21}e_2+C^3_{21}e_3) = \mu(C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3, 0) \;\Rightarrow\; 0=0, \\
(3.18): \quad & \mu(0, C^1_{31}e_1+C^2_{31}e_2+C^3_{31}e_3) = \mu(C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3, 0) \;\Rightarrow\; 0=0, \\
(3.19): \quad & \mu(0, C^1_{22}e_1+C^2_{22}e_2+C^3_{22}e_3) = \mu(C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3, e_1) \\
& \Rightarrow\; 0 = \mu(C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3, e_1), \\
(3.20): \quad & \mu(0, C^1_{23}e_1+C^2_{23}e_2+C^3_{23}e_3) = \mu(C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3, 0) \;\Rightarrow\; 0=0, \\
(3.21): \quad & \mu(0, C^1_{32}e_1+C^2_{32}e_2+C^3_{32}e_3) = \mu(C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3, e_1) \\
& \Rightarrow\; 0 = \mu(C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3, e_1), \\
(3.22): \quad & \mu(0, C^1_{33}e_1+C^2_{33}e_2+C^3_{33}e_3) = \mu(C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3, 0) \;\Rightarrow\; 0=0, \\
(3.23): \quad & \mu(e_1, C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3) = \mu(C^1_{21}e_1+C^2_{21}e_2+C^3_{21}e_3, 0) \\
& \Rightarrow\; \mu(e_1, C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3) = 0,
\end{aligned}$$

$$
\begin{aligned}
&(3.24): && \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_1),\\
&(3.25): && \mu(e_1, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0)\\
& && \Rightarrow \ \mu(e_1, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = 0,\\
&(3.26): && \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0)\\
& && \Rightarrow \ \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = 0,\\
&(3.27): && \mu(e_1, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0)\\
& && \Rightarrow \ \mu(e_1, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = 0,\\
&(3.28): && \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_1),\\
&(3.29): && \mu(e_1, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0)\\
& && \Rightarrow \ \mu(e_1, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = 0,\\
&(3.30): && \mu(e_1, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_1),\\
&(3.31): && \mu(e_1, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0)\\
& && \Rightarrow \ \mu(e_1, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = 0,\\
&(3.32): && \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \ \Rightarrow \ 0 = 0,\\
&(3.33): && \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_1)\\
& && \Rightarrow \ 0 = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_1),\\
&(3.34): && \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \ \Rightarrow \ 0 = 0,\\
&(3.35): && \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \ \Rightarrow \ 0 = 0,\\
&(3.36): && \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \ \Rightarrow \ 0 = 0,\\
&(3.37): && \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_1)\\
& && \Rightarrow \ 0 = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_1),\\
&(3.38): && \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \ \Rightarrow \ 0 = 0,\\
&(3.39): && \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_1)\\
& && \Rightarrow \ 0 = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_1),\\
&(3.40): && \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \ \Rightarrow \ 0 = 0.
\end{aligned}
$$

We remember the following formulas that we calculated earlier:

$$
\begin{aligned}
\mu(e_i, C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3) = {} & (C_{jk}^1 C_{i1}^1 + C_{jk}^2 C_{i2}^1 + C_{jk}^3 C_{i3}^1) e_1 + (C_{jk}^1 C_{i1}^2 + C_{jk}^2 C_{i2}^2 + C_{jk}^3 C_{i3}^2) e_2\\
& + (C_{jk}^1 C_{i1}^3 + C_{jk}^2 C_{i2}^3 + C_{jk}^3 C_{i3}^3) e_3
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3, e_i) = {} & (C_{jk}^1 C_{1i}^1 + C_{jk}^2 C_{2i}^1 + C_{jk}^3 C_{3i}^1) e_1 + (C_{jk}^1 C_{1i}^2 + C_{jk}^2 C_{2i}^2 + C_{jk}^3 C_{3i}^2) e_2\\
& + (C_{jk}^1 C_{1i}^3 + C_{jk}^2 C_{2i}^3 + C_{jk}^3 C_{3i}^3) e_3.
\end{aligned}
$$

We use these formulas to rewrite our equations.

$$
\begin{aligned}
(3.15):\quad & 0=(C^1_{11}C^1_{11}+C^2_{11}C^1_{21}+C^3_{11}C^1_{31})e_1+(C^1_{11}C^2_{11}+C^2_{11}C^2_{21}+C^3_{11}C^2_{31})e_1\\
&\qquad +(C^1_{11}C^3_{11}+C^2_{11}C^3_{21}+C^3_{11}C^3_{31})e_1,\\
(3.19):\quad & 0=(C^1_{12}C^1_{11}+C^2_{12}C^1_{21}+C^3_{12}C^1_{31})e_1+(C^1_{12}C^2_{11}+C^2_{12}C^2_{21}+C^3_{12}C^2_{31})e_2\\
&\qquad +(C^1_{12}C^3_{11}+C^2_{12}C^3_{21}+C^3_{12}C^3_{31})e_3,\\
(3.21):\quad & 0=(C^1_{13}C^1_{11}+C^2_{13}C^1_{21}+C^3_{13}C^1_{31})e_1+(C^1_{13}C^2_{11}+C^2_{13}C^2_{21}+C^3_{13}C^2_{31})e_2\\
&\qquad +(C^1_{13}C^3_{11}+C^2_{13}C^3_{21}+C^3_{13}C^3_{31})e_3,\\
(3.23):\quad & (C^1_{11}C^1_{11}+C^2_{11}C^1_{12}+C^3_{11}C^1_{13})e_1+(C^1_{11}C^2_{11}+C^2_{11}C^2_{12}+C^3_{11}C^2_{13})e_2\\
& +(C^1_{11}C^3_{11}+C^2_{11}C^3_{12}+C^3_{11}C^3_{13})e_3=0,\\
(3.24):\quad & (C^1_{12}C^1_{11}+C^2_{12}C^1_{12}+C^3_{12}C^1_{13})e_1+(C^1_{12}C^2_{11}+C^2_{12}C^2_{12}+C^3_{12}C^2_{13})e_2\\
& +(C^1_{12}C^3_{11}+C^2_{12}C^3_{12}+C^3_{12}C^3_{13})e_3\\
& =(C^1_{21}C^1_{11}+C^2_{21}C^1_{21}+C^3_{21}C^1_{31})e_1+(C^1_{21}C^2_{11}+C^2_{21}C^2_{21}+C^3_{21}C^2_{31})e_2\\
&\quad +(C^1_{21}C^3_{11}+C^2_{21}C^3_{21}+C^3_{21}C^3_{31})e_3\\
& \Rightarrow\ (C^1_{12}C^1_{11}+C^2_{12}C^1_{12}+C^3_{12}C^1_{13}-C^1_{21}C^1_{11}-C^2_{21}C^1_{21}-C^3_{21}C^1_{31})e_1\\
&\qquad +(C^1_{12}C^2_{11}+C^2_{12}C^2_{12}+C^3_{12}C^2_{13}-C^1_{21}C^2_{11}-C^2_{21}C^2_{21}-C^3_{21}C^2_{31})e_2\\
&\qquad +(C^1_{12}C^3_{11}+C^2_{12}C^3_{12}+C^3_{12}C^3_{13}-C^1_{21}C^3_{11}+C^2_{21}C^3_{21}+C^3_{21}C^3_{31})e_3=0,\\
(3.25):\quad & (C^1_{13}C^1_{11}+C^2_{13}C^1_{12}+C^3_{13}C^1_{13})e_1+(C^1_{13}C^2_{11}+C^2_{13}C^2_{12}+C^3_{13}C^2_{13})e_2\\
& +(C^1_{13}C^3_{11}+C^2_{13}C^3_{12}+C^3_{13}C^3_{13})e_3=0,\\
(3.26):\quad & (C^1_{21}C^1_{11}+C^2_{21}C^1_{12}+C^3_{21}C^1_{13})e_1+(C^1_{21}C^2_{11}+C^2_{21}C^2_{12}+C^3_{21}C^2_{13})e_2\\
& +(C^1_{21}C^3_{11}+C^2_{21}C^3_{12}+C^3_{21}C^3_{13})e_3=0,\\
(3.27):\quad & (C^1_{31}C^1_{11}+C^2_{31}C^1_{12}+C^3_{31}C^1_{13})e_1+(C^1_{31}C^2_{11}+C^2_{31}C^2_{12}+C^3_{31}C^2_{13})e_2\\
& +(C^1_{31}C^3_{11}+C^2_{31}C^3_{12}+C^3_{31}C^3_{13})e_3=0,\\
(3.28):\quad & (C^1_{22}C^1_{11}+C^2_{22}C^1_{12}+C^3_{22}C^1_{13})e_1+(C^1_{22}C^2_{11}+C^2_{22}C^2_{12}+C^3_{22}C^2_{13})e_2\\
& +(C^1_{22}C^3_{11}+C^2_{22}C^3_{12}+C^3_{22}C^3_{13})e_3\\
& =(C^1_{22}C^1_{11}+C^2_{22}C^1_{21}+C^3_{22}C^1_{31})e_1+(C^1_{22}C^2_{11}+C^2_{22}C^2_{21}+C^3_{22}C^2_{31})e_2\\
&\quad +(C^1_{22}C^3_{11}+C^2_{22}C^3_{21}+C^3_{22}C^3_{31})e_3\\
& \Rightarrow\ (C^2_{22}C^1_{12}+C^3_{22}C^1_{13}-C^2_{22}C^1_{21}-C^3_{22}C^1_{31})e_1+(C^2_{22}C^2_{12}+C^3_{22}C^2_{13}-C^2_{22}C^2_{21}-C^3_{22}C^2_{31})e_2\\
&\qquad +(C^2_{22}C^3_{12}+C^3_{22}C^3_{13}-C^2_{22}C^3_{21}-C^3_{22}C^3_{31})e_3=0,\\
(3.29):\quad & (C^1_{23}C^1_{11}+C^2_{23}C^1_{12}+C^3_{23}C^1_{13})e_1+(C^1_{23}C^2_{11}+C^2_{23}C^2_{12}+C^3_{23}C^2_{13})e_2\\
& +(C^1_{23}C^3_{11}+C^2_{23}C^3_{12}+C^3_{23}C^3_{13})e_3=0,\\
(3.30):\quad & (C^1_{32}C^1_{11}+C^2_{32}C^1_{12}+C^3_{32}C^1_{13})e_1+(C^1_{32}C^2_{11}+C^2_{32}C^2_{12}+C^3_{32}C^2_{13})e_2\\
& +(C^1_{32}C^3_{11}+C^2_{32}C^3_{12}+C^3_{32}C^3_{13})e_3\\
& =(C^1_{23}C^1_{11}+C^2_{23}C^1_{21}+C^3_{23}C^1_{31})e_1+(C^1_{23}C^2_{11}+C^2_{23}C^2_{21}+C^3_{23}C^2_{31})e_2
\end{aligned}
$$

$$
\begin{aligned}
&+(C_{23}^1C_{11}^3+C_{23}^2C_{21}^3+C_{23}^3C_{31}^3)e_3\\
&\Rightarrow\ (C_{32}^1C_{11}^1+C_{32}^2C_{12}^1+C_{32}^3C_{13}^1-C_{23}^1C_{11}^1-C_{23}^2C_{21}^1-C_{23}^3C_{31}^1)e_1\\
&\quad+(C_{32}^1C_{11}^2+C_{32}^2C_{12}^2+C_{32}^3C_{13}^2-C_{23}^1C_{11}^2-C_{23}^2C_{21}^2-C_{23}^3C_{31}^2)e_2\\
&\quad+(C_{32}^1C_{11}^3+C_{32}^2C_{12}^3+C_{32}^3C_{13}^3-C_{23}^1C_{11}^3-C_{23}^2C_{21}^3-C_{23}^3C_{31}^3)e_3=0,
\end{aligned}
$$

$$
\begin{aligned}
(3.31):\quad &(C_{33}^1C_{11}^1+C_{33}^2C_{12}^1+C_{33}^3C_{13}^1)e_1+(C_{33}^1C_{11}^2+C_{33}^2C_{12}^2+C_{33}^3C_{13}^2)e_2\\
&+(C_{33}^1C_{11}^3+C_{33}^2C_{12}^3+C_{33}^3C_{13}^3)e_3=0,\\
(3.33):\quad &0=(C_{31}^1C_{11}^1+C_{31}^2C_{21}^1+C_{31}^3C_{31}^1)e_1+(C_{31}^1C_{11}^2+C_{31}^2C_{21}^2+C_{31}^3C_{31}^2)e_2\\
&\quad+(C_{31}^1C_{11}^3+C_{31}^2C_{21}^3+C_{31}^3C_{31}^3)e_3,\\
(3.37):\quad &0=(C_{32}^1C_{11}^1+C_{32}^2C_{21}^1+C_{32}^3C_{31}^1)e_1+(C_{32}^1C_{11}^2+C_{32}^2C_{21}^2+C_{32}^3C_{31}^2)e_2\\
&\quad+(C_{32}^1C_{11}^3+C_{32}^2C_{21}^3+C_{32}^3C_{31}^3)e_3,\\
(3.39):\quad &0=(C_{33}^1C_{11}^1+C_{33}^2C_{21}^1+C_{33}^3C_{31}^1)e_1+(C_{33}^1C_{11}^2+C_{33}^2C_{21}^2+C_{33}^3C_{31}^2)e_2\\
&\quad+(C_{33}^1C_{11}^3+C_{33}^2C_{21}^3+C_{33}^3C_{31}^3)e_3.
\end{aligned}
$$

Remembering that $k_1e_1+k_2e_2+k_3e_3=0$ will only have the trivial solution $k_1=k_2=k_3=0$, we can rewrite our equations as the following system of equations:

$$C_{11}^1C_{11}^1+C_{11}^2C_{21}^1+C_{11}^3C_{31}^1=0 \tag{R3.15.1}$$
$$C_{11}^1C_{11}^2+C_{11}^2C_{21}^2+C_{11}^3C_{31}^2=0 \tag{R3.15.2}$$
$$C_{11}^1C_{11}^3+C_{11}^2C_{21}^3+C_{11}^3C_{31}^3=0 \tag{R3.15.3}$$
$$C_{12}^1C_{11}^1+C_{12}^2C_{21}^1+C_{12}^3C_{31}^1=0 \tag{R3.19.1}$$
$$C_{12}^1C_{11}^2+C_{12}^2C_{21}^2+C_{12}^3C_{31}^2=0 \tag{R3.19.2}$$
$$C_{12}^1C_{11}^3+C_{12}^2C_{21}^3+C_{12}^3C_{31}^3=0 \tag{R3.19.3}$$
$$C_{13}^1C_{11}^1+C_{13}^2C_{21}^1+C_{13}^3C_{31}^1=0 \tag{R3.21.1}$$
$$C_{13}^1C_{11}^2+C_{13}^2C_{21}^2+C_{13}^3C_{31}^2=0 \tag{R3.21.2}$$
$$C_{13}^1C_{11}^3+C_{13}^2C_{21}^3+C_{13}^3C_{31}^3=0 \tag{R3.21.3}$$
$$C_{11}^1C_{11}^1+C_{11}^2C_{12}^1+C_{11}^3C_{13}^1=0 \tag{R3.23.1}$$
$$C_{11}^1C_{11}^2+C_{11}^2C_{12}^2+C_{11}^3C_{13}^2=0 \tag{R3.23.2}$$
$$C_{11}^1C_{11}^3+C_{11}^2C_{12}^3+C_{11}^3C_{13}^3=0 \tag{R3.23.3}$$
$$C_{12}^1C_{11}^1+C_{12}^2C_{12}^1+C_{12}^3C_{13}^1-C_{21}^1C_{11}^1-C_{21}^2C_{21}^1-C_{21}^3C_{31}^1=0 \tag{R3.24.1}$$
$$C_{12}^1C_{11}^2+C_{12}^2C_{12}^2+C_{12}^3C_{13}^2-C_{21}^1C_{11}^2-C_{21}^2C_{21}^2-C_{21}^3C_{31}^2=0 \tag{R3.24.2}$$
$$C_{12}^1C_{11}^3+C_{12}^2C_{12}^3+C_{12}^3C_{13}^3-C_{21}^1C_{11}^3+C_{21}^2C_{21}^3+C_{21}^3C_{31}^3=0 \tag{R3.24.3}$$
$$C_{13}^1C_{11}^1+C_{13}^2C_{12}^1+C_{13}^3C_{13}^1=0 \tag{R3.25.1}$$
$$C_{13}^1C_{11}^2+C_{13}^2C_{12}^2+C_{13}^3C_{13}^2=0 \tag{R3.25.2}$$
$$C_{13}^1C_{11}^3+C_{13}^2C_{12}^3+C_{13}^3C_{13}^3=0 \tag{R3.25.3}$$
$$C_{21}^1C_{11}^1+C_{21}^2C_{12}^1+C_{21}^3C_{13}^1=0 \tag{R3.26.1}$$

$$C^1_{21}C^2_{11}+C^2_{21}C^2_{12}+C^3_{21}C^2_{13}=0 \quad \text{(R3.26.2)}$$
$$C^1_{21}C^3_{11}+C^2_{21}C^3_{12}+C^3_{21}C^3_{13}=0 \quad \text{(R3.26.3)}$$
$$C^1_{31}C^1_{11}+C^2_{31}C^1_{12}+C^3_{31}C^1_{13}=0 \quad \text{(R3.27.1)}$$
$$C^1_{31}C^2_{11}+C^2_{31}C^2_{12}+C^3_{31}C^2_{13}=0 \quad \text{(R3.27.2)}$$
$$C^1_{31}C^3_{11}+C^2_{31}C^3_{12}+C^3_{31}C^3_{13}=0 \quad \text{(R3.27.3)}$$
$$C^2_{22}C^1_{12}+C^3_{22}C^1_{13}-C^2_{22}C^1_{21}-C^3_{22}C^1_{31}=0 \quad \text{(R3.28.1)}$$
$$C^2_{22}C^2_{12}+C^3_{22}C^2_{13}-C^2_{22}C^2_{21}-C^3_{22}C^2_{31}=0 \quad \text{(R3.28.2)}$$
$$C^2_{22}C^3_{12}+C^3_{22}C^3_{13}-C^2_{22}C^3_{21}-C^3_{22}C^3_{31}=0 \quad \text{(R3.28.3)}$$
$$C^1_{23}C^1_{11}+C^2_{23}C^1_{12}+C^3_{23}C^1_{13}=0 \quad \text{(R3.29.1)}$$
$$C^1_{23}C^2_{11}+C^2_{23}C^2_{12}+C^3_{23}C^2_{13}=0 \quad \text{(R3.29.2)}$$
$$C^1_{23}C^3_{11}+C^2_{23}C^3_{12}+C^3_{23}C^3_{13}=0 \quad \text{(R3.29.3)}$$
$$C^1_{32}C^1_{11}+C^2_{32}C^1_{12}+C^3_{32}C^1_{13}-C^1_{23}C^1_{11}-C^2_{23}C^1_{21}-C^3_{23}C^1_{31}=0 \quad \text{(R3.30.1)}$$
$$C^1_{32}C^2_{11}+C^2_{32}C^2_{12}+C^3_{32}C^2_{13}-C^1_{23}C^2_{11}-C^2_{23}C^2_{21}-C^3_{23}C^2_{31}=0 \quad \text{(R3.30.2)}$$
$$C^1_{32}C^3_{11}+C^2_{32}C^3_{12}+C^3_{32}C^3_{13}-C^1_{23}C^3_{11}-C^2_{23}C^3_{21}-C^3_{23}C^3_{31}=0 \quad \text{(R3.30.3)}$$
$$C^1_{33}C^1_{11}+C^2_{33}C^1_{12}+C^3_{33}C^1_{13}=0 \quad \text{(R3.31.1)}$$
$$C^1_{33}C^2_{11}+C^2_{33}C^2_{12}+C^3_{33}C^2_{13}=0 \quad \text{(R3.31.2)}$$
$$C^1_{33}C^3_{11}+C^2_{33}C^3_{12}+C^3_{33}C^3_{13}=0 \quad \text{(R3.31.3)}$$
$$C^1_{31}C^1_{11}+C^2_{31}C^1_{21}+C^3_{31}C^1_{31}=0 \quad \text{(R3.33.1)}$$
$$C^1_{31}C^2_{11}+C^2_{31}C^2_{21}+C^3_{31}C^2_{31}=0 \quad \text{(R3.33.2)}$$
$$C^1_{31}C^3_{11}+C^2_{31}C^3_{21}+C^3_{31}C^3_{31}=0 \quad \text{(R3.33.3)}$$
$$C^1_{32}C^1_{11}+C^2_{32}C^1_{21}+C^3_{32}C^1_{31}=0 \quad \text{(R3.37.1)}$$
$$C^1_{32}C^2_{11}+C^2_{32}C^2_{21}+C^3_{32}C^2_{31}=0 \quad \text{(R3.37.2)}$$
$$C^1_{32}C^3_{11}+C^2_{32}C^3_{21}+C^3_{32}C^3_{31}=0 \quad \text{(R3.37.3)}$$
$$C^1_{33}C^1_{11}+C^2_{33}C^1_{21}+C^3_{33}C^1_{31}=0 \quad \text{(R3.39.1)}$$
$$C^1_{33}C^2_{11}+C^2_{33}C^2_{21}+C^3_{33}C^2_{31}=0 \quad \text{(R3.39.2)}$$
$$C^1_{33}C^3_{11}+C^2_{33}C^3_{21}+C^3_{33}C^3_{31}=0. \quad \text{(R3.39.3)}$$

3.2.3 α defined as E_{13}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha]=E_{13}=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}.$$

However, due to our results in Section 3.2.1, where we realised we would not be able to solve the entire system of equations in this project, we will only write out the system of equations for this α, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\begin{aligned}\alpha(e_1) &= 0e_1 + 0e_2 + 0e_3 = 0,\\ \alpha(e_2) &= 0e_1 + 0e_2 + 0e_3 = 0,\\ \alpha(e_3) &= 1e_1 + 0e_2 + 0e_3 = e_1.\end{aligned}$$

We can thus rewrite our equations as follows:

$$\begin{aligned}
&(3.14): && \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.15): && \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.16): && \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, e_1)\\
& && \Rightarrow\; 0 = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, e_1),\\
&(3.17): && \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.18): && \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.19): && \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.20): && \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_1)\\
& && \Rightarrow\; 0 = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_1),\\
&(3.21): && \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.22): && \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_1)\\
& && \Rightarrow\; 0 = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_1),\\
&(3.23): && \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.24): && \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.25): && \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_1)\\
& && \Rightarrow\; 0 = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_1),\\
&(3.26): && \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.27): && \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.28): && \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.29): && \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_1)\\
& && \Rightarrow\; 0 = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_1),\\
&(3.30): && \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.31): && \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_1)\\
& && \Rightarrow\; 0 = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_1),\\
&(3.32): && \mu(e_1, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0)\\
& && \Rightarrow\; \mu(e_1, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = 0,
\end{aligned}$$

$$
\begin{aligned}
(3.33):\quad & \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \\
& \Rightarrow\ \mu(e_1, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = 0, \\
(3.34):\quad & \mu(e_1, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_1), \\
(3.35):\quad & \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \\
& \Rightarrow\ \mu(e_1, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = 0, \\
(3.36):\quad & \mu(e_1, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \\
& \Rightarrow\ \mu(e_1, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = 0, \\
(3.37):\quad & \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \\
& \Rightarrow\ \mu(e_1, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = 0, \\
(3.38):\quad & \mu(e_1, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_1), \\
(3.39):\quad & \mu(e_1, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \\
& \Rightarrow\ \mu(e_1, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = 0, \\
(3.40):\quad & \mu(e_1, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_1).
\end{aligned}
$$

We remember the following formulas that we calculated earlier:

$$
\begin{aligned}
\mu(e_i, C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3) = {} & (C_{jk}^1 C_{i1}^1 + C_{jk}^2 C_{i2}^1 + C_{jk}^3 C_{i3}^1) e_1 + (C_{jk}^1 C_{i1}^2 + C_{jk}^2 C_{i2}^2 + C_{jk}^3 C_{i3}^2) e_2 \\
& + (C_{jk}^1 C_{i1}^3 + C_{jk}^2 C_{i2}^3 + C_{jk}^3 C_{i3}^3) e_3
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3, e_i) = {} & (C_{jk}^1 C_{1i}^1 + C_{jk}^2 C_{2i}^1 + C_{jk}^3 C_{3i}^1) e_1 + (C_{jk}^1 C_{1i}^2 + C_{jk}^2 C_{2i}^2 + C_{jk}^3 C_{3i}^2) e_2 \\
& + (C_{jk}^1 C_{1i}^3 + C_{jk}^2 C_{2i}^3 + C_{jk}^3 C_{3i}^3) e_3.
\end{aligned}
$$

We use these formulas to rewrite our equations.

$$
\begin{aligned}
(3.16):\quad & 0 = (C_{11}^1 C_{11}^1 + C_{11}^2 C_{21}^1 + C_{11}^3 C_{31}^1) e_1 + (C_{11}^1 C_{11}^2 + C_{11}^2 C_{21}^2 + C_{11}^3 C_{31}^2) e_2 \\
& \quad + (C_{11}^1 C_{11}^3 + C_{11}^2 C_{21}^3 + C_{11}^3 C_{31}^3) e_3, \\
(3.20):\quad & 0 = (C_{12}^1 C_{11}^1 + C_{12}^2 C_{21}^1 + C_{12}^3 C_{31}^1) e_1 + (C_{12}^1 C_{11}^2 + C_{12}^2 C_{21}^2 + C_{12}^3 C_{31}^2) e_2 \\
& \quad + (C_{12}^1 C_{11}^3 + C_{12}^2 C_{21}^3 + C_{12}^3 C_{31}^3) e_3, \\
(3.22):\quad & 0 = (C_{13}^1 C_{11}^1 + C_{13}^2 C_{21}^1 + C_{13}^3 C_{31}^1) e_1 + (C_{13}^1 C_{11}^2 + C_{13}^2 C_{21}^2 + C_{13}^3 C_{31}^2) e_2 \\
& \quad + (C_{13}^1 C_{11}^3 + C_{13}^2 C_{21}^3 + C_{13}^3 C_{31}^3) e_3, \\
(3.25):\quad & 0 = (C_{21}^1 C_{11}^1 + C_{21}^2 C_{21}^1 + C_{21}^3 C_{31}^1) e_1 + (C_{21}^1 C_{11}^2 + C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2) e_2 \\
& \quad + (C_{21}^1 C_{11}^3 + C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3) e_3, \\
(3.29):\quad & 0 = (C_{22}^1 C_{11}^1 + C_{22}^2 C_{21}^1 + C_{22}^3 C_{31}^1) e_1 + (C_{22}^1 C_{11}^2 + C_{22}^2 C_{21}^2 + C_{22}^3 C_{31}^2) e_2 \\
& \quad + (C_{22}^1 C_{11}^3 + C_{22}^2 C_{21}^3 + C_{22}^3 C_{31}^3) e_3, \\
(3.31):\quad & 0 = (C_{23}^1 C_{11}^1 + C_{23}^2 C_{21}^1 + C_{23}^3 C_{31}^1) e_1 + (C_{23}^1 C_{11}^2 + C_{23}^2 C_{21}^2 + C_{23}^3 C_{31}^2) e_2
\end{aligned}
$$

$$+(C_{23}^1C_{11}^3+C_{23}^2C_{21}^3+C_{23}^3C_{31}^3)e_3,$$

(3.32) : $(C_{11}^1C_{11}^1+C_{11}^2C_{12}^1+C_{11}^3C_{13}^1)e_1+(C_{11}^1C_{11}^2+C_{11}^2C_{12}^2+C_{11}^3C_{13}^2)e_2$
$+(C_{11}^1C_{11}^3+C_{11}^2C_{12}^3+C_{11}^3C_{13}^3)e_3=0,$

(3.33) : $(C_{12}^1C_{11}^1+C_{12}^2C_{12}^1+C_{12}^3C_{13}^1)e_1+(C_{12}^1C_{11}^2+C_{12}^2C_{12}^2+C_{12}^3C_{13}^2)e_2$
$+(C_{12}^1C_{11}^3+C_{12}^2C_{12}^3+C_{12}^3C_{13}^3)e_3=0,$

(3.34) : $(C_{13}^1C_{11}^1+C_{13}^2C_{12}^1+C_{13}^3C_{13}^1)e_1+(C_{13}^1C_{11}^2+C_{13}^2C_{12}^2+C_{13}^3C_{13}^2)e_2$
$+(C_{13}^1C_{11}^3+C_{13}^2C_{12}^3+C_{13}^3C_{13}^3)e_3$
$=(C_{31}^1C_{11}^1+C_{31}^2C_{21}^1+C_{31}^3C_{31}^1)e_1+(C_{31}^1C_{11}^2+C_{31}^2C_{21}^2+C_{31}^3C_{31}^2)e_2$
$+(C_{31}^1C_{11}^3+C_{31}^2C_{21}^3+C_{31}^3C_{31}^3)e_3$
$\Rightarrow\ (C_{13}^1C_{11}^1+C_{13}^2C_{12}^1+C_{13}^3C_{13}^1-C_{31}^1C_{11}^1-C_{31}^2C_{21}^1-C_{31}^3C_{31}^1)e_1$
$+(C_{13}^1C_{11}^2+C_{13}^2C_{12}^2+C_{13}^3C_{13}^2-C_{31}^1C_{11}^2-C_{31}^2C_{21}^2-C_{31}^3C_{31}^2)e_2$
$+(C_{13}^1C_{11}^3+C_{13}^2C_{12}^3+C_{13}^3C_{13}^3-C_{31}^1C_{11}^3-C_{31}^2C_{21}^3-C_{31}^3C_{31}^3)e_3=0,$

(3.35) : $(C_{21}^1C_{11}^1+C_{21}^2C_{12}^1+C_{21}^3C_{13}^1)e_1+(C_{21}^1C_{11}^2+C_{21}^2C_{12}^2+C_{21}^3C_{13}^2)e_2$
$+(C_{21}^1C_{11}^3+C_{21}^2C_{12}^3+C_{21}^3C_{13}^3)e_3=0,$

(3.36) : $(C_{31}^1C_{11}^1+C_{31}^2C_{12}^1+C_{31}^3C_{13}^1)e_1+(C_{31}^1C_{11}^2+C_{31}^2C_{12}^2+C_{31}^3C_{13}^2)e_2$
$+(C_{31}^1C_{11}^3+C_{31}^2C_{12}^3+C_{31}^3C_{13}^3)e_3=0,$

(3.37) : $(C_{22}^1C_{11}^1+C_{22}^2C_{12}^1+C_{22}^3C_{13}^1)e_1+(C_{22}^1C_{11}^2+C_{22}^2C_{12}^2+C_{22}^3C_{13}^2)e_2$
$+(C_{22}^1C_{11}^3+C_{22}^2C_{12}^3+C_{22}^3C_{13}^3)e_3=0,$

(3.38) : $(C_{23}^1C_{11}^1+C_{23}^2C_{12}^1+C_{23}^3C_{13}^1)e_1+(C_{23}^1C_{11}^2+C_{23}^2C_{12}^2+C_{23}^3C_{13}^2)e_2$
$+(C_{23}^1C_{11}^3+C_{23}^2C_{12}^3+C_{23}^3C_{13}^3)e_3$
$=(C_{32}^1C_{11}^1+C_{32}^2C_{21}^1+C_{32}^3C_{31}^1)e_1+(C_{32}^1C_{11}^2+C_{32}^2C_{21}^2+C_{32}^3C_{31}^2)e_2$
$+(C_{32}^1C_{11}^3+C_{32}^2C_{21}^3+C_{32}^3C_{31}^3)e_3$
$\Rightarrow\ (C_{23}^1C_{11}^1+C_{23}^2C_{12}^1+C_{23}^3C_{13}^1-C_{32}^1C_{11}^1-C_{32}^2C_{21}^1-C_{32}^3C_{31}^1)e_1$
$+(C_{23}^1C_{11}^2+C_{23}^2C_{12}^2+C_{23}^3C_{13}^2-C_{32}^1C_{11}^2-C_{32}^2C_{21}^2-C_{32}^3C_{31}^2)e_2$
$+(C_{23}^1C_{11}^3+C_{23}^2C_{12}^3+C_{23}^3C_{13}^3-C_{32}^1C_{11}^3-C_{32}^2C_{21}^3-C_{32}^3C_{31}^3)e_3=0,$

(3.39) : $(C_{32}^1C_{11}^1+C_{32}^2C_{12}^1+C_{32}^3C_{13}^1)e_1+(C_{32}^1C_{11}^2+C_{32}^2C_{12}^2+C_{32}^3C_{13}^2)e_2$
$+(C_{32}^1C_{11}^3+C_{32}^2C_{12}^3+C_{32}^3C_{13}^3)e_3=0,$

(3.40) : $(C_{33}^1C_{11}^1+C_{33}^2C_{12}^1+C_{33}^3C_{13}^1)e_1+(C_{33}^1C_{11}^2+C_{33}^2C_{12}^2+C_{33}^3C_{13}^2)e_2$
$+(C_{33}^1C_{11}^3+C_{33}^2C_{12}^3+C_{33}^3C_{13}^3)e_3$
$=(C_{33}^1C_{11}^1+C_{33}^2C_{21}^1+C_{33}^3C_{31}^1)e_1+(C_{33}^1C_{11}^2+C_{33}^2C_{21}^2+C_{33}^3C_{31}^2)e_2$
$+(C_{33}^1C_{11}^3+C_{33}^2C_{21}^3+C_{33}^3C_{31}^3)e_3$
$\Rightarrow\ (C_{33}^2C_{12}^1+C_{33}^3C_{13}^1-C_{33}^2C_{21}^1-C_{33}^3C_{31}^1)e_1+(C_{33}^2C_{12}^2+C_{33}^3C_{13}^2-C_{33}^2C_{21}^2-C_{33}^3C_{31}^2)e_2$
$+(C_{33}^2C_{12}^3+C_{33}^3C_{13}^3-C_{33}^2C_{21}^3-C_{33}^3C_{31}^3)e_3=0.$

Remembering that $k_1 e_1 + k_2 e_2 + k_3 e_3 = 0$ will only have the trivial solution $k_1 = k_2 = k_3 = 0$, we can rewrite our equations as the following system of equations:

$$C^1_{11}C^1_{11} + C^2_{11}C^1_{21} + C^3_{11}C^1_{31} = 0 \quad \text{(R3.16.1)}$$
$$C^1_{11}C^2_{11} + C^2_{11}C^2_{21} + C^3_{11}C^2_{31} = 0 \quad \text{(R3.16.2)}$$
$$C^1_{11}C^3_{11} + C^2_{11}C^3_{21} + C^3_{11}C^3_{31} = 0 \quad \text{(R3.16.3)}$$
$$C^1_{12}C^1_{11} + C^2_{12}C^1_{21} + C^3_{12}C^1_{31} = 0 \quad \text{(R3.20.1)}$$
$$C^1_{12}C^2_{11} + C^2_{12}C^2_{21} + C^3_{12}C^2_{31} = 0 \quad \text{(R3.20.2)}$$
$$C^1_{12}C^3_{11} + C^2_{12}C^3_{21} + C^3_{12}C^3_{31} = 0 \quad \text{(R3.20.3)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{21} + C^3_{13}C^1_{31} = 0 \quad \text{(R3.22.1)}$$
$$C^1_{13}C^2_{11} + C^2_{13}C^2_{21} + C^3_{13}C^2_{31} = 0 \quad \text{(R3.22.2)}$$
$$C^1_{13}C^3_{11} + C^2_{13}C^3_{21} + C^3_{13}C^3_{31} = 0 \quad \text{(R3.22.3)}$$
$$C^1_{21}C^1_{11} + C^2_{21}C^1_{21} + C^3_{21}C^1_{31} = 0 \quad \text{(R3.25.1)}$$
$$C^1_{21}C^2_{11} + C^2_{21}C^2_{21} + C^3_{21}C^2_{31} = 0 \quad \text{(R3.25.2)}$$
$$C^1_{21}C^3_{11} + C^2_{21}C^3_{21} + C^3_{21}C^3_{31} = 0 \quad \text{(R3.25.3)}$$
$$C^1_{22}C^1_{11} + C^2_{22}C^1_{21} + C^3_{22}C^1_{31} = 0 \quad \text{(R3.29.1)}$$
$$C^1_{22}C^2_{11} + C^2_{22}C^2_{21} + C^3_{22}C^2_{31} = 0 \quad \text{(R3.29.2)}$$
$$C^1_{22}C^3_{11} + C^2_{22}C^3_{21} + C^3_{22}C^3_{31} = 0 \quad \text{(R3.29.3)}$$
$$C^1_{23}C^1_{11} + C^2_{23}C^1_{21} + C^3_{23}C^1_{31} = 0 \quad \text{(R3.31.1)}$$
$$C^1_{23}C^2_{11} + C^2_{23}C^2_{21} + C^3_{23}C^2_{31} = 0 \quad \text{(R3.31.2)}$$
$$C^1_{23}C^3_{11} + C^2_{23}C^3_{21} + C^3_{23}C^3_{31} = 0 \quad \text{(R3.31.3)}$$
$$C^1_{11}C^1_{11} + C^2_{11}C^1_{12} + C^3_{11}C^1_{13} = 0 \quad \text{(R3.32.1)}$$
$$C^1_{11}C^2_{11} + C^2_{11}C^2_{12} + C^3_{11}C^2_{13} = 0 \quad \text{(R3.32.2)}$$
$$C^1_{11}C^3_{11} + C^2_{11}C^3_{12} + C^3_{11}C^3_{13} = 0 \quad \text{(R3.32.3)}$$
$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} + C^3_{12}C^1_{13} = 0 \quad \text{(R3.33.1)}$$
$$C^1_{12}C^2_{11} + C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \quad \text{(R3.33.2)}$$
$$C^1_{12}C^3_{11} + C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \quad \text{(R3.33.3)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} + C^3_{13}C^1_{13} - C^1_{31}C^1_{11} - C^2_{31}C^1_{21} - C^3_{31}C^1_{31} = 0 \quad \text{(R3.34.1)}$$
$$C^1_{13}C^2_{11} + C^2_{13}C^2_{12} + C^3_{13}C^2_{13} - C^1_{31}C^2_{11} - C^2_{31}C^2_{21} - C^3_{31}C^2_{31} = 0 \quad \text{(R3.34.2)}$$
$$C^1_{13}C^3_{11} + C^2_{13}C^3_{12} + C^3_{13}C^3_{13} - C^1_{31}C^3_{11} - C^2_{31}C^3_{21} - C^3_{31}C^3_{31} = 0 \quad \text{(R3.34.3)}$$
$$C^1_{21}C^1_{11} + C^2_{21}C^1_{12} + C^3_{21}C^1_{13} = 0 \quad \text{(R3.35.1)}$$
$$C^1_{21}C^2_{11} + C^2_{21}C^2_{12} + C^3_{21}C^2_{13} = 0 \quad \text{(R3.35.2)}$$
$$C^1_{21}C^3_{11} + C^2_{21}C^3_{12} + C^3_{21}C^3_{13} = 0 \quad \text{(R3.35.3)}$$
$$C^1_{31}C^1_{11} + C^2_{31}C^1_{12} + C^3_{31}C^1_{13} = 0 \quad \text{(R3.36.1)}$$

$$C_{31}^1C_{11}^2 + C_{31}^2C_{12}^2 + C_{31}^3C_{13}^2 = 0 \quad \text{(R3.36.2)}$$
$$C_{31}^1C_{11}^3 + C_{31}^2C_{12}^3 + C_{31}^3C_{13}^3 = 0 \quad \text{(R3.36.3)}$$
$$C_{22}^1C_{11}^1 + C_{22}^2C_{12}^1 + C_{22}^3C_{13}^1 = 0 \quad \text{(R3.37.1)}$$
$$C_{22}^1C_{11}^2 + C_{22}^2C_{12}^2 + C_{22}^3C_{13}^2 = 0 \quad \text{(R3.37.2)}$$
$$C_{22}^1C_{11}^3 + C_{22}^2C_{12}^3 + C_{22}^3C_{13}^3 = 0 \quad \text{(R3.37.3)}$$
$$C_{23}^1C_{11}^1 + C_{23}^2C_{12}^1 + C_{23}^3C_{13}^1 - C_{32}^1C_{11}^1 - C_{32}^2C_{21}^1 - C_{32}^3C_{31}^1 = 0 \quad \text{(R3.38.1)}$$
$$C_{23}^1C_{11}^2 + C_{23}^2C_{12}^2 + C_{23}^3C_{13}^2 - C_{32}^1C_{11}^2 - C_{32}^2C_{21}^2 - C_{32}^3C_{31}^2 = 0 \quad \text{(R3.38.2)}$$
$$C_{23}^1C_{11}^3 + C_{23}^2C_{12}^3 + C_{23}^3C_{13}^3 - C_{32}^1C_{11}^3 - C_{32}^2C_{21}^3 - C_{32}^3C_{31}^3 = 0 \quad \text{(R3.38.3)}$$
$$C_{32}^1C_{11}^1 + C_{32}^2C_{12}^1 + C_{32}^3C_{13}^1 = 0 \quad \text{(R3.39.1)}$$
$$C_{32}^1C_{11}^2 + C_{32}^2C_{12}^2 + C_{32}^3C_{13}^2 = 0 \quad \text{(R3.39.2)}$$
$$C_{32}^1C_{11}^3 + C_{32}^2C_{12}^3 + C_{32}^3C_{13}^3 = 0 \quad \text{(R3.39.3)}$$
$$C_{33}^2C_{12}^1 + C_{33}^3C_{13}^1 - C_{33}^2C_{21}^1 - C_{33}^3C_{31}^1 = 0 \quad \text{(R3.40.1)}$$
$$C_{33}^2C_{12}^2 + C_{33}^3C_{13}^2 - C_{33}^2C_{21}^2 - C_{33}^3C_{31}^2 = 0 \quad \text{(R3.40.2)}$$
$$C_{33}^2C_{12}^3 + C_{33}^3C_{13}^3 - C_{33}^2C_{21}^3 - C_{33}^3C_{31}^3 = 0. \quad \text{(R3.40.3)}$$

3.2.4 α defined as E_{21}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha] = E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

However, due to our results in Section 3.2.1, where we realised we would not be able to solve the entire system of equations in this project, we will only write out the system of equations for this α, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\begin{aligned}
\alpha(e_1) &= 0e_1 + 1e_2 + 0e_3 = e_2,\\
\alpha(e_2) &= 0e_1 + 0e_2 + 0e_3 = 0,\\
\alpha(e_3) &= 0e_1 + 0e_2 + 0e_3 = 0.
\end{aligned}$$

We can thus rewrite our equations as follows:

$$\begin{aligned}
(3.14):\quad & \mu(e_2, C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3) = \mu(C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3, e_2),\\
(3.15):\quad & \mu(e_2, C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3) = \mu(C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3, 0)\\
& \Rightarrow\ \mu(e_2, C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3) = 0,\\
(3.16):\quad & \mu(e_2, C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3) = \mu(C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3, 0)\\
& \Rightarrow\ \mu(e_2, C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3) = 0,\\
(3.17):\quad & \mu(e_2, C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3) = \mu(C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3, e_2),
\end{aligned}$$

$$
\begin{aligned}
(3.18): &\quad \mu(e_2, C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3) = \mu(C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3, e_2),\\
(3.19): &\quad \mu(e_2, C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3) = \mu(C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3, 0)\\
&\quad \Rightarrow\ \mu(e_2, C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3) = 0,\\
(3.20): &\quad \mu(e_2, C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3) = \mu(C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3, 0)\\
&\quad \Rightarrow\ \mu(e_2, C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3) = 0,\\
(3.21): &\quad \mu(e_2, C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3) = \mu(C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3, 0)\\
&\quad \Rightarrow\ \mu(e_2, C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3) = 0,\\
(3.22): &\quad \mu(e_2, C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3) = \mu(C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3, 0)\\
&\quad \Rightarrow\ \mu(e_2, C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3) = 0,\\
(3.23): &\quad \mu(0, C_{11}^1e_1+C_{11}^2e_2+C_{11}^3e_3) = \mu(C_{21}^1e_1+C_{21}^2e_2+C_{21}^3e_3, e_2)\\
&\quad \Rightarrow\ 0 = \mu(C_{21}^1e_1+C_{21}^2e_2+C_{21}^3e_3, e_2),\\
(3.24): &\quad \mu(0, C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3) = \mu(C_{21}^1e_1+C_{21}^2e_2+C_{21}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.25): &\quad \mu(0, C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3) = \mu(C_{21}^1e_1+C_{21}^2e_2+C_{21}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.26): &\quad \mu(0, C_{21}^1e_1+C_{21}^2e_2+C_{21}^3e_3) = \mu(C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3, e_2)\\
&\quad \Rightarrow\ 0 = \mu(C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3, e_2),\\
(3.27): &\quad \mu(0, C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3) = \mu(C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3, e_2)\\
&\quad \Rightarrow\ 0 = \mu(C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3, e_2),\\
(3.28): &\quad \mu(0, C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3) = \mu(C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.29): &\quad \mu(0, C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3) = \mu(C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.30): &\quad \mu(0, C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3) = \mu(C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.31): &\quad \mu(0, C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3) = \mu(C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.32): &\quad \mu(0, C_{11}^1e_1+C_{11}^2e_2+C_{11}^3e_3) = \mu(C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3, e_2)\\
&\quad \Rightarrow\ 0 = \mu(C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3, e_2),\\
(3.33): &\quad \mu(0, C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3) = \mu(C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.34): &\quad \mu(0, C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3) = \mu(C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.35): &\quad \mu(0, C_{21}^1e_1+C_{21}^2e_2+C_{21}^3e_3) = \mu(C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3, e_2)\\
&\quad \Rightarrow\ 0 = \mu(C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3, e_2),\\
(3.36): &\quad \mu(0, C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3) = \mu(C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3, e_2)\\
&\quad \Rightarrow\ 0 = \mu(C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3, e_2),\\
(3.37): &\quad \mu(0, C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3) = \mu(C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.38): &\quad \mu(0, C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3) = \mu(C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.39): &\quad \mu(0, C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3) = \mu(C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3, 0) \ \Rightarrow\ 0=0,\\
(3.40): &\quad \mu(0, C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3) = \mu(C_{33}^1e_1+C_{33}^2e_2+C_{33}^3e_3, 0) \ \Rightarrow\ 0=0.
\end{aligned}
$$

We remember the following formulas that we calculated earlier:

$$\mu(e_i, C^1_{jk}e_1 + C^2_{jk}e_2 + C^3_{jk}e_3) = (C^1_{jk}C^1_{i1} + C^2_{jk}C^1_{i2} + C^3_{jk}C^1_{i3})e_1 + (C^1_{jk}C^2_{i1} + C^2_{jk}C^2_{i2} + C^3_{jk}C^2_{i3})e_2 + (C^1_{jk}C^3_{i1} + C^2_{jk}C^3_{i2} + C^3_{jk}C^3_{i3})e_3$$

and

$$\mu(C^1_{jk}e_1 + C^2_{jk}e_2 + C^3_{jk}e_3, e_i) = (C^1_{jk}C^1_{1i} + C^2_{jk}C^1_{2i} + C^3_{jk}C^1_{3i})e_1 + (C^1_{jk}C^2_{1i} + C^2_{jk}C^2_{2i} + C^3_{jk}C^2_{3i})e_2 + (C^1_{jk}C^3_{1i} + C^2_{jk}C^3_{2i} + C^3_{jk}C^3_{3i})e_3.$$

We use these formulas to rewrite our equations.

$$\begin{aligned}
(3.14):\quad & (C^1_{11}C^1_{21} + C^2_{11}C^1_{22} + C^3_{11}C^1_{23})e_1 + (C^1_{11}C^2_{21} + C^2_{11}C^2_{22} + C^3_{11}C^2_{23})e_2 \\
& + (C^1_{11}C^3_{21} + C^2_{11}C^3_{22} + C^3_{11}C^3_{23})e_3 \\
& = (C^1_{11}C^1_{12} + C^2_{11}C^1_{22} + C^3_{11}C^1_{32})e_1 + (C^1_{11}C^2_{12} + C^2_{11}C^2_{22} + C^3_{11}C^2_{32})e_2 \\
& \quad + (C^1_{11}C^3_{12} + C^2_{11}C^3_{22} + C^3_{11}C^3_{32})e_3 \\
& \Rightarrow \ (C^1_{11}C^1_{21} + C^3_{11}C^1_{23} - C^1_{11}C^1_{12} - C^3_{11}C^1_{32})e_1 + (C^1_{11}C^2_{21} + C^3_{11}C^2_{23} - C^1_{11}C^2_{21} - C^3_{11}C^2_{23})e_2 \\
& \quad + (C^1_{11}C^3_{21} + C^3_{11}C^3_{23} - C^1_{11}C^3_{12} - C^3_{11}C^3_{32})e_3 = 0, \\
(3.15):\quad & (C^1_{12}C^1_{21} + C^2_{12}C^1_{22} + C^3_{12}C^1_{23})e_1 + (C^1_{12}C^2_{21} + C^2_{12}C^2_{22} + C^3_{12}C^2_{23})e_2 \\
& + (C^1_{12}C^3_{21} + C^2_{12}C^3_{22} + C^3_{12}C^3_{23})e_3 = 0, \\
(3.16):\quad & (C^1_{13}C^1_{21} + C^2_{13}C^1_{22} + C^3_{13}C^1_{23})e_1 + (C^1_{13}C^2_{21} + C^2_{13}C^2_{22} + C^3_{13}C^2_{23})e_2 \\
& + (C^1_{13}C^3_{21} + C^2_{13}C^3_{22} + C^3_{13}C^3_{23})e_3 = 0, \\
(3.17):\quad & (C^1_{21}C^1_{21} + C^2_{21}C^1_{22} + C^3_{21}C^1_{23})e_1 + (C^1_{21}C^2_{21} + C^2_{21}C^2_{22} + C^3_{21}C^2_{23})e_2 \\
& + (C^1_{21}C^3_{21} + C^2_{21}C^3_{22} + C^3_{21}C^3_{23})e_3 \\
& = (C^1_{12}C^1_{12} + C^2_{12}C^1_{22} + C^3_{12}C^1_{32})e_1 + (C^1_{12}C^2_{12} + C^2_{12}C^2_{22} + C^3_{12}C^2_{32})e_2 \\
& \quad + (C^1_{12}C^3_{12} + C^2_{12}C^3_{22} + C^3_{12}C^3_{32})e_3 \\
& \Rightarrow \ (C^1_{21}C^1_{21} + C^2_{21}C^1_{22} + C^3_{21}C^1_{23} - C^1_{12}C^1_{12} - C^2_{12}C^1_{22} - C^3_{12}C^1_{32})e_1 \\
& \quad + (C^1_{21}C^2_{21} + C^2_{21}C^2_{22} + C^3_{21}C^2_{23} - C^1_{12}C^2_{12} - C^2_{12}C^2_{22} - C^3_{12}C^2_{32})e_2 \\
& \quad + (C^1_{21}C^3_{21} + C^2_{21}C^3_{22} + C^3_{21}C^3_{23} - C^1_{12}C^3_{12} - C^2_{12}C^3_{22} - C^3_{12}C^3_{32})e_3 = 0, \\
(3.18):\quad & (C^1_{31}C^1_{21} + C^2_{31}C^1_{22} + C^3_{31}C^1_{23})e_1 + (C^1_{31}C^2_{21} + C^2_{31}C^2_{22} + C^3_{31}C^2_{23})e_2 \\
& + (C^1_{31}C^3_{21} + C^2_{31}C^3_{22} + C^3_{31}C^3_{23})e_3 \\
& = (C^1_{13}C^1_{12} + C^2_{13}C^1_{22} + C^3_{13}C^1_{32})e_1 + (C^1_{13}C^2_{12} + C^2_{13}C^2_{22} + C^3_{13}C^2_{32})e_2 \\
& \quad + (C^1_{13}C^3_{12} + C^2_{13}C^3_{22} + C^3_{13}C^3_{32})e_3 \\
& \Rightarrow \ (C^1_{31}C^1_{21} + C^2_{31}C^1_{22} + C^3_{31}C^1_{23} - C^1_{13}C^1_{12} - C^2_{13}C^1_{22} - C^3_{13}C^1_{32})e_1 \\
& \quad + (C^1_{31}C^2_{21} + C^2_{31}C^2_{22} + C^3_{31}C^2_{23} - C^1_{13}C^2_{12} - C^2_{13}C^2_{22} - C^3_{13}C^2_{32})e_2 \\
& \quad + (C^1_{31}C^3_{21} + C^2_{31}C^3_{22} + C^3_{31}C^3_{23} - C^1_{13}C^3_{12} - C^2_{13}C^3_{22} - C^3_{13}C^3_{32})e_3 = 0, \\
(3.19):\quad & (C^1_{22}C^1_{21} + C^2_{22}C^1_{22} + C^3_{22}C^1_{23})e_1 + (C^1_{22}C^2_{21} + C^2_{22}C^2_{22} + C^3_{22}C^2_{23})e_2
\end{aligned}$$

$$+(C^1_{22}C^3_{21}+C^2_{22}C^3_{22}+C^3_{22}C^3_{23})e_3=0,$$

$$(3.20):\quad (C^1_{23}C^1_{21}+C^2_{23}C^1_{22}+C^3_{23}C^1_{23})e_1+(C^1_{23}C^2_{21}+C^2_{23}C^2_{22}+C^3_{23}C^2_{23})e_2$$
$$+(C^1_{23}C^3_{21}+C^2_{23}C^3_{22}+C^3_{23}C^3_{23})e_3=0,$$

$$(3.21):\quad (C^1_{32}C^1_{21}+C^2_{32}C^1_{22}+C^3_{32}C^1_{23})e_1+(C^1_{32}C^2_{21}+C^2_{32}C^2_{22}+C^3_{32}C^2_{23})e_2$$
$$+(C^1_{32}C^3_{21}+C^2_{32}C^3_{22}+C^3_{32}C^3_{23})e_3=0,$$

$$(3.22):\quad (C^1_{33}C^1_{21}+C^2_{33}C^1_{22}+C^3_{33}C^1_{23})e_1+(C^1_{33}C^2_{21}+C^2_{33}C^2_{22}+C^3_{33}C^2_{23})e_2$$
$$+(C^1_{33}C^3_{21}+C^2_{33}C^3_{22}+C^3_{33}C^3_{23})e_3=0,$$

$$(3.23):\quad 0=(C^1_{21}C^1_{12}+C^2_{21}C^1_{22}+C^3_{21}C^1_{32})e_1+(C^1_{21}C^2_{12}+C^2_{21}C^2_{22}+C^3_{21}C^2_{32})e_2$$
$$+(C^1_{21}C^3_{12}+C^2_{21}C^3_{22}+C^3_{21}C^3_{32})e_3,$$

$$(3.26):\quad 0=(C^1_{22}C^1_{12}+C^2_{22}C^1_{22}+C^3_{22}C^1_{32})e_1+(C^1_{22}C^2_{12}+C^2_{22}C^2_{22}+C^3_{22}C^2_{32})e_2$$
$$+(C^1_{22}C^3_{12}+C^2_{22}C^3_{22}+C^3_{22}C^3_{32})e_3,$$

$$(3.27):\quad 0=(C^1_{23}C^1_{12}+C^2_{23}C^1_{22}+C^3_{23}C^1_{32})e_1+(C^1_{23}C^2_{12}+C^2_{23}C^2_{22}+C^3_{23}C^2_{32})e_2$$
$$+(C^1_{23}C^3_{12}+C^2_{23}C^3_{22}+C^3_{23}C^3_{32})e_3,$$

$$(3.32):\quad 0=(C^1_{31}C^1_{12}+C^2_{31}C^1_{22}+C^3_{31}C^1_{32})e_1+(C^1_{31}C^2_{12}+C^2_{31}C^2_{22}+C^3_{31}C^2_{32})e_2$$
$$+(C^1_{31}C^3_{12}+C^2_{31}C^3_{22}+C^3_{31}C^3_{32})e_3,$$

$$(3.35):\quad 0=(C^1_{32}C^1_{12}+C^2_{32}C^1_{22}+C^3_{32}C^1_{32})e_1+(C^1_{32}C^2_{12}+C^2_{32}C^2_{22}+C^3_{32}C^2_{32})e_2$$
$$+(C^1_{32}C^3_{12}+C^2_{32}C^3_{22}+C^3_{32}C^3_{32})e_3,$$

$$(3.36):\quad 0=(C^1_{33}C^1_{12}+C^2_{33}C^1_{22}+C^3_{33}C^1_{32})e_1+(C^1_{33}C^2_{12}+C^2_{33}C^2_{22}+C^3_{33}C^2_{32})e_2$$
$$+(C^1_{33}C^3_{12}+C^2_{33}C^3_{22}+C^3_{33}C^3_{32})e_3.$$

