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BACHELOR'S DEGREE PROJECT IN MATHEMATICS

# **Solving the Gleason Problem using Partition of Unity**

by

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**Abstract**

The Gleason problem has been proven to be a complicated issue to tackle. In this thesis we will conclude that a domain, $\Omega \subset \mathbb{R}^n$, has Gleason $R$-property at any point $p \in \Omega$, where $R(\Omega) \subset C^\infty(\Omega)$ is the ring of functions that are real analytic in $p$. First, we investigate function spaces and give them fitting norms. Afterwards, we build a bump function that is then used to construct a smooth partition of unity on $\mathbb{R}^n$. Finally, we show that some of the function spaces, introduced earlier, have the Gleason property. Ultimately, we use our smooth partition of unity in order to prove that the statement above holds for domains in $\mathbb{R}^2$. Subsequently, with the same reasoning one can prove that the statement also holds for domains $\Omega \subset \mathbb{R}^n$.

# Contents

# Chapter 1

# Introduction

There are many situations where one can find the use for smooth partition of unity. It can for example be useful when solving PDEs if one might want to prove the trace theorem, see [3]. However, in this thesis we will show how smooth partition of unity can be useful when investigating a version of the Gleason problem, see *Section 4*.

## 1.1 Background

A.M. Gleason introduced what is today known as the Gleason problem. The problem appeared in his article [6], where Gleason was studying maximal ideals of a Banach algebra. In this article he proved that if a point with a maximal ideal is finitely generated, then every function on this algebra is holomorphic on some neighborhood of this point. However, the question arose of how to determine whether a maximal ideal is finitely generated or not. Gleason concluded his article with a query, wondering if the maximal ideal of functions that vanish on the origin in the space of analytical functions on the open unit ball is finitely generated. The issue of determining this query in different spaces is today known as the Gleason problem. The first person to solve Gleasons query was a Russian mathematician called Z.L. Leibenzon [7]. He proved that for every convex domain in $\mathbb{C}^n$ with a boundary that is in $C^2$ has the Gleason property. Many situations of the Gleason problem are still open discussions. One of those issues is the dilemma of whether a Hartogs domain possesses the Gleason $H^\infty$-property.

## 1.2 Goal, Purpose and Problem Formulation

The purpose of this thesis is to construct a smooth partition of unity that can later be used to solve a version of the Gleason problem. When discussing the Gleason problem, an important aspect to research is the topic of function spaces. Therefore we will delve into the subject of function spaces and include some examples of these spaces as well as their appropriate norms. Afterwards, in order to solve a version of the Gleason problem with the use of partition of unity, we will construct a bump function that can be used to create our smooth partition of unity. Finally, we will delve into the topic of the Gleason problem. The function spaces that were introduced earlier will here be shown to have the Gleason property on certain domains. Then, with the use of our smooth partition of unity we will ultimately prove *Theorem 12*, which states that a domain, $\Omega \subset \mathbb{R}^2$, has Gleason $R$-property at any point $p \in \Omega$, where $R(\Omega) \subset C^\infty(\Omega)$ is the ring of functions that are real analytic in $p$.

## 1.3 Literature review

In this section we present some of the literature that was used in this thesis. The article by Backlund and Fällström [1] introduces a brief history and definition of the Gleason problem. It focuses on proving the fact that bounded pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$ with a boundary in $C^2$ have the Gleason $\mathcal{A}$-property. This contribution provides insight on the properties and behavior of these domains, which is useful in various areas of mathematics.

When it comes to finding definitions and theorems with proofs, the following books have been useful. *Principles of Mathematical Analysis* [12], published by Rudin, is often referred to as "Baby Rudin" and it covers subjects such as sequences, series, continuity, differentiation, integration and metric spaces. The book contains proper mathematical proofs and incorporates the foundations of mathematical analysis. It is a useful tool for finding well defined definitions and theorems with solid proofs. Additionally, *Functional Analysis* [13], which is also published by Rudin, provides an introduction to functional analysis, which is important for mathematics that deals with vector spaces of functions. The book covers topics such as Banach spaces, Hilbert spaces, spectral theory and more. Likewise to his previous work mentioned above, Rudin presents the theory of functional analysis in a well defined and concise manner. Furthermore, *Partial Differential Equations* [3], published by Evans, is a highly regarded publication that provides the theory, applications and solution techniques of PDEs. It also covers topics such as Soblev spaces, traces, compactness and maximum principles. Similarly to the previously named books, Evans presents many useful theorems with solid proofs.

When delving into smooth partition of unity, the work of Lloyd [10] contains some useful material on the topic. Although, the article focuses on smooth partitions over manifolds, some main theorems and definitions can be used for coverings in $\mathbb{R}^n$. The paper builds upon previous research on the smoothness properties of topological linear spaces. They ultimately provide the criteria such that smooth partitions of unity exists on manifolds that are constructed on topological linear spaces.

## 1.4 Notations

In this thesis, we will use the notation $x$ to abbreviate $(x_1, x_2, \ldots, x_n)$ for points in $\mathbb{R}^n$ and in a similar way, we write

$$\int_{\mathbb{R}^n} f(x)dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n.$$

## 1.5 Table of sets

| General notations of sets and spaces | |
|---|---|
| $\Omega$ | General sets in $\mathbb{R}^n$ |
| $M$ | Sets in $\mathbb{R}^n$ equipped with a metric, defined in *Section 2* |
| $F$