Remembering that $k_1e_1+k_2e_2+k_3e_3=0$ will only have the trivial solution $k_1=k_2=k_3=0$, we can rewrite our equations as the following system of equations:

$$C^1_{11}C^1_{21}+C^3_{11}C^1_{23}-C^1_{11}C^1_{12}-C^3_{11}C^1_{32}=0 \qquad \text{(R3.14.1)}$$
$$C^1_{11}C^2_{21}+C^3_{11}C^2_{23}-C^1_{11}C^2_{21}-C^3_{11}C^2_{23}=0 \qquad \text{(R3.14.2)}$$
$$C^1_{11}C^3_{21}+C^3_{11}C^3_{23}-C^1_{11}C^3_{12}-C^3_{11}C^3_{32}=0 \qquad \text{(R3.14.3)}$$
$$C^1_{12}C^1_{21}+C^2_{12}C^1_{22}+C^3_{12}C^1_{23}=0 \qquad \text{(R3.15.1)}$$
$$C^1_{12}C^2_{21}+C^2_{12}C^2_{22}+C^3_{12}C^2_{23}=0 \qquad \text{(R3.15.2)}$$
$$C^1_{12}C^3_{21}+C^2_{12}C^3_{22}+C^3_{12}C^3_{23}=0 \qquad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{21}+C^2_{13}C^1_{22}+C^3_{13}C^1_{23}=0 \qquad \text{(R3.16.1)}$$
$$C^1_{13}C^2_{21}+C^2_{13}C^2_{22}+C^3_{13}C^2_{23}=0 \qquad \text{(R3.16.2)}$$
$$C^1_{13}C^3_{21}+C^2_{13}C^3_{22}+C^3_{13}C^3_{23}=0 \qquad \text{(R3.16.3)}$$
$$C^1_{21}C^1_{21}+C^2_{21}C^1_{22}+C^3_{21}C^1_{23}-C^1_{12}C^1_{12}-C^2_{12}C^1_{22}-C^3_{12}C^1_{32}=0 \qquad \text{(R3.17.1)}$$
$$C^1_{21}C^2_{21}+C^2_{21}C^2_{22}+C^3_{21}C^2_{23}-C^1_{12}C^2_{12}-C^2_{12}C^2_{22}-C^3_{12}C^2_{32}=0 \qquad \text{(R3.17.2)}$$
$$C^1_{21}C^3_{21}+C^2_{21}C^3_{22}+C^3_{21}C^3_{23}-C^1_{12}C^3_{12}-C^2_{12}C^3_{22}-C^3_{12}C^3_{32}=0 \qquad \text{(R3.17.3)}$$

$$C_{31}^1C_{21}^1+C_{31}^2C_{22}^1+C_{31}^3C_{23}^1-C_{13}^1C_{12}^1-C_{13}^2C_{22}^1-C_{13}^3C_{32}^1=0 \quad \text{(R3.18.1)}$$
$$C_{31}^1C_{21}^2+C_{31}^2C_{22}^2+C_{31}^3C_{23}^2-C_{13}^1C_{12}^2-C_{13}^2C_{22}^2-C_{13}^3C_{32}^2=0 \quad \text{(R3.18.2)}$$
$$C_{31}^1C_{21}^3+C_{31}^2C_{22}^3+C_{31}^3C_{23}^3-C_{13}^1C_{12}^3-C_{13}^2C_{22}^3-C_{13}^3C_{32}^3=0 \quad \text{(R3.18.3)}$$
$$C_{22}^1C_{21}^1+C_{22}^2C_{22}^1+C_{22}^3C_{23}^1=0 \quad \text{(R3.19.1)}$$
$$C_{22}^1C_{21}^2+C_{22}^2C_{22}^2+C_{22}^3C_{23}^2=0 \quad \text{(R3.19.2)}$$
$$C_{22}^1C_{21}^3+C_{22}^2C_{22}^3+C_{22}^3C_{23}^3=0 \quad \text{(R3.19.3)}$$
$$C_{23}^1C_{21}^1+C_{23}^2C_{22}^1+C_{23}^3C_{23}^1=0 \quad \text{(R3.20.1)}$$
$$C_{23}^1C_{21}^2+C_{23}^2C_{22}^2+C_{23}^3C_{23}^2=0 \quad \text{(R3.20.2)}$$
$$C_{23}^1C_{21}^3+C_{23}^2C_{22}^3+C_{23}^3C_{23}^3=0 \quad \text{(R3.20.3)}$$
$$C_{32}^1C_{21}^1+C_{32}^2C_{22}^1+C_{32}^3C_{23}^1=0 \quad \text{(R3.21.1)}$$
$$C_{32}^1C_{21}^2+C_{32}^2C_{22}^2+C_{32}^3C_{23}^2=0 \quad \text{(R3.21.2)}$$
$$C_{32}^1C_{21}^3+C_{32}^2C_{22}^3+C_{32}^3C_{23}^3=0 \quad \text{(R3.21.3)}$$
$$C_{33}^1C_{21}^1+C_{33}^2C_{22}^1+C_{33}^3C_{23}^1=0 \quad \text{(R3.22.1)}$$
$$C_{33}^1C_{21}^2+C_{33}^2C_{22}^2+C_{33}^3C_{23}^2=0 \quad \text{(R3.22.2)}$$
$$C_{33}^1C_{21}^3+C_{33}^2C_{22}^3+C_{33}^3C_{23}^3=0 \quad \text{(R3.22.3)}$$
$$C_{21}^1C_{12}^1+C_{21}^2C_{22}^1+C_{21}^3C_{32}^1=0 \quad \text{(R3.23.1)}$$
$$C_{21}^1C_{12}^2+C_{21}^2C_{22}^2+C_{21}^3C_{32}^2=0 \quad \text{(R3.23.2)}$$
$$C_{21}^1C_{12}^3+C_{21}^2C_{22}^3+C_{21}^3C_{32}^3=0 \quad \text{(R3.23.3)}$$
$$C_{22}^1C_{12}^1+C_{22}^2C_{22}^1+C_{22}^3C_{32}^1=0 \quad \text{(R3.26.1)}$$
$$C_{22}^1C_{12}^2+C_{22}^2C_{22}^2+C_{22}^3C_{32}^2=0 \quad \text{(R3.26.2)}$$
$$C_{22}^1C_{12}^3+C_{22}^2C_{22}^3+C_{22}^3C_{32}^3=0 \quad \text{(R3.26.3)}$$
$$C_{23}^1C_{12}^1+C_{23}^2C_{22}^1+C_{23}^3C_{32}^1=0 \quad \text{(R3.27.1)}$$
$$C_{23}^1C_{12}^2+C_{23}^2C_{22}^2+C_{23}^3C_{32}^2=0 \quad \text{(R3.27.2)}$$
$$C_{23}^1C_{12}^3+C_{23}^2C_{22}^3+C_{23}^3C_{32}^3=0 \quad \text{(R3.27.3)}$$
$$C_{31}^1C_{12}^1+C_{31}^2C_{22}^1+C_{31}^3C_{32}^1=0 \quad \text{(R3.32.1)}$$
$$C_{31}^1C_{12}^2+C_{31}^2C_{22}^2+C_{31}^3C_{32}^2=0 \quad \text{(R3.32.2)}$$
$$C_{31}^1C_{12}^3+C_{31}^2C_{22}^3+C_{31}^3C_{32}^3=0 \quad \text{(R3.32.3)}$$
$$C_{32}^1C_{12}^1+C_{32}^2C_{22}^1+C_{32}^3C_{32}^1=0 \quad \text{(R3.35.1)}$$
$$C_{32}^1C_{12}^2+C_{32}^2C_{22}^2+C_{32}^3C_{32}^2=0 \quad \text{(R3.35.2)}$$
$$C_{32}^1C_{12}^3+C_{32}^2C_{22}^3+C_{32}^3C_{32}^3=0 \quad \text{(R3.35.3)}$$
$$C_{33}^1C_{12}^1+C_{33}^2C_{22}^1+C_{33}^3C_{32}^1=0 \quad \text{(R3.36.1)}$$
$$C_{33}^1C_{12}^2+C_{33}^2C_{22}^2+C_{33}^3C_{32}^2=0 \quad \text{(R3.36.2)}$$
$$C_{33}^1C_{12}^3+C_{33}^2C_{22}^3+C_{33}^3C_{32}^3=0. \quad \text{(R3.36.3)}$$

3.2.5 α defined as E_{22}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha] = E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

However, due to our results in Section 3.2.1, where we realised we would not be able to solve the entire system of equations in this project, we will only write out the system of equations for this α, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\begin{aligned} \alpha(e_1) &= 0e_1 + 0e_2 + 0e_3 = 0, \\ \alpha(e_2) &= 0e_1 + 1e_2 + 0e_3 = e_2, \\ \alpha(e_3) &= 0e_1 + 0e_2 + 0e_3 = 0. \end{aligned}$$

We can thus rewrite our equations as follows:

$$\begin{aligned}
(3.14): &\quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0) \;\Rightarrow\; 0 = 0, \\
(3.15): &\quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, e_2) \\
&\quad \Rightarrow\; 0 = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, e_2), \\
(3.16): &\quad \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0) \;\Rightarrow\; 0 = 0, \\
(3.17): &\quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0) \;\Rightarrow\; 0 = 0, \\
(3.18): &\quad \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0) \;\Rightarrow\; 0 = 0, \\
(3.19): &\quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_2) \\
&\quad \Rightarrow\; 0 = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_2), \\
(3.20): &\quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0) \;\Rightarrow\; 0 = 0, \\
(3.21): &\quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_2) \\
&\quad \Rightarrow\; 0 = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_2), \\
(3.22): &\quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0) \;\Rightarrow\; 0 = 0, \\
(3.23): &\quad \mu(e_2, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \\
&\quad \Rightarrow\; \mu(e_2, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = 0, \\
(3.24): &\quad \mu(e_2, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_2), \\
(3.25): &\quad \mu(e_2, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \\
&\quad \Rightarrow\; \mu(e_2, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = 0, \\
(3.26): &\quad \mu(e_2, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \\
&\quad \Rightarrow\; \mu(e_2, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = 0, \\
(3.27): &\quad \mu(e_2, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \\
&\quad \Rightarrow\; \mu(e_2, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = 0,
\end{aligned}$$

$$
\begin{aligned}
(3.28):\quad & \mu(e_2, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_2),\\
(3.29):\quad & \mu(e_2, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0)\\
& \Rightarrow\ \mu(e_2, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = 0,\\
(3.30):\quad & \mu(e_2, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_2),\\
(3.31):\quad & \mu(e_2, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0)\\
& \Rightarrow\ \mu(e_2, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = 0,\\
(3.32):\quad & \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.33):\quad & \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_2)\\
& \Rightarrow\ 0 = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_2),\\
(3.34):\quad & \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.35):\quad & \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.36):\quad & \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.37):\quad & \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_2)\\
& \Rightarrow\ 0 = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_2),\\
(3.38):\quad & \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.39):\quad & \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_2)\\
& \Rightarrow\ 0 = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_2),\\
(3.40):\quad & \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \ \Rightarrow\ 0 = 0.
\end{aligned}
$$

We remember the following formulas that we calculated earlier:

$$
\begin{aligned}
\mu(e_i, C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3) = {} & (C_{jk}^1 C_{i1}^1 + C_{jk}^2 C_{i2}^1 + C_{jk}^3 C_{i3}^1) e_1 + (C_{jk}^1 C_{i1}^2 + C_{jk}^2 C_{i2}^2 + C_{jk}^3 C_{i3}^2) e_2\\
& + (C_{jk}^1 C_{i1}^3 + C_{jk}^2 C_{i2}^3 + C_{jk}^3 C_{i3}^3) e_3
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3, e_i) = {} & (C_{jk}^1 C_{1i}^1 + C_{jk}^2 C_{2i}^1 + C_{jk}^3 C_{3i}^1) e_1 + (C_{jk}^1 C_{1i}^2 + C_{jk}^2 C_{2i}^2 + C_{jk}^3 C_{3i}^2) e_2\\
& + (C_{jk}^1 C_{1i}^3 + C_{jk}^2 C_{2i}^3 + C_{jk}^3 C_{3i}^3) e_3.
\end{aligned}
$$

We use these formulas to rewrite our equations.

$$
\begin{aligned}
(3.15):\quad & 0 = (C_{11}^1 C_{12}^1 + C_{11}^2 C_{22}^1 + C_{11}^3 C_{32}^1) e_1 + (C_{11}^1 C_{12}^2 + C_{11}^2 C_{22}^2 + C_{11}^3 C_{32}^2) e_1\\
& \quad + (C_{11}^1 C_{12}^3 + C_{11}^2 C_{22}^3 + C_{11}^3 C_{32}^3) e_1,\\
(3.19):\quad & 0 = (C_{12}^1 C_{12}^1 + C_{12}^2 C_{22}^1 + C_{12}^3 C_{32}^1) e_1 + (C_{12}^1 C_{12}^2 + C_{12}^2 C_{22}^2 + C_{12}^3 C_{32}^2) e_2\\
& \quad + (C_{12}^1 C_{12}^3 + C_{12}^2 C_{22}^3 + C_{12}^3 C_{32}^3) e_3,\\
(3.21):\quad & 0 = (C_{13}^1 C_{12}^1 + C_{13}^2 C_{22}^1 + C_{13}^3 C_{32}^1) e_1 + (C_{13}^1 C_{12}^2 + C_{13}^2 C_{22}^2 + C_{13}^3 C_{32}^2) e_2\\
& \quad + (C_{13}^1 C_{12}^3 + C_{13}^2 C_{22}^3 + C_{13}^3 C_{32}^3) e_3,
\end{aligned}
$$

$$(3.23): \quad (C_{11}^1C_{21}^1 + C_{11}^2C_{22}^1 + C_{11}^3C_{23}^1)e_1 + (C_{11}^1C_{21}^2 + C_{11}^2C_{22}^2 + C_{11}^3C_{23}^2)e_2 + (C_{11}^1C_{21}^3 + C_{11}^2C_{22}^3 + C_{11}^3C_{23}^3)e_3 = 0,$$

$$(3.24): \quad (C_{12}^1C_{21}^1 + C_{12}^2C_{22}^1 + C_{12}^3C_{23}^1)e_1 + (C_{12}^1C_{21}^2 + C_{12}^2C_{22}^2 + C_{12}^3C_{23}^2)e_2 + (C_{12}^1C_{21}^3 + C_{12}^2C_{22}^3 + C_{12}^3C_{23}^3)e_3$$
$$= (C_{21}^1C_{12}^1 + C_{21}^2C_{22}^1 + C_{21}^3C_{32}^1)e_1 + (C_{21}^1C_{12}^2 + C_{21}^2C_{22}^2 + C_{21}^3C_{32}^2)e_2 + (C_{21}^1C_{12}^3 + C_{21}^2C_{22}^3 + C_{21}^3C_{32}^3)e_3$$
$$\Rightarrow \quad (C_{12}^2C_{22}^1 + C_{12}^3C_{23}^1 - C_{21}^2C_{22}^1 - C_{21}^3C_{32}^1)e_1 + (C_{12}^1C_{21}^2 + C_{12}^2C_{22}^2 + C_{12}^3C_{23}^2 - C_{21}^1C_{12}^2 - C_{21}^2C_{22}^2 - C_{21}^3C_{32}^2)e_2 + (C_{12}^1C_{21}^3 + C_{12}^2C_{22}^3 + C_{12}^3C_{23}^3 - C_{21}^1C_{12}^3 - C_{21}^2C_{22}^3 - C_{21}^3C_{32}^3)e_3 = 0,$$

$$(3.25): \quad (C_{13}^1C_{21}^1 + C_{13}^2C_{22}^1 + C_{13}^3C_{23}^1)e_1 + (C_{13}^1C_{21}^2 + C_{13}^2C_{22}^2 + C_{13}^3C_{23}^2)e_2 + (C_{13}^1C_{21}^3 + C_{13}^2C_{22}^3 + C_{13}^3C_{23}^3)e_3 = 0,$$

$$(3.26): \quad (C_{21}^1C_{21}^1 + C_{21}^2C_{22}^1 + C_{21}^3C_{23}^1)e_1 + (C_{21}^1C_{21}^2 + C_{21}^2C_{22}^2 + C_{21}^3C_{23}^2)e_2 + (C_{21}^1C_{21}^3 + C_{21}^2C_{22}^3 + C_{21}^3C_{23}^3)e_3 = 0,$$

$$(3.27): \quad (C_{31}^1C_{21}^1 + C_{31}^2C_{22}^1 + C_{31}^3C_{23}^1)e_1 + (C_{31}^1C_{21}^2 + C_{31}^2C_{22}^2 + C_{31}^3C_{23}^2)e_2 + (C_{31}^1C_{21}^3 + C_{31}^2C_{22}^3 + C_{31}^3C_{23}^3)e_3 = 0,$$

$$(3.28): \quad (C_{22}^1C_{21}^1 + C_{22}^2C_{22}^1 + C_{22}^3C_{23}^1)e_1 + (C_{22}^1C_{21}^2 + C_{22}^2C_{22}^2 + C_{22}^3C_{23}^2)e_2 + (C_{22}^1C_{21}^3 + C_{22}^2C_{22}^3 + C_{22}^3C_{23}^3)e_3$$
$$= (C_{22}^1C_{12}^1 + C_{22}^2C_{22}^1 + C_{22}^3C_{32}^1)e_1 + (C_{22}^1C_{12}^2 + C_{22}^2C_{22}^2 + C_{22}^3C_{32}^2)e_2 + (C_{22}^1C_{12}^3 + C_{22}^2C_{22}^3 + C_{22}^3C_{32}^3)e_3$$
$$\Rightarrow \quad (C_{22}^1C_{21}^1 + C_{22}^3C_{23}^1 - C_{22}^1C_{12}^1 - C_{22}^3C_{32}^1)e_1 + (C_{22}^1C_{21}^2 + C_{22}^3C_{23}^2 - C_{22}^1C_{12}^2 - C_{22}^3C_{32}^2)e_2 + (C_{22}^1C_{21}^3 + C_{22}^3C_{23}^3 - C_{22}^1C_{12}^3 - C_{22}^3C_{32}^3)e_3 = 0,$$

$$(3.29): \quad (C_{23}^1C_{21}^1 + C_{23}^2C_{22}^1 + C_{23}^3C_{23}^1)e_1 + (C_{23}^1C_{21}^2 + C_{23}^2C_{22}^2 + C_{23}^3C_{23}^2)e_2 + (C_{23}^1C_{21}^3 + C_{23}^2C_{22}^3 + C_{23}^3C_{23}^3)e_3 = 0,$$

$$(3.30): \quad (C_{32}^1C_{21}^1 + C_{32}^2C_{22}^1 + C_{32}^3C_{23}^1)e_1 + (C_{32}^1C_{21}^2 + C_{32}^2C_{22}^2 + C_{32}^3C_{23}^2)e_2 + (C_{32}^1C_{21}^3 + C_{32}^2C_{22}^3 + C_{32}^3C_{23}^3)e_3$$
$$= (C_{23}^1C_{12}^1 + C_{23}^2C_{22}^1 + C_{23}^3C_{32}^1)e_1 + (C_{23}^1C_{12}^2 + C_{23}^2C_{22}^2 + C_{23}^3C_{32}^2)e_2 + (C_{23}^1C_{12}^3 + C_{23}^2C_{22}^3 + C_{23}^3C_{32}^3)e_3$$
$$\Rightarrow \quad (C_{32}^1C_{21}^1 + C_{32}^2C_{22}^1 + C_{32}^3C_{23}^1 - C_{23}^1C_{12}^1 - C_{23}^2C_{22}^1 - C_{23}^3C_{32}^1)e_1 + (C_{32}^1C_{21}^2 + C_{32}^2C_{22}^2 + C_{32}^3C_{23}^2 - C_{23}^1C_{12}^2 - C_{23}^2C_{22}^2 - C_{23}^3C_{32}^2)e_2 + (C_{32}^1C_{21}^3 + C_{32}^2C_{22}^3 - C_{23}^1C_{12}^3 - C_{23}^2C_{22}^3)e_3 = 0,$$

$$(3.31): \quad (C_{33}^1C_{21}^1 + C_{33}^2C_{22}^1 + C_{33}^3C_{23}^1)e_1 + (C_{33}^1C_{21}^2 + C_{33}^2C_{22}^2 + C_{33}^3C_{23}^2)e_2 + (C_{33}^1C_{21}^3 + C_{33}^2C_{22}^3 + C_{33}^3C_{23}^3)e_3 = 0,$$

$$(3.33): \quad 0 = (C_{31}^1C_{12}^1 + C_{31}^2C_{22}^1 + C_{31}^3C_{32}^1)e_1 + (C_{31}^1C_{12}^2 + C_{31}^2C_{22}^2 + C_{31}^3C_{32}^2)e_2$$

$$+(C^1_{31}C^3_{12}+C^2_{31}C^3_{22}+C^3_{31}C^3_{32})e_3,$$
$$(3.37): \quad 0=(C^1_{32}C^1_{12}+C^2_{32}C^1_{22}+C^3_{32}C^1_{32})e_1+(C^1_{32}C^2_{12}+C^2_{32}C^2_{22}+C^3_{32}C^2_{32})e_2$$
$$+(C^1_{32}C^3_{12}+C^2_{32}C^3_{22}+C^3_{32}C^3_{32})e_3,$$
$$(3.39): \quad 0=(C^1_{33}C^1_{12}+C^2_{33}C^1_{22}+C^3_{33}C^1_{32})e_1+(C^1_{33}C^2_{12}+C^2_{33}C^2_{22}+C^3_{33}C^2_{32})e_2$$
$$+(C^1_{33}C^3_{12}+C^2_{33}C^3_{22}+C^3_{33}C^3_{32})e_3.$$

Remembering that $k_1e_1+k_2e_2+k_3e_3=0$ will only have the trivial solution $k_1=k_2=k_3=0$, we can rewrite our equations as the following system of equations:

$$C^1_{11}C^1_{12}+C^2_{11}C^1_{22}+C^3_{11}C^1_{32}=0 \qquad \text{(R3.15.1)}$$
$$C^1_{11}C^2_{12}+C^2_{11}C^2_{22}+C^3_{11}C^2_{32}=0 \qquad \text{(R3.15.2)}$$
$$C^1_{11}C^3_{12}+C^2_{11}C^3_{22}+C^3_{11}C^3_{32}=0 \qquad \text{(R3.15.3)}$$
$$C^1_{12}C^1_{12}+C^2_{12}C^1_{22}+C^3_{12}C^1_{32}=0 \qquad \text{(R3.19.1)}$$
$$C^1_{12}C^2_{12}+C^2_{12}C^2_{22}+C^3_{12}C^2_{32}=0 \qquad \text{(R3.19.2)}$$
$$C^1_{12}C^3_{12}+C^2_{12}C^3_{22}+C^3_{12}C^3_{32}=0 \qquad \text{(R3.19.3)}$$
$$C^1_{13}C^1_{12}+C^2_{13}C^1_{22}+C^3_{13}C^1_{32}=0 \qquad \text{(R3.21.1)}$$
$$C^1_{13}C^2_{12}+C^2_{13}C^2_{22}+C^3_{13}C^2_{32}=0 \qquad \text{(R3.21.2)}$$
$$C^1_{13}C^3_{12}+C^2_{13}C^3_{22}+C^3_{13}C^3_{32}=0 \qquad \text{(R3.21.3)}$$
$$C^1_{11}C^1_{21}+C^2_{11}C^1_{22}+C^3_{11}C^1_{23}=0 \qquad \text{(R3.23.1)}$$
$$C^1_{11}C^2_{21}+C^2_{11}C^2_{22}+C^3_{11}C^2_{23}=0 \qquad \text{(R3.23.2)}$$
$$C^1_{11}C^3_{21}+C^2_{11}C^3_{22}+C^3_{11}C^3_{23}=0 \qquad \text{(R3.23.3)}$$
$$C^2_{12}C^1_{22}+C^3_{12}C^1_{23}-C^2_{21}C^1_{22}-C^3_{21}C^1_{32}=0 \qquad \text{(R3.24.1)}$$
$$C^1_{12}C^2_{21}+C^2_{12}C^2_{22}+C^3_{12}C^2_{23}-C^1_{21}C^2_{12}-C^2_{21}C^2_{22}-C^3_{21}C^2_{32}=0 \qquad \text{(R3.24.2)}$$
$$C^1_{12}C^3_{21}+C^2_{12}C^3_{22}+C^3_{12}C^3_{23}-C^1_{21}C^3_{12}-C^2_{21}C^3_{22}-C^3_{21}C^3_{32}=0 \qquad \text{(R3.24.3)}$$
$$C^1_{13}C^1_{21}+C^2_{13}C^1_{22}+C^3_{13}C^1_{23}=0 \qquad \text{(R3.25.1)}$$
$$C^1_{13}C^2_{21}+C^2_{13}C^2_{22}+C^3_{13}C^2_{23}=0 \qquad \text{(R3.25.2)}$$
$$C^1_{13}C^3_{21}+C^2_{13}C^3_{22}+C^3_{13}C^3_{23}=0 \qquad \text{(R3.25.3)}$$
$$C^1_{21}C^1_{21}+C^2_{21}C^1_{22}+C^3_{21}C^1_{23}=0 \qquad \text{(R3.26.1)}$$
$$C^1_{21}C^2_{21}+C^2_{21}C^2_{22}+C^3_{21}C^2_{23}=0 \qquad \text{(R3.26.2)}$$
$$C^1_{21}C^3_{21}+C^2_{21}C^3_{22}+C^3_{21}C^3_{23}=0 \qquad \text{(R3.26.3)}$$
$$C^1_{31}C^1_{21}+C^2_{31}C^1_{22}+C^3_{31}C^1_{23}=0 \qquad \text{(R3.27.1)}$$
$$C^1_{31}C^2_{21}+C^2_{31}C^2_{22}+C^3_{31}C^2_{23}=0 \qquad \text{(R3.27.2)}$$
$$C^1_{31}C^3_{21}+C^2_{31}C^3_{22}+C^3_{31}C^3_{23}=0 \qquad \text{(R3.27.3)}$$
$$C^1_{22}C^1_{21}+C^3_{22}C^1_{23}-C^1_{22}C^1_{12}-C^3_{22}C^1_{32}=0 \qquad \text{(R3.28.1)}$$
$$C^1_{22}C^2_{21}+C^3_{22}C^2_{23}-C^1_{22}C^2_{12}-C^3_{22}C^2_{32}=0 \qquad \text{(R3.28.2)}$$

$$C_{22}^1C_{21}^3 + C_{22}^3C_{23}^3 - C_{22}^1C_{12}^3 - C_{22}^3C_{32}^3 = 0 \tag{R3.28.3}$$
$$C_{23}^1C_{21}^1 + C_{23}^2C_{22}^1 + C_{23}^3C_{23}^1 = 0 \tag{R3.29.1}$$
$$C_{23}^1C_{21}^2 + C_{23}^2C_{22}^2 + C_{23}^3C_{23}^2 = 0 \tag{R3.29.2}$$
$$C_{23}^1C_{21}^3 + C_{23}^2C_{22}^3 + C_{23}^3C_{23}^3 = 0 \tag{R3.29.3}$$
$$C_{32}^1C_{21}^1 + C_{32}^2C_{22}^1 + C_{32}^3C_{23}^1 - C_{23}^1C_{12}^1 - C_{23}^2C_{22}^1 - C_{23}^3C_{32}^1 = 0 \tag{R3.30.1}$$
$$C_{32}^1C_{21}^2 + C_{32}^2C_{22}^2 + C_{32}^3C_{23}^2 - C_{23}^1C_{12}^2 - C_{23}^2C_{22}^2 - C_{23}^3C_{32}^2 = 0 \tag{R3.30.2}$$
$$C_{32}^1C_{21}^3 + C_{32}^2C_{22}^3 - C_{23}^1C_{12}^3 - C_{23}^2C_{22}^3 = 0 \tag{R3.30.3}$$
$$C_{33}^1C_{21}^1 + C_{33}^2C_{22}^1 + C_{33}^3C_{23}^1 = 0 \tag{R3.31.1}$$
$$C_{33}^1C_{21}^2 + C_{33}^2C_{22}^2 + C_{33}^3C_{23}^2 = 0 \tag{R3.31.2}$$
$$C_{33}^1C_{21}^3 + C_{33}^2C_{22}^3 + C_{33}^3C_{23}^3 = 0 \tag{R3.31.3}$$
$$C_{31}^1C_{12}^1 + C_{31}^2C_{22}^1 + C_{31}^3C_{32}^1 = 0 \tag{R3.33.1}$$
$$C_{31}^1C_{12}^2 + C_{31}^2C_{22}^2 + C_{31}^3C_{32}^2 = 0 \tag{R3.33.2}$$
$$C_{31}^1C_{12}^3 + C_{31}^2C_{22}^3 + C_{31}^3C_{32}^3 = 0 \tag{R3.33.3}$$
$$C_{32}^1C_{12}^1 + C_{32}^2C_{22}^1 + C_{32}^3C_{32}^1 = 0 \tag{R3.37.1}$$
$$C_{32}^1C_{12}^2 + C_{32}^2C_{22}^2 + C_{32}^3C_{32}^2 = 0 \tag{R3.37.2}$$
$$C_{32}^1C_{12}^3 + C_{32}^2C_{22}^3 + C_{32}^3C_{32}^3 = 0 \tag{R3.37.3}$$
$$C_{33}^1C_{12}^1 + C_{33}^2C_{22}^1 + C_{33}^3C_{32}^1 = 0 \tag{R3.39.1}$$
$$C_{33}^1C_{12}^2 + C_{33}^2C_{22}^2 + C_{33}^3C_{32}^2 = 0 \tag{R3.39.2}$$
$$C_{33}^1C_{12}^3 + C_{33}^2C_{22}^3 + C_{33}^3C_{32}^3 = 0. \tag{R3.39.3}$$

3.2.6 α defined as E_{23}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha] = E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

However, due to our results in Section 3.2.1, where we realised we would not be able to solve the entire system of equations in this project, we will only write out the system of equations for this α, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\alpha(e_1) = 0e_1 + 0e_2 + 0e_3 = 0,$$
$$\alpha(e_2) = 0e_1 + 0e_2 + 0e_3 = 0,$$
$$\alpha(e_3) = 0e_1 + 1e_2 + 0e_3 = e_2.$$

We can thus rewrite our equations as follows:

$$
\begin{aligned}
&(3.14): && \mu(0, C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3) = \mu(C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.15): && \mu(0, C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3) = \mu(C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.16): && \mu(0, C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3) = \mu(C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3, e_2)\\
& && \Rightarrow\; 0 = \mu(C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3, e_2),\\
&(3.17): && \mu(0, C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3) = \mu(C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.18): && \mu(0, C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3) = \mu(C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.19): && \mu(0, C_{22}^1e_1 + C_{22}^2e_2 + C_{22}^3e_3) = \mu(C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.20): && \mu(0, C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3) = \mu(C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3, e_2)\\
& && \Rightarrow\; 0 = \mu(C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3, e_2),\\
&(3.21): && \mu(0, C_{32}^1e_1 + C_{32}^2e_2 + C_{32}^3e_3) = \mu(C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.22): && \mu(0, C_{33}^1e_1 + C_{33}^2e_2 + C_{33}^3e_3) = \mu(C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3, e_2)\\
& && \Rightarrow\; 0 = \mu(C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3, e_2),\\
&(3.23): && \mu(0, C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3) = \mu(C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.24): && \mu(0, C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3) = \mu(C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.25): && \mu(0, C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3) = \mu(C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3, e_2)\\
& && \Rightarrow\; 0 = \mu(C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3, e_2),\\
&(3.26): && \mu(0, C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3) = \mu(C_{22}^1e_1 + C_{22}^2e_2 + C_{22}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.27): && \mu(0, C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3) = \mu(C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.28): && \mu(0, C_{22}^1e_1 + C_{22}^2e_2 + C_{22}^3e_3) = \mu(C_{22}^1e_1 + C_{22}^2e_2 + C_{22}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.29): && \mu(0, C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3) = \mu(C_{22}^1e_1 + C_{22}^2e_2 + C_{22}^3e_3, e_2)\\
& && \Rightarrow\; 0 = \mu(C_{22}^1e_1 + C_{22}^2e_2 + C_{22}^3e_3, e_2),\\
&(3.30): && \mu(0, C_{32}^1e_1 + C_{32}^2e_2 + C_{32}^3e_3) = \mu(C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.31): && \mu(0, C_{33}^1e_1 + C_{33}^2e_2 + C_{33}^3e_3) = \mu(C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3, e_2)\\
& && \Rightarrow\; 0 = \mu(C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3, e_2),\\
&(3.32): && \mu(e_2, C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3) = \mu(C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3, 0)\\
& && \Rightarrow\; \mu(e_2, C_{11}^1e_1 + C_{11}^2e_2 + C_{11}^3e_3) = 0,\\
&(3.33): && \mu(e_2, C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3) = \mu(C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3, 0)\\
& && \Rightarrow\; \mu(e_2, C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3) = 0,\\
&(3.34): && \mu(e_2, C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3) = \mu(C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3, e_2),\\
&(3.35): && \mu(e_2, C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3) = \mu(C_{32}^1e_1 + C_{32}^2e_2 + C_{32}^3e_3, 0)\\
& && \Rightarrow\; \mu(e_2, C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3) = 0,\\
&(3.36): && \mu(e_2, C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3) = \mu(C_{33}^1e_1 + C_{33}^2e_2 + C_{33}^3e_3, 0)
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow\ \mu(e_2, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = 0,\\
(3.37):\quad &\mu(e_2, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0)\\
&\Rightarrow\ \mu(e_2, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = 0,\\
(3.38):\quad &\mu(e_2, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_2),\\
(3.39):\quad &\mu(e_2, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0)\\
&\Rightarrow\ \mu(e_2, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = 0,\\
(3.40):\quad &\mu(e_2, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_2).
\end{aligned}
$$

We remember the following formulas that we calculated earlier:

$$
\begin{aligned}
\mu(e_i, C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3) = {} & (C_{jk}^1 C_{i1}^1 + C_{jk}^2 C_{i2}^1 + C_{jk}^3 C_{i3}^1)e_1 + (C_{jk}^1 C_{i1}^2 + C_{jk}^2 C_{i2}^2 + C_{jk}^3 C_{i3}^2)e_2\\
& + (C_{jk}^1 C_{i1}^3 + C_{jk}^2 C_{i2}^3 + C_{jk}^3 C_{i3}^3)e_3
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3, e_i) = {} & (C_{jk}^1 C_{1i}^1 + C_{jk}^2 C_{2i}^1 + C_{jk}^3 C_{3i}^1)e_1 + (C_{jk}^1 C_{1i}^2 + C_{jk}^2 C_{2i}^2 + C_{jk}^3 C_{3i}^2)e_2\\
& + (C_{jk}^1 C_{1i}^3 + C_{jk}^2 C_{2i}^3 + C_{jk}^3 C_{3i}^3)e_3.
\end{aligned}
$$

We use these formulas to rewrite our equations.

$$
\begin{aligned}
(3.16):\quad & 0 = (C_{11}^1 C_{12}^1 + C_{11}^2 C_{22}^1 + C_{11}^3 C_{32}^1)e_1 + (C_{11}^1 C_{12}^2 + C_{11}^2 C_{22}^2 + C_{11}^3 C_{32}^2)e_2\\
&\quad + (C_{11}^1 C_{12}^3 + C_{11}^2 C_{22}^3 + C_{11}^3 C_{32}^3)e_3,\\
(3.20):\quad & 0 = (C_{12}^1 C_{12}^1 + C_{12}^2 C_{22}^1 + C_{12}^3 C_{32}^1)e_1 + (C_{12}^1 C_{12}^2 + C_{12}^2 C_{22}^2 + C_{12}^3 C_{32}^2)e_2\\
&\quad + (C_{12}^1 C_{12}^3 + C_{12}^2 C_{22}^3 + C_{12}^3 C_{32}^3)e_3,\\
(3.22):\quad & 0 = (C_{13}^1 C_{12}^1 + C_{13}^2 C_{22}^1 + C_{13}^3 C_{32}^1)e_1 + (C_{13}^1 C_{12}^2 + C_{13}^2 C_{22}^2 + C_{13}^3 C_{32}^2)e_2\\
&\quad + (C_{13}^1 C_{12}^3 + C_{13}^2 C_{22}^3 + C_{13}^3 C_{32}^3)e_3,\\
(3.25):\quad & 0 = (C_{21}^1 C_{12}^1 + C_{21}^2 C_{22}^1 + C_{21}^3 C_{32}^1)e_1 + (C_{21}^1 C_{12}^2 + C_{21}^2 C_{22}^2 + C_{21}^3 C_{32}^2)e_2\\
&\quad + (C_{21}^1 C_{12}^3 + C_{21}^2 C_{22}^3 + C_{21}^3 C_{32}^3)e_3,\\
(3.29):\quad & 0 = (C_{22}^1 C_{12}^1 + C_{22}^2 C_{22}^1 + C_{22}^3 C_{32}^1)e_1 + (C_{22}^1 C_{12}^2 + C_{22}^2 C_{22}^2 + C_{22}^3 C_{32}^2)e_2\\
&\quad + (C_{22}^1 C_{12}^3 + C_{22}^2 C_{22}^3 + C_{22}^3 C_{32}^3)e_3,\\
(3.31):\quad & 0 = (C_{23}^1 C_{12}^1 + C_{23}^2 C_{22}^1 + C_{23}^3 C_{32}^1)e_1 + (C_{23}^1 C_{12}^2 + C_{23}^2 C_{22}^2 + C_{23}^3 C_{32}^2)e_2\\
&\quad + (C_{23}^1 C_{12}^3 + C_{23}^2 C_{22}^3 + C_{23}^3 C_{32}^3)e_3,\\
(3.32):\quad & (C_{11}^1 C_{21}^1 + C_{11}^2 C_{22}^1 + C_{11}^3 C_{23}^1)e_1 + (C_{11}^1 C_{21}^2 + C_{11}^2 C_{22}^2 + C_{11}^3 C_{23}^2)e_2\\
& + (C_{11}^1 C_{21}^3 + C_{11}^2 C_{22}^3 + C_{11}^3 C_{23}^3)e_3 = 0,\\
(3.33):\quad & (C_{12}^1 C_{21}^1 + C_{12}^2 C_{22}^1 + C_{12}^3 C_{23}^1)e_1 + (C_{12}^1 C_{21}^2 + C_{12}^2 C_{22}^2 + C_{12}^3 C_{23}^2)e_2\\
& + (C_{12}^1 C_{21}^3 + C_{12}^2 C_{22}^3 + C_{12}^3 C_{23}^3)e_3 = 0,\\
(3.34):\quad & (C_{13}^1 C_{21}^1 + C_{13}^2 C_{22}^1 + C_{13}^3 C_{23}^1)e_1 + (C_{13}^1 C_{21}^2 + C_{13}^2 C_{22}^2 + C_{13}^3 C_{23}^2)e_2
\end{aligned}
$$

$$
\begin{aligned}
&+(C_{13}^1C_{21}^3+C_{13}^2C_{22}^3+C_{13}^3C_{23}^3)e_3\\
&=(C_{31}^1C_{12}^1+C_{31}^2C_{22}^1+C_{31}^3C_{32}^1)e_1+(C_{31}^1C_{12}^2+C_{31}^2C_{22}^2+C_{31}^3C_{32}^2)e_2\\
&\quad+(C_{31}^1C_{12}^3+C_{31}^2C_{22}^3+C_{31}^3C_{32}^3)e_3\\
&\Rightarrow\quad (C_{13}^1C_{21}^1+C_{13}^2C_{22}^1+C_{13}^3C_{23}^1-C_{31}^1C_{12}^1-C_{31}^2C_{22}^1-C_{31}^3C_{32}^1)e_1\\
&\qquad+(C_{13}^1C_{21}^2+C_{13}^2C_{22}^2+C_{13}^3C_{23}^2-C_{31}^1C_{12}^2-C_{31}^2C_{22}^2-C_{31}^3C_{32}^2)e_2\\
&\qquad+(C_{13}^1C_{21}^3+C_{13}^2C_{22}^3+C_{13}^3C_{23}^3-C_{31}^1C_{12}^3-C_{31}^2C_{22}^3-C_{31}^3C_{32}^3)e_3=0,
\end{aligned}
$$

$$
\begin{aligned}
(3.35):\quad &(C_{21}^1C_{21}^1+C_{21}^2C_{22}^1+C_{21}^3C_{23}^1)e_1+(C_{21}^1C_{21}^2+C_{21}^2C_{22}^2+C_{21}^3C_{23}^2)e_2\\
&+(C_{21}^1C_{21}^3+C_{21}^2C_{22}^3+C_{21}^3C_{23}^3)e_3=0,\\
(3.36):\quad &(C_{31}^1C_{21}^1+C_{31}^2C_{22}^1+C_{31}^3C_{23}^1)e_1+(C_{31}^1C_{21}^2+C_{31}^2C_{22}^2+C_{31}^3C_{23}^2)e_2\\
&+(C_{31}^1C_{21}^3+C_{31}^2C_{22}^3+C_{31}^3C_{23}^3)e_3=0,\\
(3.37):\quad &(C_{22}^1C_{21}^1+C_{22}^2C_{22}^1+C_{22}^3C_{23}^1)e_1+(C_{22}^1C_{21}^2+C_{22}^2C_{22}^2+C_{22}^3C_{23}^2)e_2\\
&+(C_{22}^1C_{21}^3+C_{22}^2C_{22}^3+C_{22}^3C_{23}^3)e_3=0,\\
(3.38):\quad &(C_{23}^1C_{21}^1+C_{23}^2C_{22}^1+C_{23}^3C_{23}^1)e_1+(C_{23}^1C_{21}^2+C_{23}^2C_{22}^2+C_{23}^3C_{23}^2)e_2\\
&+(C_{23}^1C_{21}^3+C_{23}^2C_{22}^3+C_{23}^3C_{23}^3)e_3\\
&=(C_{32}^1C_{12}^1+C_{32}^2C_{22}^1+C_{32}^3C_{32}^1)e_1+(C_{32}^1C_{12}^2+C_{32}^2C_{22}^2+C_{32}^3C_{32}^2)e_2\\
&\quad+(C_{32}^1C_{12}^3+C_{32}^2C_{22}^3+C_{32}^3C_{32}^3)e_3\\
&\Rightarrow\quad (C_{23}^1C_{21}^1+C_{23}^2C_{22}^1+C_{23}^3C_{23}^1-C_{32}^1C_{12}^1-C_{32}^2C_{22}^1-C_{32}^3C_{32}^1)e_1\\
&\qquad+(C_{23}^1C_{21}^2+C_{23}^2C_{22}^2+C_{23}^3C_{23}^2-C_{32}^1C_{12}^2-C_{32}^2C_{22}^2-C_{32}^3C_{32}^2)e_2\\
&\qquad+(C_{23}^1C_{21}^3+C_{23}^2C_{22}^3+C_{23}^3C_{23}^3-C_{32}^1C_{12}^3-C_{32}^2C_{22}^3-C_{32}^3C_{32}^3)e_3=0,\\
(3.39):\quad &(C_{32}^1C_{21}^1+C_{32}^2C_{22}^1+C_{32}^3C_{23}^1)e_1+(C_{32}^1C_{21}^2+C_{32}^2C_{22}^2+C_{32}^3C_{23}^2)e_2\\
&+(C_{32}^1C_{21}^3+C_{32}^2C_{22}^3+C_{32}^3C_{23}^3)e_3=0,\\
(3.40):\quad &(C_{33}^1C_{21}^1+C_{33}^2C_{22}^1+C_{33}^3C_{23}^1)e_1+(C_{33}^1C_{21}^2+C_{33}^2C_{22}^2+C_{33}^3C_{23}^2)e_2\\
&+(C_{33}^1C_{21}^3+C_{33}^2C_{22}^3+C_{33}^3C_{23}^3)e_3\\
&=(C_{33}^1C_{12}^1+C_{33}^2C_{22}^1+C_{33}^3C_{32}^1)e_1+(C_{33}^1C_{12}^2+C_{33}^2C_{22}^2+C_{33}^3C_{32}^2)e_2\\
&\quad+(C_{33}^1C_{12}^3+C_{33}^2C_{22}^3+C_{33}^3C_{32}^3)e_3\\
&\Rightarrow\quad (C_{33}^1C_{21}^1+C_{33}^3C_{23}^1-C_{33}^1C_{12}^1-C_{33}^3C_{32}^1)e_1+(C_{33}^1C_{21}^2+C_{33}^3C_{23}^2-C_{33}^1C_{12}^2-C_{33}^3C_{32}^2)e_2\\
&\qquad+(C_{33}^1C_{21}^3+C_{33}^3C_{23}^3-C_{33}^1C_{12}^3-C_{33}^3C_{32}^3)e_3=0.
\end{aligned}
$$

Remembering that $k_1e_1+k_2e_2+k_3e_3=0$ will only have the trivial solution $k_1=k_2=k_3=0$, we can rewrite our equations as the following system of equations:

$$C_{11}^1C_{12}^1+C_{11}^2C_{22}^1+C_{11}^3C_{32}^1=0 \qquad \text{(R3.16.1)}$$
$$C_{11}^1C_{12}^2+C_{11}^2C_{22}^2+C_{11}^3C_{32}^2=0 \qquad \text{(R3.16.2)}$$
$$C_{11}^1C_{12}^3+C_{11}^2C_{22}^3+C_{11}^3C_{32}^3=0 \qquad \text{(R3.16.3)}$$
$$C_{12}^1C_{12}^1+C_{12}^2C_{22}^1+C_{12}^3C_{32}^1=0 \qquad \text{(R3.20.1)}$$

$$C_{12}^1 C_{12}^2 + C_{12}^2 C_{22}^2 + C_{12}^3 C_{32}^2 = 0 \quad \text{(R3.20.2)}$$
$$C_{12}^1 C_{12}^3 + C_{12}^2 C_{22}^3 + C_{12}^3 C_{32}^3 = 0 \quad \text{(R3.20.3)}$$
$$C_{13}^1 C_{12}^1 + C_{13}^2 C_{22}^1 + C_{13}^3 C_{32}^1 = 0 \quad \text{(R3.22.1)}$$
$$C_{13}^1 C_{12}^2 + C_{13}^2 C_{22}^2 + C_{13}^3 C_{32}^2 = 0 \quad \text{(R3.22.2)}$$
$$C_{13}^1 C_{12}^3 + C_{13}^2 C_{22}^3 + C_{13}^3 C_{32}^3 = 0 \quad \text{(R3.22.3)}$$
$$C_{21}^1 C_{12}^1 + C_{21}^2 C_{22}^1 + C_{21}^3 C_{32}^1 = 0 \quad \text{(R3.25.1)}$$
$$C_{21}^1 C_{12}^2 + C_{21}^2 C_{22}^2 + C_{21}^3 C_{32}^2 = 0 \quad \text{(R3.25.2)}$$
$$C_{21}^1 C_{12}^3 + C_{21}^2 C_{22}^3 + C_{21}^3 C_{32}^3 = 0 \quad \text{(R3.25.3)}$$
$$C_{22}^1 C_{12}^1 + C_{22}^2 C_{22}^1 + C_{22}^3 C_{32}^1 = 0 \quad \text{(R3.29.1)}$$
$$C_{22}^1 C_{12}^2 + C_{22}^2 C_{22}^2 + C_{22}^3 C_{32}^2 = 0 \quad \text{(R3.29.2)}$$
$$C_{22}^1 C_{12}^3 + C_{22}^2 C_{22}^3 + C_{22}^3 C_{32}^3 = 0 \quad \text{(R3.29.3)}$$
$$C_{23}^1 C_{12}^1 + C_{23}^2 C_{22}^1 + C_{23}^3 C_{32}^1 = 0 \quad \text{(R3.31.1)}$$
$$C_{23}^1 C_{12}^2 + C_{23}^2 C_{22}^2 + C_{23}^3 C_{32}^2 = 0 \quad \text{(R3.31.2)}$$
$$C_{23}^1 C_{12}^3 + C_{23}^2 C_{22}^3 + C_{23}^3 C_{32}^3 = 0 \quad \text{(R3.31.3)}$$
$$C_{11}^1 C_{21}^1 + C_{11}^2 C_{22}^1 + C_{11}^3 C_{23}^1 = 0 \quad \text{(R3.32.1)}$$
$$C_{11}^1 C_{21}^2 + C_{11}^2 C_{22}^2 + C_{11}^3 C_{23}^2 = 0 \quad \text{(R3.32.2)}$$
$$C_{11}^1 C_{21}^3 + C_{11}^2 C_{22}^3 + C_{11}^3 C_{23}^3 = 0 \quad \text{(R3.32.3)}$$
$$C_{12}^1 C_{21}^1 + C_{12}^2 C_{22}^1 + C_{12}^3 C_{23}^1 = 0 \quad \text{(R3.33.1)}$$
$$C_{12}^1 C_{21}^2 + C_{12}^2 C_{22}^2 + C_{12}^3 C_{23}^2 = 0 \quad \text{(R3.33.2)}$$
$$C_{12}^1 C_{21}^3 + C_{12}^2 C_{22}^3 + C_{12}^3 C_{23}^3 = 0 \quad \text{(R3.33.3)}$$
$$C_{13}^1 C_{21}^1 + C_{13}^2 C_{22}^1 + C_{13}^3 C_{23}^1 - C_{31}^1 C_{12}^1 - C_{31}^2 C_{22}^1 - C_{31}^3 C_{32}^1 = 0 \quad \text{(R3.34.1)}$$
$$C_{13}^1 C_{21}^2 + C_{13}^2 C_{22}^2 + C_{13}^3 C_{23}^2 - C_{31}^1 C_{12}^2 - C_{31}^2 C_{22}^2 - C_{31}^3 C_{32}^2 = 0 \quad \text{(R3.34.2)}$$
$$C_{13}^1 C_{21}^3 + C_{13}^2 C_{22}^3 + C_{13}^3 C_{23}^3 - C_{31}^1 C_{12}^3 - C_{31}^2 C_{22}^3 - C_{31}^3 C_{32}^3 = 0 \quad \text{(R3.34.3)}$$
$$C_{21}^1 C_{21}^1 + C_{21}^2 C_{22}^1 + C_{21}^3 C_{23}^1 = 0 \quad \text{(R3.35.1)}$$
$$C_{21}^1 C_{21}^2 + C_{21}^2 C_{22}^2 + C_{21}^3 C_{23}^2 = 0 \quad \text{(R3.35.2)}$$
$$C_{21}^1 C_{21}^3 + C_{21}^2 C_{22}^3 + C_{21}^3 C_{23}^3 = 0 \quad \text{(R3.35.3)}$$
$$C_{31}^1 C_{21}^1 + C_{31}^2 C_{22}^1 + C_{31}^3 C_{23}^1 = 0 \quad \text{(R3.36.1)}$$
$$C_{31}^1 C_{21}^2 + C_{31}^2 C_{22}^2 + C_{31}^3 C_{23}^2 = 0 \quad \text{(R3.36.2)}$$
$$C_{31}^1 C_{21}^3 + C_{31}^2 C_{22}^3 + C_{31}^3 C_{23}^3 = 0 \quad \text{(R3.36.3)}$$
$$C_{22}^1 C_{21}^1 + C_{22}^2 C_{22}^1 + C_{22}^3 C_{23}^1 = 0 \quad \text{(R3.37.1)}$$
$$C_{22}^1 C_{21}^2 + C_{22}^2 C_{22}^2 + C_{22}^3 C_{23}^2 = 0 \quad \text{(R3.37.2)}$$
$$C_{22}^1 C_{21}^3 + C_{22}^2 C_{22}^3 + C_{22}^3 C_{23}^3 = 0 \quad \text{(R3.37.3)}$$
$$C_{23}^1 C_{21}^1 + C_{23}^2 C_{22}^1 + C_{23}^3 C_{23}^1 - C_{32}^1 C_{12}^1 - C_{32}^2 C_{22}^1 - C_{32}^3 C_{32}^1 = 0 \quad \text{(R3.38.1)}$$

$$C_{23}^1C_{21}^2+C_{23}^2C_{22}^2+C_{23}^3C_{23}^2-C_{32}^1C_{12}^2-C_{32}^2C_{22}^2-C_{32}^3C_{32}^2=0 \quad \text{(R3.38.2)}$$
$$C_{23}^1C_{21}^3+C_{23}^2C_{22}^3+C_{23}^3C_{23}^3-C_{32}^1C_{12}^3-C_{32}^2C_{22}^3-C_{32}^3C_{32}^3=0 \quad \text{(R3.38.3)}$$
$$C_{32}^1C_{21}^1+C_{32}^2C_{22}^1+C_{32}^3C_{23}^1=0 \quad \text{(R3.39.1)}$$
$$C_{32}^1C_{21}^2+C_{32}^2C_{22}^2+C_{32}^3C_{23}^2=0 \quad \text{(R3.39.2)}$$
$$C_{32}^1C_{21}^3+C_{32}^2C_{22}^3+C_{32}^3C_{23}^3=0 \quad \text{(R3.39.3)}$$
$$C_{33}^1C_{21}^1+C_{33}^3C_{23}^1-C_{33}^1C_{12}^1-C_{33}^3C_{32}^1=0 \quad \text{(R3.40.1)}$$
$$C_{33}^1C_{21}^2+C_{33}^3C_{23}^2-C_{33}^1C_{12}^2-C_{33}^3C_{32}^2=0 \quad \text{(R3.40.2)}$$
$$C_{33}^1C_{21}^3+C_{33}^3C_{23}^3-C_{33}^1C_{12}^3-C_{33}^3C_{32}^3=0 \quad \text{(R3.40.3).}$$

3.2.7 α defined as E_{31}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha]=E_{31}=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}.$$

However, due to our results in Section 3.2.1, where we realised we would not be able to solve the entire system of equations in this project, we will only write out the system of equations for this α, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\begin{aligned}\alpha(e_1)&=0e_1+0e_2+1e_3=e_3,\\ \alpha(e_2)&=0e_1+0e_2+0e_3=0,\\ \alpha(e_3)&=0e_1+0e_2+0e_3=0.\end{aligned}$$

We can thus rewrite our equations as follows:

$$\begin{aligned}
(3.14):\quad & \mu(e_3, C_{11}^1e_1+C_{11}^2e_2+C_{11}^3e_3)=\mu(C_{11}^1e_1+C_{11}^2e_2+C_{11}^3e_3, e_3),\\
(3.15):\quad & \mu(e_3, C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3)=\mu(C_{11}^1e_1+C_{11}^2e_2+C_{11}^3e_3, 0)\\
& \Rightarrow\ \mu(e_3, C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3)=0,\\
(3.16):\quad & \mu(e_3, C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3)=\mu(C_{11}^1e_1+C_{11}^2e_2+C_{11}^3e_3, 0)\\
& \Rightarrow\ \mu(e_3, C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3)=0,\\
(3.17):\quad & \mu(e_3, C_{21}^1e_1+C_{21}^2e_2+C_{21}^3e_3)=\mu(C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3, e_3),\\
(3.18):\quad & \mu(e_3, C_{31}^1e_1+C_{31}^2e_2+C_{31}^3e_3)=\mu(C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3, e_3),\\
(3.19):\quad & \mu(e_3, C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3)=\mu(C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3, 0)\\
& \Rightarrow\ \mu(e_3, C_{22}^1e_1+C_{22}^2e_2+C_{22}^3e_3)=0,\\
(3.20):\quad & \mu(e_3, C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3)=\mu(C_{12}^1e_1+C_{12}^2e_2+C_{12}^3e_3, 0)\\
& \Rightarrow\ \mu(e_3, C_{23}^1e_1+C_{23}^2e_2+C_{23}^3e_3)=0,\\
(3.21):\quad & \mu(e_3, C_{32}^1e_1+C_{32}^2e_2+C_{32}^3e_3)=\mu(C_{13}^1e_1+C_{13}^2e_2+C_{13}^3e_3, 0)
\end{aligned}$$

$$\Rightarrow \ \mu(e_3, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = 0,$$

$$(3.22): \quad \mu(e_3, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0)$$
$$\Rightarrow \ \mu(e_3, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = 0,$$

$$(3.23): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_3)$$
$$\Rightarrow \ 0 = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_3),$$

$$(3.24): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.25): \quad \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.26): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_3)$$
$$\Rightarrow \ 0 = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_3),$$

$$(3.27): \quad \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_3)$$
$$\Rightarrow \ 0 = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_3),$$

$$(3.28): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.29): \quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.30): \quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.31): \quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.32): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_3)$$
$$\Rightarrow \ 0 = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_3),$$

$$(3.33): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.34): \quad \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.35): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_3)$$
$$\Rightarrow \ 0 = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_3),$$

$$(3.36): \quad \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_3)$$
$$\Rightarrow \ 0 = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_3),$$

$$(3.37): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.38): \quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.39): \quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \ \Rightarrow \ 0 = 0,$$

$$(3.40): \quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0) \ \Rightarrow \ 0 = 0.$$

We remember the following formulas that we calculated earlier:

$$\mu(e_i, C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3) = (C_{jk}^1 C_{i1}^1 + C_{jk}^2 C_{i2}^1 + C_{jk}^3 C_{i3}^1) e_1 + (C_{jk}^1 C_{i1}^2 + C_{jk}^2 C_{i2}^2 + C_{jk}^3 C_{i3}^2) e_2$$
$$+ (C_{jk}^1 C_{i1}^3 + C_{jk}^2 C_{i2}^3 + C_{jk}^3 C_{i3}^3) e_3$$

and

$$\mu(C_{jk}^1 e_1 + C_{jk}^2 e_2 + C_{jk}^3 e_3, e_i) = (C_{jk}^1 C_{1i}^1 + C_{jk}^2 C_{2i}^1 + C_{jk}^3 C_{3i}^1) e_1 + (C_{jk}^1 C_{1i}^2 + C_{jk}^2 C_{2i}^2 + C_{jk}^3 C_{3i}^2) e_2$$

$$+(C^1_{jk}C^3_{1i}+C^2_{jk}C^3_{2i}+C^3_{jk}C^3_{3i})e_3.$$

We use these formulas to rewrite our equations.

$$\begin{aligned}
(3.14):\quad & (C^1_{11}C^1_{31}+C^2_{11}C^1_{32}+C^3_{11}C^1_{33})e_1+(C^1_{11}C^2_{31}+C^2_{11}C^2_{32}+C^3_{11}C^2_{33})e_2\\
&+(C^1_{11}C^3_{31}+C^2_{11}C^3_{32}+C^3_{11}C^3_{33})e_3\\
&=(C^1_{11}C^1_{13}+C^2_{11}C^1_{23}+C^3_{11}C^1_{33})e_1+(C^1_{11}C^2_{13}+C^2_{11}C^2_{23}+C^3_{11}C^2_{33})e_2\\
&\quad+(C^1_{11}C^3_{13}+C^2_{11}C^3_{23}+C^3_{11}C^3_{33})e_3\\
&\Rightarrow\ (C^1_{11}C^1_{31}+C^2_{11}C^1_{32}-C^1_{11}C^1_{13}-C^2_{11}C^1_{23})e_1+(C^1_{11}C^2_{31}+C^2_{11}C^2_{32}-C^1_{11}C^2_{13}-C^2_{11}C^2_{23})e_2\\
&\qquad+(C^1_{11}C^3_{31}+C^2_{11}C^3_{32}-C^1_{11}C^3_{13}-C^2_{11}C^3_{23})e_3=0,\\
(3.15):\quad & (C^1_{12}C^1_{31}+C^2_{12}C^1_{32}+C^3_{12}C^1_{33})e_1+(C^1_{12}C^2_{31}+C^2_{12}C^2_{32}+C^3_{12}C^2_{33})e_2\\
&+(C^1_{12}C^3_{31}+C^2_{12}C^3_{32}+C^3_{12}C^3_{33})e_3=0,\\
(3.16):\quad & (C^1_{13}C^1_{31}+C^2_{13}C^1_{32}+C^3_{13}C^1_{33})e_1+(C^1_{13}C^2_{31}+C^2_{13}C^2_{32}+C^3_{13}C^2_{33})e_2\\
&+(C^1_{13}C^3_{31}+C^2_{13}C^3_{32}+C^3_{13}C^3_{33})e_3=0,\\
(3.17):\quad & (C^1_{21}C^1_{31}+C^2_{21}C^1_{32}+C^3_{21}C^1_{33})e_1+(C^1_{21}C^2_{31}+C^2_{21}C^2_{32}+C^3_{21}C^2_{33})e_2\\
&+(C^1_{21}C^3_{31}+C^2_{21}C^3_{32}+C^3_{21}C^3_{33})e_3\\
&=(C^1_{12}C^1_{13}+C^2_{12}C^1_{23}+C^3_{12}C^1_{33})e_1+(C^1_{12}C^2_{13}+C^2_{12}C^2_{23}+C^3_{12}C^2_{33})e_2\\
&\quad+(C^1_{12}C^3_{13}+C^2_{12}C^3_{23}+C^3_{12}C^3_{33})e_3\\
&\Rightarrow\ (C^1_{21}C^1_{31}+C^2_{21}C^1_{32}+C^3_{21}C^1_{33}-C^1_{12}C^1_{13}-C^2_{12}C^1_{23}-C^3_{12}C^1_{33})e_1\\
&\qquad+(C^1_{21}C^2_{31}+C^2_{21}C^2_{32}+C^3_{21}C^2_{33}-C^1_{12}C^2_{13}-C^2_{12}C^2_{23}-C^3_{12}C^2_{33})e_2\\
&\qquad+(C^1_{21}C^3_{31}+C^2_{21}C^3_{32}+C^3_{21}C^3_{33}-C^1_{12}C^3_{13}-C^2_{12}C^3_{23}-C^3_{12}C^3_{33})e_3=0,\\
(3.18):\quad & (C^1_{31}C^1_{31}+C^2_{31}C^1_{32}+C^3_{31}C^1_{33})e_1+(C^1_{31}C^2_{31}+C^2_{31}C^2_{32}+C^3_{31}C^2_{33})e_2\\
&+(C^1_{31}C^3_{31}+C^2_{31}C^3_{32}+C^3_{31}C^3_{33})e_3\\
&=(C^1_{13}C^1_{13}+C^2_{13}C^1_{23}+C^3_{13}C^1_{33})e_1+(C^1_{13}C^2_{13}+C^2_{13}C^2_{23}+C^3_{13}C^2_{33})e_2\\
&\quad+(C^1_{13}C^3_{13}+C^2_{13}C^3_{23}+C^3_{13}C^3_{33})e_3\\
&\Rightarrow\ (C^1_{31}C^1_{31}+C^2_{31}C^1_{32}+C^3_{31}C^1_{33}-C^1_{13}C^1_{13}-C^2_{13}C^1_{23}-C^3_{13}C^1_{33})e_1\\
&\qquad+(C^1_{31}C^2_{31}+C^2_{31}C^2_{32}+C^3_{31}C^2_{33}-C^1_{13}C^2_{13}-C^2_{13}C^2_{23}-C^3_{13}C^2_{33})e_2\\
&\qquad+(C^1_{31}C^3_{31}+C^2_{31}C^3_{32}+C^3_{31}C^3_{33}-C^1_{13}C^3_{13}-C^2_{13}C^3_{23}-C^3_{13}C^3_{33})e_3=0,\\
(3.19):\quad & (C^1_{22}C^1_{31}+C^2_{22}C^1_{32}+C^3_{22}C^1_{33})e_1+(C^1_{22}C^2_{31}+C^2_{22}C^2_{32}+C^3_{22}C^2_{33})e_2\\
&+(C^1_{22}C^3_{31}+C^2_{22}C^3_{32}+C^3_{22}C^3_{33})e_3=0,\\
(3.20):\quad & (C^1_{23}C^1_{31}+C^2_{23}C^1_{32}+C^3_{23}C^1_{33})e_1+(C^1_{23}C^2_{31}+C^2_{23}C^2_{32}+C^3_{23}C^2_{33})e_2\\
&+(C^1_{23}C^3_{31}+C^2_{23}C^3_{32}+C^3_{23}C^3_{33})e_3=0,\\
(3.21):\quad & (C^1_{32}C^1_{31}+C^2_{32}C^1_{32}+C^3_{32}C^1_{33})e_1+(C^1_{32}C^2_{31}+C^2_{32}C^2_{32}+C^3_{32}C^2_{33})e_2\\
&+(C^1_{32}C^3_{31}+C^2_{32}C^3_{32}+C^3_{32}C^3_{33})e_3=0,\\
(3.22):\quad & (C^1_{33}C^1_{31}+C^2_{33}C^1_{32}+C^3_{33}C^1_{33})e_1+(C^1_{33}C^2_{31}+C^2_{33}C^2_{32}+C^3_{33}C^2_{33})e_2
\end{aligned}$$

$$+ (C^1_{33}C^3_{31} + C^2_{33}C^3_{32} + C^3_{33}C^3_{33})e_3 = 0,$$

$$(3.23): \quad 0 = (C^1_{21}C^1_{13} + C^2_{21}C^1_{23} + C^3_{21}C^1_{33})e_1 + (C^1_{21}C^2_{13} + C^2_{21}C^2_{23} + C^3_{21}C^2_{33})e_2 + (C^1_{21}C^3_{13} + C^2_{21}C^3_{23} + C^3_{21}C^3_{33})e_3,$$

$$(3.26): \quad 0 = (C^1_{22}C^1_{13} + C^2_{22}C^1_{23} + C^3_{22}C^1_{33})e_1 + (C^1_{22}C^2_{13} + C^2_{22}C^2_{23} + C^3_{22}C^2_{33})e_2 + (C^1_{22}C^3_{13} + C^2_{22}C^3_{23} + C^3_{22}C^3_{33})e_3,$$

$$(3.27): \quad 0 = (C^1_{23}C^1_{13} + C^2_{23}C^1_{23} + C^3_{23}C^1_{33})e_1 + (C^1_{23}C^2_{13} + C^2_{23}C^2_{23} + C^3_{23}C^2_{33})e_2 + (C^1_{23}C^3_{13} + C^2_{23}C^3_{23} + C^3_{23}C^3_{33})e_3,$$

$$(3.32): \quad 0 = (C^1_{31}C^1_{13} + C^2_{31}C^1_{23} + C^3_{31}C^1_{33})e_1 + (C^1_{31}C^2_{13} + C^2_{31}C^2_{23} + C^3_{31}C^2_{33})e_2 + (C^1_{31}C^3_{13} + C^2_{31}C^3_{23} + C^3_{31}C^3_{33})e_3,$$

$$(3.35): \quad 0 = (C^1_{32}C^1_{13} + C^2_{32}C^1_{23} + C^3_{32}C^1_{33})e_1 + (C^1_{32}C^2_{13} + C^2_{32}C^2_{23} + C^3_{32}C^2_{33})e_2 + (C^1_{32}C^3_{13} + C^2_{32}C^3_{23} + C^3_{32}C^3_{33})e_3,$$

$$(3.36): \quad 0 = (C^1_{33}C^1_{13} + C^2_{33}C^1_{23} + C^3_{33}C^1_{33})e_1 + (C^1_{33}C^2_{13} + C^2_{33}C^2_{23} + C^3_{33}C^2_{33})e_2 + (C^1_{33}C^3_{13} + C^2_{33}C^3_{23} + C^3_{33}C^3_{33})e_3.$$

Remembering that $k_1e_1 + k_2e_2 + k_3e_3 = 0$ will only have the trivial solution $k_1 = k_2 = k_3 = 0$, we can rewrite our equations as the following system of equations:

$$C^1_{11}C^1_{31} + C^2_{11}C^1_{32} - C^1_{11}C^1_{13} - C^2_{11}C^1_{23} = 0 \quad \text{(R3.14.1)}$$
$$C^1_{11}C^2_{31} + C^2_{11}C^2_{32} - C^1_{11}C^2_{13} - C^2_{11}C^2_{23} = 0 \quad \text{(R3.14.2)}$$
$$C^1_{11}C^3_{31} + C^2_{11}C^3_{32} - C^1_{11}C^3_{13} - C^2_{11}C^3_{23} = 0 \quad \text{(R3.14.3)}$$
$$C^1_{12}C^1_{31} + C^2_{12}C^1_{32} + C^3_{12}C^1_{33} = 0 \quad \text{(R3.15.1)}$$
$$C^1_{12}C^2_{31} + C^2_{12}C^2_{32} + C^3_{12}C^2_{33} = 0 \quad \text{(R3.15.2)}$$
$$C^1_{12}C^3_{31} + C^2_{12}C^3_{32} + C^3_{12}C^3_{33} = 0 \quad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{31} + C^2_{13}C^1_{32} + C^3_{13}C^1_{33} = 0 \quad \text{(R3.16.1)}$$
$$C^1_{13}C^2_{31} + C^2_{13}C^2_{32} + C^3_{13}C^2_{33} = 0 \quad \text{(R3.16.2)}$$
$$C^1_{13}C^3_{31} + C^2_{13}C^3_{32} + C^3_{13}C^3_{33} = 0 \quad \text{(R3.16.3)}$$
$$C^1_{21}C^1_{31} + C^2_{21}C^1_{32} + C^3_{21}C^1_{33} - C^1_{12}C^1_{13} - C^2_{12}C^1_{23} - C^3_{12}C^1_{33} = 0 \quad \text{(R3.17.1)}$$
$$C^1_{21}C^2_{31} + C^2_{21}C^2_{32} + C^3_{21}C^2_{33} - C^1_{12}C^2_{13} - C^2_{12}C^2_{23} - C^3_{12}C^2_{33} = 0 \quad \text{(R3.17.2)}$$
$$C^1_{21}C^3_{31} + C^2_{21}C^3_{32} + C^3_{21}C^3_{33} - C^1_{12}C^3_{13} - C^2_{12}C^3_{23} - C^3_{12}C^3_{33} = 0 \quad \text{(R3.17.3)}$$
$$C^1_{31}C^1_{31} + C^2_{31}C^1_{32} + C^3_{31}C^1_{33} - C^1_{13}C^1_{13} - C^2_{13}C^1_{23} - C^3_{13}C^1_{33} = 0 \quad \text{(R3.18.1)}$$
$$C^1_{31}C^2_{31} + C^2_{31}C^2_{32} + C^3_{31}C^2_{33} - C^1_{13}C^2_{13} - C^2_{13}C^2_{23} - C^3_{13}C^2_{33} = 0 \quad \text{(R3.18.2)}$$
$$C^1_{31}C^3_{31} + C^2_{31}C^3_{32} + C^3_{31}C^3_{33} - C^1_{13}C^3_{13} - C^2_{13}C^3_{23} - C^3_{13}C^3_{33} = 0 \quad \text{(R3.18.3)}$$
$$C^1_{22}C^1_{31} + C^2_{22}C^1_{32} + C^3_{22}C^1_{33} = 0 \quad \text{(R3.19.1)}$$
$$C^1_{22}C^2_{31} + C^2_{22}C^2_{32} + C^3_{22}C^2_{33} = 0 \quad \text{(R3.19.2)}$$
$$C^1_{22}C^3_{31} + C^2_{22}C^3_{32} + C^3_{22}C^3_{33} = 0 \quad \text{(R3.19.3)}$$

$$C_{23}^1C_{31}^1+C_{23}^2C_{32}^1+C_{23}^3C_{33}^1=0 \quad \text{(R3.20.1)}$$
$$C_{23}^1C_{31}^2+C_{23}^2C_{32}^2+C_{23}^3C_{33}^2=0 \quad \text{(R3.20.2)}$$
$$C_{23}^1C_{31}^3+C_{23}^2C_{32}^3+C_{23}^3C_{33}^3=0 \quad \text{(R3.20.3)}$$
$$C_{32}^1C_{31}^1+C_{32}^2C_{32}^1+C_{32}^3C_{33}^1=0 \quad \text{(R3.21.1)}$$
$$C_{32}^1C_{31}^2+C_{32}^2C_{32}^2+C_{32}^3C_{33}^2=0 \quad \text{(R3.21.2)}$$
$$C_{32}^1C_{31}^3+C_{32}^2C_{32}^3+C_{32}^3C_{33}^3=0 \quad \text{(R3.21.3)}$$
$$C_{33}^1C_{31}^1+C_{33}^2C_{32}^1+C_{33}^3C_{33}^1=0 \quad \text{(R3.22.1)}$$
$$C_{33}^1C_{31}^2+C_{33}^2C_{32}^2+C_{33}^3C_{33}^2=0 \quad \text{(R3.22.2)}$$
$$C_{33}^1C_{31}^3+C_{33}^2C_{32}^3+C_{33}^3C_{33}^3=0 \quad \text{(R3.22.3)}$$
$$C_{21}^1C_{13}^1+C_{21}^2C_{23}^1+C_{21}^3C_{33}^1=0 \quad \text{(R3.23.1)}$$
$$C_{21}^1C_{13}^2+C_{21}^2C_{23}^2+C_{21}^3C_{33}^2=0 \quad \text{(R3.23.2)}$$
$$C_{21}^1C_{13}^3+C_{21}^2C_{23}^3+C_{21}^3C_{33}^3=0 \quad \text{(R3.23.3)}$$
$$C_{22}^1C_{13}^1+C_{22}^2C_{23}^1+C_{22}^3C_{33}^1=0 \quad \text{(R3.26.1)}$$
$$C_{22}^1C_{13}^2+C_{22}^2C_{23}^2+C_{22}^3C_{33}^2=0 \quad \text{(R3.26.2)}$$
$$C_{22}^1C_{13}^3+C_{22}^2C_{23}^3+C_{22}^3C_{33}^3=0 \quad \text{(R3.26.3)}$$
$$C_{23}^1C_{13}^1+C_{23}^2C_{23}^1+C_{23}^3C_{33}^1=0 \quad \text{(R3.27.1)}$$
$$C_{23}^1C_{13}^2+C_{23}^2C_{23}^2+C_{23}^3C_{33}^2=0 \quad \text{(R3.27.2)}$$
$$C_{23}^1C_{13}^3+C_{23}^2C_{23}^3+C_{23}^3C_{33}^3=0 \quad \text{(R3.27.3)}$$
$$C_{31}^1C_{13}^1+C_{31}^2C_{23}^1+C_{31}^3C_{33}^1=0 \quad \text{(R3.32.1)}$$
$$C_{31}^1C_{13}^2+C_{31}^2C_{23}^2+C_{31}^3C_{33}^2=0 \quad \text{(R3.32.2)}$$
$$C_{31}^1C_{13}^3+C_{31}^2C_{23}^3+C_{31}^3C_{33}^3=0 \quad \text{(R3.32.3)}$$
$$C_{32}^1C_{13}^1+C_{32}^2C_{23}^1+C_{32}^3C_{33}^1=0 \quad \text{(R3.35.1)}$$
$$C_{32}^1C_{13}^2+C_{32}^2C_{23}^2+C_{32}^3C_{33}^2=0 \quad \text{(R3.35.2)}$$
$$C_{32}^1C_{13}^3+C_{32}^2C_{23}^3+C_{32}^3C_{33}^3=0 \quad \text{(R3.35.3)}$$
$$C_{33}^1C_{13}^1+C_{33}^2C_{23}^1+C_{33}^3C_{33}^1=0 \quad \text{(R3.36.1)}$$
$$C_{33}^1C_{13}^2+C_{33}^2C_{23}^2+C_{33}^3C_{33}^2=0 \quad \text{(R3.36.2)}$$
$$C_{33}^1C_{13}^3+C_{33}^2C_{23}^3+C_{33}^3C_{33}^3=0. \quad \text{(R3.36.3)}$$

3.2.8 α defined as E_{32}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha]=E_{32}=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}.$$

However, due to our results in Section 3.2.1, where we realised we would not be able to solve the entire system of equations in this project, we will only write out the system of equations for this α, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\begin{aligned}
\alpha(e_1) &= 0e_1 + 0e_2 + 0e_3 = 0,\\
\alpha(e_2) &= 0e_1 + 0e_2 + 1e_3 = e_3,\\
\alpha(e_3) &= 0e_1 + 0e_2 + 0e_3 = 0.
\end{aligned}$$

We can thus rewrite our equations as follows:

$$\begin{aligned}
&(3.14): \quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.15): \quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, e_3)\\
&\qquad\quad \Rightarrow\; 0 = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, e_3),\\
&(3.16): \quad \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.17): \quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.18): \quad \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.19): \quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_3)\\
&\qquad\quad \Rightarrow\; 0 = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_3),\\
&(3.20): \quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.21): \quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_3)\\
&\qquad\quad \Rightarrow\; 0 = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_3),\\
&(3.22): \quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0) \;\Rightarrow\; 0 = 0,\\
&(3.23): \quad \mu(e_3, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0)\\
&\qquad\quad \Rightarrow\; \mu(e_3, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = 0,\\
&(3.24): \quad \mu(e_3, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_3),\\
&(3.25): \quad \mu(e_3, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0)\\
&\qquad\quad \Rightarrow\; \mu(e_3, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = 0,\\
&(3.26): \quad \mu(e_3, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0)\\
&\qquad\quad \Rightarrow\; \mu(e_3, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = 0,\\
&(3.27): \quad \mu(e_3, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0)\\
&\qquad\quad \Rightarrow\; \mu(e_3, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = 0,\\
&(3.28): \quad \mu(e_3, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_3),\\
&(3.29): \quad \mu(e_3, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0)\\
&\qquad\quad \Rightarrow\; \mu(e_3, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = 0,\\
&(3.30): \quad \mu(e_3, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_3),\\
&(3.31): \quad \mu(e_3, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0)
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow\ \ \mu(e_3, C^1_{33}e_1 + C^2_{33}e_2 + C^3_{33}e_3) = 0,\\
(3.32):\quad & \mu(0, C^1_{11}e_1 + C^2_{11}e_2 + C^3_{11}e_3) = \mu(C^1_{31}e_1 + C^2_{31}e_2 + C^3_{31}e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.33):\quad & \mu(0, C^1_{12}e_1 + C^2_{12}e_2 + C^3_{12}e_3) = \mu(C^1_{31}e_1 + C^2_{31}e_2 + C^3_{31}e_3, e_3)\\
&\Rightarrow\ \ 0 = \mu(C^1_{31}e_1 + C^2_{31}e_2 + C^3_{31}e_3, e_3),\\
(3.34):\quad & \mu(0, C^1_{13}e_1 + C^2_{13}e_2 + C^3_{13}e_3) = \mu(C^1_{31}e_1 + C^2_{31}e_2 + C^3_{31}e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.35):\quad & \mu(0, C^1_{21}e_1 + C^2_{21}e_2 + C^3_{21}e_3) = \mu(C^1_{32}e_1 + C^2_{32}e_2 + C^3_{32}e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.36):\quad & \mu(0, C^1_{31}e_1 + C^2_{31}e_2 + C^3_{31}e_3) = \mu(C^1_{33}e_1 + C^2_{33}e_2 + C^3_{33}e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.37):\quad & \mu(0, C^1_{22}e_1 + C^2_{22}e_2 + C^3_{22}e_3) = \mu(C^1_{32}e_1 + C^2_{32}e_2 + C^3_{32}e_3, e_3)\\
&\Rightarrow\ \ 0 = \mu(C^1_{32}e_1 + C^2_{32}e_2 + C^3_{32}e_3, e_3),\\
(3.38):\quad & \mu(0, C^1_{23}e_1 + C^2_{23}e_2 + C^3_{23}e_3) = \mu(C^1_{32}e_1 + C^2_{32}e_2 + C^3_{32}e_3, 0) \ \Rightarrow\ 0 = 0,\\
(3.39):\quad & \mu(0, C^1_{32}e_1 + C^2_{32}e_2 + C^3_{32}e_3) = \mu(C^1_{33}e_1 + C^2_{33}e_2 + C^3_{33}e_3, e_3)\\
&\Rightarrow\ \ 0 = \mu(C^1_{33}e_1 + C^2_{33}e_2 + C^3_{33}e_3, e_3),\\
(3.40):\quad & \mu(0, C^1_{33}e_1 + C^2_{33}e_2 + C^3_{33}e_3) = \mu(C^1_{33}e_1 + C^2_{33}e_2 + C^3_{33}e_3, 0) \ \Rightarrow\ 0 = 0.
\end{aligned}
$$

We remember the following formulas that we calculated earlier:

$$
\begin{aligned}
\mu(e_i, C^1_{jk}e_1 + C^2_{jk}e_2 + C^3_{jk}e_3) = {} & (C^1_{jk}C^1_{i1} + C^2_{jk}C^1_{i2} + C^3_{jk}C^1_{i3})e_1 + (C^1_{jk}C^2_{i1} + C^2_{jk}C^2_{i2} + C^3_{jk}C^2_{i3})e_2\\
& + (C^1_{jk}C^3_{i1} + C^2_{jk}C^3_{i2} + C^3_{jk}C^3_{i3})e_3
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(C^1_{jk}e_1 + C^2_{jk}e_2 + C^3_{jk}e_3, e_i) = {} & (C^1_{jk}C^1_{1i} + C^2_{jk}C^1_{2i} + C^3_{jk}C^1_{3i})e_1 + (C^1_{jk}C^2_{1i} + C^2_{jk}C^2_{2i} + C^3_{jk}C^2_{3i})e_2\\
& + (C^1_{jk}C^3_{1i} + C^2_{jk}C^3_{2i} + C^3_{jk}C^3_{3i})e_3.
\end{aligned}
$$

We use these formulas to rewrite our equations.

$$
\begin{aligned}
(3.15):\quad & 0 = (C^1_{11}C^1_{13} + C^2_{11}C^1_{23} + C^3_{11}C^1_{33})e_1 + (C^1_{11}C^2_{13} + C^2_{11}C^2_{23} + C^3_{11}C^2_{33})e_1\\
&\quad + (C^1_{11}C^3_{13} + C^2_{11}C^3_{23} + C^3_{11}C^3_{33})e_1,\\
(3.19):\quad & 0 = (C^1_{12}C^1_{13} + C^2_{12}C^1_{23} + C^3_{12}C^1_{33})e_1 + (C^1_{12}C^2_{13} + C^2_{12}C^2_{23} + C^3_{12}C^2_{33})e_2\\
&\quad + (C^1_{12}C^3_{13} + C^2_{12}C^3_{23} + C^3_{12}C^3_{33})e_3,\\
(3.21):\quad & 0 = (C^1_{13}C^1_{13} + C^2_{13}C^1_{23} + C^3_{13}C^1_{33})e_1 + (C^1_{13}C^2_{13} + C^2_{13}C^2_{23} + C^3_{13}C^2_{33})e_2\\
&\quad + (C^1_{13}C^3_{13} + C^2_{13}C^3_{23} + C^3_{13}C^3_{33})e_3,\\
(3.23):\quad & (C^1_{11}C^1_{31} + C^2_{11}C^1_{32} + C^3_{11}C^1_{33})e_1 + (C^1_{11}C^2_{31} + C^2_{11}C^2_{32} + C^3_{11}C^2_{33})e_2\\
& + (C^1_{11}C^3_{31} + C^2_{11}C^3_{32} + C^3_{11}C^3_{33})e_3 = 0,\\
(3.24):\quad & (C^1_{12}C^1_{31} + C^2_{12}C^1_{32} + C^3_{12}C^1_{33})e_1 + (C^1_{12}C^2_{31} + C^2_{12}C^2_{32} + C^3_{12}C^2_{33})e_2\\
& + (C^1_{12}C^3_{31} + C^2_{12}C^3_{32} + C^3_{12}C^3_{33})e_3\\
& = (C^1_{21}C^1_{13} + C^2_{21}C^1_{23} + C^3_{21}C^1_{33})e_1 + (C^1_{21}C^2_{13} + C^2_{21}C^2_{23} + C^3_{21}C^2_{33})e_2
\end{aligned}
$$

$$+(C_{21}^1C_{13}^3+C_{21}^2C_{23}^3+C_{21}^3C_{33}^3)e_3$$

$$\Rightarrow\ (C_{12}^1C_{31}^1+C_{12}^2C_{32}^1+C_{12}^3C_{33}^1-C_{21}^1C_{13}^1-C_{21}^2C_{23}^1-C_{21}^3C_{33}^1)e_1$$

$$+(C_{12}^1C_{31}^2+C_{12}^2C_{32}^2+C_{12}^3C_{33}^2-C_{21}^1C_{13}^2-C_{21}^2C_{23}^2-C_{21}^3C_{33}^2)e_2$$

$$+(C_{12}^1C_{31}^3+C_{12}^2C_{32}^3+C_{12}^3C_{33}^3-C_{21}^1C_{13}^3-C_{21}^2C_{23}^3-C_{21}^3C_{33}^3)e_3=0,$$

(3.25) : $$(C_{13}^1C_{31}^1+C_{13}^2C_{32}^1+C_{13}^3C_{33}^1)e_1+(C_{13}^1C_{31}^2+C_{13}^2C_{32}^2+C_{13}^3C_{33}^2)e_2$$

$$+(C_{13}^1C_{31}^3+C_{13}^2C_{32}^3+C_{13}^3C_{33}^3)e_3=0,$$

(3.26) : $$(C_{21}^1C_{31}^1+C_{21}^2C_{32}^1+C_{21}^3C_{33}^1)e_1+(C_{21}^1C_{31}^2+C_{21}^2C_{32}^2+C_{21}^3C_{33}^2)e_2$$

$$+(C_{21}^1C_{31}^3+C_{21}^2C_{32}^3+C_{21}^3C_{33}^3)e_3=0,$$

(3.27) : $$(C_{31}^1C_{31}^1+C_{31}^2C_{32}^1+C_{31}^3C_{33}^1)e_1+(C_{31}^1C_{31}^2+C_{31}^2C_{32}^2+C_{31}^3C_{33}^2)e_2$$

$$+(C_{31}^1C_{31}^3+C_{31}^2C_{32}^3+C_{31}^3C_{33}^3)e_3=0,$$

(3.28) : $$(C_{22}^1C_{31}^1+C_{22}^2C_{32}^1+C_{22}^3C_{33}^1)e_1+(C_{22}^1C_{31}^2+C_{22}^2C_{32}^2+C_{22}^3C_{33}^2)e_2$$

$$+(C_{22}^1C_{31}^3+C_{22}^2C_{32}^3+C_{22}^3C_{33}^3)e_3$$

$$=(C_{22}^1C_{13}^1+C_{22}^2C_{23}^1+C_{22}^3C_{33}^1)e_1+(C_{22}^1C_{13}^2+C_{22}^2C_{23}^2+C_{22}^3C_{33}^2)e_2$$

$$+(C_{22}^1C_{13}^3+C_{22}^2C_{23}^3+C_{22}^3C_{33}^3)e_3$$

$$\Rightarrow\ (C_{22}^1C_{31}^1+C_{22}^2C_{32}^1-C_{22}^1C_{13}^1-C_{22}^2C_{23}^1)e_1+(C_{22}^1C_{31}^2+C_{22}^2C_{32}^2-C_{22}^1C_{13}^2-C_{22}^2C_{23}^2)e_2$$

$$+(C_{22}^1C_{31}^3+C_{22}^2C_{32}^3-C_{22}^1C_{13}^3-C_{22}^2C_{23}^3)e_3=0,$$

(3.29) : $$(C_{23}^1C_{31}^1+C_{23}^2C_{32}^1+C_{23}^3C_{33}^1)e_1+(C_{23}^1C_{31}^2+C_{23}^2C_{32}^2+C_{23}^3C_{33}^2)e_2$$

$$+(C_{23}^1C_{31}^3+C_{23}^2C_{32}^3+C_{23}^3C_{33}^3)e_3=0,$$

(3.30) : $$(C_{32}^1C_{31}^1+C_{32}^2C_{32}^1+C_{32}^3C_{33}^1)e_1+(C_{32}^1C_{31}^2+C_{32}^2C_{32}^2+C_{32}^3C_{33}^2)e_2$$

$$+(C_{32}^1C_{31}^3+C_{32}^2C_{32}^3+C_{32}^3C_{33}^3)e_3$$

$$=(C_{23}^1C_{13}^1+C_{23}^2C_{23}^1+C_{23}^3C_{33}^1)e_1+(C_{23}^1C_{13}^2+C_{23}^2C_{23}^2+C_{23}^3C_{33}^2)e_2$$

$$+(C_{23}^1C_{13}^3+C_{23}^2C_{23}^3+C_{23}^3C_{33}^3)e_3$$

$$\Rightarrow\ (C_{32}^1C_{31}^1+C_{32}^2C_{32}^1+C_{32}^3C_{33}^1-C_{23}^1C_{13}^1+C_{23}^2C_{23}^1+C_{23}^3C_{33}^1)e_1$$

$$+(C_{32}^1C_{31}^2+C_{32}^2C_{32}^2+C_{32}^3C_{33}^2-C_{23}^1C_{13}^2+C_{23}^2C_{23}^2+C_{23}^3C_{33}^2)e_2$$

$$+(C_{32}^1C_{31}^3+C_{32}^2C_{32}^3+C_{32}^3C_{33}^3-C_{23}^1C_{13}^3+C_{23}^2C_{23}^3+C_{23}^3C_{33}^3)e_3=0,$$

(3.31) : $$(C_{33}^1C_{31}^1+C_{33}^2C_{32}^1+C_{33}^3C_{33}^1)e_1+(C_{33}^1C_{31}^2+C_{33}^2C_{32}^2+C_{33}^3C_{33}^2)e_2$$

$$+(C_{33}^1C_{31}^3+C_{33}^2C_{32}^3+C_{33}^3C_{33}^3)e_3=0,$$

(3.33) : $$0=(C_{31}^1C_{13}^1+C_{31}^2C_{23}^1+C_{31}^3C_{33}^1)e_1+(C_{31}^1C_{13}^2+C_{31}^2C_{23}^2+C_{31}^3C_{33}^2)e_2$$

$$+(C_{31}^1C_{13}^3+C_{31}^2C_{23}^3+C_{31}^3C_{33}^3)e_3,$$

(3.37) : $$0=(C_{32}^1C_{13}^1+C_{32}^2C_{23}^1+C_{32}^3C_{33}^1)e_1+(C_{32}^1C_{13}^2+C_{32}^2C_{23}^2+C_{32}^3C_{33}^2)e_2$$

$$+(C_{32}^1C_{13}^3+C_{32}^2C_{23}^3+C_{32}^3C_{33}^3)e_3,$$

(3.39) : $$0=(C_{33}^1C_{13}^1+C_{33}^2C_{23}^1+C_{33}^3C_{33}^1)e_1+(C_{33}^1C_{13}^2+C_{33}^2C_{23}^2+C_{33}^3C_{33}^2)e_2$$

$$+(C_{33}^1C_{13}^3+C_{33}^2C_{23}^3+C_{33}^3C_{33}^3)e_3.$$

Remembering that $k_1e_1+k_2e_2+k_3e_3=0$ will only have the trivial solution $k_1=k_2=k_3=0$, we can rewrite our equations as the following system of equations:

$$C^1_{11}C^1_{13}+C^2_{11}C^1_{23}+C^3_{11}C^1_{33}=0 \qquad \text{(R3.15.1)}$$
$$C^1_{11}C^2_{13}+C^2_{11}C^2_{23}+C^3_{11}C^2_{33}=0 \qquad \text{(R3.15.2)}$$
$$C^1_{11}C^3_{13}+C^2_{11}C^3_{23}+C^3_{11}C^3_{33}=0 \qquad \text{(R3.15.3)}$$
$$C^1_{12}C^1_{13}+C^2_{12}C^1_{23}+C^3_{12}C^1_{33}=0 \qquad \text{(R3.19.1)}$$
$$C^1_{12}C^2_{13}+C^2_{12}C^2_{23}+C^3_{12}C^2_{33}=0 \qquad \text{(R3.19.2)}$$
$$C^1_{12}C^3_{13}+C^2_{12}C^3_{23}+C^3_{12}C^3_{33}=0 \qquad \text{(R3.19.3)}$$
$$C^1_{13}C^1_{13}+C^2_{13}C^1_{23}+C^3_{13}C^1_{33}=0 \qquad \text{(R3.21.1)}$$
$$C^1_{13}C^2_{13}+C^2_{13}C^2_{23}+C^3_{13}C^2_{33}=0 \qquad \text{(R3.21.2)}$$
$$C^1_{13}C^3_{13}+C^2_{13}C^3_{23}+C^3_{13}C^3_{33}=0 \qquad \text{(R3.21.3)}$$
$$C^1_{11}C^1_{31}+C^2_{11}C^1_{32}+C^3_{11}C^1_{33}=0 \qquad \text{(R3.23.1)}$$
$$C^1_{11}C^2_{31}+C^2_{11}C^2_{32}+C^3_{11}C^2_{33}=0 \qquad \text{(R3.23.2)}$$
$$C^1_{11}C^3_{31}+C^2_{11}C^3_{32}+C^3_{11}C^3_{33}=0 \qquad \text{(R3.23.3)}$$
$$C^1_{12}C^1_{31}+C^2_{12}C^1_{32}+C^3_{12}C^1_{33}-C^1_{21}C^1_{13}-C^2_{21}C^1_{23}-C^3_{21}C^1_{33}=0 \qquad \text{(R3.24.1)}$$
$$C^1_{12}C^2_{31}+C^2_{12}C^2_{32}+C^3_{12}C^2_{33}-C^1_{21}C^2_{13}-C^2_{21}C^2_{23}-C^3_{21}C^2_{33}=0 \qquad \text{(R3.24.2)}$$
$$C^1_{12}C^3_{31}+C^2_{12}C^3_{32}+C^3_{12}C^3_{33}-C^1_{21}C^3_{13}-C^2_{21}C^3_{23}-C^3_{21}C^3_{33}=0 \qquad \text{(R3.24.3)}$$
$$C^1_{13}C^1_{31}+C^2_{13}C^1_{32}+C^3_{13}C^1_{33}=0 \qquad \text{(R3.25.1)}$$
$$C^1_{13}C^2_{31}+C^2_{13}C^2_{32}+C^3_{13}C^2_{33}=0 \qquad \text{(R3.25.2)}$$
$$C^1_{13}C^3_{31}+C^2_{13}C^3_{32}+C^3_{13}C^3_{33}=0 \qquad \text{(R3.25.3)}$$
$$C^1_{21}C^1_{31}+C^2_{21}C^1_{32}+C^3_{21}C^1_{33}=0 \qquad \text{(R3.26.1)}$$
$$C^1_{21}C^2_{31}+C^2_{21}C^2_{32}+C^3_{21}C^2_{33}=0 \qquad \text{(R3.26.2)}$$
$$C^1_{21}C^3_{31}+C^2_{21}C^3_{32}+C^3_{21}C^3_{33}=0 \qquad \text{(R3.26.3)}$$
$$C^1_{31}C^1_{31}+C^2_{31}C^1_{32}+C^3_{31}C^1_{33}=0 \qquad \text{(R3.27.1)}$$
$$C^1_{31}C^2_{31}+C^2_{31}C^2_{32}+C^3_{31}C^2_{33}=0 \qquad \text{(R3.27.2)}$$
$$C^1_{31}C^3_{31}+C^2_{31}C^3_{32}+C^3_{31}C^3_{33}=0 \qquad \text{(R3.27.3)}$$
$$C^1_{22}C^1_{31}+C^2_{22}C^1_{32}-C^1_{22}C^1_{13}-C^2_{22}C^1_{23}=0 \qquad \text{(R3.28.1)}$$
$$C^1_{22}C^2_{31}+C^2_{22}C^2_{32}-C^1_{22}C^2_{13}-C^2_{22}C^2_{23}=0 \qquad \text{(R3.28.2)}$$
$$C^1_{22}C^3_{31}+C^2_{22}C^3_{32}-C^1_{22}C^3_{13}-C^2_{22}C^3_{23}=0 \qquad \text{(R3.28.3)}$$
$$C^1_{23}C^1_{31}+C^2_{23}C^1_{32}+C^3_{23}C^1_{33}=0 \qquad \text{(R3.29.1)}$$
$$C^1_{23}C^2_{31}+C^2_{23}C^2_{32}+C^3_{23}C^2_{33}=0 \qquad \text{(R3.29.2)}$$
$$C^1_{23}C^3_{31}+C^2_{23}C^3_{32}+C^3_{23}C^3_{33}=0 \qquad \text{(R3.29.3)}$$
$$C^1_{32}C^1_{31}+C^2_{32}C^1_{32}+C^3_{32}C^1_{33}-C^1_{23}C^1_{13}+C^2_{23}C^1_{23}+C^3_{23}C^1_{33}=0 \qquad \text{(R3.30.1)}$$

$$C^1_{32}C^2_{31}+C^2_{32}C^2_{32}+C^3_{32}C^2_{33}-C^1_{23}C^2_{13}+C^2_{23}C^2_{23}+C^3_{23}C^2_{33}=0 \quad \text{(R3.30.2)}$$
$$C^1_{32}C^3_{31}+C^2_{32}C^3_{32}+C^3_{32}C^3_{33}-C^1_{23}C^3_{13}+C^2_{23}C^3_{23}+C^3_{23}C^3_{33}=0 \quad \text{(R3.30.3)}$$
$$C^1_{33}C^1_{31}+C^2_{33}C^1_{32}+C^3_{33}C^1_{33}=0 \quad \text{(R3.31.1)}$$
$$C^1_{33}C^2_{31}+C^2_{33}C^2_{32}+C^3_{33}C^2_{33}=0 \quad \text{(R3.31.2)}$$
$$C^1_{33}C^3_{31}+C^2_{33}C^3_{32}+C^3_{33}C^3_{33}=0 \quad \text{(R3.31.3)}$$
$$C^1_{31}C^1_{13}+C^2_{31}C^1_{23}+C^3_{31}C^1_{33}=0 \quad \text{(R3.33.1)}$$
$$C^1_{31}C^2_{13}+C^2_{31}C^2_{23}+C^3_{31}C^2_{33}=0 \quad \text{(R3.33.2)}$$
$$C^1_{31}C^3_{13}+C^2_{31}C^3_{23}+C^3_{31}C^3_{33}=0 \quad \text{(R3.33.3)}$$
$$C^1_{32}C^1_{13}+C^2_{32}C^1_{23}+C^3_{32}C^1_{33}=0 \quad \text{(R3.37.1)}$$
$$C^1_{32}C^2_{13}+C^2_{32}C^2_{23}+C^3_{32}C^2_{33}=0 \quad \text{(R3.37.2)}$$
$$C^1_{32}C^3_{13}+C^2_{32}C^3_{23}+C^3_{32}C^3_{33}=0 \quad \text{(R3.37.3)}$$
$$C^1_{33}C^1_{13}+C^2_{33}C^1_{23}+C^3_{33}C^1_{33}=0 \quad \text{(R3.39.1)}$$
$$C^1_{33}C^2_{13}+C^2_{33}C^2_{23}+C^3_{33}C^2_{33}=0 \quad \text{(R3.39.2)}$$
$$C^1_{33}C^3_{13}+C^2_{33}C^3_{23}+C^3_{33}C^3_{33}=0. \quad \text{(R3.39.3)}$$

3.2.9 α defined as E_{33}

We now want to investigate what the hom-associative algebras will look like in three dimensions for

$$[\alpha]=E_{33}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

However, due to our results in Section 3.2.1, where we realised we would not be able to solve the entire system of equations in this project, we will only write out the system of equations for this α, not try to solve it. Now, using formulas (3.41), (3.42) and (3.43), we get

$$\begin{aligned}\alpha(e_1)&=0e_1+0e_2+0e_3=0,\\ \alpha(e_2)&=0e_1+0e_2+0e_3=0,\\ \alpha(e_3)&=0e_1+0e_2+1e_3=e_3.\end{aligned}$$

We can thus rewrite our equations as follows:

$$\begin{aligned}
(3.14):&\quad \mu(0,C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3)=\mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3,0) \;\Rightarrow\; 0=0,\\
(3.15):&\quad \mu(0,C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3)=\mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3,0) \;\Rightarrow\; 0=0,\\
(3.16):&\quad \mu(0,C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3)=\mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3,e_3)\\
&\quad \Rightarrow\; 0=\mu(C^1_{11}e_1+C^2_{11}e_2+C^3_{11}e_3,e_3),\\
(3.17):&\quad \mu(0,C^1_{21}e_1+C^2_{21}e_2+C^3_{21}e_3)=\mu(C^1_{12}e_1+C^2_{12}e_2+C^3_{12}e_3,0) \;\Rightarrow\; 0=0,\\
(3.18):&\quad \mu(0,C^1_{31}e_1+C^2_{31}e_2+C^3_{31}e_3)=\mu(C^1_{13}e_1+C^2_{13}e_2+C^3_{13}e_3,0) \;\Rightarrow\; 0=0,
\end{aligned}$$

$$(3.19):\quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.20):\quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_3)$$
$$\Rightarrow\; 0 = \mu(C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3, e_3),$$

$$(3.21):\quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.22):\quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_3)$$
$$\Rightarrow\; 0 = \mu(C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3, e_3),$$

$$(3.23):\quad \mu(0, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.24):\quad \mu(0, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.25):\quad \mu(0, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_3)$$
$$\Rightarrow\; 0 = \mu(C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3, e_3),$$

$$(3.26):\quad \mu(0, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.27):\quad \mu(0, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.28):\quad \mu(0, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.29):\quad \mu(0, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_3)$$
$$\Rightarrow\; 0 = \mu(C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3, e_3),$$

$$(3.30):\quad \mu(0, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, 0) \;\Rightarrow\; 0 = 0,$$

$$(3.31):\quad \mu(0, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_3)$$
$$\Rightarrow\; 0 = \mu(C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3, e_3),$$

$$(3.32):\quad \mu(e_3, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0)$$
$$\Rightarrow\; \mu(e_3, C_{11}^1 e_1 + C_{11}^2 e_2 + C_{11}^3 e_3) = 0,$$

$$(3.33):\quad \mu(e_3, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, 0)$$
$$\Rightarrow\; \mu(e_3, C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3) = 0,$$

$$(3.34):\quad \mu(e_3, C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3) = \mu(C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3, e_3),$$

$$(3.35):\quad \mu(e_3, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0)$$
$$\Rightarrow\; \mu(e_3, C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3) = 0,$$

$$(3.36):\quad \mu(e_3, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0)$$
$$\Rightarrow\; \mu(e_3, C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3) = 0,$$

$$(3.37):\quad \mu(e_3, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, 0)$$
$$\Rightarrow\; \mu(e_3, C_{22}^1 e_1 + C_{22}^2 e_2 + C_{22}^3 e_3) = 0,$$

$$(3.38):\quad \mu(e_3, C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3) = \mu(C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3, e_3),$$

$$(3.39):\quad \mu(e_3, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, 0)$$
$$\Rightarrow\; \mu(e_3, C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3) = 0,$$

$$(3.40):\quad \mu(e_3, C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3) = \mu(C_{33}^1 e_1 + C_{33}^2 e_2 + C_{33}^3 e_3, e_3).$$

We remember the following formulas that we calculated earlier:

$$\mu(e_i, C^1_{jk}e_1 + C^2_{jk}e_2 + C^3_{jk}e_3) = (C^1_{jk}C^1_{i1} + C^2_{jk}C^1_{i2} + C^3_{jk}C^1_{i3})e_1 + (C^1_{jk}C^2_{i1} + C^2_{jk}C^2_{i2} + C^3_{jk}C^2_{i3})e_2 + (C^1_{jk}C^3_{i1} + C^2_{jk}C^3_{i2} + C^3_{jk}C^3_{i3})e_3$$

and

$$\mu(C^1_{jk}e_1 + C^2_{jk}e_2 + C^3_{jk}e_3, e_i) = (C^1_{jk}C^1_{1i} + C^2_{jk}C^1_{2i} + C^3_{jk}C^1_{3i})e_1 + (C^1_{jk}C^2_{1i} + C^2_{jk}C^2_{2i} + C^3_{jk}C^2_{3i})e_2 + (C^1_{jk}C^3_{1i} + C^2_{jk}C^3_{2i} + C^3_{jk}C^3_{3i})e_3.$$

We use these formulas to rewrite our equations.

$$(3.16): \quad 0 = (C^1_{11}C^1_{13} + C^2_{11}C^1_{23} + C^3_{11}C^1_{33})e_1 + (C^1_{11}C^2_{13} + C^2_{11}C^2_{23} + C^3_{11}C^2_{33})e_2 + (C^1_{11}C^3_{13} + C^2_{11}C^3_{23} + C^3_{11}C^3_{33})e_3,$$

$$(3.20): \quad 0 = (C^1_{12}C^1_{13} + C^2_{12}C^1_{23} + C^3_{12}C^1_{33})e_1 + (C^1_{12}C^2_{13} + C^2_{12}C^2_{23} + C^3_{12}C^2_{33})e_2 + (C^1_{12}C^3_{13} + C^2_{12}C^3_{23} + C^3_{12}C^3_{33})e_3,$$

$$(3.22): \quad 0 = (C^1_{13}C^1_{13} + C^2_{13}C^1_{23} + C^3_{13}C^1_{33})e_1 + (C^1_{13}C^2_{13} + C^2_{13}C^2_{23} + C^3_{13}C^2_{33})e_2 + (C^1_{13}C^3_{13} + C^2_{13}C^3_{23} + C^3_{13}C^3_{33})e_3,$$

$$(3.25): \quad 0 = (C^1_{21}C^1_{13} + C^2_{21}C^1_{23} + C^3_{21}C^1_{33})e_1 + (C^1_{21}C^2_{13} + C^2_{21}C^2_{23} + C^3_{21}C^2_{33})e_2 + (C^1_{21}C^3_{13} + C^2_{21}C^3_{23} + C^3_{21}C^3_{33})e_3,$$

$$(3.29): \quad 0 = (C^1_{22}C^1_{13} + C^2_{22}C^1_{23} + C^3_{22}C^1_{33})e_1 + (C^1_{22}C^2_{13} + C^2_{22}C^2_{23} + C^3_{22}C^2_{33})e_2 + (C^1_{22}C^3_{13} + C^2_{22}C^3_{23} + C^3_{22}C^3_{33})e_3,$$

$$(3.31): \quad 0 = (C^1_{23}C^1_{13} + C^2_{23}C^1_{23} + C^3_{23}C^1_{33})e_1 + (C^1_{23}C^2_{13} + C^2_{23}C^2_{23} + C^3_{23}C^2_{33})e_2 + (C^1_{23}C^3_{13} + C^2_{23}C^3_{23} + C^3_{23}C^3_{33})e_3,$$

$$(3.32): \quad (C^1_{11}C^1_{31} + C^2_{11}C^1_{32} + C^3_{11}C^1_{33})e_1 + (C^1_{11}C^2_{31} + C^2_{11}C^2_{32} + C^3_{11}C^2_{33})e_2 + (C^1_{11}C^3_{31} + C^2_{11}C^3_{32} + C^3_{11}C^3_{33})e_3 = 0,$$

$$(3.33): \quad (C^1_{12}C^1_{31} + C^2_{12}C^1_{32} + C^3_{12}C^1_{33})e_1 + (C^1_{12}C^2_{31} + C^2_{12}C^2_{32} + C^3_{12}C^2_{33})e_2 + (C^1_{12}C^3_{31} + C^2_{12}C^3_{32} + C^3_{12}C^3_{33})e_3 = 0,$$

$$\begin{aligned}(3.34): \quad & (C^1_{13}C^1_{31} + C^2_{13}C^1_{32} + C^3_{13}C^1_{33})e_1 + (C^1_{13}C^2_{31} + C^2_{13}C^2_{32} + C^3_{13}C^2_{33})e_2 \\ & + (C^1_{13}C^3_{31} + C^2_{13}C^3_{32} + C^3_{13}C^3_{33})e_3 \\ & = (C^1_{31}C^1_{13} + C^2_{31}C^1_{23} + C^3_{31}C^1_{33})e_1 + (C^1_{31}C^2_{13} + C^2_{31}C^2_{23} + C^3_{31}C^2_{33})e_2 \\ & \quad + (C^1_{31}C^3_{13} + C^2_{31}C^3_{23} + C^3_{31}C^3_{33})e_3 \\ & \Rightarrow \quad (C^2_{13}C^1_{32} + C^3_{13}C^1_{33} - C^2_{31}C^1_{23} - C^3_{31}C^1_{33})e_1 \\ & \qquad + (C^1_{13}C^2_{31} + C^2_{13}C^2_{32} + C^3_{13}C^2_{33} - C^1_{31}C^2_{13} - C^2_{31}C^2_{23} - C^3_{31}C^2_{33})e_2 \\ & \qquad + (C^1_{13}C^3_{31} + C^2_{13}C^3_{32} + C^3_{13}C^3_{33} - C^1_{31}C^3_{13} - C^2_{31}C^3_{23} - C^3_{31}C^3_{33})e_3 = 0,\end{aligned}$$

$$(3.35): \quad (C^1_{21}C^1_{31} + C^2_{21}C^1_{32} + C^3_{21}C^1_{33})e_1 + (C^1_{21}C^2_{31} + C^2_{21}C^2_{32} + C^3_{21}C^2_{33})e_2 + (C^1_{21}C^3_{31} + C^2_{21}C^3_{32} + C^3_{21}C^3_{33})e_3 = 0,$$

$$
\begin{aligned}
(3.36): \quad & (C_{31}^1 C_{31}^1 + C_{31}^2 C_{32}^1 + C_{31}^3 C_{33}^1)e_1 + (C_{31}^1 C_{31}^2 + C_{31}^2 C_{32}^2 + C_{31}^3 C_{33}^2)e_2 \\
& + (C_{31}^1 C_{31}^3 + C_{31}^2 C_{32}^3 + C_{31}^3 C_{33}^3)e_3 = 0, \\
(3.37): \quad & (C_{22}^1 C_{31}^1 + C_{22}^2 C_{32}^1 + C_{22}^3 C_{33}^1)e_1 + (C_{22}^1 C_{31}^2 + C_{22}^2 C_{32}^2 + C_{22}^3 C_{33}^2)e_2 \\
& + (C_{22}^1 C_{31}^3 + C_{22}^2 C_{32}^3 + C_{22}^3 C_{33}^3)e_3 = 0, \\
(3.38): \quad & (C_{23}^1 C_{31}^1 + C_{23}^2 C_{32}^1 + C_{23}^3 C_{33}^1)e_1 + (C_{23}^1 C_{31}^2 + C_{23}^2 C_{32}^2 + C_{23}^3 C_{33}^2)e_2 \\
& + (C_{23}^1 C_{31}^3 + C_{23}^2 C_{32}^3 + C_{23}^3 C_{33}^3)e_3 \\
& = (C_{32}^1 C_{13}^1 + C_{32}^2 C_{23}^1 + C_{32}^3 C_{33}^1)e_1 + (C_{32}^1 C_{13}^2 + C_{32}^2 C_{23}^2 + C_{32}^3 C_{33}^2)e_2 \\
& \quad + (C_{32}^1 C_{13}^3 + C_{32}^2 C_{23}^3 + C_{32}^3 C_{33}^3)e_3 \\
& \Rightarrow \quad (C_{23}^1 C_{31}^1 + C_{23}^2 C_{32}^1 + C_{23}^3 C_{33}^1 - C_{32}^1 C_{13}^1 - C_{32}^2 C_{23}^1 - C_{32}^3 C_{33}^1)e_1 \\
& \qquad + (C_{23}^1 C_{31}^2 + C_{23}^3 C_{33}^2 - C_{32}^1 C_{13}^2 - C_{32}^3 C_{33}^2)e_2 \\
& \qquad + (C_{23}^1 C_{31}^3 + C_{23}^2 C_{32}^3 + C_{23}^3 C_{33}^3 - C_{32}^1 C_{13}^3 - C_{32}^2 C_{23}^3 - C_{32}^3 C_{33}^3)e_3 = 0, \\
(3.39): \quad & (C_{32}^1 C_{31}^1 + C_{32}^2 C_{32}^1 + C_{32}^3 C_{33}^1)e_1 + (C_{32}^1 C_{31}^2 + C_{32}^2 C_{32}^2 + C_{32}^3 C_{33}^2)e_2 \\
& + (C_{32}^1 C_{31}^3 + C_{32}^2 C_{32}^3 + C_{32}^3 C_{33}^3)e_3 = 0, \\
(3.40): \quad & (C_{33}^1 C_{31}^1 + C_{33}^2 C_{32}^1 + C_{33}^3 C_{33}^1)e_1 + (C_{33}^1 C_{31}^2 + C_{33}^2 C_{32}^2 + C_{33}^3 C_{33}^2)e_2 \\
& + (C_{33}^1 C_{31}^3 + C_{33}^2 C_{32}^3 + C_{33}^3 C_{33}^3)e_3 \\
& = (C_{33}^1 C_{13}^1 + C_{33}^2 C_{23}^1 + C_{33}^3 C_{33}^1)e_1 + (C_{33}^1 C_{13}^2 + C_{33}^2 C_{23}^2 + C_{33}^3 C_{33}^2)e_2 \\
& \quad + (C_{33}^1 C_{13}^3 + C_{33}^2 C_{23}^3 + C_{33}^3 C_{33}^3)e_3 \\
& \Rightarrow \quad (C_{33}^1 C_{31}^1 + C_{33}^2 C_{32}^1 - C_{33}^1 C_{13}^1 - C_{33}^2 C_{23}^1)e_1 + (C_{33}^1 C_{31}^2 + C_{33}^2 C_{32}^2 - C_{33}^1 C_{13}^2 - C_{33}^2 C_{23}^2)e_2 \\
& \qquad + (C_{33}^1 C_{31}^3 + C_{33}^2 C_{32}^3 - C_{33}^1 C_{13}^3 - C_{33}^2 C_{23}^3)e_3 = 0.
\end{aligned}
$$

Remembering that $k_1e_1 + k_2e_2 + k_3e_3 = 0$ will only have the trivial solution $k_1 = k_2 = k_3 = 0$, we can rewrite our equations as the following system of equations:

$$C_{11}^1 C_{13}^1 + C_{11}^2 C_{23}^1 + C_{11}^3 C_{33}^1 = 0 \qquad \text{(R3.16.1)}$$
$$C_{11}^1 C_{13}^2 + C_{11}^2 C_{23}^2 + C_{11}^3 C_{33}^2 = 0 \qquad \text{(R3.16.2)}$$
$$C_{11}^1 C_{13}^3 + C_{11}^2 C_{23}^3 + C_{11}^3 C_{33}^3 = 0 \qquad \text{(R3.16.3)}$$
$$C_{12}^1 C_{13}^1 + C_{12}^2 C_{23}^1 + C_{12}^3 C_{33}^1 = 0 \qquad \text{(R3.20.1)}$$
$$C_{12}^1 C_{13}^2 + C_{12}^2 C_{23}^2 + C_{12}^3 C_{33}^2 = 0 \qquad \text{(R3.20.2)}$$
$$C_{12}^1 C_{13}^3 + C_{12}^2 C_{23}^3 + C_{12}^3 C_{33}^3 = 0 \qquad \text{(R3.20.3)}$$
$$C_{13}^1 C_{13}^1 + C_{13}^2 C_{23}^1 + C_{13}^3 C_{33}^1 = 0 \qquad \text{(R3.22.1)}$$
$$C_{13}^1 C_{13}^2 + C_{13}^2 C_{23}^2 + C_{13}^3 C_{33}^2 = 0 \qquad \text{(R3.22.2)}$$
$$C_{13}^1 C_{13}^3 + C_{13}^2 C_{23}^3 + C_{13}^3 C_{33}^3 = 0 \qquad \text{(R3.22.3)}$$
$$C_{21}^1 C_{13}^1 + C_{21}^2 C_{23}^1 + C_{21}^3 C_{33}^1 = 0 \qquad \text{(R3.25.1)}$$
$$C_{21}^1 C_{13}^2 + C_{21}^2 C_{23}^2 + C_{21}^3 C_{33}^2 = 0 \qquad \text{(R3.25.2)}$$
$$C_{21}^1 C_{13}^3 + C_{21}^2 C_{23}^3 + C_{21}^3 C_{33}^3 = 0 \qquad \text{(R3.25.3)}$$

$$C^1_{22}C^1_{13} + C^2_{22}C^1_{23} + C^3_{22}C^1_{33} = 0 \qquad \text{(R3.29.1)}$$

$$C^1_{22}C^2_{13} + C^2_{22}C^2_{23} + C^3_{22}C^2_{33} = 0 \qquad \text{(R3.29.2)}$$

$$C^1_{22}C^3_{13} + C^2_{22}C^3_{23} + C^3_{22}C^3_{33} = 0 \qquad \text{(R3.29.3)}$$

$$C^1_{23}C^1_{13} + C^2_{23}C^1_{23} + C^3_{23}C^1_{33} = 0 \qquad \text{(R3.31.1)}$$

$$C^1_{23}C^2_{13} + C^2_{23}C^2_{23} + C^3_{23}C^2_{33} = 0 \qquad \text{(R3.31.2)}$$

$$C^1_{23}C^3_{13} + C^2_{23}C^3_{23} + C^3_{23}C^3_{33} = 0 \qquad \text{(R3.31.3)}$$

$$C^1_{11}C^1_{31} + C^2_{11}C^1_{32} + C^3_{11}C^1_{33} = 0 \qquad \text{(R3.32.1)}$$

$$C^1_{11}C^2_{31} + C^2_{11}C^2_{32} + C^3_{11}C^2_{33} = 0 \qquad \text{(R3.32.2)}$$

$$C^1_{11}C^3_{31} + C^2_{11}C^3_{32} + C^3_{11}C^3_{33} = 0 \qquad \text{(R3.32.3)}$$

$$C^1_{12}C^1_{31} + C^2_{12}C^1_{32} + C^3_{12}C^1_{33} = 0 \qquad \text{(R3.33.1)}$$

$$C^1_{12}C^2_{31} + C^2_{12}C^2_{32} + C^3_{12}C^2_{33} = 0 \qquad \text{(R3.33.2)}$$

$$C^1_{12}C^3_{31} + C^2_{12}C^3_{32} + C^3_{12}C^3_{33} = 0 \qquad \text{(R3.33.3)}$$

$$C^2_{13}C^1_{32} + C^3_{13}C^1_{33} - C^2_{31}C^1_{23} - C^3_{31}C^1_{33} = 0 \qquad \text{(R3.34.1)}$$

$$C^1_{13}C^2_{31} + C^2_{13}C^2_{32} + C^3_{13}C^2_{33} - C^1_{31}C^2_{13} - C^2_{31}C^2_{23} - C^3_{31}C^2_{33} = 0 \qquad \text{(R3.34.2)}$$

$$C^1_{13}C^3_{31} + C^2_{13}C^3_{32} + C^3_{13}C^3_{33} - C^1_{31}C^3_{13} - C^2_{31}C^3_{23} - C^3_{31}C^3_{33} = 0 \qquad \text{(R3.34.3)}$$

$$C^1_{21}C^1_{31} + C^2_{21}C^1_{32} + C^3_{21}C^1_{33} = 0 \qquad \text{(R3.35.1)}$$

$$C^1_{21}C^2_{31} + C^2_{21}C^2_{32} + C^3_{21}C^2_{33} = 0 \qquad \text{(R3.35.2)}$$

$$C^1_{21}C^3_{31} + C^2_{21}C^3_{32} + C^3_{21}C^3_{33} = 0 \qquad \text{(R3.35.3)}$$

$$C^1_{31}C^1_{31} + C^2_{31}C^1_{32} + C^3_{31}C^1_{33} = 0 \qquad \text{(R3.36.1)}$$

$$C^1_{31}C^2_{31} + C^2_{31}C^2_{32} + C^3_{31}C^2_{33} = 0 \qquad \text{(R3.36.2)}$$

$$C^1_{31}C^3_{31} + C^2_{31}C^3_{32} + C^3_{31}C^3_{33} = 0 \qquad \text{(R3.36.3)}$$

$$C^1_{22}C^1_{31} + C^2_{22}C^1_{32} + C^3_{22}C^1_{33} = 0 \qquad \text{(R3.37.1)}$$

$$C^1_{22}C^2_{31} + C^2_{22}C^2_{32} + C^3_{22}C^2_{33} = 0 \qquad \text{(R3.37.2)}$$

$$C^1_{22}C^3_{31} + C^2_{22}C^3_{32} + C^3_{22}C^3_{33} = 0 \qquad \text{(R3.37.3)}$$

$$C^1_{23}C^1_{31} + C^2_{23}C^1_{32} + C^3_{23}C^1_{33} - C^1_{32}C^1_{13} - C^2_{32}C^1_{23} - C^3_{32}C^1_{33} = 0 \qquad \text{(R3.38.1)}$$

$$C^1_{23}C^2_{31} + C^3_{23}C^2_{33} - C^1_{32}C^2_{13} - C^3_{32}C^2_{33} = 0 \qquad \text{(R3.38.2)}$$

$$C^1_{23}C^3_{31} + C^2_{23}C^3_{32} + C^3_{23}C^3_{33} - C^1_{32}C^3_{13} - C^2_{32}C^3_{23} - C^3_{32}C^3_{33} = 0 \qquad \text{(R3.38.3)}$$

$$C^1_{32}C^1_{31} + C^2_{32}C^1_{32} + C^3_{32}C^1_{33} = 0 \qquad \text{(R3.39.1)}$$

$$C^1_{32}C^2_{31} + C^2_{32}C^2_{32} + C^3_{32}C^2_{33} = 0 \qquad \text{(R3.39.2)}$$

$$C^1_{32}C^3_{31} + C^2_{32}C^3_{32} + C^3_{32}C^3_{33} = 0 \qquad \text{(R3.39.3)}$$

$$C^1_{33}C^1_{31} + C^2_{33}C^1_{32} - C^1_{33}C^1_{13} - C^2_{33}C^1_{23} = 0 \qquad \text{(R3.40.1)}$$

$$C^1_{33}C^2_{31} + C^2_{33}C^2_{32} - C^1_{33}C^2_{13} - C^2_{33}C^2_{23} = 0 \qquad \text{(R3.40.2)}$$

$$C^1_{33}C^3_{31} + C^2_{33}C^3_{32} - C^1_{33}C^3_{13} - C^2_{33}C^3_{23} = 0. \qquad \text{(R3.40.3)}$$

3.2.10 Comparison

When we look at the system of equations we get when α is defined as E_{11}, E_{22} or E_{33}, we see that they have a very clear connection. In fact, if we look at the system for E_{11}, and for each index of C_{ij}^k we change 1 to 2, change 2 to 3, and change 3 to 1, we get the system of equations for E_{22}. That is, we perform the permutation that in cyclic notation can be written as $\sigma_3 = (1, 2, 3)$ on each of the indices. If we perform the same permutation on all of the indices in the system of equations for E_{22}, we get the system of equations for E_{33}. This means that by just finding the solutions for one of these systems of equations, it is very easy to use these solutions to also acquire the solutions for both of the other systems as well. For example, if we solve the system for E_{11}, the specific values of constant C_{12}^3 in each solution will simply be transferred to constant C_{23}^1 in the solutions for the system of equations for E_{22}, and constant C_{31}^2 in the solutions for the system of equations for E_{33}. Note that the indices in the actual solutions have to be permutated as well, so if say $C_{12}^3 = C_{21}^2$ then after permutating we get $C_{23}^1 = C_{32}^3$.

Looking at the system of equations we get when α is defined as E_{12}, we see that if we perform this same permutation of the indices on each of the structure constants, we actually get the system of equations for E_{23}, and if we permutate the indices again we get the system of equations for E_{31}. If we instead look at the system of equations we get when α is defined as E_{13}, we see that if we permutate the indices once we get the system of equations for E_{21}, and if we permutate the indices twice we get the system of equations for E_{32}. We denote the system of equations given when α is defined as E_{ij} as A_{ij}, and define σ_3 such that $\sigma_3(A_{ij}) = A_{\sigma_3(i),\sigma_3(j)}$, and we can then sum up our results as follows:

$$\begin{aligned}
\sigma_3(\sigma_3(A_{11})) &= \sigma_3(A_{22}) = A_{33},\\
\sigma_3(\sigma_3(A_{12})) &= \sigma_3(A_{23}) = A_{31},\\
\sigma_3(\sigma_3(A_{13})) &= \sigma_3(A_{21}) = A_{32}.
\end{aligned}$$

Thus, if we for example perform the permutation on the system of equations A_{23}, it is easy to find which system of equations it will turn into, because we $\sigma_3(A_{23}) = A_{\sigma_3(2),\sigma_3(3)} = A_{31}$. Furthermore, we remember that E_{ij} is the matrix a one in place ij of the matrix, and the rest of the matrix consists of zeroes. This means that E_{11}, E_{22} and E_{33} denote the different matrix units with a one on the main diagonal. If we then look at the second case, with E_{12}, E_{23} and E_{31}, we see that these three matrix units have their ones on a kind of diagonal of their own, with E_{12} and E_{23} being on the superdiagonal of the matrix, and if you imagine the continuation of that diagonal then you must go down in the matrix to E_{31}. In the same way, E_{13}, E_{21} and E_{32} have ones on a kind of diagonal. Of course, this is due to the fact that the E_{ij}:s in each of these groups being all possible permutations of ij with σ_3.

If we instead look at two α:s with ones on different such, there is no way to get the solutions for one of them simply by knowing the solutions to the other. This is because while we can of course transform one of the systems of equations into the other by changing the indices of the structure constants in a certain way, we do not change all of the structure constants in the same way. For example, when α is defined as E_{11} and we want to get α defined as E_{13}, in every term $C_{ij}^k C_{rs}^t$, if i or j is equal to 1 it will be changed to 3, and if it is equal to 3 it will be changed to 1. On the other hand, k, r, s and t will remain unchanged. Clearly, while there is a connection, it is not one we can use to our advantage.

Now, we are interested in knowing *why* we get this connection between the E_{ij} with ones on the three diagonals, and why there is no connection between the ones with ones on different diagonals. To figure this out, we go back to the beginning of Section 3.2 and look at how we created the systems of equations.

For each α, the equations in the original system of equations that will be $0 = 0$ are the ones where $\mu(\alpha(e_i), \mu(e_j, e_k)) = \mu(\mu(e_i, e_j), \alpha(e_k))$ has $\alpha(e_i) = \alpha(e_k) = 0$, where $i, j, k = \{1, 2, 3\}$. This will be true for two values of i and two values of k. Since j can be any of the three values, the number of equations for which this is true will be $2 \cdot 3 \cdot 2 = 12$. Since we have $3^3 = 27$ total number of equations, this means we will end up with a system of equations with 15 equations, which we can verify that we indeed did for every different α. Note that once we started solving the system of equations, we turned each equation $k_1e_1 + k_2e_2 + k_3e_3 = 0$ into equations $k_1 = 0$, $k_2 = 0$ and $k_3 = 0$, so we got a system with $15 \cdot 3 = 45$ equations. However, at the moment we are only interested in the original system of equations. Before we move on though, we can also note that if would look at n dimensions instead of three dimensions, our system of equations would have $n^3 - (n-1)n(n-1) = n^3 - (n^2 - n)(n-1) = n^3 - n^3 + n^2 + n^2 - n = n(2n-1)$ equations, or, once we start solving and get rid of the $e_1, e_2, \ldots, e_n$, even $n(2n-1) \cdot n = n^2(2n-1)$ equations. Clearly, we would be dealing with very large system of equations.

Now, most equations will have either the left or right hand side equal to zero, but the ones that end up "keeping" the equations on both sides are the ones for which $\alpha(e_i) \neq 0$ and $\alpha(e_k) \neq 0$. Thus we must have $i = k$, and they must be equal to the one value that keeps it from being zero, which means there is only one choice for i, one choice for k, and since there are no constraints on j there are three different choices for j. Consequently, there will only be $1 \cdot 1 \cdot 3 = 3$ equations that "keep" the equations on both sides, and end up becoming the longer equations in the system of equations.

To try to see the connections between the different E_{ij}, we begin by looking at the equations we discussed above, that "keep" the equations on both sides. We remember the formulas below:

$$\begin{aligned}
\alpha(e_1) &= a_{11}e_1 + a_{21}e_2 + a_{31}e_3, \\
\alpha(e_2) &= a_{12}e_1 + a_{22}e_2 + a_{32}e_3, \\
\alpha(e_3) &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3.
\end{aligned}$$

If we begin looking at E_{11}, we will have $\alpha(e_1) = e_1$, $\alpha(e_2) = 0$ and $\alpha(e_3) = 0$. Thus, the equations that will "keep" both sides are the ones that have $i = k = 1$.

$$\begin{aligned}
\mu(\alpha(e_1), \mu(e_1, e_1)) = \mu(\mu(e_1, e_1), \alpha(e_1)) \quad &\Rightarrow \quad \mu(e_1, \mu(e_1, e_1)) = \mu(\mu(e_1, e_1), e_1), \\
\mu(\alpha(e_1), \mu(e_2, e_1)) = \mu(\mu(e_1, e_2), \alpha(e_1)) \quad &\Rightarrow \quad \mu(e_1, \mu(e_2, e_1)) = \mu(\mu(e_1, e_2), e_1), \\
\mu(\alpha(e_1), \mu(e_3, e_1)) = \mu(\mu(e_1, e_3), \alpha(e_1)) \quad &\Rightarrow \quad \mu(e_1, \mu(e_3, e_1)) = \mu(\mu(e_1, e_3), e_1).
\end{aligned}$$

Since the only thing that changes between these three equations is the e_j, we will let $t = \{1, 2, 3\}$, so that for all such t

$$\mu(\alpha(e_1), \mu(e_t, e_1)) = \mu(\mu(e_1, e_t), \alpha(e_1)) \quad \Rightarrow \quad \mu(e_1, \mu(e_t, e_1)) = \mu(\mu(e_1, e_t), e_1)$$

means the same thing as the three equations above. We now do the same with all other E_{ij}, and get the following for all $t = \{1,2,3\}$:

$$\begin{aligned}
E_{11}: &\ \mu(\alpha(e_1),\mu(e_t,e_1)) = \mu(\mu(e_1,e_t),\alpha(e_1)) &\Rightarrow&\ \mu(e_1,\mu(e_t,e_1)) = \mu(\mu(e_1,e_t),e_1),\\
E_{12}: &\ \mu(\alpha(e_2),\mu(e_t,e_2)) = \mu(\mu(e_2,e_t),\alpha(e_2)) &\Rightarrow&\ \mu(e_1,\mu(e_t,e_2)) = \mu(\mu(e_2,e_t),e_1),\\
E_{13}: &\ \mu(\alpha(e_3),\mu(e_t,e_3)) = \mu(\mu(e_3,e_t),\alpha(e_3)) &\Rightarrow&\ \mu(e_1,\mu(e_t,e_3)) = \mu(\mu(e_3,e_t),e_1),\\
E_{21}: &\ \mu(\alpha(e_1),\mu(e_t,e_1)) = \mu(\mu(e_1,e_t),\alpha(e_1)) &\Rightarrow&\ \mu(e_2,\mu(e_t,e_1)) = \mu(\mu(e_1,e_t),e_2),\\
E_{22}: &\ \mu(\alpha(e_2),\mu(e_t,e_2)) = \mu(\mu(e_2,e_t),\alpha(e_2)) &\Rightarrow&\ \mu(e_2,\mu(e_t,e_2)) = \mu(\mu(e_2,e_t),e_2),\\
E_{23}: &\ \mu(\alpha(e_3),\mu(e_t,e_3)) = \mu(\mu(e_3,e_t),\alpha(e_3)) &\Rightarrow&\ \mu(e_2,\mu(e_t,e_3)) = \mu(\mu(e_3,e_t),e_2),\\
E_{31}: &\ \mu(\alpha(e_1),\mu(e_t,e_1)) = \mu(\mu(e_1,e_t),\alpha(e_1)) &\Rightarrow&\ \mu(e_3,\mu(e_t,e_1)) = \mu(\mu(e_1,e_t),e_3),\\
E_{32}: &\ \mu(\alpha(e_2),\mu(e_t,e_2)) = \mu(\mu(e_2,e_t),\alpha(e_2)) &\Rightarrow&\ \mu(e_3,\mu(e_t,e_2)) = \mu(\mu(e_2,e_t),e_3),\\
E_{33}: &\ \mu(\alpha(e_3),\mu(e_t,e_3)) = \mu(\mu(e_3,e_t),\alpha(e_3)) &\Rightarrow&\ \mu(e_3,\mu(e_t,e_3)) = \mu(\mu(e_3,e_t),e_3),
\end{aligned}$$

Here, if we look at the equations after we have rewritten them to remove the α, it is easy to see that if we perform our permutation σ_3 on the equations for E_{11} we get the equations for E_{22}, and if we perform the permutation again we get the equations for E_{33}. In the same way, if we perform the permutation on the equations for E_{12} we get first the equations for E_{23} and then for E_{31}, and if we perform the permutation on the equations for E_{13} we get first the equations for E_{21} and then for E_{32}. This is consistent with the results we got earlier, where S_{ij} is the system of equations you get when α is defined as E_{ij}, and $\sigma_3(A_{ij})$ is the system of equations you get when α is defined as $\sigma_3(E_{ij})$.

Now, we would like to generalise this to any other dimension. We look at dimension n and let α be defined as the n-dimensional E_{ab}. Looking at the definition of $\alpha(e_i)$, we see that for α defined as E_{ab} we will always have $\alpha(e_b) = e_a$, while α applied to every other base vector will be zero. This means that we will get

$$E_{ab}:\ \mu(\alpha(e_b),\mu(e_t,e_b)) = \mu(\mu(e_b,e_t),\alpha(e_b)) \ \Rightarrow\ \mu(e_a,\mu(e_t,e_b)) = \mu(\mu(e_b,e_t),e_a),$$

for all $t = \{1,2,3\}$. We will get back to these equations later, but for now we move on to the other equations in the system of equations; the ones that instead have either a zero left hand side or a zero right hand side.

We begin with the three-dimensional ones that have a zero left hand side. Since we have $\mu(\alpha(e_i),\mu(e_j,e_k)) = \mu(\mu(e_i,e_j),\alpha(e_k))$, that means that we must have $\alpha(e_i) = 0$ and $\alpha(e_k) \neq 0$. This gives two possible values of i and one possible value of k for each E_{ij}. If we look at it the same way as before, we get the following:

$$\begin{aligned}
E_{11}: &\ \mu(\alpha(e_2),\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),\alpha(e_1)) &\Rightarrow&\ \mu(0,\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),e_1),\\
&\ \mu(\alpha(e_3),\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),\alpha(e_1)) &\Rightarrow&\ \mu(0,\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),e_1),\\
E_{12}: &\ \mu(\alpha(e_1),\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),\alpha(e_2)) &\Rightarrow&\ \mu(0,\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),e_1),\\
&\ \mu(\alpha(e_3),\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),\alpha(e_2)) &\Rightarrow&\ \mu(0,\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),e_1),\\
E_{13}: &\ \mu(\alpha(e_1),\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),\alpha(e_3)) &\Rightarrow&\ \mu(0,\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),e_1),\\
&\ \mu(\alpha(e_2),\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),\alpha(e_3)) &\Rightarrow&\ \mu(0,\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),e_1),
\end{aligned}$$

$$\begin{aligned}
E_{21}: \ & \mu(\alpha(e_2),\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),\alpha(e_1)) && \Rightarrow && \mu(0,\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),e_2), \\
& \mu(\alpha(e_3),\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),\alpha(e_1)) && \Rightarrow && \mu(0,\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),e_2), \\
E_{22}: \ & \mu(\alpha(e_1),\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),\alpha(e_2)) && \Rightarrow && \mu(0,\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),e_2), \\
& \mu(\alpha(e_3),\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),\alpha(e_2)) && \Rightarrow && \mu(0,\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),e_2), \\
E_{23}: \ & \mu(\alpha(e_1),\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),\alpha(e_3)) && \Rightarrow && \mu(0,\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),e_2), \\
& \mu(\alpha(e_2),\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),\alpha(e_3)) && \Rightarrow && \mu(0,\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),e_2), \\
E_{31}: \ & \mu(\alpha(e_2),\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),\alpha(e_1)) && \Rightarrow && \mu(0,\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),e_3), \\
& \mu(\alpha(e_3),\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),\alpha(e_1)) && \Rightarrow && \mu(0,\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),e_3), \\
E_{32}: \ & \mu(\alpha(e_1),\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),\alpha(e_2)) && \Rightarrow && \mu(0,\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),e_3), \\
& \mu(\alpha(e_3),\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),\alpha(e_2)) && \Rightarrow && \mu(0,\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),e_3), \\
E_{33}: \ & \mu(\alpha(e_1),\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),\alpha(e_3)) && \Rightarrow && \mu(0,\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),e_3), \\
& \mu(\alpha(e_2),\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),\alpha(e_3)) && \Rightarrow && \mu(0,\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),e_3),
\end{aligned}$$

for all $t = \{1,2,3\}$. Remember that $\mu(0,\mu(e_i,e_j)) = 0$. Now, we look at E_{11}. The equations with a zero left hand side, when α is defined as E_{11}, will be $0 = \mu(\mu(e_2,e_t),e_1)$ and $0 = \mu(\mu(e_3,e_t),e_1)$. If we permutate all of the indices in these equations, we get $0 = \mu(\mu(e_3,e_t),e_2)$ and $0 = \mu(\mu(e_1,e_t),e_2)$. Looking at our results above, we see that these are exactly the equations with a zero left hand side when α is defined as E_{22}. Looking through our list in this way, we see that we get the exact same results as before; if we permutate the indices in the equations for E_{11} once we get the equations for E_{22}, and if we permutate them twice we get the equations for E_{33}, and in the same way if we permutate the indices in the equations for E_{12} once and then twice we get first the equations for E_{23} and then for E_{31}, and finally if we permutate the indices in the equations for E_{13} once and then twice we get first the equations for E_{21} and then E_{32}.

To generalise this to any dimension, we will again look at dimension n and let α be defined as the n-dimensional E_{ab}. We remember that when α is defined as E_{ab}, then $\alpha(e_b) = e_a$. Thus, to get the equations with a zero left hand side, we must have $k = b$ and $i \neq b$. Thus, for all $t = \{1,2,\ldots,n\}$ and $s_b = \{1,2,\ldots,b-1,b+1,\ldots,n-1,n\}$ we get

$$E_{ab}: \ \mu(\alpha(e_{s_b}),\mu(e_t,e_b)) = \mu(\mu(e_{s_b},e_t),\alpha(e_b)) \ \Rightarrow \ \mu(0,\mu(e_t,e_b)) = \mu(\mu(e_{s_b},e_t),e_a).$$

Now all that is left to look at are the equations with a zero right hand side in three dimensions. Again, since we have $\mu(\alpha(e_i),\mu(e_j,e_k)) = \mu(\mu(e_i,e_j),\alpha(e_k))$, this means that we must have $\alpha(e_i) \neq 0$ and $\alpha(e_k) = 0$. This gives us only one possible value of i and two possible values of k for each different α. Looking at each E_{ij} in turn, for all $t = \{1,2,3\}$ we get:

$$\begin{aligned}
E_{11}: \ & \mu(\alpha(e_1),\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),\alpha(e_2)) && \Rightarrow && \mu(e_1,\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),0), \\
& \mu(\alpha(e_1),\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),\alpha(e_3)) && \Rightarrow && \mu(e_1,\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),0), \\
E_{12}: \ & \mu(\alpha(e_2),\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),\alpha(e_1)) && \Rightarrow && \mu(e_1,\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),0), \\
& \mu(\alpha(e_2),\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),\alpha(e_3)) && \Rightarrow && \mu(e_1,\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),0), \\
E_{13}: \ & \mu(\alpha(e_3),\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),\alpha(e_1)) && \Rightarrow && \mu(e_1,\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),0), \\
& \mu(\alpha(e_3),\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),\alpha(e_2)) && \Rightarrow && \mu(e_1,\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),0),
\end{aligned}$$

$$
\begin{aligned}
E_{21}: \ & \mu(\alpha(e_1),\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),\alpha(e_2)) &\Rightarrow& \ \mu(e_2,\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),0),\\
& \mu(\alpha(e_1),\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),\alpha(e_3)) &\Rightarrow& \ \mu(e_2,\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),0),\\
E_{22}: \ & \mu(\alpha(e_2),\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),\alpha(e_1)) &\Rightarrow& \ \mu(e_2,\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),0),\\
& \mu(\alpha(e_2),\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),\alpha(e_3)) &\Rightarrow& \ \mu(e_2,\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),0),\\
E_{23}: \ & \mu(\alpha(e_3),\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),\alpha(e_1)) &\Rightarrow& \ \mu(e_2,\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),0),\\
& \mu(\alpha(e_3),\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),\alpha(e_2)) &\Rightarrow& \ \mu(e_2,\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),0),\\
E_{31}: \ & \mu(\alpha(e_1),\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),\alpha(e_2)) &\Rightarrow& \ \mu(e_3,\mu(e_t,e_2)) = \mu(\mu(e_1,e_t),0),\\
& \mu(\alpha(e_1),\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),\alpha(e_3)) &\Rightarrow& \ \mu(e_3,\mu(e_t,e_3)) = \mu(\mu(e_1,e_t),0),\\
E_{32}: \ & \mu(\alpha(e_2),\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),\alpha(e_1)) &\Rightarrow& \ \mu(e_3,\mu(e_t,e_1)) = \mu(\mu(e_2,e_t),0),\\
& \mu(\alpha(e_2),\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),\alpha(e_3)) &\Rightarrow& \ \mu(e_3,\mu(e_t,e_3)) = \mu(\mu(e_2,e_t),0),\\
E_{33}: \ & \mu(\alpha(e_3),\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),\alpha(e_1)) &\Rightarrow& \ \mu(e_3,\mu(e_t,e_1)) = \mu(\mu(e_3,e_t),0),\\
& \mu(\alpha(e_3),\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),\alpha(e_2)) &\Rightarrow& \ \mu(e_3,\mu(e_t,e_2)) = \mu(\mu(e_3,e_t),0).
\end{aligned}
$$

To generalise this we again look at dimension n and let α be defined as the n-dimensional E_{ab}. Since we then, as before, have $\alpha(e_b) = e_a$, to get the equations with a zero right hand side, we must have $i = b$ and $k \neq b$. Thus, for all $t = \{1,2,\ldots,n\}$ and $s_b = \{1,2,\ldots,b-1,b+1,\ldots,n-1,n\}$ we get

$$
E_{ab}: \ \mu(\alpha(e_b),\mu(e_t,e_{s_b})) = \mu(\mu(e_b,e_t),\alpha(e_{s_b})) \ \Rightarrow \ \mu(e_a,\mu(e_t,e_{s_b})) = \mu(\mu(e_b,e_t),0).
$$

To sum up our results, we have found that in dimension n, with α defined as the n-dimensional E_{ab}, the system of equations will consist of the following equations:

$$
\begin{aligned}
\mu(e_a,\mu(e_t,e_b)) &= \mu(\mu(e_b,e_t),e_a),\\
0 &= \mu(\mu(e_{s_b},e_t),e_a),\\
\mu(e_a,\mu(e_t,e_{s_b})) &= 0.
\end{aligned}
$$

To clearly see why this gives us the results with only having to permutate the indices of the structure constants in the results for one α to get the results for a certain other α, we rewrite these equations so they are written on the form where we can actually see the structure constants. For simplicity, we begin with each of the nonzero terms separately and then put them back in the system of equations:

$$
\begin{aligned}
\mu(e_a,\mu(e_t,e_b)) &= \mu\left(e_a,\sum_{k=1}^{n} C_{tb}^{k} e_k\right) = \sum_{k=1}^{n} C_{tb}^{k}\mu(e_a,e_k) = \sum_{k=1}^{n} C_{tb}^{k}\sum_{l=1}^{n} C_{ak}^{l} e_l = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{tb}^{k} C_{ak}^{l} e_l,\\
\mu(\mu(e_b,e_t),e_a) &= \mu\left(\sum_{k=1}^{n} C_{bt}^{k} e_k, e_a\right) = \sum_{k=1}^{n} C_{bt}^{k}\mu(e_k,e_a) = \sum_{k=1}^{n} C_{bt}^{k}\sum_{l=1}^{n} C_{ka}^{l} e_l = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{bt}^{k} C_{ka}^{l} e_l,\\
\mu(\mu(e_{s_b},e_t),e_a) &= \mu\left(\sum_{k=1}^{n} C_{s_b t}^{k} e_k, e_a\right) = \sum_{k=1}^{n} C_{s_b t}^{k}\mu(e_k,e_a) = \sum_{k=1}^{n} C_{s_b t}^{k}\sum_{l=1}^{n} C_{ka}^{l} e_l = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{s_b t}^{k} C_{ka}^{l} e_l,\\
\mu(e_a,\mu(e_t,e_{s_b})) &= \mu\left(e_a,\sum_{k=1}^{n} C_{t s_b}^{k} e_k\right) = \sum_{k=1}^{n} C_{t s_b}^{k}\mu(e_a,e_k) = \sum_{k=1}^{n} C_{t s_b}^{k}\sum_{l=1}^{n} C_{ak}^{l} e_l = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{t s_b}^{k} C_{ak}^{l} e_l.
\end{aligned}
$$

Thus, the system of equations will consist of the following equations:

$$\sum_{k=1}^{n}\sum_{l=1}^{n} C_{tb}^{k} C_{ak}^{l} e_l = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{bt}^{k} C_{ka}^{l} e_l$$

$$0 = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{s_b t}^{k} C_{ka}^{l} e_l$$

$$\sum_{k=1}^{n}\sum_{l=1}^{n} C_{ts_b}^{k} C_{ak}^{l} e_l = 0,$$

which we can rewrite as

$$\left(\sum_{k=1}^{n}\sum_{l=1}^{n} C_{tb}^{k} C_{ak}^{l} - \sum_{k=1}^{n}\sum_{l=1}^{n} C_{bt}^{k} C_{ka}^{l}\right) e_l = \sum_{k=1}^{n}\sum_{l=1}^{n} (C_{tb}^{k} C_{ak}^{l} - C_{bt}^{k} C_{ka}^{l}) e_l = 0$$

$$\sum_{k=1}^{n}\sum_{l=1}^{n} C_{s_b t}^{k} C_{ka}^{l} e_l = 0$$

$$\sum_{k=1}^{n}\sum_{l=1}^{n} C_{ts_b}^{k} C_{ak}^{l} e_l = 0.$$

Remembering that since $\{e_1, e_2, e_3\}$ is an independent set, $k_1e_1 + k_2e_2 + k_3e_3$ will only have the trivial solution $k_1 = k_2 = k_3 = 0$, and we can therefore rewrite our system of equations as follows:

$$\sum_{k=1}^{n} (C_{tb}^{k} C_{ak}^{1} - C_{bt}^{k} C_{ka}^{1}) = 0$$

$$\sum_{k=1}^{n} (C_{tb}^{k} C_{ak}^{2} - C_{bt}^{k} C_{ka}^{2}) = 0$$

$$\vdots$$

$$\sum_{k=1}^{n} (C_{tb}^{k} C_{ak}^{n} - C_{bt}^{k} C_{ka}^{n}) = 0$$

$$\sum_{k=1}^{n} C_{s_b t}^{k} C_{ka}^{1} = 0$$

$$\sum_{k=1}^{n} C_{s_b t}^{k} C_{ka}^{2} = 0$$

$$\vdots$$

$$\sum_{k=1}^{n} C_{s_b t}^{k} C_{ka}^{n} = 0$$

$$\sum_{k=1}^{n} C_{ts_b}^{k} C_{ak}^{1} = 0$$
$$\sum_{k=1}^{n} C_{ts_b}^{k} C_{ak}^{2} = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} C_{ts_b}^{k} C_{ak}^{n} = 0.$$

Now, we know that this is the system of equations we get when α is defined as E_{ab}. We now want to see what happens when α is instead defined as $E_{\sigma_n(a),\sigma_n(b)}$. This will turn our system of equations into

$$\sum_{k=1}^{n} (C_{t,\sigma_n(b)}^{k} C_{\sigma_n(a),k}^{1} - C_{\sigma_n(b),t}^{k} C_{k,\sigma_n(a)}^{1}) = 0$$
$$\sum_{k=1}^{n} (C_{t,\sigma_n(b)}^{k} C_{\sigma_n(a),k}^{2} - C_{\sigma_n(b),t}^{k} C_{k,\sigma_n(a)}^{2}) = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} (C_{t,\sigma_n(b)}^{k} C_{\sigma_n(a),k}^{n} - C_{\sigma_n(b),t}^{k} C_{k,\sigma_n(a)}^{n}) = 0$$
$$\sum_{k=1}^{n} C_{s_{\sigma_n(b)}t}^{k} C_{k,\sigma_n(a)}^{1} = 0$$
$$\sum_{k=1}^{n} C_{s_{\sigma_n(b)}t}^{k} C_{k,\sigma_n(a)}^{2} = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} C_{s_{\sigma_n(b)}t}^{k} C_{k,\sigma_n(a)}^{n} = 0$$
$$\sum_{k=1}^{n} C_{ts_{\sigma_n(b)}}^{k} C_{\sigma_n(a),k}^{1} = 0$$
$$\sum_{k=1}^{n} C_{ts_{\sigma_n(b)}}^{k} C_{\sigma_n(a),k}^{2} = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} C_{ts_{\sigma_n(b)}}^{k} C_{\sigma_n(a),k}^{n} = 0.$$

Theorem 2. *Let $\mathbb{F}$ be an algebraically closed field of characteristic 0, let V be an n-dimensional linear space over $\mathbb{F}$, let α be a homomorphism from V to V, and μ a bilinear map from $V \times V$ to V. Then let $\{e_1, e_2, \ldots, e_n\}$ be a basis for V, let $\alpha(e_i) = \sum_{j=1}^{n} a_{ji} e_j$ and $\mu(e_i, e_j) = \sum_{k=1}^{n} C_{ij}^k e_k$, and finally let $\sigma_n = (1, 2, \ldots, n)$ be a permutation that permutates the indices of its input.*

If the different solutions, consisting of which values of the structure constants C_{ij}^k for which this will be a hom-associative algebra, have already been found for α defined as the n-dimensional E_{ab}, these solutions can be used to find the solutions for α defined as $\sigma_n(E_{ab}) = E_{\sigma_n(a),\sigma_n(b)} = E_{a+1,b+1}$ by simply using the same permutation on every structure constant in each of the solutions.

Looking back at our discussion from two dimensions, in Section 3.1.5, we remember that we thought there might be a connection between the α:s with ones on the different diagonals – in that case the main diagonal and the anti-diagonal. Based on what we now know, we can conclude that yes, there is a connection between the α:s with ones on the main diagonal, but not on the anti-diagonal, and the reason there seemed to be in the two-dimensional case is because E_{12} and E_{21} have ones on a diagonal in two dimensions, in the way we discussed at the beginning of this section for three dimensions.

3.3 Dimension 3 Mapped to Dimension 1

Since we have concluded that we will not be able to find all of the solutions to the systems of equations in three dimensions, we want to find a way to find at least some solutions instead. To do this, we map the systems of equations in three dimensions to only one dimension, by restricting the range for multiplication μ. Specifically, we let the structure constants $C_{ij}^k = 0$ for k equal to two of the three possible values – 1, 2 and 3. We begin with letting $k = 2$ and $k = 3$, which means that the only structure constants that will remain nonzero are the ones on the form C_{ij}^1. However, if we do this for E_{11} and then permutate the solutions, the only nonzero structure constants will be on the form C_{ij}^2 for E_{22}, and on the form C_{ij}^3 for E_{33}. Thus, to be able to compare the solutions, we cannot only look at the case when C_{ij}^1 are the nonzero structure constants for E_{11}, but also the cases where first C_{ij}^2 and then C_{ij}^3 are the only nonzero structure constants for E_{11}. Then, when we permutate the results, we will get all three of these cases for α defined as E_{22} and E_{33} as well. Of course, this exact same thing holds for α defined as E_{12}, E_{23} and E_{31}, and α defined as E_{13}, E_{21} and E_{32} as well.

Now, thanks to Theorem 2, we do not have to find the solutions for every α, but only three ones such that if we permutate the indices of the matrix units, we get each of the remaining matrix units as well. We choose to use E_{11}, E_{12} and E_{13} here.

Thus, what we are going to do is that for α defined as E_{11} we will first look at the case where $C_{ij}^k = 0$ for $k = 2$ and $k = 3$, then for $k = 3$ and $k = 1$, and finally for $k = 1$ and $k = 2$. We will then permutate the results according to Theorem 2 to get the results for α defined as E_{22} and E_{33}. For each of these matrix units we are then going to find the commutator tables for the hom-Lie admissible algebras associated to these hom-associative algebras. Finally, we will do the exact same thing with α defined as E_{12} and, after permutating these results, for E_{23} and

E_{31}, and then the same thing again for α defined as E_{13} and, after permutating these results, for E_{21} and E_{32}.

3.3.1 α defined as E_{11}, E_{22} and E_{33}

We begin by looking at the system of equations for α defined as E_{11}. We found earlier, in Section 3.2.1, that this looks as follows:

$$C_{11}^2C_{12}^1+C_{11}^3C_{13}^1-C_{11}^2C_{21}^1-C_{11}^3C_{31}^1=0 \quad \text{(R3.14.1)}$$
$$C_{11}^2C_{12}^2+C_{11}^3C_{13}^2-C_{11}^2C_{21}^2-C_{11}^3C_{31}^2=0 \quad \text{(R3.14.2)}$$
$$C_{11}^2C_{12}^3+C_{11}^3C_{13}^3-C_{11}^2C_{21}^3-C_{11}^3C_{31}^3=0 \quad \text{(R3.14.3)}$$
$$C_{12}^1C_{11}^1+C_{12}^2C_{12}^1+C_{12}^3C_{13}^1=0 \quad \text{(R3.15.1)}$$
$$C_{12}^1C_{11}^2+C_{12}^2C_{12}^2+C_{12}^3C_{13}^2=0 \quad \text{(R3.15.2)}$$
$$C_{12}^1C_{11}^3+C_{12}^2C_{12}^3+C_{12}^3C_{13}^3=0 \quad \text{(R3.15.3)}$$
$$C_{13}^1C_{11}^1+C_{13}^2C_{12}^1+C_{13}^3C_{13}^1=0 \quad \text{(R3.16.1)}$$
$$C_{13}^1C_{11}^2+C_{13}^2C_{12}^2+C_{13}^3C_{13}^2=0 \quad \text{(R3.16.2)}$$
$$C_{13}^1C_{11}^3+C_{13}^2C_{12}^3+C_{13}^3C_{13}^3=0 \quad \text{(R3.16.3)}$$
$$C_{21}^1C_{11}^1+C_{21}^2C_{12}^1+C_{21}^3C_{13}^1-C_{12}^1C_{11}^1-C_{12}^2C_{21}^1-C_{12}^3C_{31}^1=0 \quad \text{(R3.17.1)}$$
$$C_{21}^1C_{11}^2+C_{21}^2C_{12}^2+C_{21}^3C_{13}^2-C_{12}^1C_{11}^2-C_{12}^2C_{21}^2-C_{12}^3C_{31}^2=0 \quad \text{(R3.17.2)}$$
$$C_{21}^1C_{11}^3+C_{21}^2C_{12}^3+C_{21}^3C_{13}^3-C_{12}^1C_{11}^3-C_{12}^2C_{21}^3-C_{12}^3C_{31}^3=0 \quad \text{(R3.17.3)}$$
$$C_{31}^1C_{11}^1+C_{31}^2C_{12}^1+C_{31}^3C_{13}^1-C_{13}^1C_{11}^1-C_{13}^2C_{21}^1-C_{13}^3C_{31}^1=0 \quad \text{(R3.18.1)}$$
$$C_{31}^1C_{11}^2+C_{31}^2C_{12}^2+C_{31}^3C_{13}^2-C_{13}^1C_{11}^2-C_{13}^2C_{21}^2-C_{13}^3C_{31}^2=0 \quad \text{(R3.18.2)}$$
$$C_{31}^1C_{11}^3+C_{31}^2C_{12}^3+C_{31}^3C_{13}^3-C_{13}^1C_{11}^3-C_{13}^2C_{21}^3-C_{13}^3C_{31}^3=0 \quad \text{(R3.18.3)}$$
$$C_{22}^1C_{11}^1+C_{22}^2C_{12}^1+C_{22}^3C_{13}^1=0 \quad \text{(R3.19.1)}$$
$$C_{22}^1C_{11}^2+C_{22}^2C_{12}^2+C_{22}^3C_{13}^2=0 \quad \text{(R3.19.2)}$$
$$C_{22}^1C_{11}^3+C_{22}^2C_{12}^3+C_{22}^3C_{13}^3=0 \quad \text{(R3.19.3)}$$
$$C_{23}^1C_{11}^1+C_{23}^2C_{12}^1+C_{23}^3C_{13}^1=0 \quad \text{(R3.20.1)}$$
$$C_{23}^1C_{11}^2+C_{23}^2C_{12}^2+C_{23}^3C_{13}^2=0 \quad \text{(R3.20.2)}$$
$$C_{23}^1C_{11}^3+C_{23}^2C_{12}^3+C_{23}^3C_{13}^3=0 \quad \text{(R3.20.3)}$$
$$C_{32}^1C_{11}^1+C_{32}^2C_{12}^1+C_{32}^3C_{13}^1=0 \quad \text{(R3.21.1)}$$
$$C_{32}^1C_{11}^2+C_{32}^2C_{12}^2+C_{32}^3C_{13}^2=0 \quad \text{(R3.21.2)}$$
$$C_{32}^1C_{11}^3+C_{32}^2C_{12}^3+C_{32}^3C_{13}^3=0 \quad \text{(R3.21.3)}$$
$$C_{33}^1C_{11}^1+C_{33}^2C_{12}^1+C_{33}^3C_{13}^1=0 \quad \text{(R3.22.1)}$$
$$C_{33}^1C_{11}^2+C_{33}^2C_{12}^2+C_{33}^3C_{13}^2=0 \quad \text{(R3.22.2)}$$
$$C_{33}^1C_{11}^3+C_{33}^2C_{12}^3+C_{33}^3C_{13}^3=0 \quad \text{(R3.22.3)}$$

$$C^1_{21}C^1_{11} + C^2_{21}C^1_{21} + C^3_{21}C^1_{31} = 0 \qquad \text{(R3.23.1)}$$
$$C^1_{21}C^2_{11} + C^2_{21}C^2_{21} + C^3_{21}C^2_{31} = 0 \qquad \text{(R3.23.2)}$$
$$C^1_{21}C^3_{11} + C^2_{21}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^1_{22}C^1_{11} + C^2_{22}C^1_{21} + C^3_{22}C^1_{31} = 0 \qquad \text{(R3.26.1)}$$
$$C^1_{22}C^2_{11} + C^2_{22}C^2_{21} + C^3_{22}C^2_{31} = 0 \qquad \text{(R3.26.2)}$$
$$C^1_{22}C^3_{11} + C^2_{22}C^3_{21} + C^3_{22}C^3_{31} = 0 \qquad \text{(R3.26.3)}$$
$$C^1_{23}C^1_{11} + C^2_{23}C^1_{21} + C^3_{23}C^1_{31} = 0 \qquad \text{(R3.27.1)}$$
$$C^1_{23}C^2_{11} + C^2_{23}C^2_{21} + C^3_{23}C^2_{31} = 0 \qquad \text{(R3.27.2)}$$
$$C^1_{23}C^3_{11} + C^2_{23}C^3_{21} + C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^1_{31}C^1_{11} + C^2_{31}C^1_{21} + C^3_{31}C^1_{31} = 0 \qquad \text{(R3.32.1)}$$
$$C^1_{31}C^2_{11} + C^2_{31}C^2_{21} + C^3_{31}C^2_{31} = 0 \qquad \text{(R3.32.2)}$$
$$C^1_{31}C^3_{11} + C^2_{31}C^3_{21} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} + C^2_{32}C^1_{21} + C^3_{32}C^1_{31} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{32}C^2_{11} + C^2_{32}C^2_{21} + C^3_{32}C^2_{31} = 0 \qquad \text{(R3.35.2)}$$
$$C^1_{32}C^3_{11} + C^2_{32}C^3_{21} + C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11} + C^2_{33}C^1_{21} + C^3_{33}C^1_{31} = 0 \qquad \text{(R3.36.1)}$$
$$C^1_{33}C^2_{11} + C^2_{33}C^2_{21} + C^3_{33}C^2_{31} = 0 \qquad \text{(R3.36.2)}$$
$$C^1_{33}C^3_{11} + C^2_{33}C^3_{21} + C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

Setting all C^k_{ij} equal to zero for $k = 2$ and $k = 3$, our system of equations is reduced to

$$C^1_{12}C^1_{11} = 0 \qquad \text{(R3.15.1)}$$
$$C^1_{13}C^1_{11} = 0 \qquad \text{(R3.16.1)}$$
$$C^1_{21}C^1_{11} - C^1_{12}C^1_{11} = 0 \qquad \text{(R3.17.1)}$$
$$C^1_{31}C^1_{11} - C^1_{13}C^1_{11} = 0 \qquad \text{(R3.18.1)}$$
$$C^1_{22}C^1_{11} = 0 \qquad \text{(R3.19.1)}$$
$$C^1_{23}C^1_{11} = 0 \qquad \text{(R3.20.1)}$$
$$C^1_{32}C^1_{11} = 0 \qquad \text{(R3.21.1)}$$
$$C^1_{33}C^1_{11} = 0 \qquad \text{(R3.22.1)}$$
$$C^1_{21}C^1_{11} = 0 \qquad \text{(R3.23.1)}$$
$$C^1_{22}C^1_{11} = 0 \qquad \text{(R3.26.1)}$$
$$C^1_{23}C^1_{11} = 0 \qquad \text{(R3.27.1)}$$
$$C^1_{31}C^1_{11} = 0 \qquad \text{(R3.32.1)}$$
$$C^1_{32}C^1_{11} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{33}C^1_{11} = 0. \qquad \text{(R3.36.1)}$$

It is easy to see that this system of equations only has two solutions – either $C^1_{11} = 0$, or $C^1_{12} = C^1_{13} = C^1_{21} = C^1_{22} = C^1_{23} = C^1_{31} = C^1_{32} = C^1_{33} = 0$. Of course, all of these variables in the second solution could be equal to zero at the same time as $C^1_{11} = 0$, but that would only be a special case of both solutions, so we do not care about that. Thus, we can now move on to instead setting all C^k_{ij} equal to zero for $k = 3$ and $k = 1$, which reduces our system of equations to

$$C^2_{11}C^2_{12} - C^2_{11}C^2_{21} = 0 \quad \text{(R3.14.2)}$$
$$C^2_{12}C^2_{12} = 0 \quad \text{(R3.15.2)}$$
$$C^2_{13}C^2_{12} = 0 \quad \text{(R3.16.2)}$$
$$C^2_{21}C^2_{12} - C^2_{12}C^2_{21} = 0 \quad \text{(R3.17.1.2)}$$
$$C^2_{31}C^2_{12} - C^2_{13}C^2_{21} = 0 \quad \text{(R3.18.2)}$$
$$C^2_{22}C^2_{12} = 0 \quad \text{(R3.19.2)}$$
$$C^2_{23}C^2_{12} = 0 \quad \text{(R3.20.2)}$$
$$C^2_{32}C^2_{12} = 0 \quad \text{(R3.21.2)}$$
$$C^2_{33}C^2_{12} = 0 \quad \text{(R3.22.2)}$$
$$C^2_{21}C^2_{21} = 0 \quad \text{(R3.23.2)}$$
$$C^2_{22}C^2_{21} = 0 \quad \text{(R3.26.2)}$$
$$C^2_{23}C^2_{21} = 0 \quad \text{(R3.27.2)}$$
$$C^2_{31}C^2_{21} = 0 \quad \text{(R3.32.2)}$$
$$C^2_{32}C^2_{21} = 0 \quad \text{(R3.35.2)}$$
$$C^2_{33}C^2_{21} = 0. \quad \text{(R3.36.2)}$$

The only solution to equations (R3.15.2) and (R3.23.2) are $C^2_{12} = 0$ and $C^2_{21} = 0$, respectively, and we see that this also solves the rest of the system of equations. Finally setting all C^k_{ij} equal to zero for $k = 1$ and $k = 2$, our system of equations is instead reduced to

$$C^3_{11}C^3_{13} - C^3_{11}C^3_{31} = 0 \quad \text{(R3.14.3)}$$
$$C^3_{12}C^3_{13} = 0 \quad \text{(R3.15.3)}$$
$$C^3_{13}C^3_{13} = 0 \quad \text{(R3.16.3)}$$
$$C^3_{21}C^3_{13} - C^3_{12}C^3_{31} = 0 \quad \text{(R3.17.3)}$$
$$C^3_{31}C^3_{13} - C^3_{13}C^3_{31} = 0 \quad \text{(R3.18.3)}$$
$$C^3_{22}C^3_{13} = 0 \quad \text{(R3.19.3)}$$
$$C^3_{23}C^3_{13} = 0 \quad \text{(R3.20.3)}$$
$$C^3_{32}C^3_{13} = 0 \quad \text{(R3.21.3)}$$
$$C^3_{33}C^3_{13} = 0 \quad \text{(R3.22.3)}$$
$$C^3_{21}C^3_{31} = 0 \quad \text{(R3.23.3)}$$

$$C^3_{22}C^3_{31} = 0 \qquad \text{(R3.26.3)}$$
$$C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

The only solution to equations (R3.16.3) and (R3.32.3) are $C^3_{13} = 0$ and $C^3_{31} = 0$, respectively, and we see that these also solve the rest of the system of equations. We sum our results up in Table 3.21. Note that since we got some of solutions from when we initially mapped the system of equations to one dimension by setting some of the structure constants to zero, we let those zeroes be written as normal, while all values of the structure constants that we got from solving the system of equations are written in bold, to be able to easily differentiate between them. Permutating the solutions, we get the solutions for when α is defined as E_{22} in Table 3.22, and the solutions for when α is defined as E_{33} in Table 3.23.

Again, these hom-associative algebras are hom-Lie admissible with commutator $[\cdot,\cdot]$, so we now want to find these commutators. Since we are in three dimensions, we must calculate $[e_i, e_j]$ for $i, j = \{1,2,3\}$ for each α to find the hom-Lie admissible algebras. By definition, $[e_1,e_1] = [e_2,e_2] = [e_3,e_3] = 0$, so we only have to calculate the other combinations of i and j. Remembering that

$$\mu(e_i,e_j) = \sum_{k=1}^{3} C^k_{ij}e_k = C^1_{ij}e_1 + C^2_{ij}e_2 + C^3_{ij}e_3,$$

and $[e_i,e_j] = \mu(e_i,e_j) - \mu(e_j,e_i)$, we can now move on to calculating the hom-Lie admissible algebras for each solution to E_{11}, E_{22} and E_{33}. We begin with E_{11}, and first look at solution 1.

$$\begin{aligned}
\mu(e_1,e_2) &= C^1_{12}e_1 + C^2_{12}e_2 + C^3_{12}e_3 = C^1_{12}e_1 + 0e_2 + 0e_3 = C^1_{12}e_1,\\
\mu(e_1,e_3) &= C^1_{13}e_1 + C^2_{13}e_2 + C^3_{13}e_3 = C^1_{13}e_1 + 0e_2 + 0e_3 = C^1_{13}e_1,\\
\mu(e_2,e_1) &= C^1_{21}e_1 + C^2_{21}e_2 + C^3_{21}e_3 = C^1_{21}e_1 + 0_2 + 0e_3 = C^1_{21}e_1,\\
\mu(e_2,e_3) &= C^1_{23}e_1 + C^2_{23}e_2 + C^3_{23}e_3 = C^1_{23}e_1 + 0e_2 + 0e_3 = C^1_{23}e_1,\\
\mu(e_3,e_1) &= C^1_{31}e_1 + C^2_{31}e_2 + C^3_{31}e_3 = C^1_{31}e_1 + 0e_2 + 0e_3 = C^1_{31}e_1,\\
\mu(e_3,e_2) &= C^1_{32}e_1 + C^2_{32}e_2 + C^3_{32}e_3 = C^1_{32}e_1 + 0e_2 + 0e_3 = C^1_{32}e_1.
\end{aligned}$$

This gives us

$$\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2) - \mu(e_2,e_1) = C^1_{12}e_1 - C^1_{21}e_1 = (C^1_{12} - C^1_{21})e_1,\\
[e_1,e_3] &= \mu(e_1,e_3) - \mu(e_3,e_1) = C^1_{13}e_1 - C^1_{31}e_1 = (C^1_{13} - C^1_{31})e_1,\\
[e_2,e_1] &= -[e_1,e_2] = -(C^1_{12} - C^1_{21})e_1 = (C^1_{21} - C^1_{12})e_1,\\
[e_2,e_3] &= \mu(e_2,e_3) - \mu(e_3,e_2) = C^1_{23}e_1 - C^1_{32}e_1 = (C^1_{23} - C^1_{32})e_1,\\
[e_3,e_1] &= -[e_1,e_3] = -(C^1_{13} - C^1_{31})e_1 = (C^1_{31} - C^1_{13})e_1,\\
[e_3,e_2] &= -[e_2,e_3] = -(C^1_{23} - C^1_{32})e_1 = (C^1_{32} - C^1_{23})e_1.
\end{aligned}$$

	Soln 1	Soln 2	Soln 3	Soln 4
C_{11}^1	**0**	**free**	0	0
C_{11}^2	0	0	**free**	0
C_{11}^3	0	0	0	**free**
C_{12}^1	**free**	**0**	0	0
C_{12}^2	0	0	**0**	0
C_{12}^3	0	0	0	**free**
C_{13}^1	**free**	**0**	0	0
C_{13}^2	0	0	**free**	0
C_{13}^3	0	0	0	**0**
C_{21}^1	**free**	**0**	0	0
C_{21}^2	0	0	**0**	0
C_{21}^3	0	0	0	**free**
C_{22}^1	**free**	**0**	0	0
C_{22}^2	0	0	**free**	0
C_{22}^3	0	0	0	**free**
C_{23}^1	**free**	**0**	0	0
C_{23}^2	0	0	**free**	0
C_{23}^3	0	0	0	**free**
C_{31}^1	**free**	**0**	0	0
C_{31}^2	0	0	**free**	0
C_{31}^3	0	0	0	**0**
C_{32}^1	**free**	**0**	0	0
C_{32}^2	0	0	**free**	0
C_{32}^3	0	0	0	**free**
C_{33}^1	**free**	**0**	0	0
C_{33}^2	0	0	**free**	0
C_{33}^3	0	0	0	**free**

Table 3.21: The values of the structure constants that give hom-associative algebras when α is defined as E_{11}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	Soln 1	Soln 2	Soln 3	Soln 4
C_{11}^{1}	0	0	0	**free**
C_{11}^{2}	**free**	**0**	0	0
C_{11}^{3}	0	0	**free**	0
C_{12}^{1}	0	0	0	**0**
C_{12}^{2}	**free**	**0**	0	0
C_{12}^{3}	0	0	**free**	0
C_{13}^{1}	0	0	0	**free**
C_{13}^{2}	**free**	**0**	0	0
C_{13}^{3}	0	0	**free**	0
C_{21}^{1}	0	0	0	**0**
C_{21}^{2}	**free**	**0**	0	0
C_{21}^{3}	0	0	**free**	0
C_{22}^{1}	0	0	0	**free**
C_{22}^{2}	**0**	**free**	0	0
C_{22}^{3}	0	0	**free**	0
C_{23}^{1}	0	0	0	**free**
C_{23}^{2}	**free**	**0**	0	0
C_{23}^{3}	0	0	**0**	0
C_{31}^{1}	0	0	0	**free**
C_{31}^{2}	**free**	**0**	0	0
C_{31}^{3}	0	0	**free**	0
C_{32}^{1}	0	0	0	**free**
C_{32}^{2}	**free**	**0**	0	0
C_{32}^{3}	0	0	**0**	0
C_{33}^{1}	0	0	0	**free**
C_{33}^{2}	**free**	**0**	0	0
C_{33}^{3}	0	0	**free**	0

Table 3.22: The values of the structure constants that give hom-associative algebras when α is defined as E_{22}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	Soln 1	Soln 2	Soln 3	Soln 4
C_{11}^{1}	0	0	**free**	0
C_{11}^{2}	0	0	0	**free**
C_{11}^{3}	**free**	**0**	0	0
C_{12}^{1}	0	0	**free**	0
C_{12}^{2}	0	0	0	**free**
C_{12}^{3}	**free**	**0**	0	0
C_{13}^{1}	0	0	**0**	0
C_{13}^{2}	0	0	0	**free**
C_{13}^{3}	**free**	**0**	0	0
C_{21}^{1}	0	0	**free**	0
C_{21}^{2}	0	0	0	**free**
C_{21}^{3}	**free**	**0**	0	0
C_{22}^{1}	0	0	**free**	0
C_{22}^{2}	0	0	0	**free**
C_{22}^{3}	**free**	**0**	0	0
C_{23}^{1}	0	0	**free**	0
C_{23}^{2}	0	0	0	**0**
C_{23}^{3}	**free**	**0**	0	0
C_{31}^{1}	0	0	**0**	0
C_{31}^{2}	0	0	0	**free**
C_{31}^{3}	**free**	**0**	0	0
C_{32}^{1}	0	0	**free**	0
C_{32}^{2}	0	0	0	**0**
C_{32}^{3}	**free**	**0**	0	0
C_{33}^{1}	0	0	**free**	0
C_{33}^{2}	0	0	0	**free**
C_{33}^{3}	**0**	**free**	0	0

Table 3.23: The values of the structure constants that give hom-associative algebras when α is defined as E_{33}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

Moving on to solution 2, we get

$$\begin{aligned}
\mu(e_1,e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_1,e_3) &= C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_3) &= C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_1) &= C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_2) &= C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0.
\end{aligned}$$

This gives us

$$\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2) - \mu(e_2,e_1) = 0 - 0 = 0,\\
[e_1,e_3] &= \mu(e_1,e_3) - \mu(e_3,e_1) = 0 - 0 = 0,\\
[e_2,e_1] &= -[e_1,e_2] = -0 = 0,\\
[e_2,e_3] &= \mu(e_2,e_3) - \mu(e_3,e_2) = 0 - 0 = 0,\\
[e_3,e_1] &= -[e_1,e_3] = -0 = 0,\\
[e_3,e_2] &= -[e_2,e_3] = -0 = 0.
\end{aligned}$$

Looking at solution 3, we then get

$$\begin{aligned}
\mu(e_1,e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_1,e_3) &= C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = 0e_1 + C_{13}^2 e_2 + 0e_3 = C_{13}^2 e_2,\\
\mu(e_2,e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_3) &= C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = 0e_1 + C_{23}^2 e_2 + 0e_3 = C_{23}^2 e_2,\\
\mu(e_3,e_1) &= C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = 0e_1 + C_{31}^2 e_2 + 0e_3 = C_{31}^2 e_2,\\
\mu(e_3,e_2) &= C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = 0e_1 + C_{32}^2 e_2 + 0e_3 = C_{32}^2 e_2.
\end{aligned}$$

This gives us

$$\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2) - \mu(e_2,e_1) = 0 - 0 = 0,\\
[e_1,e_3] &= \mu(e_1,e_3) - \mu(e_3,e_1) = C_{13}^2 e_2 - C_{31}^2 e_2 = (C_{13}^2 - C_{31}^2) e_2,\\
[e_2,e_1] &= -[e_1,e_2] = -0 = 0,\\
[e_2,e_3] &= \mu(e_2,e_3) - \mu(e_3,e_2) = C_{23}^2 e_2 - C_{32}^2 e_2 = (C_{23}^2 - C_{32}^2) e_2,\\
[e_3,e_1] &= -[e_1,e_3] = -(C_{13}^2 - C_{31}^2) e_2 = (C_{31}^2 - C_{13}^2) e_2,\\
[e_3,e_2] &= -[e_2,e_3] = -(C_{23}^2 - C_{32}^2) e_2 = (C_{32}^2 - C_{23}^2) e_2.
\end{aligned}$$

Finally moving on to solution 4, we get

$$
\begin{aligned}
\mu(e_1,e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0e_1 + 0e_2 + C_{12}^3 e_3 = C_{12}^3 e_3,\\
\mu(e_1,e_3) &= C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0e_1 + 0e_2 + C_{21}^3 e_3 = C_{21}^3 e_3,\\
\mu(e_2,e_3) &= C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = 0e_1 + 0e_2 + C_{23}^3 e_3 = C_{23}^3 e_3,\\
\mu(e_3,e_1) &= C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_2) &= C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = 0e_1 + 0e_2 + C_{32}^3 e_3 = C_{32}^3 e_3.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2) - \mu(e_2,e_1) = C_{12}^3 e_3 - C_{21}^3 e_3 = (C_{12}^3 - C_{21}^3)e_3,\\
[e_1,e_3] &= \mu(e_1,e_3) - \mu(e_3,e_1) = 0 - 0 = 0,\\
[e_2,e_1] &= -[e_1,e_2] = -(C_{12}^3 - C_{21}^3)e_3 = (C_{21}^3 - C_{12}^3)e_3,\\
[e_2,e_3] &= \mu(e_2,e_3) - \mu(e_3,e_2) = C_{23}^3 e_3 - C_{32}^3 e_3 = (C_{23}^3 - C_{32}^3)e_3,\\
[e_3,e_1] &= -[e_1,e_3] = -0 = 0,\\
[e_3,e_2] &= -[e_2,e_3] = -(C_{23}^3 - C_{32}^3)e_3 = (C_{32}^3 - C_{23}^3)e_3.
\end{aligned}
$$

We write the commutator table for each of these solutions in Table 3.24.

We can now move on to E_{22}. We once again begin with solution 1, and thus get

$$
\begin{aligned}
\mu(e_1,e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0e_1 + C_{12}^2 e_2 + 0e_3 = C_{12}^2 e_2,\\
\mu(e_1,e_3) &= C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = 0e_1 + C_{13}^2 e_2 + 0e_3 = C_{13}^2 e_2,\\
\mu(e_2,e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0e_1 + C_{21}^2 e_2 + 0e_3 = C_{21}^2 e_2,\\
\mu(e_2,e_3) &= C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = 0e_1 + C_{23}^2 e_2 + 0e_3 = C_{23}^2 e_2,\\
\mu(e_3,e_1) &= C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = 0e_1 + C_{31}^2 e_2 + 0e_3 = C_{31}^2 e_2,\\
\mu(e_3,e_2) &= C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = 0e_1 + C_{32}^2 e_2 + 0e_3 = C_{32}^2 e_2.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2) - \mu(e_2,e_1) = C_{12}^2 e_2 - C_{21}^2 e_2 = (C_{12}^2 - C_{21}^2)e_2,\\
[e_1,e_3] &= \mu(e_1,e_3) - \mu(e_3,e_1) = C_{13}^2 e_2 - C_{31}^2 e_2 = (C_{13}^2 - C_{31}^2)e_2,\\
[e_2,e_1] &= -[e_1,e_2] = -(C_{12}^2 - C_{21}^2)e_2 = (C_{21}^2 - C_{12}^2)e_2,\\
[e_2,e_3] &= \mu(e_2,e_3) - \mu(e_3,e_2) = C_{23}^2 e_2 - C_{32}^2 e_2 = (C_{23}^2 - C_{32}^2)e_2,\\
[e_3,e_1] &= -[e_1,e_3] = -(C_{13}^2 - C_{31}^2)e_2 = (C_{31}^2 - C_{13}^2)e_2,\\
[e_3,e_2] &= -[e_2,e_3] = -(C_{23}^2 - C_{32}^2)e_2 = (C_{32}^2 - C_{23}^2)e_2.
\end{aligned}
$$

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^1 - C_{21}^1)e_1$	$(C_{13}^1 - C_{31}^1)e_1$
e_2	$(C_{21}^1 - C_{12}^1)e_1$	0	$(C_{23}^1 - C_{32}^1)e_1$
e_3	$(C_{31}^1 - C_{13}^1)e_1$	$(C_{32}^1 - C_{23}^1)e_1$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	0
e_2	0	0	0
e_3	0	0	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	$(C_{13}^2 - C_{31}^2)e_2$
e_2	0	0	$(C_{23}^2 - C_{32}^2)e_2$
e_3	$(C_{31}^2 - C_{13}^2)e_2$	$(C_{32}^2 - C_{23}^2)e_2$	0

Soln 4

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^3 - C_{21}^3)e_3$	0
e_2	$(C_{21}^3 - C_{12}^3)e_3$	0	$(C_{23}^3 - C_{32}^3)e_3$
e_3	0	$(C_{32}^3 - C_{23}^3)e_3$	0

Table 3.24: The commutator tables for the four different hom-Lie admissible algebras with α defined as E_{11} and the structure constants defined as in solution 1, solution 2, solution 3 and solution 4, in three dimensions mapped to one dimension

Moving on to solution 2, we get

$$\begin{aligned}
\mu(e_1,e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_1,e_3) &= C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_3) &= C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_1) &= C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_2) &= C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0.
\end{aligned}$$

This gives us

$$\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2) - \mu(e_2,e_1) = 0 - 0 = 0,\\
[e_1,e_3] &= \mu(e_1,e_3) - \mu(e_3,e_1) = 0 - 0 = 0,\\
[e_2,e_1] &= -[e_1,e_2] = -0 = 0,\\
[e_2,e_3] &= \mu(e_2,e_3) - \mu(e_3,e_2) = 0 - 0 = 0,
\end{aligned}$$

$$[e_3, e_1] = -[e_1, e_3] = -0 = 0,$$
$$[e_3, e_2] = -[e_2, e_3] = -0 = 0.$$

Then looking at solution 3, we get

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0e_1 + 0e_2 + C_{12}^3 e_3 = C_{12}^3 e_3,$$
$$\mu(e_1, e_3) = C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = 0e_1 + 0e_2 + C_{13}^3 e_3 = C_{13}^3 e_3,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0e_1 + 0e_2 + C_{21}^3 e_3 = C_{21}^3 e_3,$$
$$\mu(e_2, e_3) = C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,$$
$$\mu(e_3, e_1) = C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = 0e_1 + 0e_2 + C_{31}^3 e_3 = C_{31}^3 e_3,$$
$$\mu(e_3, e_2) = C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0.$$

This gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^3 e_3 - C_{21}^3 e_3 = (C_{12}^3 - C_{21}^3) e_3,$$
$$[e_1, e_3] = \mu(e_1, e_3) - \mu(e_3, e_1) = C_{13}^3 e_3 - C_{31}^3 e_3 = (C_{13}^3 - C_{31}^3) e_3,$$
$$[e_2, e_1] = -[e_1, e_2] = -(C_{12}^3 - C_{21}^3) e_3 = (C_{21}^3 - C_{12}^3) e_3,$$
$$[e_2, e_3] = \mu(e_2, e_3) - \mu(e_3, e_2) = 0 - 0 = 0,$$
$$[e_3, e_1] = -[e_1, e_3] = -(C_{13}^3 - C_{31}^3) e_3 = (C_{31}^3 - C_{13}^3) e_3,$$
$$[e_3, e_2] = -[e_2, e_3] = -0 = 0.$$

Finally looking at solution 4, we get

$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,$$
$$\mu(e_1, e_3) = C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = C_{13}^1 e_1 + 0e_2 + 0e_3 = C_{13}^1 e_1,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0e_1 + 0e_2 + 0e_3 = 0,$$
$$\mu(e_2, e_3) = C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = C_{23}^1 e_1 + 0e_2 + 0e_3 = C_{23}^1 e_1,$$
$$\mu(e_3, e_1) = C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = C_{31}^1 e_1 + 0e_2 + 0e_3 = C_{31}^1 e_1,$$
$$\mu(e_3, e_2) = C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = C_{32}^1 e_1 + 0e_2 + 0e_3 = C_{32}^1 e_1.$$

This gives us

$$[e_1, e_2] = \mu(e_1, e_2) - \mu(e_2, e_1) = 0 - 0 = 0,$$
$$[e_1, e_3] = \mu(e_1, e_3) - \mu(e_3, e_1) = C_{13}^1 e_1 - C_{31}^1 e_1 = (C_{13}^1 - C_{31}^1) e_1,$$
$$[e_2, e_1] = -[e_1, e_2] = -0 = 0,$$
$$[e_2, e_3] = \mu(e_2, e_3) - \mu(e_3, e_2) = C_{23}^1 e_1 - C_{32}^1 e_1 = (C_{23}^1 - C_{32}^1) e_1,$$
$$[e_3, e_1] = -[e_1, e_3] = -(C_{13}^1 - C_{31}^1) e_1 = (C_{31}^1 - C_{13}^1) e_1,$$
$$[e_3, e_2] = -[e_2, e_3] = -(C_{23}^1 - C_{32}^1) e_1 = (C_{32}^1 - C_{23}^1) e_1.$$

We write the commutator table for each of these solutions in Table 3.25.

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C^2_{12} - C^2_{21})e_2$	$(C^2_{13} - C^2_{31})e_2$
e_2	$(C^2_{21} - C^2_{12})e_2$	0	$(C^2_{23} - C^2_{32})e_2$
e_3	$(C^2_{31} - C^2_{13})e_2$	$(C^2_{32} - C^2_{23})e_2$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	0
e_2	0	0	0
e_3	0	0	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C^3_{12} - C^3_{21})e_3$	$(C^3_{13} - C^3_{31})e_3$
e_2	$(C^3_{21} - C^3_{12})e_3$	0	0
e_3	$(C^3_{31} - C^3_{13})e_3$	0	0

Soln 4

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	$(C^1_{13} - C^1_{31})e_1$
e_2	0	0	$(C^1_{23} - C^1_{32})e_1$
e_3	$(C^1_{31} - C^1_{13})e_1$	$(C^1_{32} - C^1_{23})e_1$	0

Table 3.25: The commutator tables for the four different hom-Lie admissible algebras with α defined as E_{22} and the structure constants defined as in solution 1, solution 2, solution 3 and solution 4, in three dimensions mapped to one dimension

Finally, we look at E_{33}. Just like before, we begin with solution 1, and we get

$$\begin{aligned}
\mu(e_1,e_2) &= C^1_{12}e_1 + C^2_{12}e_2 + C^3_{12}e_3 = 0e_1 + 0e_2 + C^3_{12}e_3 = C^3_{12}e_3,\\
\mu(e_1,e_3) &= C^1_{13}e_1 + C^2_{13}e_2 + C^3_{13}e_3 = 0e_1 + 0e_2 + C^3_{13}e_3 = C^3_{13}e_3,\\
\mu(e_2,e_1) &= C^1_{21}e_1 + C^2_{21}e_2 + C^3_{21}e_3 = 0e_1 + 0e_2 + C^3_{21}e_3 = C^3_{21}e_3,\\
\mu(e_2,e_3) &= C^1_{23}e_1 + C^2_{23}e_2 + C^3_{23}e_3 = 0e_1 + 0e_2 + C^3_{23}e_3 = C^3_{23}e_3,\\
\mu(e_3,e_1) &= C^1_{31}e_1 + C^2_{31}e_2 + C^3_{31}e_3 = 0e_1 + 0e_2 + C^3_{31}e_3 = C^3_{31}e_3,\\
\mu(e_3,e_2) &= C^1_{32}e_1 + C^2_{32}e_2 + C^3_{32}e_3 = 0e_1 + 0e_2 + C^3_{32}e_3 = C^3_{32}e_3.
\end{aligned}$$

This gives us

$$
\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2)-\mu(e_2,e_1) = C_{12}^3e_3 - C_{21}^3e_3 = (C_{12}^3 - C_{21}^3)e_3,\\
[e_1,e_3] &= \mu(e_1,e_3)-\mu(e_3,e_1) = C_{13}^3e_3 - C_{31}^3e_3 = (C_{13}^3 - C_{31}^3)e_3,\\
[e_2,e_1] &= -[e_1,e_2] = -(C_{12}^3 - C_{21}^3)e_3 = (C_{21}^3 - C_{12}^3)e_3,\\
[e_2,e_3] &= \mu(e_2,e_3)-\mu(e_3,e_2) = C_{23}^3e_3 - C_{32}^3e_3 = (C_{23}^3 - C_{32}^3)e_3,\\
[e_3,e_1] &= -[e_1,e_3] = -(C_{13}^3 - C_{31}^3)e_3 = (C_{31}^3 - C_{13}^3)e_3,\\
[e_3,e_2] &= -[e_2,e_3] = -(C_{23}^3 - C_{32}^3)e_3 = (C_{32}^3 - C_{23}^3)e_3.
\end{aligned}
$$

Then looking at solution 2, we get

$$
\begin{aligned}
\mu(e_1,e_2) &= C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_1,e_3) &= C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_1) &= C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_3) &= C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_1) &= C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_2) &= C_{32}^1e_1 + C_{32}^2e_2 + C_{32}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
[e_1,e_2] &= \mu(e_1,e_2)-\mu(e_2,e_1) = 0-0 = 0,\\
[e_1,e_3] &= \mu(e_1,e_3)-\mu(e_3,e_1) = 0-0 = 0,\\
[e_2,e_1] &= -[e_1,e_2] = -0 = 0,\\
[e_2,e_3] &= \mu(e_2,e_3)-\mu(e_3,e_2) = 0-0 = 0,\\
[e_3,e_1] &= -[e_1,e_3] = -0 = 0,\\
[e_3,e_2] &= -[e_2,e_3] = -0 = 0.
\end{aligned}
$$

Moving on to solution 3, we get

$$
\begin{aligned}
\mu(e_1,e_2) &= C_{12}^1e_1 + C_{12}^2e_2 + C_{12}^3e_3 = C_{12}^1e_1 + 0e_2 + 0e_3 = C_{12}^1e_1,\\
\mu(e_1,e_3) &= C_{13}^1e_1 + C_{13}^2e_2 + C_{13}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_2,e_1) &= C_{21}^1e_1 + C_{21}^2e_2 + C_{21}^3e_3 = C_{21}^1e_1 + 0e_2 + 0e_3 = C_{21}^1e_1,\\
\mu(e_2,e_3) &= C_{23}^1e_1 + C_{23}^2e_2 + C_{23}^3e_3 = C_{23}^1e_1 + 0e_2 + 0e_3 = C_{23}^1e_1,\\
\mu(e_3,e_1) &= C_{31}^1e_1 + C_{31}^2e_2 + C_{31}^3e_3 = 0e_1 + 0e_2 + 0e_3 = 0,\\
\mu(e_3,e_2) &= C_{32}^1e_1 + C_{32}^2e_2 + C_{32}^3e_3 = C_{32}^1e_1 + 0e_2 + 0e_3 = C_{32}^1e_1.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
[e_1, e_2] &= \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^1 e_1 - C_{21}^1 e_1 = (C_{12}^1 - C_{21}^1) e_1, \\
[e_1, e_3] &= \mu(e_1, e_3) - \mu(e_3, e_1) = 0 - 0 = 0, \\
[e_2, e_1] &= -[e_1, e_2] = -(C_{12}^1 - C_{21}^1) e_1 = (C_{21}^1 - C_{12}^1) e_1, \\
[e_2, e_3] &= \mu(e_2, e_3) - \mu(e_3, e_2) = C_{23}^1 e_1 - C_{32}^1 e_1 = (C_{23}^1 - C_{32}^1) e_1, \\
[e_3, e_1] &= -[e_1, e_3] = -0 = 0, \\
[e_3, e_2] &= -[e_2, e_3] = -(C_{23}^1 - C_{32}^1) e_1 = (C_{32}^1 - C_{23}^1) e_1.
\end{aligned}
$$

Finally looking at solution 4, we get

$$
\begin{aligned}
\mu(e_1, e_2) &= C_{12}^1 e_1 + C_{12}^2 e_2 + C_{12}^3 e_3 = 0 e_1 + C_{12}^2 e_2 + 0 e_3 = C_{12}^2 e_2, \\
\mu(e_1, e_3) &= C_{13}^1 e_1 + C_{13}^2 e_2 + C_{13}^3 e_3 = 0 e_1 + C_{13}^2 e_2 + 0 e_3 = C_{13}^2 e_2, \\
\mu(e_2, e_1) &= C_{21}^1 e_1 + C_{21}^2 e_2 + C_{21}^3 e_3 = 0 e_1 + C_{21}^2 e_2 + 0 e_3 = C_{21}^2 e_2, \\
\mu(e_2, e_3) &= C_{23}^1 e_1 + C_{23}^2 e_2 + C_{23}^3 e_3 = 0 e_1 + 0 e_2 + 0 e_3 = 0, \\
\mu(e_3, e_1) &= C_{31}^1 e_1 + C_{31}^2 e_2 + C_{31}^3 e_3 = 0 e_1 + C_{31}^2 e_2 + 0 e_3 = C_{31}^2 e_2, \\
\mu(e_3, e_2) &= C_{32}^1 e_1 + C_{32}^2 e_2 + C_{32}^3 e_3 = 0 e_1 + 0 e_2 + 0 e_3 = 0.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
[e_1, e_2] &= \mu(e_1, e_2) - \mu(e_2, e_1) = C_{12}^2 e_2 - C_{21}^2 e_2 = (C_{12}^2 - C_{21}^2) e_2, \\
[e_1, e_3] &= \mu(e_1, e_3) - \mu(e_3, e_1) = C_{13}^2 e_2 - C_{31}^2 e_2 = (C_{13}^2 - C_{31}^2) e_2, \\
[e_2, e_1] &= -[e_1, e_2] = -(C_{12}^2 - C_{21}^2) e_2 = (C_{21}^2 - C_{12}^2) e_2, \\
[e_2, e_3] &= \mu(e_2, e_3) - \mu(e_3, e_2) = 0 - 0 = 0, \\
[e_3, e_1] &= -[e_1, e_3] = -(C_{13}^2 - C_{31}^2) e_2 = (C_{31}^2 - C_{13}^2) e_2, \\
[e_3, e_2] &= -[e_2, e_3] = -0 = 0.
\end{aligned}
$$

We write the commutator table for each of these solutions in Table 3.26.

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^3 - C_{21}^3)e_3$	$(C_{13}^3 - C_{31}^3)e_3$
e_2	$(C_{21}^3 - C_{12}^3)e_3$	0	$(C_{23}^3 - C_{32}^3)e_3$
e_3	$(C_{31}^3 - C_{13}^3)e_3$	$(C_{32}^3 - C_{23}^3)e_3$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	0
e_2	0	0	0
e_3	0	0	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^1 - C_{21}^1)e_1$	0
e_2	$(C_{21}^1 - C_{12}^1)e_1$	0	$(C_{23}^1 - C_{32}^1)e_1$
e_3	0	$(C_{32}^1 - C_{23}^1)e_1$	0

Soln 4

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^2 - C_{21}^2)e_2$	$(C_{13}^2 - C_{31}^2)e_2$
e_2	$(C_{21}^2 - C_{12}^2)e_2$	0	0
e_3	$(C_{31}^2 - C_{13}^2)e_2$	0	0

Table 3.26: The commutator tables for the four different hom-Lie admissible algebras with α defined as E_{33} and the structure constants defined as in solution 1, solution 2, solution 3 and solution 4, in three dimensions mapped to one dimension

3.3.2 α defined as E_{12}, E_{23} and E_{31}

We begin by looking at the system of equations for α defined as E_{12}. We found earlier, in Section 3.2.2 that this looks as follows:

$$C_{11}^1C_{11}^1 + C_{11}^2C_{21}^1 + C_{11}^3C_{31}^1 = 0 \tag{R3.15.1}$$
$$C_{11}^1C_{11}^2 + C_{11}^2C_{21}^2 + C_{11}^3C_{31}^2 = 0 \tag{R3.15.2}$$
$$C_{11}^1C_{11}^3 + C_{11}^2C_{21}^3 + C_{11}^3C_{31}^3 = 0 \tag{R3.15.3}$$
$$C_{12}^1C_{11}^1 + C_{12}^2C_{21}^1 + C_{12}^3C_{31}^1 = 0 \tag{R3.19.1}$$
$$C_{12}^1C_{11}^2 + C_{12}^2C_{21}^2 + C_{12}^3C_{31}^2 = 0 \tag{R3.19.2}$$
$$C_{12}^1C_{11}^3 + C_{12}^2C_{21}^3 + C_{12}^3C_{31}^3 = 0 \tag{R3.19.3}$$
$$C_{13}^1C_{11}^1 + C_{13}^2C_{21}^1 + C_{13}^3C_{31}^1 = 0 \tag{R3.21.1}$$
$$C_{13}^1C_{11}^2 + C_{13}^2C_{21}^2 + C_{13}^3C_{31}^2 = 0 \tag{R3.21.2}$$
$$C_{13}^1C_{11}^3 + C_{13}^2C_{21}^3 + C_{13}^3C_{31}^3 = 0 \tag{R3.21.3}$$
$$C_{11}^1C_{11}^1 + C_{11}^2C_{12}^1 + C_{11}^3C_{13}^1 = 0 \tag{R3.23.1}$$

$$C_{11}^1 C_{11}^2 + C_{11}^2 C_{12}^2 + C_{11}^3 C_{13}^2 = 0 \quad \text{(R3.23.2)}$$

$$C_{11}^1 C_{11}^3 + C_{11}^2 C_{12}^3 + C_{11}^3 C_{13}^3 = 0 \quad \text{(R3.23.3)}$$

$$C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1 + C_{12}^3 C_{13}^1 - C_{21}^1 C_{11}^1 - C_{21}^2 C_{21}^1 - C_{21}^3 C_{31}^1 = 0 \quad \text{(R3.24.1)}$$

$$C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 + C_{12}^3 C_{13}^2 - C_{21}^1 C_{11}^2 - C_{21}^2 C_{21}^2 - C_{21}^3 C_{31}^2 = 0 \quad \text{(R3.24.2)}$$

$$C_{12}^1 C_{11}^3 + C_{12}^2 C_{12}^3 + C_{12}^3 C_{13}^3 - C_{21}^1 C_{11}^3 + C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \quad \text{(R3.24.3)}$$

$$C_{13}^1 C_{11}^1 + C_{13}^2 C_{12}^1 + C_{13}^3 C_{13}^1 = 0 \quad \text{(R3.25.1)}$$

$$C_{13}^1 C_{11}^2 + C_{13}^2 C_{12}^2 + C_{13}^3 C_{13}^2 = 0 \quad \text{(R3.25.2)}$$

$$C_{13}^1 C_{11}^3 + C_{13}^2 C_{12}^3 + C_{13}^3 C_{13}^3 = 0 \quad \text{(R3.25.3)}$$

$$C_{21}^1 C_{11}^1 + C_{21}^2 C_{12}^1 + C_{21}^3 C_{13}^1 = 0 \quad \text{(R3.26.1)}$$

$$C_{21}^1 C_{11}^2 + C_{21}^2 C_{12}^2 + C_{21}^3 C_{13}^2 = 0 \quad \text{(R3.26.2)}$$

$$C_{21}^1 C_{11}^3 + C_{21}^2 C_{12}^3 + C_{21}^3 C_{13}^3 = 0 \quad \text{(R3.26.3)}$$

$$C_{31}^1 C_{11}^1 + C_{31}^2 C_{12}^1 + C_{31}^3 C_{13}^1 = 0 \quad \text{(R3.27.1)}$$

$$C_{31}^1 C_{11}^2 + C_{31}^2 C_{12}^2 + C_{31}^3 C_{13}^2 = 0 \quad \text{(R3.27.2)}$$

$$C_{31}^1 C_{11}^3 + C_{31}^2 C_{12}^3 + C_{31}^3 C_{13}^3 = 0 \quad \text{(R3.27.3)}$$

$$C_{22}^2 C_{12}^1 + C_{22}^3 C_{13}^1 - C_{22}^2 C_{21}^1 - C_{22}^3 C_{31}^1 = 0 \quad \text{(R3.28.1)}$$

$$C_{22}^2 C_{12}^2 + C_{22}^3 C_{13}^2 - C_{22}^2 C_{21}^2 - C_{22}^3 C_{31}^2 = 0 \quad \text{(R3.28.2)}$$

$$C_{22}^2 C_{12}^3 + C_{22}^3 C_{13}^3 - C_{22}^2 C_{21}^3 - C_{22}^3 C_{31}^3 = 0 \quad \text{(R3.28.3)}$$

$$C_{23}^1 C_{11}^1 + C_{23}^2 C_{12}^1 + C_{23}^3 C_{13}^1 = 0 \quad \text{(R3.29.1)}$$

$$C_{23}^1 C_{11}^2 + C_{23}^2 C_{12}^2 + C_{23}^3 C_{13}^2 = 0 \quad \text{(R3.29.2)}$$

$$C_{23}^1 C_{11}^3 + C_{23}^2 C_{12}^3 + C_{23}^3 C_{13}^3 = 0 \quad \text{(R3.29.3)}$$

$$C_{32}^1 C_{11}^1 + C_{32}^2 C_{12}^1 + C_{32}^3 C_{13}^1 - C_{23}^1 C_{11}^1 - C_{23}^2 C_{21}^1 - C_{23}^3 C_{31}^1 = 0 \quad \text{(R3.30.1)}$$

$$C_{32}^1 C_{11}^2 + C_{32}^2 C_{12}^2 + C_{32}^3 C_{13}^2 - C_{23}^1 C_{11}^2 - C_{23}^2 C_{21}^2 - C_{23}^3 C_{31}^2 = 0 \quad \text{(R3.30.2)}$$

$$C_{32}^1 C_{11}^3 + C_{32}^2 C_{12}^3 + C_{32}^3 C_{13}^3 - C_{23}^1 C_{11}^3 - C_{23}^2 C_{21}^3 - C_{23}^3 C_{31}^3 = 0 \quad \text{(R3.30.3)}$$

$$C_{33}^1 C_{11}^1 + C_{33}^2 C_{12}^1 + C_{33}^3 C_{13}^1 = 0 \quad \text{(R3.31.1)}$$

$$C_{33}^1 C_{11}^2 + C_{33}^2 C_{12}^2 + C_{33}^3 C_{13}^2 = 0 \quad \text{(R3.31.2)}$$

$$C_{33}^1 C_{11}^3 + C_{33}^2 C_{12}^3 + C_{33}^3 C_{13}^3 = 0 \quad \text{(R3.31.3)}$$

$$C_{31}^1 C_{11}^1 + C_{31}^2 C_{21}^1 + C_{31}^3 C_{31}^1 = 0 \quad \text{(R3.33.1)}$$

$$C_{31}^1 C_{11}^2 + C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = 0 \quad \text{(R3.33.2)}$$

$$C_{31}^1 C_{11}^3 + C_{31}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \quad \text{(R3.33.3)}$$

$$C_{32}^1 C_{11}^1 + C_{32}^2 C_{21}^1 + C_{32}^3 C_{31}^1 = 0 \quad \text{(R3.37.1)}$$

$$C_{32}^1 C_{11}^2 + C_{32}^2 C_{21}^2 + C_{32}^3 C_{31}^2 = 0 \quad \text{(R3.37.2)}$$

$$C_{32}^1 C_{11}^3 + C_{32}^2 C_{21}^3 + C_{32}^3 C_{31}^3 = 0 \quad \text{(R3.37.3)}$$

$$C_{33}^1 C_{11}^1 + C_{33}^2 C_{21}^1 + C_{33}^3 C_{31}^1 = 0 \quad \text{(R3.39.1)}$$

$$C_{33}^1C_{11}^2 + C_{33}^2C_{21}^2 + C_{33}^3C_{31}^2 = 0 \tag{R3.39.2}$$
$$C_{33}^1C_{11}^3 + C_{33}^2C_{21}^3 + C_{33}^3C_{31}^3 = 0. \tag{R3.39.3}$$

Setting all C_{ij}^k equal to zero for $k = 2$ and $k = 3$, our system of equations is reduced to

$$C_{11}^1C_{11}^1 = 0 \tag{R3.15.1}$$
$$C_{12}^1C_{11}^1 = 0 \tag{R3.19.1}$$
$$C_{13}^1C_{11}^1 = 0 \tag{R3.21.1}$$
$$C_{11}^1C_{11}^1 = 0 \tag{R3.23.1}$$
$$C_{12}^1C_{11}^1 - C_{21}^1C_{11}^1 = 0 \tag{R3.24.1}$$
$$C_{13}^1C_{11}^1 = 0 \tag{R3.25.1}$$
$$C_{21}^1C_{11}^1 = 0 \tag{R3.26.1}$$
$$C_{31}^1C_{11}^1 = 0 \tag{R3.27.1}$$
$$C_{23}^1C_{11}^1 = 0 \tag{R3.29.1}$$
$$C_{32}^1C_{11}^1 - C_{23}^1C_{11}^1 = 0 \tag{R3.30.1}$$
$$C_{33}^1C_{11}^1 = 0 \tag{R3.31.1}$$
$$C_{31}^1C_{11}^1 = 0 \tag{R3.33.1}$$
$$C_{32}^1C_{11}^1 = 0 \tag{R3.37.1}$$
$$C_{33}^1C_{11}^1 = 0. \tag{R3.39.1}$$

The only solution to equation (R3.23.1) is $C_{11}^1 = 0$, and this clearly solves the rest of the equations as well. Moving on, we set all C_{ij}^k equal to zero for $k = 3$ and $k = 1$. Our system of equations is then reduced to

$$C_{11}^2C_{21}^2 = 0 \tag{R3.15.2}$$
$$C_{12}^2C_{21}^2 = 0 \tag{R3.19.2}$$
$$C_{13}^2C_{21}^2 = 0 \tag{R3.21.2}$$
$$C_{11}^2C_{12}^2 = 0 \tag{R3.23.2}$$
$$C_{12}^2C_{12}^2 - C_{21}^2C_{21}^2 = 0 \tag{R3.24.2}$$
$$C_{13}^2C_{12}^2 = 0 \tag{R3.25.2}$$
$$C_{21}^2C_{12}^2 = 0 \tag{R3.26.2}$$
$$C_{31}^2C_{12}^2 = 0 \tag{R3.27.2}$$
$$C_{22}^2C_{12}^2 - C_{22}^2C_{21}^2 = 0 \tag{R3.28.2}$$
$$C_{23}^2C_{12}^2 = 0 \tag{R3.29.2}$$
$$C_{32}^2C_{12}^2 - C_{23}^2C_{21}^2 = 0 \tag{R3.30.2}$$
$$C_{33}^2C_{12}^2 = 0 \tag{R3.31.2}$$

$$C^2_{31}C^2_{21} = 0 \tag{R3.33.2}$$
$$C^2_{32}C^2_{21} = 0 \tag{R3.37.2}$$
$$C^2_{33}C^2_{21} = 0. \tag{R3.39.2}$$

Looking at (R3.19.2) we see that the two possible solutions are $C^2_{12} = 0$ and $C^2_{21} = 0$. However, if $C^2_{12} = 0$ then equation (R3.24.2) becomes $-C^2_{21}C^2_{21} = 0$, which only has the solution $C^2_{21} = 0$, and if instead $C^2_{21} = 0$, equation (R3.24.2) becomes $C^2_{12}C^2_{12} = 0$ which only has the solution $C^2_{12} = 0$. Thus, we clearly must have both $C^2_{12} = 0$ and $C^2_{21} = 0$. Looking at the system of equations, we see that this solves all of the remaining equations as well.

Finally, we set $C^k_{ij} = 0$ for $k = 1$ and $k = 2$. This reduces our system of equations to

$$C^3_{11}C^3_{31} = 0 \tag{R3.15.3}$$
$$C^3_{12}C^3_{31} = 0 \tag{R3.19.3}$$
$$C^3_{13}C^3_{31} = 0 \tag{R3.21.3}$$
$$C^3_{11}C^3_{13} = 0 \tag{R3.23.3}$$
$$C^3_{12}C^3_{13} + C^3_{21}C^3_{31} = 0 \tag{R3.24.3}$$
$$C^3_{13}C^3_{13} = 0 \tag{R3.25.3}$$
$$C^3_{21}C^3_{13} = 0 \tag{R3.26.3}$$
$$C^3_{31}C^3_{13} = 0 \tag{R3.27.3}$$
$$C^3_{22}C^3_{13} - C^3_{22}C^3_{31} = 0 \tag{R3.28.3}$$
$$C^3_{23}C^3_{13} = 0 \tag{R3.29.3}$$
$$C^3_{32}C^3_{13} - C^3_{23}C^3_{31} = 0 \tag{R3.30.3}$$
$$C^3_{33}C^3_{13} = 0 \tag{R3.31.3}$$
$$C^3_{31}C^3_{31} = 0 \tag{R3.33.3}$$
$$C^3_{32}C^3_{31} = 0 \tag{R3.37.3}$$
$$C^3_{33}C^3_{31} = 0. \tag{R3.39.3}$$

The only solution to equation (R3.25.3) is $C^3_{13} = 0$ and the only solution to equation (R3.33.3) is $C^3_{31} = 0$. We see that this also solves the rest of the system of equations. We sum up all of our results in Table 3.27.

	Soln 1	Soln 2	Soln 3
C_{11}^1	**0**	0	0
C_{11}^2	0	**free**	0
C_{11}^3	0	0	**free**
C_{12}^1	**free**	0	0
C_{12}^2	0	**0**	0
C_{12}^3	0	0	**free**
C_{13}^1	**free**	0	0
C_{13}^2	0	**free**	0
C_{13}^3	0	0	**0**
C_{21}^1	**free**	0	0
C_{21}^2	0	**0**	0
C_{21}^3	0	0	**free**
C_{22}^1	**free**	0	0
C_{22}^2	0	**free**	0
C_{22}^3	0	0	**free**
C_{23}^1	**free**	0	0
C_{23}^2	0	**free**	0
C_{23}^3	0	0	**free**
C_{31}^1	**free**	0	0
C_{31}^2	0	**free**	0
C_{31}^3	0	0	**0**
C_{32}^1	**free**	0	0
C_{32}^2	0	**free**	0
C_{32}^3	0	0	**free**
C_{33}^1	**free**	0	0
C_{33}^2	0	**free**	0
C_{33}^3	0	0	**free**

Table 3.27: The values of the structure constants that give hom-associative algebras when α is defined as E_{12}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

Permutating the solutions, we get the solutions for when α is defined as E_{23} in Table 3.28, and the solutions for when α is defined as E_{31} in Table 3.29.

	Soln 1	Soln 2	Soln 3
C_{11}^1	0	0	**free**
C_{11}^2	**free**	0	0
C_{11}^3	0	**free**	0
C_{12}^1	0	0	**0**
C_{12}^2	**free**	0	0
C_{12}^3	0	**free**	0
C_{13}^1	0	0	**free**
C_{13}^2	**free**	0	0
C_{13}^3	0	**free**	0
C_{21}^1	0	0	**0**
C_{21}^2	**free**	0	0
C_{21}^3	0	**free**	0
C_{22}^1	0	0	**free**
C_{22}^2	**0**	0	0
C_{22}^3	0	**free**	0
C_{23}^1	0	0	**free**
C_{23}^2	**free**	0	0
C_{23}^3	0	**0**	0
C_{31}^1	0	0	**free**
C_{31}^2	**free**	0	0
C_{31}^3	0	**free**	0
C_{32}^1	0	0	**free**
C_{32}^2	**free**	0	0
C_{32}^3	0	**0**	0
C_{33}^1	0	0	**free**
C_{33}^2	**free**	0	0
C_{33}^3	0	**free**	0

Table 3.28: The values of the structure constants that give hom-associative algebras when α is defined as E_{23}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	Soln 1	Soln 2	Soln 3
C_{11}^{1}	0	**free**	0
C_{11}^{2}	0	0	**free**
C_{11}^{3}	**free**	0	0
C_{12}^{1}	0	**free**	0
C_{12}^{2}	0	0	**free**
C_{12}^{3}	**free**	0	0
C_{13}^{1}	0	**0**	0
C_{13}^{2}	0	0	**free**
C_{13}^{3}	**free**	0	0
C_{21}^{1}	0	**free**	0
C_{21}^{2}	0	0	**free**
C_{21}^{3}	**free**	0	0
C_{22}^{1}	0	**free**	0
C_{22}^{2}	0	0	**free**
C_{22}^{3}	**free**	0	0
C_{23}^{1}	0	**free**	0
C_{23}^{2}	0	0	**0**
C_{23}^{3}	**free**	0	0
C_{31}^{1}	0	**0**	0
C_{31}^{2}	0	0	**free**
C_{31}^{3}	**free**	0	0
C_{32}^{1}	0	**free**	0
C_{32}^{2}	0	0	**0**
C_{32}^{3}	**free**	0	0
C_{33}^{1}	0	**free**	0
C_{33}^{2}	0	0	**free**
C_{33}^{3}	**0**	0	0

Table 3.29: The values of the structure constants that give hom-associative algebras when α is defined as E_{31}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

We see that the solutions for E_{12}, shown in Table 3.27, are exactly the same as solutions 1, 3 and 4 for E_{11}, shown in Table 3.21. Therefore, we can immediately use the results for E_{11} to find the commutators for the hom-Lie admissible algebras associated to the hom-associative algebras where α is defined as E_{12}, by copying the commutator tables found for solutions 1, 3 and 4. Thus, our results for E_{12} are given in Table 3.30.

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C^1_{12}-C^1_{21})e_1$	$(C^1_{13}-C^1_{31})e_1$
e_2	$(C^1_{21}-C^1_{12})e_1$	0	$(C^1_{23}-C^1_{32})e_1$
e_3	$(C^1_{31}-C^1_{13})e_1$	$(C^1_{32}-C^1_{23})e_1$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	$(C^2_{13}-C^2_{31})e_2$
e_2	0	0	$(C^2_{23}-C^2_{32})e_2$
e_3	$(C^2_{31}-C^2_{13})e_2$	$(C^2_{32}-C^2_{23})e_2$	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C^3_{12}-C^3_{21})e_3$	0
e_2	$(C^3_{21}-C^3_{12})e_3$	0	$(C^3_{23}-C^3_{32})e_3$
e_3	0	$(C^3_{32}-C^3_{23})e_3$	0

Table 3.30: The commutator tables for the three different hom-Lie admissible algebras with α defined as E_{12} and the structure constants defined as in solution 1, solution 2 and solution 3, in three dimensions mapped to one dimension

Looking at the solutions for the hom-associative algebras when α is defined as E_{23} and E_{31}, given in Table 3.28 and Table 3.29, respectively, we see that the solutions for E_{23} are identical to solutions 1, 3 and 4 for E_{22}, given in Table 3.22, and the solutions for E_{31} are identical to solutions 1, 3 and 4 for E_{33}, given in Table 3.23. Thus, just as for E_{12}, we can copy the solutions we got for the commutators belonging to the hom-Lie admissible algebras associated to the hom-associative algebras found when α is defined as E_{22} and E_{33}, given in Table 3.25 and Table 3.26. The multiplications for the hom-Lie admissible algebras for E_{23} are given in Table 3.31, and the ones for E_{31} are given in Table 3.32.

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^2 - C_{21}^2)e_2$	$(C_{13}^2 - C_{31}^2)e_2$
e_2	$(C_{21}^2 - C_{12}^2)e_2$	0	$(C_{23}^2 - C_{32}^2)e_2$
e_3	$(C_{31}^2 - C_{13}^2)e_2$	$(C_{32}^2 - C_{23}^2)e_2$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^3 - C_{21}^3)e_3$	$(C_{13}^3 - C_{31}^3)e_3$
e_2	$(C_{21}^3 - C_{12}^3)e_3$	0	0
e_3	$(C_{31}^3 - C_{13}^3)e_3$	0	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	$(C_{13}^1 - C_{31}^1)e_1$
e_2	0	0	$(C_{23}^1 - C_{32}^1)e_1$
e_3	$(C_{31}^1 - C_{13}^1)e_1$	$(C_{32}^1 - C_{23}^1)e_1$	0

Table 3.31: The commutator tables for the three different hom-Lie admissible algebras with α defined as E_{23} and the structure constants defined as in solution 1, solution 2 and solution 3, in three dimensions mapped to one dimension

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^3 - C_{21}^3)e_3$	$(C_{13}^3 - C_{31}^3)e_3$
e_2	$(C_{21}^3 - C_{12}^3)e_3$	0	$(C_{23}^3 - C_{32}^3)e_3$
e_3	$(C_{31}^3 - C_{13}^3)e_3$	$(C_{32}^3 - C_{23}^3)e_3$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^1 - C_{21}^1)e_1$	0
e_2	$(C_{21}^1 - C_{12}^1)e_1$	0	$(C_{23}^1 - C_{32}^1)e_1$
e_3	0	$(C_{32}^1 - C_{23}^1)e_1$	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^2 - C_{21}^2)e_2$	$(C_{13}^2 - C_{31}^2)e_2$
e_2	$(C_{21}^2 - C_{12}^2)e_2$	0	0
e_3	$(C_{31}^2 - C_{13}^2)e_2$	0	0

Table 3.32: The commutator tables for the three different hom-Lie admissible algebras with α defined as E_{31} and the structure constants defined as in solution 1, solution 2 and solution 3, in three dimensions mapped to one dimension

3.3.3 α defined as E_{13}, E_{21} and E_{32}

We begin by looking at the system of equations for α defined as E_{13}. We found earlier, in Section 3.2.3, that this looks as follows:

$$C_{11}^1 C_{11}^1 + C_{11}^2 C_{21}^1 + C_{11}^3 C_{31}^1 = 0 \tag{R3.16.1}$$
$$C_{11}^1 C_{11}^2 + C_{11}^2 C_{21}^2 + C_{11}^3 C_{31}^2 = 0 \tag{R3.16.2}$$
$$C_{11}^1 C_{11}^3 + C_{11}^2 C_{21}^3 + C_{11}^3 C_{31}^3 = 0 \tag{R3.16.3}$$
$$C_{12}^1 C_{11}^1 + C_{12}^2 C_{21}^1 + C_{12}^3 C_{31}^1 = 0 \tag{R3.20.1}$$
$$C_{12}^1 C_{11}^2 + C_{12}^2 C_{21}^2 + C_{12}^3 C_{31}^2 = 0 \tag{R3.20.2}$$
$$C_{12}^1 C_{11}^3 + C_{12}^2 C_{21}^3 + C_{12}^3 C_{31}^3 = 0 \tag{R3.20.3}$$
$$C_{13}^1 C_{11}^1 + C_{13}^2 C_{21}^1 + C_{13}^3 C_{31}^1 = 0 \tag{R3.22.1}$$
$$C_{13}^1 C_{11}^2 + C_{13}^2 C_{21}^2 + C_{13}^3 C_{31}^2 = 0 \tag{R3.22.2}$$
$$C_{13}^1 C_{11}^3 + C_{13}^2 C_{21}^3 + C_{13}^3 C_{31}^3 = 0 \tag{R3.22.3}$$

$$C_{21}^{1}C_{11}^{1}+C_{21}^{2}C_{21}^{1}+C_{21}^{3}C_{31}^{1}=0 \quad \text{(R3.25.1)}$$
$$C_{21}^{1}C_{11}^{2}+C_{21}^{2}C_{21}^{2}+C_{21}^{3}C_{31}^{2}=0 \quad \text{(R3.25.2)}$$
$$C_{21}^{1}C_{11}^{3}+C_{21}^{2}C_{21}^{3}+C_{21}^{3}C_{31}^{3}=0 \quad \text{(R3.25.3)}$$
$$C_{22}^{1}C_{11}^{1}+C_{22}^{2}C_{21}^{1}+C_{22}^{3}C_{31}^{1}=0 \quad \text{(R3.29.1)}$$
$$C_{22}^{1}C_{11}^{2}+C_{22}^{2}C_{21}^{2}+C_{22}^{3}C_{31}^{2}=0 \quad \text{(R3.29.2)}$$
$$C_{22}^{1}C_{11}^{3}+C_{22}^{2}C_{21}^{3}+C_{22}^{3}C_{31}^{3}=0 \quad \text{(R3.29.3)}$$
$$C_{23}^{1}C_{11}^{1}+C_{23}^{2}C_{21}^{1}+C_{23}^{3}C_{31}^{1}=0 \quad \text{(R3.31.1)}$$
$$C_{23}^{1}C_{11}^{2}+C_{23}^{2}C_{21}^{2}+C_{23}^{3}C_{31}^{2}=0 \quad \text{(R3.31.2)}$$
$$C_{23}^{1}C_{11}^{3}+C_{23}^{2}C_{21}^{3}+C_{23}^{3}C_{31}^{3}=0 \quad \text{(R3.31.3)}$$
$$C_{11}^{1}C_{11}^{1}+C_{11}^{2}C_{12}^{1}+C_{11}^{3}C_{13}^{1}=0 \quad \text{(R3.32.1)}$$
$$C_{11}^{1}C_{11}^{2}+C_{11}^{2}C_{12}^{2}+C_{11}^{3}C_{13}^{2}=0 \quad \text{(R3.32.2)}$$
$$C_{11}^{1}C_{11}^{3}+C_{11}^{2}C_{12}^{3}+C_{11}^{3}C_{13}^{3}=0 \quad \text{(R3.32.3)}$$
$$C_{12}^{1}C_{11}^{1}+C_{12}^{2}C_{12}^{1}+C_{12}^{3}C_{13}^{1}=0 \quad \text{(R3.33.1)}$$
$$C_{12}^{1}C_{11}^{2}+C_{12}^{2}C_{12}^{2}+C_{12}^{3}C_{13}^{2}=0 \quad \text{(R3.33.2)}$$
$$C_{12}^{1}C_{11}^{3}+C_{12}^{2}C_{12}^{3}+C_{12}^{3}C_{13}^{3}=0 \quad \text{(R3.33.3)}$$
$$C_{13}^{1}C_{11}^{1}+C_{13}^{2}C_{12}^{1}+C_{13}^{3}C_{13}^{1}-C_{31}^{1}C_{11}^{1}-C_{31}^{2}C_{21}^{1}-C_{31}^{3}C_{31}^{1}=0 \quad \text{(R3.34.1)}$$
$$C_{13}^{1}C_{11}^{2}+C_{13}^{2}C_{12}^{2}+C_{13}^{3}C_{13}^{2}-C_{31}^{1}C_{11}^{2}-C_{31}^{2}C_{21}^{2}-C_{31}^{3}C_{31}^{2}=0 \quad \text{(R3.34.2)}$$
$$C_{13}^{1}C_{11}^{3}+C_{13}^{2}C_{12}^{3}+C_{13}^{3}C_{13}^{3}-C_{31}^{1}C_{11}^{3}-C_{31}^{2}C_{21}^{3}-C_{31}^{3}C_{31}^{3}=0 \quad \text{(R3.34.3)}$$
$$C_{21}^{1}C_{11}^{1}+C_{21}^{2}C_{12}^{1}+C_{21}^{3}C_{13}^{1}=0 \quad \text{(R3.35.1)}$$
$$C_{21}^{1}C_{11}^{2}+C_{21}^{2}C_{12}^{2}+C_{21}^{3}C_{13}^{2}=0 \quad \text{(R3.35.2)}$$
$$C_{21}^{1}C_{11}^{3}+C_{21}^{2}C_{12}^{3}+C_{21}^{3}C_{13}^{3}=0 \quad \text{(R3.35.3)}$$
$$C_{31}^{1}C_{11}^{1}+C_{31}^{2}C_{12}^{1}+C_{31}^{3}C_{13}^{1}=0 \quad \text{(R3.36.1)}$$
$$C_{31}^{1}C_{11}^{2}+C_{31}^{2}C_{12}^{2}+C_{31}^{3}C_{13}^{2}=0 \quad \text{(R3.36.2)}$$
$$C_{31}^{1}C_{11}^{3}+C_{31}^{2}C_{12}^{3}+C_{31}^{3}C_{13}^{3}=0 \quad \text{(R3.36.3)}$$
$$C_{22}^{1}C_{11}^{1}+C_{22}^{2}C_{12}^{1}+C_{22}^{3}C_{13}^{1}=0 \quad \text{(R3.37.1)}$$
$$C_{22}^{1}C_{11}^{2}+C_{22}^{2}C_{12}^{2}+C_{22}^{3}C_{13}^{2}=0 \quad \text{(R3.37.2)}$$
$$C_{22}^{1}C_{11}^{3}+C_{22}^{2}C_{12}^{3}+C_{22}^{3}C_{13}^{3}=0 \quad \text{(R3.37.3)}$$
$$C_{23}^{1}C_{11}^{1}+C_{23}^{2}C_{12}^{1}+C_{23}^{3}C_{13}^{1}-C_{32}^{1}C_{11}^{1}-C_{32}^{2}C_{21}^{1}-C_{32}^{3}C_{31}^{1}=0 \quad \text{(R3.38.1)}$$
$$C_{23}^{1}C_{11}^{2}+C_{23}^{2}C_{12}^{2}+C_{23}^{3}C_{13}^{2}-C_{32}^{1}C_{11}^{2}-C_{32}^{2}C_{21}^{2}-C_{32}^{3}C_{31}^{2}=0 \quad \text{(R3.38.2)}$$
$$C_{23}^{1}C_{11}^{3}+C_{23}^{2}C_{12}^{3}+C_{23}^{3}C_{13}^{3}-C_{32}^{1}C_{11}^{3}-C_{32}^{2}C_{21}^{3}-C_{32}^{3}C_{31}^{3}=0 \quad \text{(R3.38.3)}$$
$$C_{32}^{1}C_{11}^{1}+C_{32}^{2}C_{12}^{1}+C_{32}^{3}C_{13}^{1}=0 \quad \text{(R3.39.1)}$$
$$C_{32}^{1}C_{11}^{2}+C_{32}^{2}C_{12}^{2}+C_{32}^{3}C_{13}^{2}=0 \quad \text{(R3.39.2)}$$
$$C_{32}^{1}C_{11}^{3}+C_{32}^{2}C_{12}^{3}+C_{32}^{3}C_{13}^{3}=0 \quad \text{(R3.39.3)}$$

$$C^2_{33}C^1_{12} + C^3_{33}C^1_{13} - C^2_{33}C^1_{21} - C^3_{33}C^1_{31} = 0 \quad \text{(R3.40.1)}$$
$$C^2_{33}C^2_{12} + C^3_{33}C^2_{13} - C^2_{33}C^2_{21} - C^3_{33}C^2_{31} = 0 \quad \text{(R3.40.2)}$$
$$C^2_{33}C^3_{12} + C^3_{33}C^3_{13} - C^2_{33}C^3_{21} - C^3_{33}C^3_{31} = 0. \quad \text{(R3.40.3)}$$

Setting $C^k_{ij} = 0$ for $k = 2$ and $k = 3$, our system of equations is reduced to

$$C^1_{11}C^1_{11} = 0 \quad \text{(R3.16.1)}$$
$$C^1_{12}C^1_{11} = 0 \quad \text{(R3.20.1)}$$
$$C^1_{13}C^1_{11} = 0 \quad \text{(R3.22.1)}$$
$$C^1_{21}C^1_{11} = 0 \quad \text{(R3.25.1)}$$
$$C^1_{22}C^1_{11} = 0 \quad \text{(R3.29.1)}$$
$$C^1_{23}C^1_{11} = 0 \quad \text{(R3.31.1)}$$
$$C^1_{11}C^1_{11} = 0 \quad \text{(R3.32.1)}$$
$$C^1_{12}C^1_{11} = 0 \quad \text{(R3.33.1)}$$
$$C^1_{13}C^1_{11} - C^1_{31}C^1_{11} = 0 \quad \text{(R3.34.1)}$$
$$C^1_{21}C^1_{11} = 0 \quad \text{(R3.35.1)}$$
$$C^1_{31}C^1_{11} = 0 \quad \text{(R3.36.1)}$$
$$C^1_{22}C^1_{11} = 0 \quad \text{(R3.37.1)}$$
$$C^1_{23}C^1_{11} - C^1_{32}C^1_{11} = 0 \quad \text{(R3.38.1)}$$
$$C^1_{32}C^1_{11} = 0. \quad \text{(R3.39.1)}$$

The only solution to the first equation is $C^1_{11} = 0$, and we see that this also solves the rest of the system of equations. We can then move on to setting $C^k_{ij} = 0$ for $k = 3$ and $k = 1$. This reduces our system of equations to

$$C^2_{11}C^2_{21} = 0 \quad \text{(R3.16.2)}$$
$$C^2_{13}C^2_{21} = 0 \quad \text{(R3.22.2)}$$
$$C^2_{21}C^2_{21} = 0 \quad \text{(R3.25.2)}$$
$$C^2_{22}C^2_{21} = 0 \quad \text{(R3.29.2)}$$
$$C^2_{23}C^2_{21} = 0 \quad \text{(R3.31.2)}$$
$$C^2_{11}C^2_{12} = 0 \quad \text{(R3.32.2)}$$
$$C^2_{12}C^2_{12} = 0 \quad \text{(R3.33.2)}$$
$$C^2_{13}C^2_{12} - C^2_{31}C^2_{21} = 0 \quad \text{(R3.34.2)}$$
$$C^2_{21}C^2_{12} = 0 \quad \text{(R3.35.2)}$$
$$C^2_{31}C^2_{12} = 0 \quad \text{(R3.36.2)}$$
$$C^2_{22}C^2_{12} = 0 \quad \text{(R3.37.2)}$$

$$C^2_{23}C^2_{12} - C^2_{32}C^2_{21} = 0 \quad \text{(R3.38.2)}$$
$$C^2_{32}C^2_{12} = 0 \quad \text{(R3.39.2)}$$
$$C^2_{33}C^2_{12} - C^2_{33}C^2_{21} = 0. \quad \text{(R3.40.2)}$$

The only solution to equation (R3.33.2) is $C^2_{12} = 0$ and the only solution to equation (R3.25.2) is $C^2_{21} = 0$. We see that this also solves the the remaining equations, and can thus move on to finally setting $C^k_{ij} = 0$ for $k = 1$ and $k = 2$. This reduces our system of equations to

$$C^3_{11}C^3_{31} = 0 \quad \text{(R3.16.3)}$$
$$C^3_{12}C^3_{31} = 0 \quad \text{(R3.20.3)}$$
$$C^3_{13}C^3_{31} = 0 \quad \text{(R3.22.3)}$$
$$C^3_{21}C^3_{31} = 0 \quad \text{(R3.25.3)}$$
$$C^3_{22}C^3_{31} = 0 \quad \text{(R3.29.3)}$$
$$C^3_{23}C^3_{31} = 0 \quad \text{(R3.31.3)}$$
$$C^3_{11}C^3_{13} = 0 \quad \text{(R3.32.3)}$$
$$C^3_{12}C^3_{13} = 0 \quad \text{(R3.33.3)}$$
$$C^3_{13}C^3_{13} - C^3_{31}C^3_{31} = 0 \quad \text{(R3.34.3)}$$
$$C^3_{21}C^3_{13} = 0 \quad \text{(R3.35.3)}$$
$$C^3_{31}C^3_{13} = 0 \quad \text{(R3.36.3)}$$
$$C^3_{22}C^3_{13} = 0 \quad \text{(R3.37.3)}$$
$$C^3_{23}C^3_{13} - C^3_{32}C^3_{31} = 0 \quad \text{(R3.38.3)}$$
$$C^3_{32}C^3_{13} = 0 \quad \text{(R3.39.3)}$$
$$C^3_{33}C^3_{13} - C^3_{33}C^3_{31} = 0. \quad \text{(R3.40.3)}$$

The only solutions to equation (R3.36.2) are $C^3_{13} = 0$ and $C^3_{31} = 0$, but due to equation (R3.34.3) both of these variables need to be zero if one of them is, and hence we must have $C^3_{13} = 0$ and $C^3_{31} = 0$. Looking at the system of equations, it is clear that this solves the remaining equations as well. We are thus done, and can sum up our solutions in Table 3.33. Permutating the solutions, we get the solutions for when α is defined as E_{21} in Table 3.34, and the solutions for when α is defined as E_{32} in Table 3.35.

Now, we again want to find the commutators $[\cdot,\cdot]$ for the hom-Lie admissible algebras associated to the hom-associative algebras found above. However, looking at our solutions for the hom-associative algebras above, we see that the solutions for E_{13} are exactly the same as the ones for E_{12}, and the ones for E_{21} and E_{32} are identical to the ones for E_{23} and E_{31}, respectively. Thus, we can simply use the results for the hom-Lie admissible algebras for these values of α directly. The commutator tables for the commutators belonging to the hom-Lie admissible algebras associated to the hom-associative algebras found when α is defined as E_{13}, E_{21} and E_{32} are given in Table 3.36, Table 3.37 and 3.38, respectively.

	Soln 1	Soln 2	Soln 3
C_{11}^1	**0**	0	0
C_{11}^2	0	**free**	0
C_{11}^3	0	0	**free**
C_{12}^1	**free**	0	0
C_{12}^2	0	**0**	0
C_{12}^3	0	0	**free**
C_{13}^1	**free**	0	0
C_{13}^2	0	**free**	0
C_{13}^3	0	0	**0**
C_{21}^1	**free**	0	0
C_{21}^2	0	**0**	0
C_{21}^3	0	0	**free**
C_{22}^1	**free**	0	0
C_{22}^2	0	**free**	0
C_{22}^3	0	0	**free**
C_{23}^1	**free**	0	0
C_{23}^2	0	**free**	0
C_{23}^3	0	0	**free**
C_{31}^1	**free**	0	0
C_{31}^2	0	**free**	0
C_{31}^3	0	0	**0**
C_{32}^1	**free**	0	0
C_{32}^2	0	**free**	0
C_{32}^3	0	0	**free**
C_{33}^1	**free**	0	0
C_{33}^2	0	**free**	0
C_{33}^3	0	0	**free**

Table 3.33: The values of the structure constants that give hom-associative algebras when α is defined as E_{13}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	Soln 1	Soln 2	Soln 3
C_{11}^1	0	0	**free**
C_{11}^2	**free**	0	0
C_{11}^3	0	**free**	0
C_{12}^1	0	0	**0**
C_{12}^2	**free**	0	0
C_{12}^3	0	**free**	0
C_{13}^1	0	0	**free**
C_{13}^2	**free**	0	0
C_{13}^3	0	**free**	0
C_{21}^1	0	0	**0**
C_{21}^2	**free**	0	0
C_{21}^3	0	**free**	0
C_{22}^1	0	0	**free**
C_{22}^2	**0**	0	0
C_{22}^3	0	**free**	0
C_{23}^1	0	0	**free**
C_{23}^2	**free**	0	0
C_{23}^3	0	**0**	0
C_{31}^1	0	0	**free**
C_{31}^2	**free**	0	0
C_{31}^3	0	**free**	0
C_{32}^1	0	0	**free**
C_{32}^2	**free**	0	0
C_{32}^3	0	**0**	0
C_{33}^1	0	0	**free**
C_{33}^2	**free**	0	0
C_{33}^3	0	**free**	0

Table 3.34: The values of the structure constants that give hom-associative algebras when α is defined as E_{21}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	Soln 1	Soln 2	Soln 3
C_{11}^1	0	**free**	0
C_{11}^2	0	0	**free**
C_{11}^3	**free**	0	0
C_{12}^1	0	**free**	0
C_{12}^2	0	0	**free**
C_{12}^3	**free**	0	0
C_{13}^1	0	**0**	0
C_{13}^2	0	0	**free**
C_{13}^3	**free**	0	0
C_{21}^1	0	**free**	0
C_{21}^2	0	0	**free**
C_{21}^3	**free**	0	0
C_{22}^1	0	**free**	0
C_{22}^2	0	0	**free**
C_{22}^3	**free**	0	0
C_{23}^1	0	**free**	0
C_{23}^2	0	0	**0**
C_{23}^3	**free**	0	0
C_{31}^1	0	**0**	0
C_{31}^2	0	0	**free**
C_{31}^3	**free**	0	0
C_{32}^1	0	**free**	0
C_{32}^2	0	0	**0**
C_{32}^3	**free**	0	0
C_{33}^1	0	**free**	0
C_{33}^2	0	0	**free**
C_{33}^3	**0**	0	0

Table 3.35: The values of the structure constants that give hom-associative algebras when α is defined as E_{32}, in three dimensions mapped to one dimension. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^1 - C_{21}^1)e_1$	$(C_{13}^1 - C_{31}^1)e_1$
e_2	$(C_{21}^1 - C_{12}^1)e_1$	0	$(C_{23}^1 - C_{32}^1)e_1$
e_3	$(C_{31}^1 - C_{13}^1)e_1$	$(C_{32}^1 - C_{23}^1)e_1$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	$(C_{13}^2 - C_{31}^2)e_2$
e_2	0	0	$(C_{23}^2 - C_{32}^2)e_2$
e_3	$(C_{31}^2 - C_{13}^2)e_2$	$(C_{32}^2 - C_{23}^2)e_2$	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^3 - C_{21}^3)e_3$	0
e_2	$(C_{21}^3 - C_{12}^3)e_3$	0	$(C_{23}^3 - C_{32}^3)e_3$
e_3	0	$(C_{32}^3 - C_{23}^3)e_3$	0

Table 3.36: The commutator tables for the three different hom-Lie admissible algebras with α defined as E_{13} and the structure constants defined as in solution 1, solution 2 and solution 3, in three dimensions mapped to one dimension

3.3.4 Comparison

As we already noted, we see that the solutions for which values of the structure constants that we get hom-associative algebras are actually exactly the same for $[\alpha] = E_{12}$ (Table 3.27) and $[\alpha] = E_{13}$ (Table 3.33), and $[\alpha] = E_{11}$ (Table 3.21) has the exact same solutions as well plus one extra where every structure constant is zero except for C_{11}^1 which is a free variable. In the same way, $[\alpha] = E_{21}$ (Table 3.34) and $[\alpha] = E_{23}$ (Table 3.28) get the exact same solutions for the structure constants, and $[\alpha] = E_{22}$ (Table 3.22) have the same solutions plus one extra solution where every structure constant is zero except for C_{22}^2 which is a free variable. Predictably, this also holds for the final three cases; $[\alpha] = E_{31}$ (Table 3.29) and $[\alpha] = E_{32}$ (Table 3.35) get the exact same solutions for the structure constants, while $[\alpha] = E_{33}$ (Table 3.29) has these same solutions plus an extra one where every structure constant is zero except for C_{33}^3 which is a free variable.

This of course determines what the commutator tables for the hom-Lie admissible algebras will look like. The three solutions where the only nonzero structure constant is first C_{11}^1, then C_{22}^2, then finally C_{33}^3, of course lead to an all zero commutator table. This means that

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C^2_{12}-C^2_{21})e_2$	$(C^2_{13}-C^2_{31})e_2$
e_2	$(C^2_{21}-C^2_{12})e_2$	0	$(C^2_{23}-C^2_{32})e_2$
e_3	$(C^2_{31}-C^2_{13})e_2$	$(C^2_{32}-C^2_{23})e_2$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C^3_{12}-C^3_{21})e_3$	$(C^3_{13}-C^3_{31})e_3$
e_2	$(C^3_{21}-C^3_{12})e_3$	0	0
e_3	$(C^3_{31}-C^3_{13})e_3$	0	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	0	$(C^1_{13}-C^1_{31})e_1$
e_2	0	0	$(C^1_{23}-C^1_{32})e_1$
e_3	$(C^1_{31}-C^1_{13})e_1$	$(C^1_{32}-C^1_{23})e_1$	0

Table 3.37: The commutator tables for the three different hom-Lie admissible algebras with α defined as E_{21} and the structure constants defined as in solution 1, solution 2 and solution 3, in three dimensions mapped to one dimension

these hom-Lie admissible algebras from $[\alpha] = E_{11}$ (Table 3.24), $[\alpha] = E_{22}$ (Table 3.25) and $[\alpha] = E_{33}$ (Table 3.26) will all be abelian. Apart from this, however, the three hom-Lie admissible algebras with $[\alpha] = E_{11}$, $[\alpha] = E_{12}$ (Table 3.30) and $[\alpha] = E_{13}$ (Table 3.36) will each have the same commutator table for each of the three matrix units, and the same is true for the hom-Lie admissible algebras with $[\alpha] = E_{21}$ (Table 3.37), $[\alpha] = E_{22}$ and $[\alpha] = E_{23}$ (Table 3.31), and again for the ones with $[\alpha] = E_{31}$ (Table 3.32), $[\alpha] = E_{32}$ (Table 3.38) and $[\alpha] = E_{33}$.

Now, what if we compare the commutator tables for for example $[\alpha] = E_{11}$, $[\alpha] = E_{22}$ and $[\alpha] = E_{33}$ (we only need one from each of the three sets of identical commutator tables, so we could for example just as well look at $[\alpha] = E_{12}$, $[\alpha] = E_{21}$ and $[\alpha] = E_{32}$)? Note that we ignore the zero commutator tables, since we have already discussed those. We remember here that the solutions for the structure constants for $[\alpha] = E_{22}$ are actually the same as the ones for $[\alpha] = E_{11}$, just with all the indices permutated according to Theorem 2, and the ones for $[\alpha] = E_{33}$ are the same as the ones for $[\alpha] = E_{22}$ just with all the indices permutated. Thus it makes sense that if we look at the commutator tables for $[\alpha] = E_{11}$ and permutate all of the indices in each table (as in for example in the table belonging to the first solution,

Soln 1

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^3 - C_{21}^3)e_3$	$(C_{13}^3 - C_{31}^3)e_3$
e_2	$(C_{21}^3 - C_{12}^3)e_3$	0	$(C_{23}^3 - C_{32}^3)e_3$
e_3	$(C_{31}^3 - C_{13}^3)e_3$	$(C_{32}^3 - C_{23}^3)e_3$	0

Soln 2

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^1 - C_{21}^1)e_1$	0
e_2	$(C_{21}^1 - C_{12}^1)e_1$	0	$(C_{23}^1 - C_{32}^1)e_1$
e_3	0	$(C_{32}^1 - C_{23}^1)e_1$	0

Soln 3

$[\cdot,\cdot]$	e_1	e_2	e_3
e_1	0	$(C_{12}^2 - C_{21}^2)e_2$	$(C_{13}^2 - C_{31}^2)e_2$
e_2	$(C_{21}^2 - C_{12}^2)e_2$	0	0
e_3	$(C_{31}^2 - C_{13}^2)e_2$	0	0

Table 3.38: The commutator tables for the three different hom-Lie admissible algebras with α defined as E_{32} and the structure constants defined as in solution 1, solution 2 and solution 3, in three dimensions mapped to one dimension

$[e_2, e_1] = (C_{21}^1 - C_{12}^1)e_1$, and permutating the indices we get $[e_3, e_2] = (C_{32}^2 - C_{23}^2)e_2$ that must be in the table belonging to the first solution for $[\alpha] = E_{22}$), we get exactly the commutator tables for the hom-Lie admissible algebras where $[\alpha] = E_{22}$. In the same way, if we permutate the indices in these commutator tables, we get exactly the ones for $[\alpha] = E_{33}$.

Another interesting thing to note is that in all solutions for the structure constants, we only got two possible results for each individual structure constant; either it had to be zero, or it was a free variable. This, in combination with C_{ij}^k only being nonzero for one value of k at a time, led to the multiplications for the hom-Lie admissible only being on two possible forms; either $[e_i, e_j] = 0$ or $[e_i, e_j] = (C_{ij}^k - C_{ji}^k)e_k$, where k here is the value for which the structure is not zero.

3.4 Dimension 3 Mapped to Dimension 2

We were not able to find all solutions to the full system of equations in three dimensions, so in the previous section, Section 3.3, we restricted the range of μ by mapping the systems of equations to only one dimension. This worked well, so we now try to restrict the range of μ a bit less by mapping the systems of equations to two dimensions. This will allow us to hopefully find even more solutions. We do this by letting $C^l_{ij} = 0$ for first $k = 3$, then $k = 1$ and finally $k = 2$, to find all possible two-dimensional solutions. Just as before, since we have concluded that you can find the solutions to α defined as E_{11}, E_{22} and E_{33} by only solving one of them, the same for E_{12}, E_{23} and E_{31}, and the same for E_{13}, E_{21} and E_{32}, we will look at these three subsets of the three-dimensional matrix units separately. We begin with α defined as E_{11}, E_{22} and E_{33}.

3.4.1 α defined as E_{11}, E_{22} and E_{33}

To solve the systems of equations for α defined as E_{11}, E_{22} and E_{33} we simply solve the one for E_{11} and then permutate the results according to Theorem 2 to get the solutions for E_{22} and E_{33}. We already found this system of equations in Section 3.2.1, so we use that here. We begin by setting all C^k_{ij} equal to zero for $k = 3$, the system of equations is reduced to

$$C^2_{11}(C^1_{12} - C^1_{21}) = 0 \qquad \text{(R3.14.1)}$$
$$C^2_{11}(C^2_{12} - C^2_{21}) = 0 \qquad \text{(R3.14.2)}$$
$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} = 0 \qquad \text{(R3.15.1)}$$
$$C^1_{12}C^2_{11} + C^2_{12}C^2_{12} = 0 \qquad \text{(R3.15.2)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} = 0 \qquad \text{(R3.16.1)}$$
$$C^1_{13}C^2_{11} + C^2_{13}C^2_{12} = 0 \qquad \text{(R3.16.2)}$$
$$(C^2_{12} + C^2_{21})(C^1_{12} - C^1_{21}) = 0 \qquad \text{(R3.17.1.2)}$$
$$(C^2_{12} + C^2_{21})(C^2_{12} - C^2_{21}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^2_{13} + C^2_{31})(C^1_{12} - C^1_{21}) = 0 \qquad \text{(R3.18.1.2)}$$
$$(C^2_{13} + C^2_{31})(C^2_{12} - C^2_{21}) = 0 \qquad \text{(R3.18.2.2)}$$
$$C^2_{22}(C^1_{12} - C^1_{21}) = 0 \qquad \text{(R3.19.1.2)}$$
$$C^2_{22}(C^2_{12} - C^2_{21}) = 0 \qquad \text{(R3.19.2.2)}$$
$$C^2_{23}(C^1_{12} - C^1_{21}) = 0 \qquad \text{(R3.20.1.2)}$$
$$C^2_{23}(C^2_{12} - C^2_{21}) = 0 \qquad \text{(R3.20.2.2)}$$
$$C^2_{32}(C^1_{12} - C^1_{21}) = 0 \qquad \text{(R3.21.1.2)}$$
$$C^2_{32}(C^2_{12} - C^2_{21}) = 0 \qquad \text{(R3.21.2.2)}$$
$$C^2_{33}(C^1_{12} - C^1_{21}) = 0 \qquad \text{(R3.22.1.2)}$$
$$C^2_{33}(C^2_{12} - C^2_{21}) = 0 \qquad \text{(R3.22.2.2)}$$

$$C^1_{21}C^1_{11} + C^2_{21}C^1_{21} = 0 \quad \text{(R3.23.1)}$$
$$C^1_{21}C^2_{11} + C^2_{21}C^2_{21} = 0 \quad \text{(R3.23.2)}$$
$$C^1_{22}C^1_{11} + C^2_{22}C^1_{21} = 0 \quad \text{(R3.26.1)}$$
$$C^1_{22}C^2_{11} + C^2_{22}C^2_{21} = 0 \quad \text{(R3.26.2)}$$
$$C^1_{23}C^1_{11} + C^2_{23}C^1_{21} = 0 \quad \text{(R3.27.1)}$$
$$C^1_{23}C^2_{11} + C^2_{23}C^2_{21} = 0 \quad \text{(R3.27.2)}$$
$$C^1_{31}C^1_{11} + C^2_{31}C^1_{21} = 0 \quad \text{(R3.32.1)}$$
$$C^1_{31}C^2_{11} + C^2_{31}C^2_{21} = 0 \quad \text{(R3.32.2)}$$
$$C^1_{32}C^1_{11} + C^2_{32}C^1_{21} = 0 \quad \text{(R3.35.1)}$$
$$C^1_{32}C^2_{11} + C^2_{32}C^2_{21} = 0 \quad \text{(R3.35.2)}$$
$$C^1_{33}C^1_{11} + C^2_{33}C^1_{21} = 0 \quad \text{(R3.36.1)}$$
$$C^1_{33}C^2_{11} + C^2_{33}C^2_{21} = 0. \quad \text{(R3.36.2)}$$

We now try to solve this system of equations. We see that one solution to equations (R3.14.1), (R3.17.1.2), (R3.18.1.2), (R3.19.1.2), (R3.20.1.2), (R3.21.1.2), and (R3.22.1.2) is $C^1_{12} = C^1_{21}$. The only other solution to all of these equations is $C^2_{11} = C^2_{12} + C^2_{21} = C^2_{13} + C^2_{31} = C^2_{22} = C^2_{23} = C^2_{32} = C^2_{33} = 0$. We look at each of these cases.

Case 1: If $C^2_{11} = C^2_{12} + C^2_{21} = C^2_{13} + C^2_{31} = C^2_{22} = C^2_{23} = C^2_{32} = C^2_{33} = 0$, the system of equations becomes

$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} = 0 \quad \text{(R3.15.1)}$$
$$C^2_{12}C^2_{12} = 0 \quad \text{(R3.15.2)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} = 0 \quad \text{(R3.16.1)}$$
$$C^2_{13}C^2_{12} = 0 \quad \text{(R3.16.2)}$$
$$C^1_{21}C^1_{11} - C^2_{12}C^1_{21} = 0 \quad \text{(R3.23.1)}$$
$$C^2_{12}C^2_{12} = 0 \quad \text{(R3.23.2)}$$
$$C^1_{22}C^1_{11} = 0 \quad \text{(R3.26.1)}$$
$$C^1_{23}C^1_{11} = 0 \quad \text{(R3.27.1)}$$
$$C^1_{31}C^1_{11} - C^2_{13}C^1_{21} = 0 \quad \text{(R3.32.1)}$$
$$C^2_{13}C^2_{12} = 0 \quad \text{(R3.32.2)}$$
$$C^1_{32}C^1_{11} = 0 \quad \text{(R3.35.1)}$$
$$C^1_{33}C^1_{11} = 0. \quad \text{(R3.36.1)}$$

The only solution to equation (R3.15.2) is $C^2_{12} = 0$. This gives us

$$C^1_{12}C^1_{11} = 0 \quad \text{(R3.15.1)}$$

$$C^1_{13}C^1_{11}+C^2_{13}C^1_{12}=0 \qquad \text{(R3.16.1)}$$
$$C^1_{21}C^1_{11}=0 \qquad \text{(R3.23.1)}$$
$$C^1_{22}C^1_{11}=0 \qquad \text{(R3.26.1)}$$
$$C^1_{23}C^1_{11}=0 \qquad \text{(R3.27.1)}$$
$$C^1_{31}C^1_{11}-C^2_{13}C^1_{21}=0 \qquad \text{(R3.32.1)}$$
$$C^1_{32}C^1_{11}=0 \qquad \text{(R3.35.1)}$$
$$C^1_{33}C^1_{11}=0. \qquad \text{(R3.36.1)}$$

We now see that one solution to equations (R3.15.1), (R3.23.1), (R3.26.1), (R3.27.1), (R3.35.1) and (R3.36.1) is $C^1_{11}=0$, and the only other solution is $C^1_{12}=C^1_{21}=C^1_{22}=C^1_{23}=C^1_{32}=C^1_{33}=0$. We look at each of these cases.

Case 1.1: If $C^1_{11}=0$, the system of equations becomes

$$C^2_{13}C^1_{12}=0 \qquad \text{(R3.16.1)}$$
$$-C^2_{13}C^1_{21}=0. \qquad \text{(R3.32.1)}$$

Now, one solution is $C^2_{13}=0$, while the other solution is $C^1_{12}=C^1_{21}=0$, and we have now solved the entire system of equations.

Case 1.2: If instead $C^1_{12}=C^1_{21}=C^1_{22}=C^1_{23}=C^1_{32}=C^1_{33}=0$ and $C^1_{11}\neq 0$, the system of equations becomes

$$C^1_{13}C^1_{11}=0 \qquad \text{(R3.16.1)}$$
$$C^1_{31}C^1_{11}=0. \qquad \text{(R3.32.1)}$$

Since we have assumed that $C^1_{11}\neq 0$, the only possible solution to these two equations is $C^1_{13}=C^1_{31}=0$, and we have now solved the entire system of equations.

Case 2: If instead $C^1_{12}=C^1_{21}$, $C^2_{11}\neq 0$, $C^2_{12}+C^2_{21}\neq 0$, $C^2_{13}+C^2_{31}\neq 0$, $C^2_{22}\neq 0$, $C^2_{23}\neq 0$, $C^2_{32}\neq 0$ and $C^2_{33}\neq 0$, the system of equations becomes

$$C^2_{11}(C^2_{12}-C^2_{21})=0 \qquad \text{(R3.14.2)}$$
$$C^1_{12}C^1_{11}+C^2_{12}C^1_{12}=0 \qquad \text{(R3.15.1)}$$
$$C^1_{12}C^2_{11}+C^2_{12}C^2_{12}=0 \qquad \text{(R3.15.2)}$$
$$C^1_{13}C^1_{11}+C^2_{13}C^1_{12}=0 \qquad \text{(R3.16.1)}$$
$$C^1_{13}C^2_{11}+C^2_{13}C^2_{12}=0 \qquad \text{(R3.16.2)}$$
$$(C^2_{12}+C^2_{21})(C^2_{12}-C^2_{21})=0 \qquad \text{(R3.17.2.2)}$$
$$(C^2_{13}+C^2_{31})(C^2_{12}-C^2_{21})=0 \qquad \text{(R3.18.2.2)}$$

$$C^2_{22}(C^2_{12} - C^2_{21}) = 0 \quad \text{(R3.19.2.2)}$$
$$C^2_{23}(C^2_{12} - C^2_{21}) = 0 \quad \text{(R3.20.2.2)}$$
$$C^2_{32}(C^2_{12} - C^2_{21}) = 0 \quad \text{(R3.21.2.2)}$$
$$C^2_{33}(C^2_{12} - C^2_{21}) = 0 \quad \text{(R3.22.2.2)}$$
$$C^1_{12}C^1_{11} + C^2_{21}C^1_{12} = 0 \quad \text{(R3.23.1)}$$
$$C^1_{12}C^2_{11} + C^2_{21}C^2_{21} = 0 \quad \text{(R3.23.2)}$$
$$C^1_{22}C^1_{11} + C^2_{22}C^1_{12} = 0 \quad \text{(R3.26.1)}$$
$$C^1_{22}C^2_{11} + C^2_{22}C^2_{21} = 0 \quad \text{(R3.26.2)}$$
$$C^1_{23}C^1_{11} + C^2_{23}C^1_{12} = 0 \quad \text{(R3.27.1)}$$
$$C^1_{23}C^2_{11} + C^2_{23}C^2_{21} = 0 \quad \text{(R3.27.2)}$$
$$C^1_{31}C^1_{11} + C^2_{31}C^1_{12} = 0 \quad \text{(R3.32.1)}$$
$$C^1_{31}C^2_{11} + C^2_{31}C^2_{21} = 0 \quad \text{(R3.32.2)}$$
$$C^1_{32}C^1_{11} + C^2_{32}C^1_{12} = 0 \quad \text{(R3.35.1)}$$
$$C^1_{32}C^2_{11} + C^2_{32}C^2_{21} = 0 \quad \text{(R3.35.2)}$$
$$C^1_{33}C^1_{11} + C^2_{33}C^1_{12} = 0 \quad \text{(R3.36.1)}$$
$$C^1_{33}C^2_{11} + C^2_{33}C^2_{21} = 0. \quad \text{(R3.36.2)}$$

Since we have assumed that $C^2_{11} \neq 0$, the only solution to equation (R3.14.2) is $C^2_{12} - C^2_{21} = 0$. This gives us

$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} = 0 \quad \text{(R3.15.1)}$$
$$C^1_{12}C^2_{11} + C^2_{12}C^2_{12} = 0 \quad \text{(R3.15.2)}$$
$$C^1_{13}C^1_{11} + C^2_{13}C^1_{12} = 0 \quad \text{(R3.16.1)}$$
$$C^1_{13}C^2_{11} + C^2_{13}C^2_{12} = 0 \quad \text{(R3.16.2)}$$
$$C^1_{12}C^1_{11} + C^2_{12}C^1_{12} = 0 \quad \text{(R3.23.1)}$$
$$C^1_{12}C^2_{11} + C^2_{12}C^2_{12} = 0 \quad \text{(R3.23.2)}$$
$$C^1_{22}C^1_{11} + C^2_{22}C^1_{12} = 0 \quad \text{(R3.26.1)}$$
$$C^1_{22}C^2_{11} + C^2_{22}C^2_{12} = 0 \quad \text{(R3.26.2)}$$
$$C^1_{23}C^1_{11} + C^2_{23}C^1_{12} = 0 \quad \text{(R3.27.1)}$$
$$C^1_{23}C^2_{11} + C^2_{23}C^2_{12} = 0 \quad \text{(R3.27.2)}$$
$$C^1_{31}C^1_{11} + C^2_{31}C^1_{12} = 0 \quad \text{(R3.32.1)}$$
$$C^1_{31}C^2_{11} + C^2_{31}C^2_{12} = 0 \quad \text{(R3.32.2)}$$
$$C^1_{32}C^1_{11} + C^2_{32}C^1_{12} = 0 \quad \text{(R3.35.1)}$$
$$C^1_{32}C^2_{11} + C^2_{32}C^2_{12} = 0 \quad \text{(R3.35.2)}$$
$$C^1_{33}C^1_{11} + C^2_{33}C^1_{12} = 0 \quad \text{(R3.36.1)}$$

$$C_{33}^1 C_{11}^2 + C_{33}^2 C_{12}^2 = 0. \qquad \text{(R3.36.2)}$$

Equation (R3.15.1) can be rewritten as $C_{12}^1 C_{11}^1 + C_{12}^2 C_{12}^1 = C_{12}^1 (C_{11}^1 + C_{12}^2) = 0$, which has the solution $C_{12}^1 = 0$ and $C_{11}^1 + C_{12}^2 = 0$. We look at each of these cases.

Case 2.1: If $C_{12}^1 = 0$, the system of equations becomes

$$\begin{aligned}
C_{12}^2 C_{12}^2 &= 0 && \text{(R3.15.2)}\\
C_{13}^1 C_{11}^1 &= 0 && \text{(R3.16.1)}\\
C_{13}^1 C_{11}^2 + C_{13}^2 C_{12}^2 &= 0 && \text{(R3.16.2)}\\
C_{12}^2 C_{12}^2 &= 0 && \text{(R3.23.2)}\\
C_{22}^1 C_{11}^1 &= 0 && \text{(R3.26.1)}\\
C_{22}^1 C_{11}^2 + C_{22}^2 C_{12}^2 &= 0 && \text{(R3.26.2)}\\
C_{23}^1 C_{11}^1 &= 0 && \text{(R3.27.1)}\\
C_{23}^1 C_{11}^2 + C_{23}^2 C_{12}^2 &= 0 && \text{(R3.27.2)}\\
C_{31}^1 C_{11}^1 &= 0 && \text{(R3.32.1)}\\
C_{31}^1 C_{11}^2 + C_{31}^2 C_{12}^2 &= 0 && \text{(R3.32.2)}\\
C_{32}^1 C_{11}^1 &= 0 && \text{(R3.35.1)}\\
C_{32}^1 C_{11}^2 + C_{32}^2 C_{12}^2 &= 0 && \text{(R3.35.2)}\\
C_{33}^1 C_{11}^1 &= 0 && \text{(R3.36.1)}\\
C_{33}^1 C_{11}^2 + C_{33}^2 C_{12}^2 &= 0. && \text{(R3.36.2)}
\end{aligned}$$

The only solution to equation (R3.15.2) is $C_{12}^2 = 0$. However, we remember that we have both assumed that $C_{12}^2 + C_{21}^2 \neq 0$ and concluded that we must have $C_{12}^2 - C_{21}^2 = 0$. If $C_{12}^2 = 0$, we will have both $C_{21}^2 \neq 0$ and $C_{21}^2 = 0$, which is clearly a contradiction. Thus, this case did not lead to a solution.

Case 2.2: If instead $C_{11}^1 + C_{12}^2 = 0$ and $C_{12}^1 \neq 0$, the system of equations becomes

$$\begin{aligned}
C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 && \text{(R3.15.2)}\\
-C_{13}^1 C_{12}^2 + C_{13}^2 C_{12}^1 &= 0 && \text{(R3.16.1)}\\
C_{13}^1 C_{11}^2 + C_{13}^2 C_{12}^2 &= 0 && \text{(R3.16.2)}\\
C_{12}^1 C_{11}^2 + C_{12}^2 C_{12}^2 &= 0 && \text{(R3.23.2)}\\
-C_{22}^1 C_{12}^2 + C_{22}^2 C_{12}^1 &= 0 && \text{(R3.26.1)}\\
C_{22}^1 C_{11}^2 + C_{22}^2 C_{12}^2 &= 0 && \text{(R3.26.2)}\\
-C_{23}^1 C_{12}^2 + C_{23}^2 C_{12}^1 &= 0 && \text{(R3.27.1)}\\
C_{23}^1 C_{11}^2 + C_{23}^2 C_{12}^2 &= 0 && \text{(R3.27.2)}\\
-C_{31}^1 C_{12}^2 + C_{31}^2 C_{12}^1 &= 0 && \text{(R3.32.1)}
\end{aligned}$$

$$C^1_{31}C^2_{11} + C^2_{31}C^2_{12} = 0 \qquad \text{(R3.32.2)}$$
$$-C^1_{32}C^2_{12} + C^2_{32}C^1_{12} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{32}C^2_{11} + C^2_{32}C^2_{12} = 0 \qquad \text{(R3.35.2)}$$
$$-C^1_{33}C^2_{12} + C^2_{33}C^1_{12} = 0 \qquad \text{(R3.36.1)}$$
$$C^1_{33}C^2_{11} + C^2_{33}C^2_{12} = 0. \qquad \text{(R3.36.2)}$$

Since we have assumed that $C^1_{12} \neq 0$, we can write the solution to equation (R3.15.2) as $C^2_{11} = -C^2_{12}C^2_{12}/C^1_{12}$. Inserting this into the system of equations gives us

$$-C^1_{13}C^2_{12} + C^2_{13}C^1_{12} = 0 \qquad \text{(R3.16.1)}$$
$$-C^1_{13}C^2_{12}C^2_{12}/C^1_{12} + C^2_{13}C^2_{12} = 0 \qquad \text{(R3.16.2)}$$
$$-C^1_{22}C^2_{12} + C^2_{22}C^1_{12} = 0 \qquad \text{(R3.26.1)}$$
$$-C^1_{22}C^2_{12}C^2_{12}/C^1_{12} + C^2_{22}C^2_{12} = 0 \qquad \text{(R3.26.2)}$$
$$-C^1_{23}C^2_{12} + C^2_{23}C^1_{12} = 0 \qquad \text{(R3.27.1)}$$
$$-C^1_{23}C^2_{12}C^2_{12}/C^1_{12} + C^2_{23}C^2_{12} = 0 \qquad \text{(R3.27.2)}$$
$$-C^1_{31}C^2_{12} + C^2_{31}C^1_{12} = 0 \qquad \text{(R3.32.1)}$$
$$-C^1_{31}C^2_{12}C^2_{12}/C^1_{12} + C^2_{31}C^2_{12} = 0 \qquad \text{(R3.32.2)}$$
$$-C^1_{32}C^2_{12} + C^2_{32}C^1_{12} = 0 \qquad \text{(R3.35.1)}$$
$$-C^1_{32}C^2_{12}C^2_{12}/C^1_{12} + C^2_{32}C^2_{12} = 0 \qquad \text{(R3.35.2)}$$
$$-C^1_{33}C^2_{12} + C^2_{33}C^1_{12} = 0 \qquad \text{(R3.36.1)}$$
$$-C^1_{33}C^2_{12}C^2_{12}/C^1_{12} + C^2_{33}C^2_{12} = 0. \qquad \text{(R3.36.2)}$$

In the same way, we rewrite the following equations:

$$\text{(R3.16.1)}: \quad C^2_{13} = C^1_{13}C^2_{12}/C^1_{12},$$
$$\text{(R3.26.1)}: \quad C^2_{22} = C^1_{22}C^2_{12}/C^1_{12},$$
$$\text{(R3.27.1)}: \quad C^2_{23} = C^1_{23}C^2_{12}/C^1_{12},$$
$$\text{(R3.32.1)}: \quad C^2_{31} = C^1_{31}C^2_{12}/C^1_{12},$$
$$\text{(R3.35.1)}: \quad C^2_{32} = C^1_{32}C^2_{12}/C^1_{12},$$
$$\text{(R3.36.1)}: \quad C^2_{33} = C^1_{33}C^2_{12}/C^1_{12}.$$

Inserting this into the remaining equations gives us

$$\begin{aligned}\text{(R3.16.2)}: \quad & -C^1_{13}C^2_{12}C^2_{12}/C^1_{12} + C^2_{13}C^2_{12} \\ & = -C^1_{13}C^2_{12}C^2_{12}/C^1_{12} + C^1_{13}C^2_{12}C^2_{12}/C^1_{12} \\ & = 0, \\ \text{(R3.26.2)}: \quad & -C^1_{22}C^2_{12}C^2_{12}/C^1_{12} + C^2_{22}C^2_{12}\end{aligned}$$

$$
\begin{aligned}
&= -C^1_{22}C^2_{12}C^2_{12}/C^1_{12} + C^1_{22}C^2_{12}C^2_{12}/C^1_{12}\\
&= 0,\\
\text{(R3.27.2)}:\quad & -C^1_{23}C^2_{12}C^2_{12}/C^1_{12} + C^2_{23}C^2_{12}\\
&= -C^1_{23}C^2_{12}C^2_{12}/C^1_{12} + C^1_{23}C^2_{12}C^2_{12}/C^1_{12}\\
&= 0,\\
\text{(R3.32.2)}:\quad & -C^1_{31}C^2_{12}C^2_{12}/C^1_{12} + C^2_{31}C^2_{12}\\
&= -C^1_{31}C^2_{12}C^2_{12}/C^1_{12} + C^1_{31}C^2_{12}C^2_{12}/C^1_{12}\\
&= 0,\\
\text{(R3.35.2)}:\quad & -C^1_{32}C^2_{12}C^2_{12}/C^1_{12} + C^2_{32}C^2_{12}\\
&= -C^1_{32}C^2_{12}C^2_{12}/C^1_{12} + C^1_{32}C^2_{12}C^2_{12}/C^1_{12}\\
&= 0,\\
\text{(R3.36.2)}:\quad & -C^1_{33}C^2_{12}C^2_{12}/C^1_{12} + C^2_{33}C^2_{12}\\
&= -C^1_{33}C^2_{12}C^2_{12}/C^1_{12} + C^1_{33}C^2_{12}C^2_{12}/C^1_{12}\\
&= 0.
\end{aligned}
$$

Clearly these solutions solved the remaining equations as well, and we have thus solved the entire system of equations.

We have now found all solutions for the case where $C^k_{ij} = 0$ when $k = 3$. We move on to the case where $C^k_{ij} = 0$ when $k = 1$.

$$C^2_{11}(C^2_{12} - C^2_{21}) + C^3_{11}(C^2_{13} - C^2_{31}) = 0 \tag{R3.14.2}$$
$$C^2_{11}(C^3_{12} - C^3_{21}) + C^3_{11}(C^3_{13} - C^3_{31}) = 0 \tag{R3.14.3}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \tag{R3.15.2}$$
$$C^2_{12}C^3_{12} + C^3_{12}C^3_{13} = 0 \tag{R3.15.3}$$
$$C^2_{13}C^2_{12} + C^3_{13}C^2_{13} = 0 \tag{R3.16.2}$$
$$C^2_{13}C^3_{12} + C^3_{13}C^3_{13} = 0 \tag{R3.16.3}$$
$$(C^2_{12} + C^2_{21})(C^2_{12} - C^2_{21}) + (C^3_{12} + C^3_{21})(C^2_{13} - C^2_{31}) = 0 \tag{R3.17.2.2}$$
$$(C^2_{12} + C^2_{21})(C^3_{12} - C^3_{21}) + (C^3_{12} + C^3_{21})(C^3_{13} - C^3_{31}) = 0 \tag{R3.17.3.2}$$
$$(C^2_{13} + C^2_{31})(C^2_{12} - C^2_{21}) + (C^3_{13} + C^3_{31})(C^2_{13} - C^2_{31}) = 0 \tag{R3.18.2.2}$$
$$(C^2_{13} + C^2_{31})(C^3_{12} - C^3_{21}) + (C^3_{13} + C^3_{31})(C^3_{13} - C^3_{31}) = 0 \tag{R3.18.3.2}$$
$$C^2_{22}(C^2_{12} - C^2_{21}) + C^3_{22}(C^2_{13} - C^2_{31}) = 0 \tag{R3.19.2.2}$$
$$C^2_{22}(C^3_{12} - C^3_{21}) + C^3_{22}(C^3_{13} - C^3_{31}) = 0 \tag{R3.19.3.2}$$
$$C^2_{23}(C^2_{12} - C^2_{21}) + C^3_{23}(C^2_{13} - C^2_{31}) = 0 \tag{R3.20.2.2}$$
$$C^2_{23}(C^3_{12} - C^3_{21}) + C^3_{23}(C^3_{13} - C^3_{31}) = 0 \tag{R3.20.3.2}$$
$$C^2_{32}(C^2_{12} - C^2_{21}) + C^3_{32}(C^2_{13} - C^2_{31}) = 0 \tag{R3.21.2.2}$$

$$C_{32}^2(C_{12}^3 - C_{21}^3) + C_{32}^3(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.21.3.2)}$$
$$C_{33}^2(C_{12}^2 - C_{21}^2) + C_{33}^3(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.22.2.2)}$$
$$C_{33}^2(C_{12}^3 - C_{21}^3) + C_{33}^3(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.22.3.2)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^2 C_{21}^2 + C_{22}^3 C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^2 C_{21}^3 + C_{22}^3 C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^2 C_{21}^2 + C_{23}^3 C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^2 C_{21}^3 + C_{23}^3 C_{31}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.32.2)}$$
$$C_{31}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^2 C_{21}^2 + C_{32}^3 C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^2 C_{21}^3 + C_{32}^3 C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^2 C_{21}^2 + C_{33}^3 C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^2 C_{21}^3 + C_{33}^3 C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

We notice that we can simplify equations (R3.19.2.2) to (R3.22.3.2) by adding equations (R3.26.2) to (R3.36.3), except for equations (R3.32.2) and (R3.32.3), in order. This gives us

$$C_{11}^2(C_{12}^2 - C_{21}^2) + C_{11}^3(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^2(C_{12}^3 - C_{21}^3) + C_{11}^3(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.14.3)}$$
$$C_{12}^2 C_{12}^2 + C_{12}^3 C_{13}^2 = 0 \qquad \text{(R3.15.2)}$$
$$C_{12}^2 C_{12}^3 + C_{12}^3 C_{13}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^2 C_{12}^2 + C_{13}^3 C_{13}^2 = 0 \qquad \text{(R3.16.2)}$$
$$C_{13}^2 C_{12}^3 + C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.16.3)}$$
$$(C_{12}^2 + C_{21}^2)(C_{12}^2 - C_{21}^2) + (C_{12}^3 + C_{21}^3)(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C_{12}^2 + C_{21}^2)(C_{12}^3 - C_{21}^3) + (C_{12}^3 + C_{21}^3)(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C_{13}^2 + C_{31}^2)(C_{12}^2 - C_{21}^2) + (C_{13}^3 + C_{31}^3)(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C_{13}^2 + C_{31}^2)(C_{12}^3 - C_{21}^3) + (C_{13}^3 + C_{31}^3)(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{22}^2 C_{12}^2 + C_{22}^3 C_{13}^2 = 0 \qquad \text{(R3.19.2.2)} + \text{(R3.26.2)}$$
$$C_{22}^2 C_{12}^3 + C_{22}^3 C_{13}^3 = 0 \qquad \text{(R3.19.3.2)} + \text{(R3.26.3)}$$
$$C_{23}^2 C_{12}^2 + C_{23}^3 C_{13}^2 = 0 \qquad \text{(R3.20.2.2)} + \text{(R3.27.2)}$$
$$C_{23}^2 C_{12}^3 + C_{23}^3 C_{13}^3 = 0 \qquad \text{(R3.20.3.2)} + \text{(R3.27.3)}$$
$$C_{32}^2 C_{12}^2 + C_{32}^3 C_{13}^2 = 0 \qquad \text{(R3.21.2.2)} + \text{(R3.35.2)}$$
$$C_{32}^2 C_{12}^3 + C_{32}^3 C_{13}^3 = 0 \qquad \text{(R3.21.3.2)} + \text{(R3.35.3)}$$

$$C_{33}^2 C_{12}^2 + C_{33}^3 C_{13}^2 = 0 \qquad \text{(R3.22.2.2)} + \text{(R3.36.2)}$$
$$C_{33}^2 C_{12}^3 + C_{33}^3 C_{13}^3 = 0 \qquad \text{(R3.22.3.2)} + \text{(R3.36.3)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^2 C_{21}^2 + C_{22}^3 C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^2 C_{21}^3 + C_{22}^3 C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^2 C_{21}^2 + C_{23}^3 C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^2 C_{21}^3 + C_{23}^3 C_{31}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.32.2)}$$
$$C_{31}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^2 C_{21}^2 + C_{32}^3 C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^2 C_{21}^3 + C_{32}^3 C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^2 C_{21}^2 + C_{33}^3 C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^2 C_{21}^3 + C_{33}^3 C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

We now rename the equations changed in the last step, so that (R3.19.2.2) + (R3.26.2) becomes ((R3.19.2.2).2) and so on. Our system of equations will then look as follows:

$$C_{11}^2 (C_{12}^2 - C_{21}^2) + C_{11}^3 (C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^2 (C_{12}^3 - C_{21}^3) + C_{11}^3 (C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.14.3)}$$
$$C_{12}^2 C_{12}^2 + C_{12}^3 C_{13}^2 = 0 \qquad \text{(R3.15.2)}$$
$$C_{12}^2 C_{12}^3 + C_{12}^3 C_{13}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^2 C_{12}^2 + C_{13}^3 C_{13}^2 = 0 \qquad \text{(R3.16.2)}$$
$$C_{13}^2 C_{12}^3 + C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.16.3)}$$
$$(C_{12}^2 + C_{21}^2)(C_{12}^2 - C_{21}^2) + (C_{12}^3 + C_{21}^3)(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C_{12}^2 + C_{21}^2)(C_{12}^3 - C_{21}^3) + (C_{12}^3 + C_{21}^3)(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C_{13}^2 + C_{31}^2)(C_{12}^2 - C_{21}^2) + (C_{13}^3 + C_{31}^3)(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C_{13}^2 + C_{31}^2)(C_{12}^3 - C_{21}^3) + (C_{13}^3 + C_{31}^3)(C_{13}^3 - C_{31}^3) = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{22}^2 C_{12}^2 + C_{22}^3 C_{13}^2 = 0 \qquad \text{(R3.19.2.2.2)}$$
$$C_{22}^2 C_{12}^3 + C_{22}^3 C_{13}^3 = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C_{23}^2 C_{12}^2 + C_{23}^3 C_{13}^2 = 0 \qquad \text{(R3.20.2.2.2)}$$
$$C_{23}^2 C_{12}^3 + C_{23}^3 C_{13}^3 = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C_{32}^2 C_{12}^2 + C_{32}^3 C_{13}^2 = 0 \qquad \text{(R3.21.2.2.2)}$$
$$C_{32}^2 C_{12}^3 + C_{32}^3 C_{13}^3 = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C_{33}^2 C_{12}^2 + C_{33}^3 C_{13}^2 = 0 \qquad \text{(R3.22.2.2.2)}$$

$$C^2_{33}C^3_{12}+C^3_{33}C^3_{13}=0 \quad \text{(R3.22.3.2.2)}$$
$$C^2_{21}C^2_{21}+C^3_{21}C^2_{31}=0 \quad \text{(R3.23.2)}$$
$$C^2_{21}C^3_{21}+C^3_{21}C^3_{31}=0 \quad \text{(R3.23.3)}$$
$$C^2_{22}C^2_{21}+C^3_{22}C^2_{31}=0 \quad \text{(R3.26.2)}$$
$$C^2_{22}C^3_{21}+C^3_{22}C^3_{31}=0 \quad \text{(R3.26.3)}$$
$$C^2_{23}C^2_{21}+C^3_{23}C^2_{31}=0 \quad \text{(R3.27.2)}$$
$$C^2_{23}C^3_{21}+C^3_{23}C^3_{31}=0 \quad \text{(R3.27.3)}$$
$$C^2_{31}C^2_{21}+C^3_{31}C^2_{31}=0 \quad \text{(R3.32.2)}$$
$$C^2_{31}C^3_{21}+C^3_{31}C^3_{31}=0 \quad \text{(R3.32.3)}$$
$$C^2_{32}C^2_{21}+C^3_{32}C^2_{31}=0 \quad \text{(R3.35.2)}$$
$$C^2_{32}C^3_{21}+C^3_{32}C^3_{31}=0 \quad \text{(R3.35.3)}$$
$$C^2_{33}C^2_{21}+C^3_{33}C^2_{31}=0 \quad \text{(R3.36.2)}$$
$$C^2_{33}C^3_{21}+C^3_{33}C^3_{31}=0. \quad \text{(R3.36.3)}$$

Now, we see that due to commutativity, we can rewrite equation (R3.16.2) as $C^2_{13}C^2_{12}+C^3_{13}C^2_{13}=C^2_{13}C^2_{12}+C^2_{13}C^3_{13}=C^2_{13}(C^2_{12}+C^3_{13})=0$. This has two solutions, $C^2_{13}=0$ and $C^2_{12}+C^3_{13}=0$. We look at each of these cases.

Case 1: If $C^2_{13}=0$, the system of equations becomes

$$C^2_{11}(C^2_{12}-C^2_{21})-C^3_{11}C^2_{31}=0 \quad \text{(R3.14.2)}$$
$$C^2_{11}(C^3_{12}-C^3_{21})+C^3_{11}(C^3_{13}-C^3_{31})=0 \quad \text{(R3.14.3)}$$
$$C^2_{12}C^2_{12}=0 \quad \text{(R3.15.2)}$$
$$C^2_{12}C^3_{12}+C^3_{12}C^3_{13}=0 \quad \text{(R3.15.3)}$$
$$C^3_{13}C^3_{13}=0 \quad \text{(R3.16.3)}$$
$$(C^2_{12}+C^2_{21})(C^2_{12}-C^2_{21})-(C^3_{12}+C^3_{21})C^2_{31}=0 \quad \text{(R3.17.2.2)}$$
$$(C^2_{12}+C^2_{21})(C^3_{12}-C^3_{21})+(C^3_{12}+C^3_{21})(C^3_{13}-C^3_{31})=0 \quad \text{(R3.17.3.2)}$$
$$C^2_{31}(C^2_{12}-C^2_{21})-(C^3_{13}+C^3_{31})C^2_{31}=0 \quad \text{(R3.18.2.2)}$$
$$C^2_{31}(C^3_{12}-C^3_{21})+(C^3_{13}+C^3_{31})(C^3_{13}-C^3_{31})=0 \quad \text{(R3.18.3.2)}$$
$$C^2_{22}C^2_{12}=0 \quad \text{(R3.19.2.2.2)}$$
$$C^2_{22}C^3_{12}+C^3_{22}C^3_{13}=0 \quad \text{(R3.19.3.2.2)}$$
$$C^2_{23}C^2_{12}=0 \quad \text{(R3.20.2.2.2)}$$
$$C^2_{23}C^3_{12}+C^3_{23}C^3_{13}=0 \quad \text{(R3.20.3.2.2)}$$
$$C^2_{32}C^2_{12}=0 \quad \text{(R3.21.2.2.2)}$$
$$C^2_{32}C^3_{12}+C^3_{32}C^3_{13}=0 \quad \text{(R3.21.3.2.2)}$$
$$C^2_{33}C^2_{12}=0 \quad \text{(R3.22.2.2.2)}$$

$$C_{33}^2 C_{12}^3 + C_{33}^3 C_{13}^3 = 0 \qquad \text{(R3.22.3.2.2)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^2 C_{21}^2 + C_{22}^3 C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^2 C_{21}^3 + C_{22}^3 C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^2 C_{21}^2 + C_{23}^3 C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^2 C_{21}^3 + C_{23}^3 C_{31}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.32.2)}$$
$$C_{31}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^2 C_{21}^2 + C_{32}^3 C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^2 C_{21}^3 + C_{32}^3 C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^2 C_{21}^2 + C_{33}^3 C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^2 C_{21}^3 + C_{33}^3 C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

The only solution to equation (R3.15.2) is $C_{12}^2 = 0$, and the only solution to equation (R3.16.3) is $C_{13}^3 = 0$. This gives us

$$-C_{11}^2 C_{21}^2 - C_{11}^3 C_{31}^2 = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^2 (C_{12}^3 - C_{21}^3) - C_{11}^3 C_{31}^3 = 0 \qquad \text{(R3.14.3)}$$
$$-C_{21}^2 C_{21}^2 - (C_{12}^3 + C_{21}^3) C_{31}^2 = 0 \qquad \text{(R3.17.2.2)}$$
$$C_{21}^2 (C_{12}^3 - C_{21}^3) - (C_{12}^3 + C_{21}^3) C_{31}^3 = 0 \qquad \text{(R3.17.3.2)}$$
$$-C_{31}^2 C_{21}^2 - C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.18.2.2)}$$
$$C_{31}^2 (C_{12}^3 - C_{21}^3) - C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{22}^2 C_{12}^3 = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C_{23}^2 C_{12}^3 = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C_{32}^2 C_{12}^3 = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C_{33}^2 C_{12}^3 = 0 \qquad \text{(R3.22.3.2.2)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^2 C_{21}^2 + C_{22}^3 C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^2 C_{21}^3 + C_{22}^3 C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^2 C_{21}^2 + C_{23}^3 C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^2 C_{21}^3 + C_{23}^3 C_{31}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.32.2)}$$
$$C_{31}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.32.3)}$$

$$C_{32}^2 C_{21}^2 + C_{32}^3 C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^2 C_{21}^3 + C_{32}^3 C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^2 C_{21}^2 + C_{33}^3 C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^2 C_{21}^3 + C_{33}^3 C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

We now see that one solution to equations (R3.19.3.2.2), (R3.20.3.2.2), (R3.21.3.2.2) and (R3.22.3.2.2) is $C_{12}^3 = 0$, and the only other solution is $C_{22}^2 = C_{23}^2 = C_{32}^2 = C_{33}^2 = 0$. We again look at each case separately.

Case 1.1: If $C_{22}^2 = C_{23}^2 = C_{32}^2 = C_{33}^2 = 0$, the system of equations becomes

$$-C_{11}^2 C_{21}^2 - C_{11}^3 C_{31}^2 = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^2 (C_{12}^3 - C_{21}^3) - C_{11}^3 C_{31}^3 = 0 \qquad \text{(R3.14.3)}$$
$$-C_{21}^2 C_{21}^2 - (C_{12}^3 + C_{21}^3) C_{31}^2 = 0 \qquad \text{(R3.17.2.2)}$$
$$C_{21}^2 (C_{12}^3 - C_{21}^3) - (C_{12}^3 + C_{21}^3) C_{31}^3 = 0 \qquad \text{(R3.17.3.2)}$$
$$-C_{31}^2 C_{21}^2 - C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.18.2.2)}$$
$$C_{31}^2 (C_{12}^3 - C_{21}^3) - C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^3 C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^3 C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^3 C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^3 C_{31}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.32.2)}$$
$$C_{31}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^3 C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^3 C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^3 C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^3 C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

Looking at equations (R3.26.2), (R3.27.2), (R3.35.2) and (R3.36.2) we see that one solution is $C_{31}^2 = 0$, and the only other solution is $C_{22}^3 = C_{23}^3 = C_{32}^3 = C_{33}^3 = 0$.

Case 1.1.1: If $C_{22}^3 = C_{23}^3 = C_{32}^3 = C_{33}^3 = 0$, the system of equations becomes

$$-C_{11}^2 C_{21}^2 - C_{11}^3 C_{31}^2 = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^2 (C_{12}^3 - C_{21}^3) - C_{11}^3 C_{31}^3 = 0 \qquad \text{(R3.14.3)}$$
$$-C_{21}^2 C_{21}^2 - (C_{12}^3 + C_{21}^3) C_{31}^2 = 0 \qquad \text{(R3.17.2.2)}$$

$$C^2_{21}(C^3_{12} - C^3_{21}) - (C^3_{12} + C^3_{21})C^3_{31} = 0 \qquad \text{(R3.17.3.2)}$$
$$-C^2_{31}C^2_{21} - C^3_{31}C^2_{31} = 0 \qquad \text{(R3.18.2.2)}$$
$$C^2_{31}(C^3_{12} - C^3_{21}) - C^3_{31}C^3_{31} = 0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{21}C^2_{21} + C^3_{21}C^2_{31} = 0 \qquad \text{(R3.23.2)}$$
$$C^2_{21}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^2_{31}C^2_{21} + C^3_{31}C^2_{31} = 0 \qquad \text{(R3.32.2)}$$
$$C^2_{31}C^3_{21} + C^3_{31}C^3_{31} = 0. \qquad \text{(R3.32.3)}$$

Since $-C^2_{31}C^2_{21} - C^3_{31}C^2_{31} = -(C^2_{31}C^2_{21} + C^3_{31}C^2_{31}) = 0 \Rightarrow C^2_{31}C^2_{21} + C^3_{31}C^2_{31} = 0$, equation (R3.18.2.2) is the same as equation (R3.32.2) and can be removed. Then, we note that, due to commutativity, we can rewrite equation (R3.32.2) as $C^2_{31}C^2_{21} + C^3_{31}C^2_{31} = C^2_{31}C^2_{21} + C^2_{31}C^3_{31} = C^2_{31}(C^2_{21} + C^3_{31}) = 0$. This clearly has two solutions, $C^2_{31} = 0$ and $C^2_{21} + C^3_{31} = 0$.

Case 1.1.1.1: If $C^2_{31} = 0$, the system of equations becomes

$$-C^2_{11}C^2_{21} = 0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}(C^3_{12} - C^3_{21}) - C^3_{11}C^3_{31} = 0 \qquad \text{(R3.14.3)}$$
$$-C^2_{21}C^2_{21} = 0 \qquad \text{(R3.17.2.2)}$$
$$C^2_{21}(C^3_{12} - C^3_{21}) - (C^3_{12} + C^3_{21})C^3_{31} = 0 \qquad \text{(R3.17.3.2)}$$
$$C^3_{31}C^3_{31} = 0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{21}C^2_{21} = 0 \qquad \text{(R3.23.2)}$$
$$C^2_{21}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^3_{31}C^3_{31} = 0. \qquad \text{(R3.32.3)}$$

The only solution to equation (R3.17.2.2) is $C^2_{21} = 0$, and the only solution to equation (R3.18.3.2) is $C^3_{31} = 0$. Then all that remains of the system of equations is

$$C^2_{11}(C^3_{12} - C^3_{21}) = 0. \qquad \text{(R3.14.3)}$$

The two possible solutions to this final equation are $C^2_{11} = 0$ and $C^3_{12} = C^3_{21}$, and we have now finished solving the entire system of equations.

Case 1.1.1.2: If instead $C^2_{31} \neq 0$ and $C^2_{21} + C^3_{31} = 0$, the system of equations becomes

$$-C^2_{11}C^2_{21} - C^3_{11}C^2_{31} = 0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}(C^3_{12} - C^3_{21}) + C^3_{11}C^2_{21} = 0 \qquad \text{(R3.14.3)}$$
$$-C^2_{21}C^2_{21} - (C^3_{12} + C^3_{21})C^2_{31} = 0 \qquad \text{(R3.17.2.2)}$$

$$C_{21}^2(C_{12}^3 - C_{21}^3) + (C_{12}^3 + C_{21}^3)C_{21}^2 = 0 \qquad \text{(R3.17.3.2)}$$
$$C_{31}^2(C_{12}^3 - C_{21}^3) - C_{21}^2 C_{21}^2 = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{31}^2 C_{21}^3 - C_{21}^2 C_{21}^2 = 0. \qquad \text{(R3.32.3)}$$

We see that equation (R3.17.3.2) can be rewritten as $C_{21}^2(C_{12}^3 - C_{21}^3) + (C_{12}^3 + C_{21}^3)C_{21}^2 = C_{21}^2 C_{12}^3 - C_{21}^2 C_{21}^3 + C_{12}^3 C_{21}^2 + C_{21}^3 C_{21}^2 = C_{21}^2 C_{12}^3 + C_{12}^3 C_{21}^2 = 2C_{12}^3 C_{21}^2 = 0$. This clearly has the two solutions, $C_{12}^3 = 0$ and $C_{21}^2 = 0$.

Case 1.1.1.2.1: If $C_{12}^3 = 0$, the system of equations becomes

$$-C_{11}^2 C_{21}^2 - C_{11}^3 C_{31}^2 = 0 \qquad \text{(R3.14.2)}$$
$$-C_{11}^2 C_{21}^3 + C_{11}^3 C_{21}^2 = 0 \qquad \text{(R3.14.3)}$$
$$-C_{21}^2 C_{21}^2 - C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.17.2.2)}$$
$$-C_{31}^2 C_{21}^3 - C_{21}^2 C_{21}^2 = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{31}^2 C_{21}^3 - C_{21}^2 C_{21}^2 = 0. \qquad \text{(R3.32.3)}$$

First of all, we note that, after a bit of rewriting, both equation (R3.17.2.2) and equation (R3.18.3.2) are actually the exact same equation as equation (R3.23.2) and can be removed. Then, we add equation (R3.23.2) to equation (R3.32.3) and get $C_{31}^2 C_{21}^3 - C_{21}^2 C_{21}^2 + C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 2C_{21}^3 C_{31}^2$. Since we have assumed that $C_{31}^2 \neq 0$, the only solution to this is $C_{21}^3 = 0$. Thus, our system of equations becomes

$$-C_{11}^2 C_{21}^2 - C_{11}^3 C_{31}^2 = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^3 C_{21}^2 = 0 \qquad \text{(R3.14.3)}$$
$$C_{21}^2 C_{21}^2 = 0. \qquad \text{(R3.23.2)}$$

Clearly, the only solution to equation (R3.32.3) is now $C_{21}^2 = 0$. Then, the only thing remaining of our system of equations is

$$-C_{11}^3 C_{31}^2 = 0. \qquad \text{(R3.14.2)}$$

Again, since we have assumed that $C_{31}^2 \neq 0$, the only solution to this is $C_{11}^3 = 0$, and we have now solved the entire system of equations.

Case 1.1.1.2.2: If instead $C_{21}^2 = 0$ and $C_{12}^3 \neq 0$, the system of equations becomes

$$\begin{aligned}
-C_{11}^3 C_{31}^2 &= 0 && \text{(R3.14.2)}\\
C_{11}^2(C_{12}^3 - C_{21}^3) &= 0 && \text{(R3.14.3)}\\
-(C_{12}^3 + C_{21}^3)C_{31}^2 &= 0 && \text{(R3.17.2.2)}\\
C_{31}^2(C_{12}^3 - C_{21}^3) &= 0 && \text{(R3.18.3.2)}\\
C_{21}^3 C_{31}^2 &= 0 && \text{(R3.23.2)}\\
C_{31}^2 C_{21}^3 &= 0. && \text{(R3.32.3)}
\end{aligned}$$

Since we have assumed that $C_{31}^2 \neq 0$, the only solution to equation (R3.32.3) id $C_{21}^3 = 0$. This gives us

$$\begin{aligned}
-C_{11}^3 C_{31}^2 &= 0 && \text{(R3.14.2)}\\
C_{11}^2 C_{12}^3 &= 0 && \text{(R3.14.3)}\\
-C_{12}^3 C_{31}^2 &= 0 && \text{(R3.17.2.2)}\\
C_{31}^2 C_{12}^3 &= 0. && \text{(R3.18.3.2)}
\end{aligned}$$

Now, we have assumed that both $C_{31}^2 \neq 0$ and $C_{12}^3 \neq 0$, which means there is no possible solution to equation (R3.18.3.2). Thus, this case did not lead to any solution.

Case 1.1.2: If instead $C_{31}^2 = 0$, $C_{22}^3 \neq 0$, $C_{23}^3 \neq 0$, $C_{32}^3 \neq 0$ and $C_{33}^3 \neq 0$, the system of equations becomes

$$\begin{aligned}
-C_{11}^2 C_{21}^2 &= 0 && \text{(R3.14.2)}\\
C_{11}^2(C_{12}^3 - C_{21}^3) - C_{11}^3 C_{31}^3 &= 0 && \text{(R3.14.3)}\\
-C_{21}^2 C_{21}^2 &= 0 && \text{(R3.17.2.2)}\\
C_{21}^2(C_{12}^3 - C_{21}^3) - (C_{12}^3 + C_{21}^3)C_{31}^3 &= 0 && \text{(R3.17.3.2)}\\
-C_{31}^3 C_{31}^3 &= 0 && \text{(R3.18.3.2)}\\
C_{21}^2 C_{21}^2 &= 0 && \text{(R3.23.2)}\\
C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 &= 0 && \text{(R3.23.3)}\\
C_{22}^3 C_{31}^3 &= 0 && \text{(R3.26.3)}\\
C_{23}^3 C_{31}^3 &= 0 && \text{(R3.27.3)}\\
C_{31}^3 C_{31}^3 &= 0 && \text{(R3.32.3)}\\
C_{32}^3 C_{31}^3 &= 0 && \text{(R3.35.3)}\\
C_{33}^3 C_{31}^3 &= 0. && \text{(R3.36.3)}
\end{aligned}$$

The only solution to equation (R3.17.2.2) is $C^2_{21} = 0$. This gives us

$$C^2_{11}(C^3_{12} - C^3_{21}) - C^3_{11}C^3_{31} = 0 \qquad \text{(R3.14.3)}$$
$$-(C^3_{12} + C^3_{21})C^3_{31} = 0 \qquad \text{(R3.17.3.2)}$$
$$-C^3_{31}C^3_{31} = 0 \qquad \text{(R3.18.3.2)}$$
$$C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^3_{22}C^3_{31} = 0 \qquad \text{(R3.26.3)}$$
$$C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

Now, the only solution to equation (R3.18.3.2) is $C^3_{31} = 0$, which means that all that remains of our system of equations is now

$$C^2_{11}(C^3_{12} - C^3_{21}) = 0. \qquad \text{(R3.14.3)}$$

Clearly, the two solutions to this final equation are $C^2_{11} = 0$ and $C^3_{12} = C^3_{21}$, and we have now solved the entire system of equations.

Case 1.2: If instead $C^3_{12} = 0$, $C^2_{22} \neq 0$, $C^2_{23} \neq 0$, $C^2_{32} \neq 0$ and $C^2_{33} \neq 0$, the system of equations becomes

$$-C^2_{11}C^2_{21} - C^3_{11}C^2_{31} = 0 \qquad \text{(R3.14.2)}$$
$$-C^2_{11}C^3_{21} - C^3_{11}C^3_{31} = 0 \qquad \text{(R3.14.3)}$$
$$-C^2_{21}C^2_{21} - C^3_{21}C^2_{31} = 0 \qquad \text{(R3.17.2.2)}$$
$$-C^2_{21}C^3_{21} - C^3_{21}C^3_{31} = 0 \qquad \text{(R3.17.3.2)}$$
$$-C^2_{31}C^2_{21} - C^3_{31}C^2_{31} = 0 \qquad \text{(R3.18.2.2)}$$
$$-C^2_{31}C^3_{21} - C^3_{31}C^3_{31} = 0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{21}C^2_{21} + C^3_{21}C^2_{31} = 0 \qquad \text{(R3.23.2)}$$
$$C^2_{21}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^2_{22}C^2_{21} + C^3_{22}C^2_{31} = 0 \qquad \text{(R3.26.2)}$$
$$C^2_{22}C^3_{21} + C^3_{22}C^3_{31} = 0 \qquad \text{(R3.26.3)}$$
$$C^2_{23}C^2_{21} + C^3_{23}C^2_{31} = 0 \qquad \text{(R3.27.2)}$$
$$C^2_{23}C^3_{21} + C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^2_{31}C^2_{21} + C^3_{31}C^2_{31} = 0 \qquad \text{(R3.32.2)}$$
$$C^2_{31}C^3_{21} + C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^2_{32}C^2_{21} + C^3_{32}C^2_{31} = 0 \qquad \text{(R3.35.2)}$$

$$C_{32}^2 C_{21}^3 + C_{32}^3 C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^2 C_{21}^2 + C_{33}^3 C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^2 C_{21}^3 + C_{33}^3 C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

We see that we can rewrite equation (R3.17.2.2) to be identical to equation (R3.23.2), and thus we can remove it. In the same way, equation (R3.17.3.2) is the same as equation (R3.23.3), equation (R3.18.2.2) is the same as equation (R3.32.2), and (R3.18.3.2) is the same as equation (R3.32.3), and we can remove these three equations too. Then, we end up with

$$-C_{11}^2 C_{21}^2 - C_{11}^3 C_{31}^2 = 0 \qquad \text{(R3.14.2)}$$
$$-C_{11}^2 C_{21}^3 - C_{11}^3 C_{31}^3 = 0 \qquad \text{(R3.14.3)}$$
$$C_{21}^2 C_{21}^2 + C_{21}^3 C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^2 C_{21}^2 + C_{22}^3 C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^2 C_{21}^3 + C_{22}^3 C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^2 C_{21}^2 + C_{23}^3 C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^2 C_{21}^3 + C_{23}^3 C_{31}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = 0 \qquad \text{(R3.32.2)}$$
$$C_{31}^2 C_{21}^3 + C_{31}^3 C_{31}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^2 C_{21}^2 + C_{32}^3 C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^2 C_{21}^3 + C_{32}^3 C_{31}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^2 C_{21}^2 + C_{33}^3 C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^2 C_{21}^3 + C_{33}^3 C_{31}^3 = 0. \qquad \text{(R3.36.3)}$$

We can rewrite equations (R3.23.3) and (R3.32.2) as $C_{21}^2 C_{21}^3 + C_{21}^3 C_{31}^3 = C_{21}^3(C_{21}^2 + C_{31}^3)$ and $C_{31}^2 C_{21}^2 + C_{31}^3 C_{31}^2 = C_{31}^2(C_{21}^2 + C_{31}^3) = 0$, respectively. This means that to solve both of these equations, we must either have $C_{21}^3 = 0$ and $C_{31}^2 = 0$, or $C_{21}^2 + C_{31}^3 = 0$. We look at each of these cases.

Case 1.2.1: If $C_{21}^3 = 0$ and $C_{31}^2 = 0$, the system of equations becomes

$$-C_{11}^2 C_{21}^2 = 0 \qquad \text{(R3.14.2)}$$
$$-C_{11}^3 C_{31}^3 = 0 \qquad \text{(R3.14.3)}$$
$$C_{21}^2 C_{21}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{22}^2 C_{21}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^3 C_{31}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^2 C_{21}^2 = 0 \qquad \text{(R3.27.2)}$$

$$C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^2_{32}C^2_{21} = 0 \qquad \text{(R3.35.2)}$$
$$C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^2_{33}C^2_{21} = 0 \qquad \text{(R3.36.2)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

We now see that the only possible solution to equation (R3.23.2) is $C^2_{21} = 0$, and the only possible solution to equation (R3.32.3) is $C^3_{31} = 0$. Looking at the rest of the system of equations, we see that this actually solves all of the remaining equations as well, and we are done.

Case 1.2.2: If instead $C^2_{21} + C^3_{31} = 0$, $C^3_{21} \neq 0$ and $C^2_{31} \neq 0$, the system of equations becomes

$$-C^2_{11}C^2_{21} - C^3_{11}C^2_{31} = 0 \qquad \text{(R3.14.2)}$$
$$-C^2_{11}C^3_{21} + C^3_{11}C^2_{21} = 0 \qquad \text{(R3.14.3)}$$
$$C^2_{21}C^2_{21} + C^3_{21}C^2_{31} = 0 \qquad \text{(R3.23.2)}$$
$$C^2_{22}C^2_{21} + C^3_{22}C^2_{31} = 0 \qquad \text{(R3.26.2)}$$
$$C^2_{22}C^3_{21} - C^3_{22}C^2_{21} = 0 \qquad \text{(R3.26.3)}$$
$$C^2_{23}C^2_{21} + C^3_{23}C^2_{31} = 0 \qquad \text{(R3.27.2)}$$
$$C^2_{23}C^3_{21} - C^3_{23}C^2_{21} = 0 \qquad \text{(R3.27.3)}$$
$$C^2_{31}C^3_{21} + C^2_{21}C^2_{21} = 0 \qquad \text{(R3.32.3)}$$
$$C^2_{32}C^2_{21} + C^3_{32}C^2_{31} = 0 \qquad \text{(R3.35.2)}$$
$$C^2_{32}C^3_{21} - C^3_{32}C^2_{21} = 0 \qquad \text{(R3.35.3)}$$
$$C^2_{33}C^2_{21} + C^3_{33}C^2_{31} = 0 \qquad \text{(R3.36.2)}$$
$$C^2_{33}C^3_{21} - C^3_{33}C^2_{21} = 0. \qquad \text{(R3.36.3)}$$

Now, we see that thanks to commutativity we can rewrite equation (R3.23.2) so that it is identical to equation (R3.32.3), and we can thus remove it since it gives no additional information. What remains of our system of equations is then

$$-C^2_{11}C^2_{21} - C^3_{11}C^2_{31} = 0 \qquad \text{(R3.14.2)}$$
$$-C^2_{11}C^3_{21} + C^3_{11}C^2_{21} = 0 \qquad \text{(R3.14.3)}$$
$$C^2_{22}C^2_{21} + C^3_{22}C^2_{31} = 0 \qquad \text{(R3.26.2)}$$
$$C^2_{22}C^3_{21} - C^3_{22}C^2_{21} = 0 \qquad \text{(R3.26.3)}$$
$$C^2_{23}C^2_{21} + C^3_{23}C^2_{31} = 0 \qquad \text{(R3.27.2)}$$
$$C^2_{23}C^3_{21} - C^3_{23}C^2_{21} = 0 \qquad \text{(R3.27.3)}$$

$$
\begin{aligned}
C_{31}^2C_{21}^3 + C_{21}^2C_{21}^2 &= 0 && \text{(R3.32.3)}\\
C_{32}^2C_{21}^2 + C_{32}^3C_{31}^2 &= 0 && \text{(R3.35.2)}\\
C_{32}^2C_{21}^3 - C_{32}^3C_{21}^2 &= 0 && \text{(R3.35.3)}\\
C_{33}^2C_{21}^2 + C_{33}^3C_{31}^2 &= 0 && \text{(R3.36.2)}\\
C_{33}^2C_{21}^3 - C_{33}^3C_{21}^2 &= 0. && \text{(R3.36.3)}
\end{aligned}
$$

Since we have assumed that $C_{21}^3 \neq 0$, we can write the solution to equation (R3.32.3) as $C_{31}^2 = -C_{21}^2C_{21}^2/C_{21}^3$. This gives us

$$
\begin{aligned}
-C_{11}^2C_{21}^2 + C_{11}^3C_{21}^2C_{21}^2/C_{21}^3 &= 0 && \text{(R3.14.2)}\\
-C_{11}^2C_{21}^3 + C_{11}^3C_{21}^2 &= 0 && \text{(R3.14.3)}\\
C_{22}^2C_{21}^2 - C_{22}^3C_{21}^2C_{21}^2/C_{21}^3 &= 0 && \text{(R3.26.2)}\\
C_{22}^2C_{21}^3 - C_{22}^3C_{21}^2 &= 0 && \text{(R3.26.3)}\\
C_{23}^2C_{21}^2 - C_{23}^3C_{21}^2C_{21}^2/C_{21}^3 &= 0 && \text{(R3.27.2)}\\
C_{23}^2C_{21}^3 - C_{23}^3C_{21}^2 &= 0 && \text{(R3.27.3)}\\
C_{32}^2C_{21}^2 - C_{32}^3C_{21}^2C_{21}^2/C_{21}^3 &= 0 && \text{(R3.35.2)}\\
C_{32}^2C_{21}^3 - C_{32}^3C_{21}^2 &= 0 && \text{(R3.35.3)}\\
C_{33}^2C_{21}^2 - C_{33}^3C_{21}^2C_{21}^2/C_{21}^3 &= 0 && \text{(R3.36.2)}\\
C_{33}^2C_{21}^3 - C_{33}^3C_{21}^2 &= 0. && \text{(R3.36.3)}
\end{aligned}
$$

Now, we again use the assumption that $C_{21}^3 \neq 0$ to rewrite the equations still having C_{21}^3 as a numerator, as seen below:

$$
\begin{aligned}
\text{(R3.14.3)}: &\quad C_{11}^2 = C_{11}^3C_{21}^2/C_{21}^3,\\
\text{(R3.26.3)}: &\quad C_{22}^2 = C_{22}^3C_{21}^2/C_{21}^3,\\
\text{(R3.27.3)}: &\quad C_{23}^2 = C_{23}^3C_{21}^2/C_{21}^3,\\
\text{(R3.35.3)}: &\quad C_{32}^2 = C_{32}^3C_{21}^2/C_{21}^3,\\
\text{(R3.36.3)}: &\quad C_{33}^2 = C_{33}^3C_{21}^2/C_{21}^3.
\end{aligned}
$$

Inserting these solutions into the remaining equations, we get

$$
\begin{aligned}
\text{(R3.14.2)}: &\quad -C_{11}^2C_{21}^2 + C_{11}^3C_{21}^2C_{21}^2/C_{21}^3\\
&\quad = -C_{11}^3C_{21}^2C_{21}^2/C_{21}^3 + C_{11}^3C_{21}^2C_{21}^2/C_{21}^3\\
&\quad = 0,\\
\text{(R3.26.2)}: &\quad C_{22}^2C_{21}^2 - C_{22}^3C_{21}^2C_{21}^2/C_{21}^3\\
&\quad = C_{22}^3C_{21}^2C_{21}^2/C_{21}^3 - C_{22}^3C_{21}^2C_{21}^2/C_{21}^3\\
&\quad = 0,\\
\text{(R3.27.2)}: &\quad C_{23}^2C_{21}^2 - C_{23}^3C_{21}^2C_{21}^2/C_{21}^3
\end{aligned}
$$

$$= C^3_{23}C^2_{21}C^2_{21}/C^3_{21} - C^3_{23}C^2_{21}C^2_{21}/C^3_{21}$$
$$= 0,$$

$$\text{(R3.35.2)}: \quad C^2_{32}C^2_{21} - C^3_{32}C^2_{21}C^2_{21}/C^3_{21}$$
$$= C^3_{32}C^2_{21}C^2_{21}/C^3_{21} - C^3_{32}C^2_{21}C^2_{21}/C^3_{21}$$
$$= 0,$$

$$\text{(R3.36.2)}: \quad C^2_{33}C^2_{21} - C^3_{33}C^2_{21}C^2_{21}/C^3_{21}$$
$$= C^3_{33}C^2_{21}C^2_{21}/C^3_{21} - C^3_{33}C^2_{21}C^2_{21}/C^3_{21}$$
$$= 0.$$

The solutions we found were clearly solutions to the final equations as well, so we have now solved the entire system of equations.

Case 2: If instead $C^2_{13} \neq 0$ and $C^2_{12} + C^3_{13} = 0$, the system of equations becomes

$$C^2_{11}(C^2_{12} - C^2_{21}) + C^3_{11}(C^2_{13} - C^2_{31}) = 0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}(C^3_{12} - C^3_{21}) + C^3_{11}(-C^2_{12} - C^3_{31}) = 0 \qquad \text{(R3.14.3)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{13}C^3_{12} + C^2_{12}C^2_{12} = 0 \qquad \text{(R3.16.3)}$$
$$(C^2_{12} + C^2_{21})(C^2_{12} - C^2_{21}) + (C^3_{12} + C^3_{21})(C^2_{13} - C^2_{31}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^2_{12} + C^2_{21})(C^3_{12} - C^3_{21}) + (C^3_{12} + C^3_{21})(-C^2_{12} - C^3_{31}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^2_{13} + C^2_{31})(C^2_{12} - C^2_{21}) + (-C^2_{12} + C^3_{31})(C^2_{13} - C^2_{31}) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C^2_{13} + C^2_{31})(C^3_{12} - C^3_{21}) + (-C^2_{12} + C^3_{31})(-C^2_{12} - C^3_{31}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{22}C^2_{12} + C^3_{22}C^2_{13} = 0 \qquad \text{(R3.19.2.2.2)}$$
$$C^2_{22}C^3_{12} - C^3_{22}C^2_{12} = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C^2_{23}C^2_{12} + C^3_{23}C^2_{13} = 0 \qquad \text{(R3.20.2.2.2)}$$
$$C^2_{23}C^3_{12} - C^3_{23}C^2_{12} = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C^2_{32}C^2_{12} + C^3_{32}C^2_{13} = 0 \qquad \text{(R3.21.2.2.2)}$$
$$C^2_{32}C^3_{12} - C^3_{32}C^2_{12} = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C^2_{33}C^2_{12} + C^3_{33}C^2_{13} = 0 \qquad \text{(R3.22.2.2.2)}$$
$$C^2_{33}C^3_{12} - C^3_{33}C^2_{12} = 0 \qquad \text{(R3.22.3.2.2)}$$
$$C^2_{21}C^2_{21} + C^3_{21}C^2_{31} = 0 \qquad \text{(R3.23.2)}$$
$$C^2_{21}C^3_{21} + C^3_{21}C^3_{31} = 0 \qquad \text{(R3.23.3)}$$
$$C^2_{22}C^2_{21} + C^3_{22}C^2_{31} = 0 \qquad \text{(R3.26.2)}$$
$$C^2_{22}C^3_{21} + C^3_{22}C^3_{31} = 0 \qquad \text{(R3.26.3)}$$
$$C^2_{23}C^2_{21} + C^3_{23}C^2_{31} = 0 \qquad \text{(R3.27.2)}$$

$$C^2_{23}C^3_{21}+C^3_{23}C^3_{31}=0 \qquad \text{(R3.27.3)}$$
$$C^2_{31}C^2_{21}+C^3_{31}C^2_{31}=0 \qquad \text{(R3.32.2)}$$
$$C^2_{31}C^3_{21}+C^3_{31}C^3_{31}=0 \qquad \text{(R3.32.3)}$$
$$C^2_{32}C^2_{21}+C^3_{32}C^2_{31}=0 \qquad \text{(R3.35.2)}$$
$$C^2_{32}C^3_{21}+C^3_{32}C^3_{31}=0 \qquad \text{(R3.35.3)}$$
$$C^2_{33}C^2_{21}+C^3_{33}C^2_{31}=0 \qquad \text{(R3.36.2)}$$
$$C^2_{33}C^3_{21}+C^3_{33}C^3_{31}=0. \qquad \text{(R3.36.3)}$$

We see that due to commutativity, equation (R3.16.3) can be rewritten to be identical to equation (R3.15.2), and it can thus be removed. Then, we can rewrite equation (R3.23.3) as $C^2_{21}C^3_{21}+C^3_{21}C^3_{31}=C^3_{21}C^2_{21}+C^3_{21}C^3_{31}=C^3_{21}(C^2_{21}+C^3_{31}=0$, and we can rewrite equation (R3.32.2) as $C^2_{31}C^2_{21}+C^3_{31}C^2_{31}=C^2_{31}C^2_{21}+C^2_{31}C^3_{31}=C^2_{31}(C^2_{21}+C^3_{31})=0$. One solution to both of these equations is $C^2_{21}+C^3_{31}=0$. If instead $C^2_{21}+C^3_{31}\neq 0$, the only possible solution is $C^3_{21}=C^2_{31}=0$.

Case 2.1: If $C^3_{21}=C^2_{31}=0$, the system of equations becomes

$$C^2_{11}(C^2_{12}-C^2_{21})+C^3_{11}C^2_{13}=0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}C^3_{12}+C^3_{11}(-C^2_{12}-C^3_{31})=0 \qquad \text{(R3.14.3)}$$
$$C^2_{12}C^2_{12}+C^3_{12}C^2_{13}=0 \qquad \text{(R3.15.2)}$$
$$(C^2_{12}+C^2_{21})(C^2_{12}-C^2_{21})+C^3_{12}C^2_{13}=0 \qquad \text{(R3.17.2.2)}$$
$$(C^2_{12}+C^2_{21})C^3_{12}+C^3_{12}(-C^2_{12}-C^3_{31})=0 \qquad \text{(R3.17.3.2)}$$
$$C^2_{13}(C^2_{12}-C^2_{21})+(-C^2_{12}+C^3_{31})C^2_{13}=0 \qquad \text{(R3.18.2.2)}$$
$$C^2_{13}C^3_{12}+(-C^2_{12}+C^3_{31})(-C^2_{12}-C^3_{31})=0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{22}C^2_{12}+C^3_{22}C^2_{13}=0 \qquad \text{(R3.19.2.2.2)}$$
$$C^2_{22}C^3_{12}-C^3_{22}C^2_{12}=0 \qquad \text{(R3.19.3.2.2)}$$
$$C^2_{23}C^2_{12}+C^3_{23}C^2_{13}=0 \qquad \text{(R3.20.2.2.2)}$$
$$C^2_{23}C^3_{12}-C^3_{23}C^2_{12}=0 \qquad \text{(R3.20.3.2.2)}$$
$$C^2_{32}C^2_{12}+C^3_{32}C^2_{13}=0 \qquad \text{(R3.21.2.2.2)}$$
$$C^2_{32}C^3_{12}-C^3_{32}C^2_{12}=0 \qquad \text{(R3.21.3.2.2)}$$
$$C^2_{33}C^2_{12}+C^3_{33}C^2_{13}=0 \qquad \text{(R3.22.2.2.2)}$$
$$C^2_{33}C^3_{12}-C^3_{33}C^2_{12}=0 \qquad \text{(R3.22.3.2.2)}$$
$$C^2_{21}C^2_{21}=0 \qquad \text{(R3.23.2)}$$
$$C^2_{22}C^2_{21}=0 \qquad \text{(R3.26.2)}$$
$$C^3_{22}C^3_{31}=0 \qquad \text{(R3.26.3)}$$
$$C^2_{23}C^2_{21}=0 \qquad \text{(R3.27.2)}$$

$$C^3_{23}C^3_{31} = 0 \qquad \text{(R3.27.3)}$$
$$C^3_{31}C^3_{31} = 0 \qquad \text{(R3.32.3)}$$
$$C^2_{32}C^2_{21} = 0 \qquad \text{(R3.35.2)}$$
$$C^3_{32}C^3_{31} = 0 \qquad \text{(R3.35.3)}$$
$$C^2_{33}C^2_{21} = 0 \qquad \text{(R3.36.2)}$$
$$C^3_{33}C^3_{31} = 0. \qquad \text{(R3.36.3)}$$

The only solution to equation (R3.23.2) is $C^2_{21} = 0$, and the only solution to equation (R3.32.3) is $C^3_{31} = 0$. This gives us

$$C^2_{11}C^2_{12} + C^3_{11}C^2_{13} = 0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}C^3_{12} - C^3_{11}C^2_{12} = 0 \qquad \text{(R3.14.3)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.17.2.2)}$$
$$C^2_{13}C^3_{12} + C^2_{12}C^2_{12} = 0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{22}C^2_{12} + C^3_{22}C^2_{13} = 0 \qquad \text{(R3.19.2.2.2)}$$
$$C^2_{22}C^3_{12} - C^3_{22}C^2_{12} = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C^2_{23}C^2_{12} + C^3_{23}C^2_{13} = 0 \qquad \text{(R3.20.2.2.2)}$$
$$C^2_{23}C^3_{12} - C^3_{23}C^2_{12} = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C^2_{32}C^2_{12} + C^3_{32}C^2_{13} = 0 \qquad \text{(R3.21.2.2.2)}$$
$$C^2_{32}C^3_{12} - C^3_{32}C^2_{12} = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C^2_{33}C^2_{12} + C^3_{33}C^2_{13} = 0 \qquad \text{(R3.22.2.2.2)}$$
$$C^2_{33}C^3_{12} - C^3_{33}C^2_{12} = 0. \qquad \text{(R3.22.3.2.2)}$$

We see that equations (R3.15.2) and (R3.17.2.2) have now both been reduced to the same equation, but due to commutativity they are actually also the same as equation (R3.18.3.2) and can both be removed. Our system of equations will then look as follows:

$$C^2_{11}C^2_{12} + C^3_{11}C^2_{13} = 0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}C^3_{12} - C^3_{11}C^2_{12} = 0 \qquad \text{(R3.14.3)}$$
$$C^2_{13}C^3_{12} + C^2_{12}C^2_{12} = 0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{22}C^2_{12} + C^3_{22}C^2_{13} = 0 \qquad \text{(R3.19.2.2.2)}$$
$$C^2_{22}C^3_{12} - C^3_{22}C^2_{12} = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C^2_{23}C^2_{12} + C^3_{23}C^2_{13} = 0 \qquad \text{(R3.20.2.2.2)}$$
$$C^2_{23}C^3_{12} - C^3_{23}C^2_{12} = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C^2_{32}C^2_{12} + C^3_{32}C^2_{13} = 0 \qquad \text{(R3.21.2.2.2)}$$

$$C^2_{32}C^3_{12} - C^3_{32}C^2_{12} = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C^2_{33}C^2_{12} + C^3_{33}C^2_{13} = 0 \qquad \text{(R3.22.2.2.2)}$$
$$C^2_{33}C^3_{12} - C^3_{33}C^2_{12} = 0. \qquad \text{(R3.22.3.2.2)}$$

Now, we can use the fact that we have assumed that $C^2_{13} \neq 0$ to write the solution to equation (R3.18.3.2) as $C^3_{12} = -C^2_{12}C^2_{12}/C^2_{13}$. Inserting this into the system of equations gives us

$$C^2_{11}C^2_{12} + C^3_{11}C^2_{13} = 0 \qquad \text{(R3.14.2)}$$
$$-C^2_{11}C^2_{12}C^2_{12}/C^2_{13} - C^3_{11}C^2_{12} = 0 \qquad \text{(R3.14.3)}$$
$$C^2_{22}C^2_{12} + C^3_{22}C^2_{13} = 0 \qquad \text{(R3.19.2.2.2)}$$
$$-C^2_{22}C^2_{12}C^2_{12}/C^2_{13} - C^3_{22}C^2_{12} = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C^2_{23}C^2_{12} + C^3_{23}C^2_{13} = 0 \qquad \text{(R3.20.2.2.2)}$$
$$-C^2_{23}C^2_{12}C^2_{12}/C^2_{13} - C^3_{23}C^2_{12} = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C^2_{32}C^2_{12} + C^3_{32}C^2_{13} = 0 \qquad \text{(R3.21.2.2.2)}$$
$$-C^2_{32}C^2_{12}C^2_{12}/C^2_{13} - C^3_{32}C^2_{12} = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C^2_{33}C^2_{12} + C^3_{33}C^2_{13} = 0 \qquad \text{(R3.22.2.2.2)}$$
$$-C^2_{33}C^2_{12}C^2_{12}/C^2_{13} - C^3_{33}C^2_{12} = 0. \qquad \text{(R3.22.3.2.2)}$$

We now write the solutions to all of the equations having C^2_{13} as a numerator in the same way:

$$\text{(R3.14.2)}: \quad C^3_{11} = -C^2_{11}C^2_{12}/C^2_{13},$$
$$\text{(R3.19.2.2.2)}: \quad C^3_{22} = -C^2_{22}C^2_{12}/C^2_{13},$$
$$\text{(R3.20.2.2.2)}: \quad C^3_{23} = -C^2_{23}C^2_{12}/C^2_{13},$$
$$\text{(R3.21.2.2.2)}: \quad C^3_{32} = -C^2_{32}C^2_{12}/C^2_{13},$$
$$\text{(R3.22.2.2.2)}: \quad C^3_{33} = -C^2_{33}C^2_{12}/C^2_{13}.$$

We now insert these solutions into the remaining equations.

$$\begin{aligned}
\text{(R3.14.3)}: \quad & -C^2_{11}C^2_{12}C^2_{12}/C^2_{13} - C^3_{11}C^2_{12} \\
& = -C^2_{11}C^2_{12}C^2_{12}/C^2_{13} + C^2_{11}C^2_{12}C^2_{12}/C^2_{13} \\
& = 0, \\
\text{(R3.19.3.2.2)}: \quad & -C^2_{22}C^2_{12}C^2_{12}/C^2_{13} - C^3_{22}C^2_{12} \\
& = -C^2_{22}C^2_{12}C^2_{12}/C^2_{13} + C^2_{22}C^2_{12}C^2_{12}/C^2_{13} \\
& = 0, \\
\text{(R3.20.3.2.2)}: \quad & -C^2_{23}C^2_{12}C^2_{12}/C^2_{13} - C^3_{23}C^2_{12} \\
& = -C^2_{23}C^2_{12}C^2_{12}/C^2_{13} + C^2_{23}C^2_{12}C^2_{12}/C^2_{13}
\end{aligned}$$

$$= 0,$$

$$\text{(R3.21.3.2.2)}: \quad -C^2_{32}C^2_{12}C^2_{12}/C^2_{13} - C^3_{32}C^2_{12}$$
$$= -C^2_{32}C^2_{12}C^2_{12}/C^2_{13} + C^2_{32}C^2_{12}C^2_{12}/C^2_{13}$$
$$= 0,$$

$$\text{(R3.22.3.2.2)}: \quad -C^2_{33}C^2_{12}C^2_{12}/C^2_{13} - C^3_{33}C^2_{12}$$
$$= -C^2_{33}C^2_{12}C^2_{12}/C^2_{13} + C^2_{33}C^2_{12}C^2_{12}/C^2_{13}$$
$$= 0.$$

Clearly, these solutions also solved the remaining equations, and thus we have now solved the entire system of equations.

Case 2.2: If instead $C^2_{21} + C^3_{31} = 0$, $C^3_{21} \neq 0$ and $C^2_{31} \neq 0$, the system of equations becomes

$$C^2_{11}(C^2_{12} - C^2_{21}) + C^3_{11}(C^2_{13} - C^2_{31}) = 0 \qquad \text{(R3.14.2)}$$
$$C^2_{11}(C^3_{12} - C^3_{21}) + C^3_{11}(-C^2_{12} + C^2_{21}) = 0 \qquad \text{(R3.14.3)}$$
$$C^2_{12}C^2_{12} + C^3_{12}C^2_{13} = 0 \qquad \text{(R3.15.2)}$$
$$(C^2_{12} + C^2_{21})(C^2_{12} - C^2_{21}) + (C^3_{12} + C^3_{21})(C^2_{13} - C^2_{31}) = 0 \qquad \text{(R3.17.2.2)}$$
$$(C^2_{12} + C^2_{21})(C^3_{12} - C^3_{21}) + (C^3_{12} + C^3_{21})(-C^2_{12} + C^2_{21}) = 0 \qquad \text{(R3.17.3.2)}$$
$$(C^2_{13} + C^2_{31})(C^2_{12} - C^2_{21}) + (-C^2_{12} - C^2_{21})(C^2_{13} - C^2_{31}) = 0 \qquad \text{(R3.18.2.2)}$$
$$(C^2_{13} + C^2_{31})(C^3_{12} - C^3_{21}) + (-C^2_{12} - C^2_{21})(-C^2_{12} + C^2_{21}) = 0 \qquad \text{(R3.18.3.2)}$$
$$C^2_{22}C^2_{12} + C^3_{22}C^2_{13} = 0 \qquad \text{(R3.19.2.2.2)}$$
$$C^2_{22}C^3_{12} - C^3_{22}C^2_{12} = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C^2_{23}C^2_{12} + C^3_{23}C^2_{13} = 0 \qquad \text{(R3.20.2.2.2)}$$
$$C^2_{23}C^3_{12} - C^3_{23}C^2_{12} = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C^2_{32}C^2_{12} + C^3_{32}C^2_{13} = 0 \qquad \text{(R3.21.2.2.2)}$$
$$C^2_{32}C^3_{12} - C^3_{32}C^2_{12} = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C^2_{33}C^2_{12} + C^3_{33}C^2_{13} = 0 \qquad \text{(R3.22.2.2.2)}$$
$$C^2_{33}C^3_{12} - C^3_{33}C^2_{12} = 0 \qquad \text{(R3.22.3.2.2)}$$
$$C^2_{21}C^2_{21} + C^3_{21}C^2_{31} = 0 \qquad \text{(R3.23.2)}$$
$$C^2_{22}C^2_{21} + C^3_{22}C^2_{31} = 0 \qquad \text{(R3.26.2)}$$
$$C^2_{22}C^3_{21} - C^3_{22}C^2_{21} = 0 \qquad \text{(R3.26.3)}$$
$$C^2_{23}C^2_{21} + C^3_{23}C^2_{31} = 0 \qquad \text{(R3.27.2)}$$
$$C^2_{23}C^3_{21} - C^3_{23}C^2_{21} = 0 \qquad \text{(R3.27.3)}$$
$$C^2_{31}C^3_{21} + C^2_{21}C^2_{21} = 0 \qquad \text{(R3.32.3)}$$
$$C^2_{32}C^2_{21} + C^3_{32}C^2_{31} = 0 \qquad \text{(R3.35.2)}$$
$$C^2_{32}C^3_{21} - C^3_{32}C^2_{21} = 0 \qquad \text{(R3.35.3)}$$

$$C_{33}^{2}C_{21}^{2} + C_{33}^{3}C_{31}^{2} = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^{2}C_{21}^{3} - C_{33}^{3}C_{21}^{2} = 0. \qquad \text{(R3.36.3)}$$

We see that equation (R3.17.3.2) has the same variables in both of its products, so it seems like it might be possible to simplify it. We rewrite it as

$$\begin{aligned}
&(C_{12}^{2} + C_{21}^{2})(C_{12}^{3} - C_{21}^{3}) + (C_{12}^{3} + C_{21}^{3})(-C_{12}^{2} + C_{21}^{2}) \\
&\quad = C_{12}^{2}C_{12}^{3} + C_{21}^{2}C_{12}^{3} - C_{12}^{2}C_{21}^{3} - C_{21}^{2}C_{21}^{3} - C_{12}^{3}C_{12}^{2} - C_{21}^{3}C_{12}^{2} + C_{12}^{3}C_{21}^{2} + C_{21}^{3}C_{21}^{2} \\
&\quad = C_{21}^{2}C_{12}^{3} + C_{12}^{3}C_{21}^{2} - C_{12}^{2}C_{21}^{3} - C_{21}^{3}C_{12}^{2} \\
&\quad = 2(C_{12}^{3}C_{21}^{2} - C_{12}^{2}C_{21}^{3}) \\
&\quad = 0,
\end{aligned}$$

and since we are working over a field with characteristic 0 this means that we must have $C_{12}^{3}C_{21}^{2} - C_{12}^{2}C_{21}^{3} = 0$. Looking at equation (R3.18.2.2), it seems like it might be possible to simplify this equation as well, so we rewrite it as

$$\begin{aligned}
&(C_{13}^{2} + C_{31}^{2})(C_{12}^{2} - C_{21}^{2}) + (-C_{12}^{2} - C_{21}^{2})(C_{13}^{2} - C_{31}^{2}) \\
&\quad = C_{13}^{2}C_{12}^{2} + C_{31}^{2}C_{12}^{2} - C_{13}^{2}C_{21}^{2} - C_{31}^{2}C_{21}^{2} - C_{12}^{2}C_{13}^{2} - C_{21}^{2}C_{13}^{2} + C_{12}^{2}C_{31}^{2} + C_{21}^{2}C_{31}^{2} \\
&\quad = C_{31}^{2}C_{12}^{2} + C_{12}^{2}C_{31}^{2} - C_{13}^{2}C_{21}^{2} - C_{21}^{2}C_{13}^{2} \\
&\quad = 2(C_{12}^{2}C_{31}^{2} - C_{13}^{2}C_{21}^{2}) \\
&\quad = 0,
\end{aligned}$$

and for the same reason as above this means that we need to have $C_{12}^{2}C_{31}^{2} - C_{13}^{2}C_{21}^{2} = 0$. Furthermore, due to commutativity, we see that equation (R3.32.3) can be rewritten to be exactly equation (R3.23.2), and it can thus be removed. Finally, equations (R3.17.2.2) and (R3.18.3.2) are made up of the exact same structure constants, so we might be able to simplify one of them by adding or subtracting the other from it. To do this, we must first expand the brackets. We begin with equation (R3.17.2.2):

$$\begin{aligned}
&(C_{12}^{2} + C_{21}^{2})(C_{12}^{2} - C_{21}^{2}) + (C_{12}^{3} + C_{21}^{3})(C_{13}^{2} - C_{31}^{2}) \\
&\quad = C_{12}^{2}C_{12}^{2} + C_{21}^{2}C_{12}^{2} - C_{12}^{2}C_{21}^{2} - C_{21}^{2}C_{21}^{2} + C_{12}^{3}C_{13}^{2} + C_{21}^{3}C_{13}^{2} - C_{12}^{3}C_{31}^{2} - C_{21}^{3}C_{31}^{2} \\
&\quad = C_{12}^{2}C_{12}^{2} - C_{21}^{2}C_{21}^{2} + C_{12}^{3}C_{13}^{2} + C_{21}^{3}C_{13}^{2} - C_{12}^{3}C_{31}^{2} - C_{21}^{3}C_{31}^{2} \\
&\quad = 0.
\end{aligned}$$

Doing the same for equation (R3.18.3.2), we get

$$\begin{aligned}
&(C_{13}^{2} + C_{31}^{2})(C_{12}^{3} - C_{21}^{3}) + (-C_{12}^{2} - C_{21}^{2})(-C_{12}^{2} + C_{21}^{2}) \\
&\quad = C_{13}^{2}C_{12}^{3} + C_{31}^{2}C_{12}^{3} - C_{13}^{2}C_{21}^{3} - C_{31}^{2}C_{21}^{3} + C_{12}^{2}C_{12}^{2} + C_{21}^{2}C_{12}^{2} - C_{12}^{2}C_{21}^{2} - C_{21}^{2}C_{21}^{2} \\
&\quad = C_{13}^{2}C_{12}^{3} + C_{31}^{2}C_{12}^{3} - C_{13}^{2}C_{21}^{3} - C_{31}^{2}C_{21}^{3} + C_{12}^{2}C_{12}^{2} - C_{21}^{2}C_{21}^{2} \\
&\quad = 0.
\end{aligned}$$

Subtracting equation (R3.18.3.2) from equation (R3.17.2.2) gives us

$$\begin{aligned}&C_{12}^2C_{12}^2 - C_{21}^2C_{21}^2 + C_{12}^3C_{13}^2 + C_{21}^3C_{13}^2 - C_{12}^3C_{31}^2 - C_{21}^3C_{31}^2\\&-(C_{13}^2C_{12}^3 + C_{31}^2C_{12}^3 - C_{13}^2C_{21}^3 - C_{31}^2C_{21}^3 + C_{12}^2C_{12}^2 - C_{21}^2C_{21}^2)\\&= 2C_{21}^3C_{13}^2 - 2C_{12}^3C_{31}^2\\&= 2(C_{21}^3C_{13}^2 - C_{12}^3C_{31}^2)\\&= 0.\end{aligned}$$

Again, this means that we must have $C_{21}^3C_{13}^2 - C_{12}^3C_{31}^2 = 0$. Now, after these changes, our system of equations will look as follows:

$$C_{11}^2(C_{12}^2 - C_{21}^2) + C_{11}^3(C_{13}^2 - C_{31}^2) = 0 \qquad \text{(R3.14.2)}$$
$$C_{11}^2(C_{12}^3 - C_{21}^3) + C_{11}^3(-C_{12}^2 + C_{21}^2) = 0 \qquad \text{(R3.14.3)}$$
$$C_{12}^2C_{12}^2 + C_{12}^3C_{13}^2 = 0 \qquad \text{(R3.15.2)}$$
$$C_{21}^3C_{13}^2 - C_{12}^3C_{31}^2 = 0 \qquad \text{(R3.17.2.2)}$$
$$C_{12}^3C_{21}^2 - C_{12}^2C_{21}^3 = 0 \qquad \text{(R3.17.3.2)}$$
$$C_{12}^2C_{31}^2 - C_{13}^2C_{21}^2 = 0 \qquad \text{(R3.18.2.2)}$$
$$(C_{13}^2 + C_{31}^2)(C_{12}^3 - C_{21}^3) + (-C_{12}^2 - C_{21}^2)(-C_{12}^2 + C_{21}^2) = 0 \qquad \text{(R3.18.3.2)}$$
$$C_{22}^2C_{12}^2 + C_{22}^3C_{13}^2 = 0 \qquad \text{(R3.19.2.2.2)}$$
$$C_{22}^2C_{12}^3 - C_{22}^3C_{12}^2 = 0 \qquad \text{(R3.19.3.2.2)}$$
$$C_{23}^2C_{12}^2 + C_{23}^3C_{13}^2 = 0 \qquad \text{(R3.20.2.2.2)}$$
$$C_{23}^2C_{12}^3 - C_{23}^3C_{12}^2 = 0 \qquad \text{(R3.20.3.2.2)}$$
$$C_{32}^2C_{12}^2 + C_{32}^3C_{13}^2 = 0 \qquad \text{(R3.21.2.2.2)}$$
$$C_{32}^2C_{12}^3 - C_{32}^3C_{12}^2 = 0 \qquad \text{(R3.21.3.2.2)}$$
$$C_{33}^2C_{12}^2 + C_{33}^3C_{13}^2 = 0 \qquad \text{(R3.22.2.2.2)}$$
$$C_{33}^2C_{12}^3 - C_{33}^3C_{12}^2 = 0 \qquad \text{(R3.22.3.2.2)}$$
$$C_{21}^2C_{21}^2 + C_{21}^3C_{31}^2 = 0 \qquad \text{(R3.23.2)}$$
$$C_{22}^2C_{21}^2 + C_{22}^3C_{31}^2 = 0 \qquad \text{(R3.26.2)}$$
$$C_{22}^2C_{21}^3 - C_{22}^3C_{21}^2 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^2C_{21}^2 + C_{23}^3C_{31}^2 = 0 \qquad \text{(R3.27.2)}$$
$$C_{23}^2C_{21}^3 - C_{23}^3C_{21}^2 = 0 \qquad \text{(R3.27.3)}$$
$$C_{32}^2C_{21}^2 + C_{32}^3C_{31}^2 = 0 \qquad \text{(R3.35.2)}$$
$$C_{32}^2C_{21}^3 - C_{32}^3C_{21}^2 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^2C_{21}^2 + C_{33}^3C_{31}^2 = 0 \qquad \text{(R3.36.2)}$$
$$C_{33}^2C_{21}^3 - C_{33}^3C_{21}^2 = 0. \qquad \text{(R3.36.3)}$$

If we avoid looking at the three longer equations at the moment, we recognise a pattern we have had before, and can therefore tell that a good next step is probably to write the solution to equation (R3.15.2) as $C^3_{12} = -C^2_{12}C^2_{12}/C^2_{13}$, since we have earlier assumed that $C^2_{13} \neq 0$, and also write the solution to equation (R3.23.2) as $C^3_{21} = -C^2_{21}C^2_{21}/C^2_{31}$, since we have assumed that $C^2_{31} \neq 0$. We can now insert these solutions into the system of equations. Beginning with equation (R3.18.3.2), we get

$$\begin{aligned}
&(C^2_{13}+C^2_{31})(C^3_{12}-C^3_{21})+(-C^2_{12}-C^2_{21})(-C^2_{12}+C^2_{21})\\
&\quad=(C^2_{13}+C^2_{31})(-C^2_{12}C^2_{12}/C^2_{13}+C^2_{21}C^2_{21}/C^2_{31})+(-C^2_{12}-C^2_{21})(-C^2_{12}+C^2_{21})\\
&\quad=-C^2_{12}C^2_{12}-C^2_{31}C^2_{12}C^2_{12}/C^2_{13}+C^2_{13}C^2_{21}C^2_{21}/C^2_{31}+C^2_{21}C^2_{21}+C^2_{12}C^2_{12}-C^2_{21}C^2_{21}\\
&\quad=-C^2_{31}C^2_{12}C^2_{12}/C^2_{13}+C^2_{13}C^2_{21}C^2_{21}/C^2_{31}.
\end{aligned}$$

The remaining equations do not require any calculations, so we write them immediately in the system of equations:

$$\begin{aligned}
C^2_{11}(C^2_{12}-C^2_{21})+C^3_{11}(C^2_{13}-C^2_{31})&=0 &&\text{(R3.14.2)}\\
C^2_{11}(-C^2_{12}C^2_{12}/C^2_{13}+C^2_{21}C^2_{21}/C^2_{31})+C^3_{11}(-C^2_{12}+C^2_{21})&=0 &&\text{(R3.14.3)}\\
-C^2_{21}C^2_{21}C^2_{13}/C^2_{31}+C^2_{12}C^2_{12}C^2_{31}/C^2_{13}&=0 &&\text{(R3.17.2.2)}\\
-C^2_{12}C^2_{12}C^2_{21}/C^2_{13}+C^2_{12}C^2_{21}C^2_{21}/C^2_{31}&=0 &&\text{(R3.17.3.2)}\\
C^2_{12}C^2_{31}-C^2_{13}C^2_{21}&=0 &&\text{(R3.18.2.2)}\\
-C^2_{31}C^2_{12}C^2_{12}/C^2_{13}+C^2_{13}C^2_{21}C^2_{21}/C^2_{31}&=0 &&\text{(R3.18.3.2)}\\
C^2_{22}C^2_{12}+C^3_{22}C^2_{13}&=0 &&\text{(R3.19.2.2.2)}\\
-C^2_{22}C^2_{12}C^2_{12}/C^2_{13}-C^3_{22}C^2_{12}&=0 &&\text{(R3.19.3.2.2)}\\
C^2_{23}C^2_{12}+C^3_{23}C^2_{13}&=0 &&\text{(R3.20.2.2.2)}\\
-C^2_{23}C^2_{12}C^2_{12}/C^2_{13}-C^3_{23}C^2_{12}&=0 &&\text{(R3.20.3.2.2)}\\
C^2_{32}C^2_{12}+C^3_{32}C^2_{13}&=0 &&\text{(R3.21.2.2.2)}\\
-C^2_{32}C^2_{12}C^2_{12}/C^2_{13}-C^3_{32}C^2_{12}&=0 &&\text{(R3.21.3.2.2)}\\
C^2_{33}C^2_{12}+C^3_{33}C^2_{13}&=0 &&\text{(R3.22.2.2.2)}\\
-C^2_{33}C^2_{12}C^2_{12}/C^2_{13}-C^3_{33}C^2_{12}&=0 &&\text{(R3.22.3.2.2)}\\
C^2_{22}C^2_{21}+C^3_{22}C^2_{31}&=0 &&\text{(R3.26.2)}\\
-C^2_{22}C^2_{21}C^2_{21}/C^2_{31}-C^3_{22}C^2_{21}&=0 &&\text{(R3.26.3)}\\
C^2_{23}C^2_{21}+C^3_{23}C^2_{31}&=0 &&\text{(R3.27.2)}\\
-C^2_{23}C^2_{21}C^2_{21}/C^2_{31}-C^3_{23}C^2_{21}&=0 &&\text{(R3.27.3)}\\
C^2_{32}C^2_{21}+C^3_{32}C^2_{31}&=0 &&\text{(R3.35.2)}\\
-C^2_{32}C^2_{21}C^2_{21}/C^2_{31}-C^3_{32}C^2_{21}&=0 &&\text{(R3.35.3)}\\
C^2_{33}C^2_{21}+C^3_{33}C^2_{31}&=0 &&\text{(R3.36.2)}
\end{aligned}$$

$$-C^2_{33}C^2_{21}C^2_{21}/C^2_{31} - C^3_{33}C^2_{21} = 0. \qquad \text{(R3.36.3)}$$

We now again use the fact that we have assumed that $C^2_{13} \neq 0$ and $C^2_{31} \neq 0$ to write the solutions to the following equations:

$$\begin{aligned}
\text{(R3.19.2.2.2)}: \quad & C^3_{22} = -C^2_{22}C^2_{12}/C^2_{13},\\
\text{(R3.20.2.2.2)}: \quad & C^3_{23} = -C^2_{23}C^2_{12}/C^2_{13},\\
\text{(R3.21.2.2.2)}: \quad & C^3_{32} = -C^2_{32}C^2_{12}/C^2_{13},\\
\text{(R3.22.2.2.2)}: \quad & C^3_{33} = -C^2_{33}C^2_{12}/C^2_{13},\\
\text{(R3.26.2)}: \quad & C^3_{22} = -C^2_{22}C^2_{21}/C^2_{31},\\
\text{(R3.27.2)}: \quad & C^3_{23} = -C^2_{23}C^2_{21}/C^2_{31},\\
\text{(R3.35.2)}: \quad & C^3_{32} = -C^2_{32}C^2_{21}/C^2_{31},\\
\text{(R3.36.2)}: \quad & C^3_{33} = C^2_{33}C^2_{21}/C^2_{31}.
\end{aligned}$$

We now see that we get two formulas for each of the four variables. Thus, for example, the formulas for C^3_{22} give us that $C^2_{22}C^2_{12}/C^2_{13} = C^2_{22}C^2_{21}/C^2_{31}$, which of course implies that $C^2_{12}/C^2_{13} = C^2_{21}/C^2_{31}$, or $C^2_{12}C^2_{31} = C^2_{21}C^2_{13}$. Inserting this, as well as all of the solutions, into the system of equations gives us

$$\begin{aligned}
C^2_{11}(C^2_{12} - C^2_{21}) + C^3_{11}(C^2_{13} - C^2_{31}) &= 0 && \text{(R3.14.2)}\\
C^2_{11}(-C^2_{12}C^2_{12}/C^2_{13} + C^2_{12}C^2_{21}/C^2_{13}) + C^3_{11}(-C^2_{12} + C^2_{21}) &= 0 && \text{(R3.14.3)}\\
-C^2_{12}C^2_{21}C^2_{13}/C^2_{13} + C^2_{12}C^2_{21}C^2_{31}/C^2_{31} &= 0 && \text{(R3.17.2.2)}\\
-C^2_{12}C^2_{12}C^2_{21}/C^2_{13} + C^2_{12}C^2_{12}C^2_{21}/C^2_{13} &= 0 && \text{(R3.17.3.2)}\\
-C^2_{31}C^2_{12}C^2_{21}/C^2_{31} + C^2_{13}C^2_{12}C^2_{21}/C^2_{13} &= 0 && \text{(R3.18.3.2)}\\
-C^2_{22}C^2_{12}C^2_{12}/C^2_{13} + C^2_{22}C^2_{12}C^2_{12}/C^2_{13} &= 0 && \text{(R3.19.3.2.2)}\\
-C^2_{23}C^2_{12}C^2_{12}/C^2_{13} + C^2_{23}C^2_{12}C^2_{12}/C^2_{13} &= 0 && \text{(R3.20.3.2.2)}\\
-C^2_{32}C^2_{12}C^2_{12}/C^2_{13} + C^2_{32}C^2_{12}C^2_{12}/C^2_{13} &= 0 && \text{(R3.21.3.2.2)}\\
-C^2_{33}C^2_{12}C^2_{12}/C^2_{13} + C^2_{33}C^2_{12}C^2_{12}/C^2_{13} &= 0 && \text{(R3.22.3.2.2)}\\
-C^2_{22}C^2_{21}C^2_{12}/C^2_{13} + C^2_{22}C^2_{12}C^2_{21}/C^2_{13} &= 0 && \text{(R3.26.3)}\\
-C^2_{23}C^2_{21}C^2_{12}/C^2_{13} + C^2_{23}C^2_{12}C^2_{21}/C^2_{13} &= 0 && \text{(R3.27.3)}\\
-C^2_{32}C^2_{21}C^2_{12}/C^2_{13} + C^2_{32}C^2_{12}C^2_{21}/C^2_{13} &= 0 && \text{(R3.35.3)}\\
-C^2_{33}C^2_{21}C^2_{12}/C^2_{13} + C^2_{33}C^2_{12}C^2_{21}/C^2_{13} &= 0. && \text{(R3.36.3)}
\end{aligned}$$

For a lot of these equations, we see that the left hand side is now zero regardless of the values of the variables, and we can thus remove them. All that remains of the system of equations is then

$$\begin{aligned}
C^2_{11}(C^2_{12} - C^2_{21}) + C^3_{11}(C^2_{13} - C^2_{31}) &= 0 && \text{(R3.14.2)}\\
C^2_{11}(-C^2_{12}C^2_{12}/C^2_{13} + C^2_{12}C^2_{21}/C^2_{13}) + C^3_{11}(-C^2_{12} + C^2_{21}) &= 0. && \text{(R3.14.3)}
\end{aligned}$$

We can rewrite equation (R3.14.3) as

$$\begin{aligned}
&C_{11}^2(-C_{12}^2C_{12}^2/C_{13}^2 + C_{12}^2C_{21}^2/C_{13}^2) + C_{11}^3(-C_{12}^2 + C_{21}^2) \\
&= C_{11}^2C_{12}^2/C_{13}^2(-C_{12}^2 + C_{21}^2) + C_{11}^3(-C_{12}^2 + C_{21}^2) \\
&= (C_{11}^2C_{12}^2/C_{13}^2 + C_{11}^3)(-C_{12}^2 + C_{21}^2) \\
&= 0.
\end{aligned}$$

We see that this has two solutions, $C_{11}^2C_{12}^2/C_{13}^2 + C_{11}^3 = 0$ and $-C_{12}^2 + C_{21}^2 = 0$.

Case 2.2.1: If $C_{11}^2C_{12}^2/C_{13}^2 + C_{11}^3 = 0$, the remaining equation in the system of equations becomes

$$\begin{aligned}
&C_{11}^2(C_{12}^2 - C_{21}^2) + C_{11}^3(C_{13}^2 - C_{31}^2) \\
&= C_{11}^2(C_{12}^2 - C_{21}^2) - C_{11}^2C_{12}^2(C_{13}^2 - C_{31}^2)/C_{13}^2 \\
&= C_{11}^2C_{12}^2 - C_{11}^2C_{21}^2 - C_{11}^2C_{12}^2 + C_{11}^2C_{12}^2C_{31}^2/C_{13}^2 \\
&= -C_{11}^2C_{21}^2 + C_{11}^2C_{21}^2C_{31}^2/C_{31}^2 \\
&= -C_{11}^2C_{21}^2 + C_{11}^2C_{21}^2 \\
&= 0.
\end{aligned}$$

Thus, we have clearly now solved the entire system of equations.

Case 2.2.2: If instead $C_{12}^2 = C_{21}^2$ and $C_{11}^2C_{12}^2/C_{13}^2 + C_{11}^3 \neq 0$, the remaining equations in the system of equations becomes

$$C_{11}^3(C_{13}^2 - C_{31}^2) = 0.$$

The two solutions to this are $C_{11}^3 = 0$ and $C_{13}^2 - C_{31}^2 = 0$, and we have now solved the entire system of equations.

We are thus now done with this case as well.

Finally, we look at the case where $C_{ij}^k = 0$ when $k = 2$.

$$\begin{aligned}
C_{11}^3(C_{13}^1 - C_{31}^1) &= 0 && \text{(R3.14.1)} \\
C_{11}^3(C_{13}^3 - C_{31}^3) &= 0 && \text{(R3.14.3)} \\
C_{12}^1C_{11}^1 + C_{12}^3C_{13}^1 &= 0 && \text{(R3.15.1)} \\
C_{12}^1C_{11}^3 + C_{12}^3C_{13}^3 &= 0 && \text{(R3.15.3)} \\
C_{13}^1C_{11}^1 + C_{13}^3C_{13}^1 &= 0 && \text{(R3.16.1)} \\
C_{13}^1C_{11}^3 + C_{13}^3C_{13}^3 &= 0 && \text{(R3.16.3)} \\
(C_{12}^3 + C_{21}^3)(C_{13}^1 - C_{31}^1) &= 0 && \text{(R3.17.1.2)} \\
(C_{12}^3 + C_{21}^3)(C_{13}^3 - C_{31}^3) &= 0 && \text{(R3.17.3.2)} \\
(C_{13}^3 + C_{31}^3)(C_{13}^1 - C_{31}^1) &= 0 && \text{(R3.18.1.2)}
\end{aligned}$$

$$(C^3_{13}+C^3_{31})(C^3_{13}-C^3_{31})=0 \quad \text{(R3.18.3.2)}$$
$$C^3_{22}(C^1_{13}-C^1_{31})=0 \quad \text{(R3.19.1.2)}$$
$$C^3_{22}(C^3_{13}-C^3_{31})=0 \quad \text{(R3.19.3.2)}$$
$$C^3_{23}(C^1_{13}-C^1_{31})=0 \quad \text{(R3.20.1.2)}$$
$$C^3_{23}(C^3_{13}-C^3_{31})=0 \quad \text{(R3.20.3.2)}$$
$$C^3_{32}(C^1_{13}-C^1_{31})=0 \quad \text{(R3.21.1.2)}$$
$$C^3_{32}(C^3_{13}-C^3_{31})=0 \quad \text{(R3.21.3.2)}$$
$$C^3_{33}(C^1_{13}-C^1_{31})=0 \quad \text{(R3.22.1.2)}$$
$$C^3_{33}(C^3_{13}-C^3_{31})=0 \quad \text{(R3.22.3.2)}$$
$$C^1_{21}C^1_{11}+C^3_{21}C^1_{31}=0 \quad \text{(R3.23.1)}$$
$$C^1_{21}C^3_{11}+C^3_{21}C^3_{31}=0 \quad \text{(R3.23.3)}$$
$$C^1_{22}C^1_{11}+C^3_{22}C^1_{31}=0 \quad \text{(R3.26.1)}$$
$$C^1_{22}C^3_{11}+C^3_{22}C^3_{31}=0 \quad \text{(R3.26.3)}$$
$$C^1_{23}C^1_{11}+C^3_{23}C^1_{31}=0 \quad \text{(R3.27.1)}$$
$$C^1_{23}C^3_{11}+C^3_{23}C^3_{31}=0 \quad \text{(R3.27.3)}$$
$$C^1_{31}C^1_{11}+C^3_{31}C^1_{31}=0 \quad \text{(R3.32.1)}$$
$$C^1_{31}C^3_{11}+C^3_{31}C^3_{31}=0 \quad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11}+C^3_{32}C^1_{31}=0 \quad \text{(R3.35.1)}$$
$$C^1_{32}C^3_{11}+C^3_{32}C^3_{31}=0 \quad \text{(R3.35.3)}$$
$$C^1_{33}C^1_{11}+C^3_{33}C^1_{31}=0 \quad \text{(R3.36.1)}$$
$$C^1_{33}C^3_{11}+C^3_{33}C^3_{31}=0. \quad \text{(R3.36.3)}$$

We see that one solution to equations (R3.14.1), (R3.17.1.2), (R3.18.1.2), (R3.19.1.2), (R3.20.1.2), (R3.21.1.2), (R3.22.1.2) is $C^1_{13}=C^1_{31}$, and another solution is $C^3_{11}=C^3_{12}+C^3_{21}=C^3_{13}+C^3_{31}=C^3_{22}=C^3_{23}=C^3_{32}=C^3_{33}=0$. We look at each of these cases.

Case 1: If $C^3_{11}=C^3_{12}+C^3_{21}=C^3_{13}+C^3_{31}=C^3_{22}=C^3_{23}=C^3_{32}=C^3_{33}=0$, the system of equations becomes

$$C^1_{12}C^1_{11}+C^3_{12}C^1_{13}=0 \quad \text{(R3.15.1)}$$
$$C^3_{12}C^3_{13}=0 \quad \text{(R3.15.3)}$$
$$C^1_{13}C^1_{11}+C^3_{13}C^1_{13}=0 \quad \text{(R3.16.1)}$$
$$C^3_{13}C^3_{13}=0 \quad \text{(R3.16.3)}$$
$$C^1_{21}C^1_{11}-C^3_{12}C^1_{31}=0 \quad \text{(R3.23.1)}$$
$$C^3_{12}C^3_{13}=0 \quad \text{(R3.23.3)}$$
$$C^1_{22}C^1_{11}=0 \quad \text{(R3.26.1)}$$

$$C^1_{23}C^1_{11} = 0 \qquad \text{(R3.27.1)}$$
$$C^1_{31}C^1_{11} - C^3_{13}C^1_{31} = 0 \qquad \text{(R3.32.1)}$$
$$C^3_{13}C^3_{13} = 0 \qquad \text{(R3.32.3)}$$
$$C^1_{32}C^1_{11} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{33}C^1_{11} = 0. \qquad \text{(R3.36.1)}$$

The only solution to equation (R3.16.3) is $C^3_{13} = 0$. This gives us

$$C^1_{12}C^1_{11} + C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$C^1_{13}C^1_{11} = 0 \qquad \text{(R3.16.1)}$$
$$C^1_{21}C^1_{11} - C^3_{12}C^1_{31} = 0 \qquad \text{(R3.23.1)}$$
$$C^1_{22}C^1_{11} = 0 \qquad \text{(R3.26.1)}$$
$$C^1_{23}C^1_{11} = 0 \qquad \text{(R3.27.1)}$$
$$C^1_{31}C^1_{11} = 0 \qquad \text{(R3.32.1)}$$
$$C^1_{32}C^1_{11} = 0 \qquad \text{(R3.35.1)}$$
$$C^1_{33}C^1_{11} = 0. \qquad \text{(R3.36.1)}$$

We now see that one solution to equations (R3.26.1), (R3.27.1), (R3.32.1), (R3.35.1) and (R3.36.1) is $C^1_{11} = 0$, and the only other solution is $C^1_{13} = C^1_{22} = C^1_{23} = C^1_{31} = C^1_{32} = C^1_{33} = 0$. We look at each of these cases.

Case 1.1: If $C^1_{11} = 0$, all that remains of the system of equations is

$$C^3_{12}C^1_{13} = 0 \qquad \text{(R3.15.1)}$$
$$-C^3_{12}C^1_{31} = 0. \qquad \text{(R3.23.1)}$$

This clearly has two solutions: $C^3_{12} = 0$ and $C^1_{13} = C^1_{31} = 0$. We have now solved the entire system of equations.

Case 1.2: If instead $C^1_{13} = C^1_{22} = C^1_{23} = C^1_{31} = C^1_{32} = C^1_{33} = 0$ and $C^1_{11} \neq 0$, the system of equations becomes

$$C^1_{12}C^1_{11} = 0 \qquad \text{(R3.15.1)}$$
$$C^1_{21}C^1_{11} = 0 \qquad \text{(R3.23.1),}$$

and since we have assumed that $C^1_{11} \neq 0$, the only possible solution is $C^1_{12} = C^1_{21} = 0$, and we have now solved the entire system of equations.

Case 2: If instead $C_{13}^1 = C_{31}^1$, $C_{11}^3 \neq 0$, $C_{12}^3 + C_{21}^3 \neq 0$, $C_{13}^3 + C_{31}^3 \neq 0$, $C_{22}^3 \neq 0$, $C_{23}^3 \neq 0$, $C_{32}^3 \neq 0$, and $C_{33}^3 \neq 0$, the system of equations becomes

$$C_{11}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.14.3)}$$
$$C_{12}^1 C_{11}^1 + C_{12}^3 C_{13}^1 = 0 \quad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 + C_{12}^3 C_{13}^3 = 0 \quad \text{(R3.15.3)}$$
$$C_{13}^1 C_{11}^1 + C_{13}^3 C_{13}^1 = 0 \quad \text{(R3.16.1)}$$
$$C_{13}^1 C_{11}^3 + C_{13}^3 C_{13}^3 = 0 \quad \text{(R3.16.3)}$$
$$(C_{12}^3 + C_{21}^3)(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.17.3.2)}$$
$$(C_{13}^3 + C_{31}^3)(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.18.3.2)}$$
$$C_{22}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.19.3.2)}$$
$$C_{23}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.20.3.2)}$$
$$C_{32}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.21.3.2)}$$
$$C_{33}^3(C_{13}^3 - C_{31}^3) = 0 \quad \text{(R3.22.3.2)}$$
$$C_{21}^1 C_{11}^1 + C_{21}^3 C_{13}^1 = 0 \quad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 + C_{21}^3 C_{31}^3 = 0 \quad \text{(R3.23.3)}$$
$$C_{22}^1 C_{11}^1 + C_{22}^3 C_{13}^1 = 0 \quad \text{(R3.26.1)}$$
$$C_{22}^1 C_{11}^3 + C_{22}^3 C_{31}^3 = 0 \quad \text{(R3.26.3)}$$
$$C_{23}^1 C_{11}^1 + C_{23}^3 C_{13}^1 = 0 \quad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 + C_{23}^3 C_{31}^3 = 0 \quad \text{(R3.27.3)}$$
$$C_{13}^1 C_{11}^1 + C_{31}^3 C_{13}^1 = 0 \quad \text{(R3.32.1)}$$
$$C_{13}^1 C_{11}^3 + C_{31}^3 C_{31}^3 = 0 \quad \text{(R3.32.3)}$$
$$C_{32}^1 C_{11}^1 + C_{32}^3 C_{13}^1 = 0 \quad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 + C_{32}^3 C_{31}^3 = 0 \quad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 + C_{33}^3 C_{13}^1 = 0 \quad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 + C_{33}^3 C_{31}^3 = 0. \quad \text{(R3.36.3)}$$

Since we have assumed that $C_{11}^3 \neq 0$, the only solution to equation (R3.14.3) is $C_{13}^3 = C_{31}^3$. This gives us

$$C_{12}^1 C_{11}^1 + C_{12}^3 C_{13}^1 = 0 \quad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 + C_{12}^3 C_{13}^3 = 0 \quad \text{(R3.15.3)}$$
$$C_{13}^1 C_{11}^1 + C_{13}^3 C_{13}^1 = 0 \quad \text{(R3.16.1)}$$
$$C_{13}^1 C_{11}^3 + C_{13}^3 C_{13}^3 = 0 \quad \text{(R3.16.3)}$$
$$C_{21}^1 C_{11}^1 + C_{21}^3 C_{13}^1 = 0 \quad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 + C_{21}^3 C_{13}^3 = 0 \quad \text{(R3.23.3)}$$

$$C_{22}^1 C_{11}^1 + C_{22}^3 C_{13}^1 = 0 \qquad \text{(R3.26.1)}$$
$$C_{22}^1 C_{11}^3 + C_{22}^3 C_{13}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^1 C_{11}^1 + C_{23}^3 C_{13}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 + C_{23}^3 C_{13}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{13}^1 C_{11}^1 + C_{13}^3 C_{13}^1 = 0 \qquad \text{(R3.32.1)}$$
$$C_{13}^1 C_{11}^3 + C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^1 C_{11}^1 + C_{32}^3 C_{13}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 + C_{32}^3 C_{13}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 + C_{33}^3 C_{13}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 + C_{33}^3 C_{13}^3 = 0. \qquad \text{(R3.36.3)}$$

Equation (R3.16.1) can be rewritten as $C_{13}^1 C_{11}^1 + C_{13}^3 C_{13}^1 = C_{13}^1(C_{11}^1 + C_{13}^3) = 0$, which has the solution $C_{13}^1 = 0$ and $C_{11}^1 + C_{13}^3 = 0$. We look at each of these cases.

Case 2.1: If $C_{13}^1 = 0$, the system of equations becomes

$$C_{12}^1 C_{11}^1 = 0 \qquad \text{(R3.15.1)}$$
$$C_{12}^1 C_{11}^3 + C_{12}^3 C_{13}^3 = 0 \qquad \text{(R3.15.3)}$$
$$C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.16.3)}$$
$$C_{21}^1 C_{11}^1 = 0 \qquad \text{(R3.23.1)}$$
$$C_{21}^1 C_{11}^3 + C_{21}^3 C_{13}^3 = 0 \qquad \text{(R3.23.3)}$$
$$C_{22}^1 C_{11}^1 = 0 \qquad \text{(R3.26.1)}$$
$$C_{22}^1 C_{11}^3 + C_{22}^3 C_{13}^3 = 0 \qquad \text{(R3.26.3)}$$
$$C_{23}^1 C_{11}^1 = 0 \qquad \text{(R3.27.1)}$$
$$C_{23}^1 C_{11}^3 + C_{23}^3 C_{13}^3 = 0 \qquad \text{(R3.27.3)}$$
$$C_{13}^3 C_{13}^3 = 0 \qquad \text{(R3.32.3)}$$
$$C_{32}^1 C_{11}^1 = 0 \qquad \text{(R3.35.1)}$$
$$C_{32}^1 C_{11}^3 + C_{32}^3 C_{13}^3 = 0 \qquad \text{(R3.35.3)}$$
$$C_{33}^1 C_{11}^1 = 0 \qquad \text{(R3.36.1)}$$
$$C_{33}^1 C_{11}^3 + C_{33}^3 C_{13}^3 = 0. \qquad \text{(R3.36.3)}$$

The only solution to equation (R3.16.3) is $C_{13}^3 = 0$. However, we remember that we have earlier assumed that both $C_{13}^3 + C_{31}^3 \neq 0$ and $C_{13}^3 = C_{31}^3$, and if $C_{13}^3 = 0$ that means we will have both $C_{31}^3 \neq 0$ and $C_{31}^3 = 0$, which is a contradiction. Thus, this case did not lead to a solution.

Case 2.2: If instead $C^1_{11}+C^3_{13}=0$ and $C^1_{13}\neq 0$, the system of equations becomes

$$-C^1_{12}C^3_{13}+C^3_{12}C^1_{13}=0 \quad \text{(R3.15.1)}$$
$$C^1_{12}C^3_{11}+C^3_{12}C^3_{13}=0 \quad \text{(R3.15.3)}$$
$$C^1_{13}C^3_{11}+C^3_{13}C^3_{13}=0 \quad \text{(R3.16.3)}$$
$$-C^1_{21}C^3_{13}+C^3_{21}C^1_{13}=0 \quad \text{(R3.23.1)}$$
$$C^1_{21}C^3_{11}+C^3_{21}C^3_{13}=0 \quad \text{(R3.23.3)}$$
$$-C^1_{22}C^3_{13}+C^3_{22}C^1_{13}=0 \quad \text{(R3.26.1)}$$
$$C^1_{22}C^3_{11}+C^3_{22}C^3_{13}=0 \quad \text{(R3.26.3)}$$
$$-C^1_{23}C^3_{13}+C^3_{23}C^1_{13}=0 \quad \text{(R3.27.1)}$$
$$C^1_{23}C^3_{11}+C^3_{23}C^3_{13}=0 \quad \text{(R3.27.3)}$$
$$C^1_{13}C^3_{11}+C^3_{13}C^3_{13}=0 \quad \text{(R3.32.3)}$$
$$-C^1_{32}C^3_{13}+C^3_{32}C^1_{13}=0 \quad \text{(R3.35.1)}$$
$$C^1_{32}C^3_{11}+C^3_{32}C^3_{13}=0 \quad \text{(R3.35.3)}$$
$$-C^1_{33}C^3_{13}+C^3_{33}C^1_{13}=0 \quad \text{(R3.36.1)}$$
$$C^1_{33}C^3_{11}+C^3_{33}C^3_{13}=0. \quad \text{(R3.36.3)}$$

Now, first of all, we see that we can remove equation (R3.32.3) since it is identical to equation (R3.16.3). Then, since we have assumed that $C^1_{13}\neq 0$, we can write the solution to equation (R3.16.3) as $C^3_{11}=-C^3_{13}C^3_{13}/C^1_{13}$. Inserting this into the system of equation gives us

$$-C^1_{12}C^3_{13}+C^3_{12}C^1_{13}=0 \quad \text{(R3.15.1)}$$
$$-C^1_{12}C^3_{13}C^3_{13}/C^1_{13}+C^3_{12}C^3_{13}=0 \quad \text{(R3.15.3)}$$
$$-C^1_{21}C^3_{13}+C^3_{21}C^1_{13}=0 \quad \text{(R3.23.1)}$$
$$-C^1_{21}C^3_{13}C^3_{13}/C^1_{13}+C^3_{21}C^3_{13}=0 \quad \text{(R3.23.3)}$$
$$-C^1_{22}C^3_{13}+C^3_{22}C^1_{13}=0 \quad \text{(R3.26.1)}$$
$$-C^1_{22}C^3_{13}C^3_{13}/C^1_{13}+C^3_{22}C^3_{13}=0 \quad \text{(R3.26.3)}$$
$$-C^1_{23}C^3_{13}+C^3_{23}C^1_{13}=0 \quad \text{(R3.27.1)}$$
$$-C^1_{23}C^3_{13}C^3_{13}/C^1_{13}+C^3_{23}C^3_{13}=0 \quad \text{(R3.27.3)}$$
$$-C^1_{32}C^3_{13}+C^3_{32}C^1_{13}=0 \quad \text{(R3.35.1)}$$
$$-C^1_{32}C^3_{13}C^3_{13}/C^1_{13}+C^3_{32}C^3_{13}=0 \quad \text{(R3.35.3)}$$
$$-C^1_{33}C^3_{13}+C^3_{33}C^1_{13}=0 \quad \text{(R3.36.1)}$$
$$-C^1_{33}C^3_{13}C^3_{13}/C^1_{13}+C^3_{33}C^3_{13}=0. \quad \text{(R3.36.3)}$$

In the same way, we write the solution to all of the equations that still include

C^1_{13}, but no divisions by this variable:

$$\begin{aligned}
(\text{R3.15.1}):\quad & C^3_{12} = C^1_{12}C^3_{13}/C^1_{13},\\
(\text{R3.23.1}):\quad & C^3_{21} = C^1_{21}C^3_{13}/C^1_{13},\\
(\text{R3.26.1}):\quad & C^3_{22} = C^1_{22}C^3_{13}/C^1_{13},\\
(\text{R3.27.1}):\quad & C^3_{23} = C^1_{23}C^3_{13}/C^1_{13},\\
(\text{R3.35.1}):\quad & C^3_{32} = C^1_{32}C^3_{13}/C^1_{13},\\
(\text{R3.36.1}):\quad & C^3_{33} = C^1_{33}C^3_{13}/C^1_{13}.
\end{aligned}$$

Inserting this into the remaining equations give us

$$\begin{aligned}
(\text{R3.15.3}):\quad & -C^1_{12}C^3_{13}C^3_{13}/C^1_{13} + C^3_{12}C^3_{13}\\
& = -C^1_{12}C^3_{13}C^3_{13}/C^1_{13} + C^1_{12}C^3_{13}C^3_{13}/C^1_{13}\\
& = 0,\\
(\text{R3.23.3}):\quad & -C^1_{21}C^3_{13}C^3_{13}/C^1_{13} + C^3_{21}C^3_{13}\\
& = -C^1_{21}C^3_{13}C^3_{13}/C^1_{13} + C^1_{21}C^3_{13}C^3_{13}/C^1_{13}\\
& = 0,\\
(\text{R3.26.3}):\quad & -C^1_{22}C^3_{13}C^3_{13}/C^1_{13} + C^3_{22}C^3_{13}\\
& = -C^1_{22}C^3_{13}C^3_{13}/C^1_{13} + C^1_{22}C^3_{13}C^3_{13}/C^1_{13}\\
& = 0,\\
(\text{R3.27.3}):\quad & -C^1_{23}C^3_{13}C^3_{13}/C^1_{13} + C^3_{23}C^3_{13}\\
& = -C^1_{23}C^3_{13}C^3_{13}/C^1_{13} + C^1_{23}C^3_{13}C^3_{13}/C^1_{13}\\
& = 0,\\
(\text{R3.35.3}):\quad & -C^1_{32}C^3_{13}C^3_{13}/C^1_{13} + C^3_{32}C^3_{13}\\
& = -C^1_{32}C^3_{13}C^3_{13}/C^1_{13} + C^1_{32}C^3_{13}C^3_{13}/C^1_{13}\\
& = 0,\\
(\text{R3.36.3}):\quad & -C^1_{33}C^3_{13}C^3_{13}/C^1_{13} + C^3_{33}C^3_{13}\\
& = -C^1_{33}C^3_{13}C^3_{13}/C^1_{13} + C^1_{33}C^3_{13}C^3_{13}/C^1_{13}\\
& = 0.
\end{aligned}$$

Clearly the solutions to the first equations also solved the remaining equations, and thus we have now solved the entire system of equations.

We have now found all possible solutions when the original system of equations is mapped to two dimensions. We sum up the results in Table 3.39, Table 3.40 and Table 3.41, since they do not fit in one table. Note that to make the tables slightly smaller, in these tables we write solution 1 as S1, solution 2 as S2, and so on, instead of Soln 1, Soln 2 etc. as we have written in previous tables.

Looking at all of the solutions, we see that solution 1 is actually just a special case of solution 5, solution 9 is a special case of solution 10, and solutions 16 and 18 are special cases of solution 17, which means that solutions 1, 9, 16 and 18 are actually superfluous and could be removed from the tables. However, since the tables are so big it might actually become more confusing if we write the tables again without these four solutions, so instead we just note here that they should be removed from the tables.

Now, we use these tables and permutate the indices of the structure constants according to Theorem 2 to get the results for when α is defined as E_{22}, shown in Table 3.42 and Table 3.43, and finally permutate again to get the results for when α is defined as E_{33}, shown in Table 3.44 and Table 3.45. Note that we removed the solutions that we noted were just special cases of other solutions.

Finally, what we would like to do is the same thing we did in two dimensions as well as three dimensions mapped to one dimension, and find the commutator tables of the commutators belonging to the hom-Lie admissible algebras associated to the hom-associative algebras we just found. However, each hom-associative algebra is represented by one of the solutions, meaning we have 15 different hom-associative algebras for α defined as E_{11}, 15 ones for α defined as E_{22}, and 15 ones for α defined as E_{33} as well. That means we would have to calculate the commutator tables for 45 hom-Lie admissible algebras, and while it would of course be possible, we feel it would be more work than it is worth considering we would still be missing the results for α defined as six out of the nine different three-dimensional matrix units, so the analysis we could do would still be far from complete.

	S1	S2	S3	S4	S5	S6	S7	S8
C_{11}^1	$\mathbf{0}$	$\mathbf{0}$	**free**	$\mathbf{-C_{12}^2}$	$\mathbf{0}$	$\mathbf{0}$	**free**	$\mathbf{-C_{13}^3}$
C_{11}^2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-C_{12}^2C_{12}^2/C_{12}^1}$	0	0	0	0
C_{11}^3	0	0	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-C_{13}^3C_{13}^3/C_{13}^1}$
C_{12}^1	**free**	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{12}^1 \neq 0}$	**free**	**free**	$\mathbf{0}$	**free**
C_{12}^2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	**free**	0	0	0	0
C_{12}^3	0	0	0	0	$\mathbf{0}$	**free**	**free**	$\mathbf{C_{12}^1C_{13}^3/C_{13}^1}$
C_{13}^1	**free**	**free**	$\mathbf{0}$	**free**	**free**	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{13}^1 \neq 0}$
C_{13}^2	$\mathbf{0}$	**free**	**free**	$\mathbf{C_{13}^1C_{12}^2/C_{12}^1}$	0	0	0	0
C_{13}^3	0	0	0	0	**free**	**free**	**free**	**free**
C_{21}^1	**free**	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{12}^1}$	**free**	**free**	$\mathbf{0}$	**free**
C_{21}^2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{12}^2}$	0	0	0	0
C_{21}^3	0	0	0	0	$\mathbf{0}$	$\mathbf{-C_{12}^3}$	$\mathbf{-C_{12}^3}$	$\mathbf{C_{21}^1C_{13}^3/C_{13}^1}$
C_{22}^1	**free**	**free**	$\mathbf{0}$	**free**	**free**	**free**	$\mathbf{0}$	**free**
C_{22}^2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{22}^1C_{12}^2/C_{12}^1}$	0	0	0	0
C_{22}^3	0	0	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{22}^1C_{13}^3/C_{13}^1}$
C_{23}^1	**free**	**free**	$\mathbf{0}$	**free**	**free**	**free**	$\mathbf{0}$	**free**
C_{23}^2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{23}^1C_{12}^2/C_{12}^1}$	0	0	0	0
C_{23}^3	0	0	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{23}^1C_{13}^3/C_{13}^1}$
C_{31}^1	**free**	**free**	$\mathbf{0}$	**free**	**free**	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{13}^1}$
C_{31}^2	$\mathbf{0}$	$\mathbf{-C_{13}^2}$	$\mathbf{-C_{13}^2}$	$\mathbf{C_{31}^1C_{12}^2/C_{12}^1}$	0	0	0	0
C_{31}^3	0	0	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{13}^3}$
C_{32}^1	**free**	**free**	$\mathbf{0}$	**free**	**free**	**free**	$\mathbf{0}$	**free**
C_{32}^2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{32}^1C_{12}^2/C_{12}^1}$	0	0	0	0
C_{32}^3	0	0	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{32}^1C_{13}^3/C_{13}^1}$
C_{33}^1	**free**	**free**	$\mathbf{0}$	**free**	**free**	**free**	$\mathbf{0}$	**free**
C_{33}^2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{33}^1C_{12}^2/C_{12}^1}$	0	0	0	0
C_{33}^3	0	0	0	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_{33}^1C_{13}^3/C_{13}^1}$

Table 3.39: The values of the structure constants that give hom-associative algebras when α is defined as E_{11}, in three dimensions mapped to two dimensions. Solutions 1-4 come from the case when we have let $C_{ij}^k = 0$ for $k = 3$, and solutions 5-8 come from the case when we have let $C_{ij}^k = 0$ for $k = 2$. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	S9	S10	S11	S12	S13	S14	S15
C_{11}^1	0	0	0	0	0	0	0
C_{11}^2	**0**	**free**	**free**	**0**	**free**	**free**	$\mathbf{C_{11}^3C_{21}^2/C_{21}^3}$
C_{11}^3	**free**	**free**	**0**	**free**	**free**	**free**	**free**
C_{12}^1	0	0	0	0	0	0	0
C_{12}^2	**0**	**0**	**0**	**0**	**0**	**0**	**0**
C_{12}^3	**free**	**free**	**0**	**free**	**free**	**0**	**0**
C_{13}^1	0	0	0	0	0	0	0
C_{13}^2	**0**	**0**	**0**	**0**	**0**	**0**	**0**
C_{13}^3	**0**	**0**	**0**	**0**	**0**	**0**	**0**
C_{21}^1	0	0	0	0	0	0	0
C_{21}^2	**0**	**0**	**0**	**0**	**0**	**0**	**free**
C_{21}^3	$\mathbf{C_{12}^3}$	**free**	**0**	**free**	$\mathbf{C_{12}^3}$	**0**	$\mathbf{C_{21}^3 \neq 0}$
C_{22}^1	0	0	0	0	0	0	0
C_{22}^2	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{22}^3C_{21}^2/C_{21}^3}$
C_{22}^3	**0**	**0**	**0**	**free**	**free**	**free**	**free**
C_{23}^1	0	0	0	0	0	0	0
C_{23}^2	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{23}^3C_{21}^2/C_{21}^3}$
C_{23}^3	**0**	**0**	**0**	**free**	**free**	**free**	**free**
C_{31}^1	0	0	0	0	0	0	0
C_{31}^2	**0**	**0**	**free**	**0**	**0**	**0**	$\mathbf{-C_{21}^2C_{21}^2/C_{21}^3}$
C_{31}^3	**0**	**0**	**0**	**0**	**0**	**0**	$\mathbf{-C_{21}^2}$
C_{32}^1	0	0	0	0	0	0	0
C_{32}^2	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{32}^3C_{21}^2/C_{21}^3}$
C_{32}^3	**0**	**0**	**0**	**free**	**free**	**free**	**free**
C_{33}^1	0	0	0	0	0	0	0
C_{33}^2	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{33}^3C_{21}^2/C_{21}^3}$
C_{33}^3	**0**	**0**	**0**	**free**	**free**	**free**	**free**

Table 3.40: The values of the structure constants that give hom-associative algebras when α is defined as E_{11}, in three dimensions mapped to two dimensions. Solutions 9-19 (continued in Table 3.41) come from the case when we have let $C_{ij}^k = 0$ for $k = 1$. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	S16	S17	S18	S19
C_{11}^1	0	0	0	0
C_{11}^2	**free**	**free**	**free**	**free**
C_{11}^3	$\mathbf{-C_{11}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{11}^2C_{12}^2/C_{13}^2}$	**0**	**free**
C_{12}^1	0	0	0	0
C_{12}^2	**free**	**free**	$\mathbf{C_{12}^2C_{31}^2/C_{13}^2}$	$\mathbf{C_{12}^2C_{31}^2/C_{13}^2}$
C_{12}^3	$\mathbf{-C_{12}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{12}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{12}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{12}^2C_{12}^2/C_{13}^2}$
C_{13}^1	0	0	0	0
C_{13}^2	**free**	**free**	**free**	**free**
C_{13}^3	$\mathbf{-C_{12}^2}$	$\mathbf{-C_{12}^2}$	$\mathbf{-C_{12}^2}$	$\mathbf{-C_{12}^2}$
C_{21}^1	0	0	0	0
C_{21}^2	**0**	$\mathbf{C_{12}^2C_{31}^2/C_{13}^2}$	$\mathbf{C_{12}^2C_{31}^2/C_{13}^2}$	$\mathbf{C_{12}^2C_{31}^2/C_{13}^2}$
C_{21}^3	**0**	$\mathbf{-C_{21}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{21}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{21}^2C_{12}^2/C_{13}^2}$
C_{22}^1	0	0	0	0
C_{22}^2	**free**	**free**	**free**	**free**
C_{22}^3	$\mathbf{-C_{22}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{22}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{22}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{22}^2C_{12}^2/C_{13}^2}$
C_{23}^1	0	0	0	0
C_{23}^2	**free**	**free**	**free**	**free**
C_{23}^3	$\mathbf{-C_{23}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{23}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{23}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{23}^2C_{12}^2/C_{13}^2}$
C_{31}^1	0	0	0	0
C_{31}^2	**0**	**free**	**free**	$\mathbf{C_{13}^2}$
C_{31}^3	**0**	$\mathbf{-C_{12}^2C_{31}^2/C_{13}^2}$	$\mathbf{-C_{12}^2C_{31}^2/C_{13}^2}$	$\mathbf{-C_{12}^2C_{31}^2/C_{13}^2}$
C_{32}^1	0	0	0	0
C_{32}^2	**free**	**free**	**free**	**free**
C_{32}^3	$\mathbf{-C_{32}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{32}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{32}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{32}^2C_{12}^2/C_{13}^2}$
C_{33}^1	0	0	0	0
C_{33}^2	**free**	**free**	**free**	**free**
C_{33}^3	$\mathbf{-C_{33}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{33}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{33}^2C_{12}^2/C_{13}^2}$	$\mathbf{-C_{33}^2C_{12}^2/C_{13}^2}$

Table 3.41: The values of the structure constants that give hom-associative algebras when α is defined as E_{11}, in three dimensions mapped to two dimensions. Solutions 9-19 (continued from Table 3.40) come from the case when we have let $C_{ij}^k = 0$ for $k = 1$. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	S2	S3	S4	S5	S6	S7	S8
C_{11}^1	0	0	0	**0**	**0**	**0**	$\mathbf{C_{11}^2C_{21}^1/C_{21}^2}$
C_{11}^2	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{11}^3	**0**	**0**	$\mathbf{C_{11}^2C_{23}^3/C_{23}^2}$	0	0	0	0
C_{12}^1	0	0	0	**0**	**0**	**0**	$\mathbf{C_{21}^1}$
C_{12}^2	**free**	**0**	**free**	**free**	**0**	**0**	$\mathbf{C_{21}^2}$
C_{12}^3	$\mathbf{-C_{21}^3}$	$\mathbf{-C_{21}^3}$	$\mathbf{C_{12}^2C_{23}^3/C_{23}^2}$	0	0	0	0
C_{13}^1	0	0	0	**0**	**0**	**0**	$\mathbf{C_{13}^2C_{21}^1/C_{21}^2}$
C_{13}^2	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{13}^3	**0**	**0**	$\mathbf{C_{13}^2C_{23}^3/C_{23}^2}$	0	0	0	0
C_{21}^1	0	0	0	**free**	**free**	**free**	**free**
C_{21}^2	**free**	**0**	**free**	**free**	**0**	**0**	$\mathbf{C_{21}^2 \neq 0}$
C_{21}^3	**free**	**free**	$\mathbf{C_{21}^2C_{23}^3/C_{23}^2}$	0	0	0	0
C_{22}^1	0	0	0	**0**	**0**	**0**	$\mathbf{-C_{21}^1C_{21}^1/C_{21}^2}$
C_{22}^2	**0**	**free**	$\mathbf{-C_{23}^3}$	**0**	**0**	**free**	$\mathbf{-C_{21}^1}$
C_{22}^3	**0**	**0**	$\mathbf{-C_{23}^3C_{23}^3/C_{23}^2}$	0	0	0	0
C_{23}^1	0	0	0	**0**	**free**	**free**	$\mathbf{C_{23}^2C_{21}^1/C_{21}^2}$
C_{23}^2	**0**	**0**	$\mathbf{C_{23}^2 \neq 0}$	**free**	**free**	**0**	**free**
C_{23}^3	**0**	**0**	**free**	0	0	0	0
C_{31}^1	0	0	0	**0**	**0**	**0**	$\mathbf{C_{31}^2C_{21}^1/C_{21}^2}$
C_{31}^2	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{31}^3	**0**	**0**	$\mathbf{C_{31}^2C_{23}^3/C_{23}^2}$	0	0	0	0
C_{32}^1	0	0	0	**0**	$\mathbf{-C_{23}^1}$	$\mathbf{-C_{23}^1}$	$\mathbf{C_{32}^2C_{21}^1/C_{21}^2}$
C_{32}^2	**0**	**0**	$\mathbf{C_{23}^2}$	**free**	**free**	**0**	**free**
C_{32}^3	**0**	**0**	$\mathbf{C_{23}^3}$	0	0	0	0
C_{33}^1	0	0	0	**0**	**0**	**0**	$\mathbf{C_{33}^2C_{21}^1/C_{21}^2}$
C_{33}^2	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{33}^3	**0**	**0**	$\mathbf{C_{33}^2C_{23}^3/C_{23}^2}$	0	0	0	0

Table 3.42: The values of the structure constants that give hom-associative algebras when α is defined as E_{22}, in three dimensions mapped to two dimensions. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	S10	S11	S12	S13	S14	S15	S17	S19
C_{11}^1	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C_{11}^3C_{23}^3/C_{21}^3}$	$\mathbf{-C_{11}^3C_{23}^3/C_{21}^3}$
C_{11}^2	0	0	0	0	0	0	0	0
C_{11}^3	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{11}^1C_{32}^3/C_{32}^1}$	**free**	**free**
C_{12}^1	**0**	**0**	**0**	**0**	**0**	$\mathbf{-C_{32}^3}$	$\mathbf{-C_{23}^3C_{12}^3/C_{21}^3}$	$\mathbf{-C_{23}^3C_{12}^3/C_{21}^3}$
C_{12}^2	0	0	0	0	0	0	0	0
C_{12}^3	**0**	**free**	**0**	**0**	**0**	$\mathbf{-C_{32}^3C_{32}^3/C_{32}^1}$	**free**	$\mathbf{C_{21}^3}$
C_{13}^1	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C_{13}^3C_{23}^3/C_{21}^3}$	$\mathbf{-C_{13}^3C_{23}^3/C_{21}^3}$
C_{13}^2	0	0	0	0	0	0	0	0
C_{13}^3	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{13}^1C_{32}^3/C_{32}^1}$	**free**	**free**
C_{21}^1	**0**	**0**	**0**	**0**	**0**	**0**	$\mathbf{-C_{23}^3}$	$\mathbf{-C_{23}^3}$
C_{21}^2	0	0	0	0	0	0	0	0
C_{21}^3	**0**	**0**	**0**	**0**	**0**	**0**	**free**	**free**
C_{22}^1	**free**	0	**free**	**free**	**free**	**free**	$\mathbf{-C_{22}^3C_{23}^3/C_{21}^3}$	**free**
C_{22}^2	0	0	0	0	0	0	0	0
C_{22}^3	**free**	**free**	**0**	**free**	**free**	$\mathbf{C_{22}^1C_{32}^3/C_{32}^1}$	**free**	**free**
C_{23}^1	**free**	**0**	**free**	**free**	**0**	**0**	$\mathbf{-C_{23}^3C_{23}^3/C_{21}^3}$	$\mathbf{-C_{23}^3C_{23}^3/C_{21}^3}$
C_{23}^2	0	0	0	0	0	0	0	0
C_{23}^3	**0**	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{23}^3C_{12}^3/C_{21}^3}$
C_{31}^1	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C_{31}^3C_{23}^3/C_{21}^3}$	$\mathbf{-C_{31}^3C_{23}^3/C_{21}^3}$
C_{31}^2	0	0	0	0	0	0	0	0
C_{31}^3	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{31}^1C_{32}^3/C_{32}^1}$	**free**	**free**
C_{32}^1	**free**	**0**	**free**	$\mathbf{C_{23}^1}$	**0**	$\mathbf{C_{32}^1 \neq 0}$	$\mathbf{-C_{32}^3C_{23}^3/C_{21}^3}$	$\mathbf{-C_{32}^3C_{23}^3/C_{21}^3}$
C_{32}^2	0	0	0	0	0	0	0	0
C_{32}^3	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{23}^3C_{12}^3/C_{21}^3}$	$\mathbf{C_{23}^3C_{12}^3/C_{21}^3}$
C_{33}^1	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C_{33}^3C_{23}^3/C_{21}^3}$	$\mathbf{-C_{33}^3C_{23}^3/C_{21}^3}$
C_{33}^2	0	0	0	0	0	0	0	0
C_{33}^3	**0**	**0**	**0**	**0**	**free**	$\mathbf{C_{33}^1C_{32}^3/C_{32}^1}$	**free**	**free**

Table 3.43: The values of the structure constants that give hom-associative algebras when α is defined as E_{22}, in three dimensions mapped to two dimensions. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	S2	S3	S4	S5	S6	S7	S8
C_{11}^1	**0**	**0**	$\mathbf{C_{11}^3C_{31}^1/C_{31}^3}$	0	0	0	0
C_{11}^2	0	0	0	**0**	**0**	**0**	$\mathbf{C_{11}^3C_{32}^2/C_{32}^3}$
C_{11}^3	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{12}^1	**0**	**0**	$\mathbf{C_{12}^3C_{31}^1/C_{31}^3}$	0	0	0	0
C_{12}^2	0	0	0	**0**	**0**	**0**	$\mathbf{C_{12}^3C_{32}^2/C_{32}^3}$
C_{12}^3	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{13}^1	**0**	**0**	$\mathbf{C_{31}^1}$	0	0	0	0
C_{13}^2	0	0	0	**0**	$\mathbf{-C_{31}^2}$	$\mathbf{-C_{31}^2}$	$\mathbf{C_{13}^3C_{32}^2/C_{32}^3}$
C_{13}^3	**0**	**0**	$\mathbf{C_{31}^3}$	**free**	**free**	**0**	**free**
C_{21}^1	**0**	**0**	$\mathbf{C_{21}^3C_{31}^1/C_{31}^3}$	0	0	0	0
C_{21}^2	0	0	0	**0**	**0**	**0**	$\mathbf{C_{21}^3C_{32}^2/C_{32}^3}$
C_{21}^3	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{22}^1	**0**	**0**	$\mathbf{C_{22}^3C_{31}^1/C_{31}^3}$	0	0	0	0
C_{22}^2	0	0	0	**0**	**0**	**0**	$\mathbf{C_{22}^3C_{32}^2/C_{32}^3}$
C_{22}^3	**free**	**0**	**free**	**free**	**free**	**0**	**free**
C_{23}^1	$\mathbf{-C_{32}^1}$	$\mathbf{-C_{32}^1}$	$\mathbf{C_{23}^3C_{31}^1/C_{31}^3}$	0	0	0	0
C_{23}^2	0	0	0	**0**	**0**	**0**	$\mathbf{C_{32}^2}$
C_{23}^3	**free**	**0**	**free**	**free**	**0**	**0**	$\mathbf{C_{32}^3}$
C_{31}^1	**0**	**0**	**free**	0	0	0	0
C_{31}^2	0	0	0	**0**	**free**	**free**	$\mathbf{C_{31}^3C_{32}^2/C_{32}^3}$
C_{31}^3	**0**	**0**	$\mathbf{C_{31}^3 \neq 0}$	**free**	**free**	**0**	**free**
C_{32}^1	**free**	**free**	$\mathbf{C_{32}^3C_{31}^1/C_{31}^3}$	0	0	0	0
C_{32}^2	0	0	0	**free**	**free**	**free**	**free**
C_{32}^3	**free**	**0**	**free**	**free**	**0**	**0**	$\mathbf{C_{32}^3 \neq 0}$
C_{33}^1	**0**	**0**	$\mathbf{-C_{31}^1C_{31}^1/C_{31}^3}$	0	0	0	0
C_{33}^2	0	0	0	**0**	**0**	**0**	$\mathbf{-C_{32}^2C_{32}^2/C_{32}^3}$
C_{33}^3	**0**	**free**	$\mathbf{-C_{31}^1}$	**0**	**0**	**free**	$\mathbf{-C_{32}^2}$

Table 3.44: The values of the structure constants that give hom-associative algebras when α is defined as E_{33}, in three dimensions mapped to two dimensions. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

	S10	S11	S12	S13	S14	S15	S17	S19
C^1_{11}	**0**	**0**	**0**	**0**	**free**	$\mathbf{C^2_{11}C^1_{13}/C^2_{13}}$	**free**	**free**
C^2_{11}	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C^1_{11}C^1_{31}/C^1_{32}}$	$\mathbf{-C^1_{11}C^1_{31}/C^1_{32}}$
C^3_{11}	0	0	0	0	0	0	0	0
C^1_{12}	**0**	**0**	**0**	**0**	**free**	$\mathbf{C^2_{12}C^1_{13}/C^2_{13}}$	**free**	**free**
C^2_{12}	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C^1_{12}C^1_{31}/C^1_{32}}$	$\mathbf{-C^1_{12}C^1_{31}/C^1_{32}}$
C^3_{12}	0	0	0	0	0	0	0	0
C^1_{13}	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C^1_{31}C^1_{23}/C^1_{32}}$	$\mathbf{C^1_{31}C^1_{23}/C^1_{32}}$
C^2_{13}	**free**	**0**	**free**	$\mathbf{C^2_{31}}$	**0**	$\mathbf{C^2_{13} \neq 0}$	$\mathbf{-C^1_{13}C^1_{31}/C^1_{32}}$	$\mathbf{-C^1_{13}C^1_{31}/C^1_{32}}$
C^3_{13}	0	0	0	0	0	0	0	0
C^1_{21}	**0**	**0**	**0**	**0**	**free**	$\mathbf{C^2_{21}C^1_{13}/C^2_{13}}$	**free**	**free**
C^2_{21}	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C^1_{21}C^1_{31}/C^1_{32}}$	$\mathbf{-C^1_{21}C^1_{31}/C^1_{32}}$
C^3_{21}	0	0	0	0	0	0	0	0
C^1_{22}	**0**	**0**	**0**	**0**	**free**	$\mathbf{C^2_{22}C^1_{13}/C^2_{13}}$	**free**	**free**
C^2_{22}	**0**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C^1_{22}C^1_{31}/C^1_{32}}$	$\mathbf{-C^1_{22}C^1_{31}/C^1_{32}}$
C^3_{22}	0	0	0	0	0	0	0	0
C^1_{23}	**0**	**free**	**0**	**0**	**0**	$\mathbf{-C^1_{13}C^1_{13}/C^2_{13}}$	**free**	$\mathbf{C^1_{32}}$
C^2_{23}	**0**	**0**	**0**	**0**	**0**	$\mathbf{-C^1_{13}}$	$\mathbf{-C^1_{31}C^1_{23}/C^1_{32}}$	$\mathbf{-C^1_{31}C^1_{23}/C^1_{32}}$
C^3_{23}	0	0	0	0	0	0	0	0
C^1_{31}	**0**	**0**	**0**	**0**	**0**	**0**	**free**	$\mathbf{C^1_{31}C^1_{23}/C^1_{32}}$
C^2_{31}	**free**	**0**	**free**	**free**	**0**	**0**	$\mathbf{-C^1_{31}C^1_{31}/C^1_{32}}$	$\mathbf{-C^1_{31}C^1_{31}/C^1_{32}}$
C^3_{31}	0	0	0	0	0	0	0	0
C^1_{32}	**0**	**0**	**0**	**0**	**0**	**0**	**free**	**free**
C^2_{32}	**0**	**0**	**0**	**0**	**0**	**0**	$\mathbf{-C^1_{31}}$	$\mathbf{-C^1_{31}}$
C^3_{32}	0	0	0	0	0	0	0	0
C^1_{33}	**free**	**free**	**0**	**free**	**free**	$\mathbf{C^2_{33}C^1_{13}/C^2_{13}}$	**free**	**free**
C^2_{33}	**free**	**0**	**free**	**free**	**free**	**free**	$\mathbf{-C^1_{33}C^1_{31}/C^1_{32}}$	**free**
C^3_{33}	0	0	0	0	0	0	0	0

Table 3.45: The values of the structure constants that give hom-associative algebras when α is defined as E_{33}, in three dimensions mapped to two dimensions. The values written in bold are the solutions from solving the system of equations, while the ones not in bold were set to zero before solving the system of equations

Chapter 4

Conclusion

4.1 Results and Discussion

We wanted to find the values of the structure constants C_{ij}^k for which our algebras became hom-associative (G_1-hom-associative). First, in two dimensions and with $[\alpha] = E_{11}$, we got the results in Table 3.2, which looked as follows: Remembering that we have $\mu(e_i, e_j) = \sum_{k=1}^{n} C_{ij}^k e_k$,

	Soln 2	Soln 4	Soln 5	Soln 6
C_{11}^1	0	0	free	$-C_{12}^2$
C_{11}^2	0	0	free	$C_{11}^2 \neq 0$
C_{12}^1	0	free	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{12}^2	0	0	0	free
C_{21}^1	0	free	0	$-C_{12}^2 C_{12}^2 / C_{11}^2$
C_{21}^2	0	0	0	C_{12}^2
C_{22}^1	free	free	0	$-C_{22}^2 C_{12}^2 / C_{11}^2$
C_{22}^2	free	0	free	free

this means that we have found four different hom-associative algebras, where μ is defined as

$$\mu(e_1, e_1) = C_{11}^1 e_1 + C_{11}^2 e_2 = 0e_1 + 0e_2 = 0,$$
$$\mu(e_1, e_2) = C_{12}^1 e_1 + C_{12}^2 e_2 = 0e_1 + 0e_2 = 0,$$
$$\mu(e_2, e_1) = C_{21}^1 e_1 + C_{21}^2 e_2 = 0e_1 + 0e_2 = ,$$
$$\mu(e_2, e_2) = C_{22}^1 e_1 + C_{22}^2 e_2.$$

for the first hom-associative algebra, defined as

$$\mu(e_1, e_1) = 0e_1 + 0e_2 = 0,$$
$$\mu(e_1, e_2) = C_{12}^1 e_1 + 0e_2 = C_{12}^1 e_1,$$

$$\mu(e_2,e_1) = C^1_{21}e_1 + 0e_2 = C^1_{21}e_1,$$
$$\mu(e_2,e_2) = C^1_{22}e_1 + 0e_2 = C^1_{22}e_1.$$

for the second hom-associative algebra, defined as

$$\mu(e_1,e_1) = C^1_{11}e_1 + C^2_{11}e_2,$$
$$\mu(e_1,e_2) = 0e_1 + 0e_2 = 0,$$
$$\mu(e_2,e_1) = 0e_1 + 0e_2 = 0,$$
$$\mu(e_2,e_2) = 0e_1 + C^2_{22}e_2 = C^2_{22}e_2.$$

for the third hom-associative algebra, as finally defined as

$$\mu(e_1,e_1) = -C^2_{12}e_1 + C^2_{11}e_2,$$
$$\mu(e_1,e_2) = -(C^2_{12}C^2_{12}/C^2_{11})e_1 + C^2_{12}e_2,$$
$$\mu(e_2,e_1) = -(C^2_{12}C^2_{12}/C^2_{11})e_1 + C^2_{12}e_2 = \mu(e_1,e_2),$$
$$\mu(e_2,e_2) = -(C^2_{22}C^2_{12}/C^2_{11})e_1 + C^2_{22}e_2.$$

for the fourth and last hom-associative algebra. For each of these hom-associative algebras we get a hom-Lie admissible (G_6-hom-associative) algebra. The tables for how $[e_i,e_j] = \mu(e_i,e_j) - \mu(e_j,e_j)$ is calculated for each of these hom-Lie admissible algebras were also calculated earlier, in Section 3.1.1. For the first, third and fourth hom-associative given above, this is given for all three hom-Lie admissible algebras in Table 3.3, and for the second one, it is given in Table 3.4.

We are not going to write out the results for α defined as each of the matrix units in both two and three dimensions is detail as for the case above, so instead we are going to sum up where to find all results in Table 4.1. In each row of the table, we first write what α is defined as and in what dimension the algebra is, then in the next column we write in what table the values of the structure constants for which this becomes hom-associative algebras can be found, and in the third and final column we write in which tables the calculations for the hom-Lie admissible algebras we get from these hom-associative algebras can be found. Note that in some cases this final column consists of multiple tables, and then it has to be checked in each individual case which of these tables belong to which of the hom-Lie admissible algebras gotten from the related table of the structure constants, which can be found by reading their captions.

In Table 4.1, we see that $[\alpha] = E_{11}$ is written as incomplete in three dimensions. This is because the problem turned out to be too big, so we ended up stopping after only finding a few solutions. Thus, the tables belonging to this case only consist of those few solutions that we did manage to find. To be able to find more solutions, we then mapped the system of equations to only one dimension, and there we found all possible solutions, as shown in the table. Then, we tried mapping to two dimensions instead, but instead the problem ended up becoming too big, so while we did find all possible hom-associative algebras for α defined s E_{11}, E_{22} and E_{33}, we never calculated $[\cdot,\cdot]$ for these hom-Lie admissible algebras, and we did continue with α defined as any of the six remaining matrix units either.

The major result we got from Section 3.2.10 was a theorem, Theorem 2, that gave us a way of using permutation to find the values of the structure constants for all hom-associative

	Structure constants	$[\cdot,\cdot]$ for hom-Lie admissible
$[\alpha] = E_{11}$ in dim2	Table 3.2	Tables 3.3, 3.4
$[\alpha] = E_{12}$ in dim2	Table 3.6	Tables 3.7, 3.8
$[\alpha] = E_{21}$ in dim2	Table 3.10	Tables 3.11, 3.12
$[\alpha] = E_{22}$ in dim2	Table 3.14	Tables 3.15, 3.16
$[\alpha] = E_{11}$ in dim3 (incomplete)	Table 3.17	Tables 3.18, 3.19, 3.20
$[\alpha] = E_{11}$ in dim3 mapped to dim1	Table 3.21	Table 3.24
$[\alpha] = E_{12}$ in dim3 mapped to dim1	Table 3.27	Table 3.30
$[\alpha] = E_{13}$ in dim3 mapped to dim1	Table 3.33	Table 3.36
$[\alpha] = E_{21}$ in dim3 mapped to dim1	Table 3.34	Table 3.37
$[\alpha] = E_{22}$ in dim3 mapped to dim1	Table 3.22	Table 3.25
$[\alpha] = E_{23}$ in dim3 mapped to dim1	Table 3.28	Table 3.31
$[\alpha] = E_{31}$ in dim3 mapped to dim1	Table 3.29	Table 3.32
$[\alpha] = E_{32}$ in dim3 mapped to dim1	Table 3.35	Table 3.38
$[\alpha] = E_{33}$ in dim3 mapped to dim1	Table 3.23	Table 3.26
$[\alpha] = E_{11}$ in dim3 mapped to dim2	Tables 3.39, 3.40, 3.41	
$[\alpha] = E_{22}$ in dim3 mapped to dim2	Tables 3.42, 3.43	
$[\alpha] = E_{33}$ in dim3 mapped to dim2	Tables 3.44, 3.45	

Table 4.1: Where you can find each of the hom-associative and hom-Lie admissible algebras found in this thesis

algebras with α defined as some matrix unit, by using the known values of the structure constants for all hom-associative algebras with α defined as a certain matrix unit, where the requirement on the matrix unit is that it has a one on the same diagonal as the first matrix unit (where the diagonals are as explained in Section 3.2.10).

In n dimensions there are a total of n^2 of the $n \times n$ matrix units. If we want to find the hom-associative algebras to α defined as each of these matrix units, as a result of the theorem we will only have to find them for one matrix unit with a one on each diagonal. For simplicity, say that we solve the systems of equations for α defined as $E_{11}, E_{12}, \ldots, E_{1n}$, making it it very easy to see that these each have a one on their own diagonal. This means that to find all hom-associative algebras in dimension n you would only have to solve n systems of equations, instead of all n^2 of them. Since we, also in Section 3.2.10, concluded that we in dimension n will end up solving systems with $n^n(2n-1)$ equations, this means that the theorem has greatly reduced the amount of work that would be needed to find all possible hom-associative algebras.

We were also able to use these results in Section 3.3, where we looked at three dimensions with the system of equations mapped to one dimensions, and Section 3.4, where we looked at three dimensions with the system of equations mapped to two dimensions. In the first case, that meant we only had to solve three systems of equations instead of nine, and in the second case it meant we only had to solve one system of equations instead of three (since we did not end up solving the remaining systems of equations at all).

Now, we look at the commutator tables for the hom-Lie admissible algebras. In two dimensions we found in Section 3.1.5 that all except one hom-Lie admissible algebra for each different α are abelian; that is, have a commutator table that is all zero. The one hom-Lie admissible algebra that does not for $[\alpha] = E_{11}$ actually has the exact same values of all structure constants, and therefore the same commutator table, as the one for $[\alpha] = E_{12}$. In the same way, the one hom-Lie admissible algebra that is not abelian for $[\alpha] = E_{21}$ actually has the exact same values of all structure constants, and therefore the same commutator table, as the one for $[\alpha] = E_{22}$.

In three dimensions mapped to one dimension, we found in Section 3.3.4 that the solutions for the structure constants that make the algebra hom-associative turn out to be exactly the same for both $[\alpha] = E_{11}$, $[\alpha] = E_{12}$ and $[\alpha] = E_{13}$, and they also turn out to be the same for both $[\alpha] = E_{21}$, $[\alpha] = E_{22}$ and $[\alpha] = E_{23}$, and also for both $[\alpha] = E_{31}$, $[\alpha] = E_{32}$ and $[\alpha] = E_{33}$, except $[\alpha] = E_{11}$, $[\alpha] = E_{22}$ and $[\alpha] = E_{33}$ each have an extra solution where every structure constant is zero except for one, C_{11}^1, C_{22}^2 and C_{33}^3 respectively, which is a free variable. Since they have the same values for the structure constants, that of course mean their hom-Lie admissible algebras have the same commutator tables as well. The extra solutions all lead to a commutator table of all zeroes, meaning these hom-Lie admissible algebras are abelian.

This means that in both two dimensions and three dimensions mapped to one dimension we have gotten at least one hom-associative algebra for each $[\alpha] = E_{ij}$ that has the exact values of the structure constants as α defined as $E_{i,j+1}, E_{i,j+2}, \ldots, E_{i,n}$. This is extra interesting since Theorem 2 tells us that we can permutate the results for $[\alpha] = E_{ij}$ to get the results for $[\alpha] = E_{i+1,j+1}$, so just by finding this solution for one α defined as a matrix unit we can find one hom-associative (and hom-Lie admissible) algebra for every other such α in that dimension. If this was the case for the full three dimensions, and higher dimensions as well, and there was a way to identify this particular solution, it would be very useful, but it is impossible to tell if it will in fact only be the case from only the calculations that have been done in this thesis.

Finally, we look at our results from three dimensions mapped to two dimensions. We see that all the solutions in three dimensions where the system of equations has been mapped to one dimension with $[\alpha] = E_{11}$, $[\alpha] = E_{22}$ and $[\alpha] = E_{33}$ are actually special cases of three dimensions where the system of equations has been mapped to two dimensions for those same three matrix units. We check the three solutions we found in three dimensions for $[\alpha] = E_{11}$ to see if there are solutions in three dimensions mapped to two dimensions that are just special cases of these solutions, but that is not the case. However, that does not necessarily mean it would not be the case if we had all of the solutions in three dimensions, since the three solutions we found might all be special cases of more general solutions. Either way, we have at least found some three-dimensional hom-associative algebras (and hom-Lie admissible algebras, even though their commutators have not been calculated), even if it might turn out that there are more general hom-associative algebras.

4.2 Future Work

In this thesis, we had to stop partway through the calculations for which values of the structure constants C_{ij}^k give us a hom-associative algebra when α is defined as E_{11} in three dimensions – see Section 3.2.1 – since we realised that the time these calculations would take to be finished would be outside of the scope of this project. Instead, we had to switch to mapping our system of equations to first one and then two dimensions, and then find the solutions we got in those cases. However, it would be very interesting to continue the calculations and find all of the solutions for the full three-dimensional system of equations, and then continue and do the same thing for α defined as each of the eight remaining three-dimensional matrix units.

Furthermore, after realising we would not be able to find all possible solutions to the full three-dimensional system of equations we mapped it to one dimension, for which we were able to find all solutions, and then two two dimensions, for which we did manage to find all possible solutions for α defined as E_{11}, E_{22} and E_{33}. However, this took more time and work than expected, so we did not continue with the remaining six matrix units. Thus, for a smaller problem than the one proposed above, with solving the full three-dimensional systems of equations, it would still be interesting to solve the three-dimensional systems of equations when mapped to two dimensions, for α defined as each of the matrix units not covered in this thesis. After doing this, it would also be interesting to find the commutator tables for the hom-Lie admissible algebras associated to the hom-associative algebras just found, which would have taken too much work to be able to accomplish in this thesis even for the hom-associative algebras we did find for α defined as E_{11}, E_{22} and E_{33}.

Another simplification we had to make in this thesis was to only look at G_1-hom-associative algebras (which are exactly the hom-associative algebras) and the G_6-hom-associative (hom-Lie admissible) algebras we got from those. To continue this work, it would be interesting to look at each of the other G-hom-associative algebras as well.

There are also different ways of finding hom-Lie admissible algebras. In this thesis, we used Proposition 5, which states that any G-hom-associative algebra is a hom-Lie admissible algebra, since we could then get hom-Lie admissible algebras from the hom-associative (G_1-hom-associative) algebras we found. However, not all hom-Lie admissible algebras are necessarily G_1-hom-associative, so it would be interesting to try to find the ones that are not as well. One way to do that would be to use Proposition 4, which instead imposes a condition on the ternary map $S(x,y,z)$ that a Hom algebra needs to fulfill in order to be hom-Lie admissible.

Finally, it would be interesting to do the same thing as we have done in this project, but with α:s that are a bit more complicated than the matrix units used here. For example, looking at α defined as some matrices with Jordan blocks bigger than 2×2, such as $E_{12}+E_{23}$, would be very interesting.

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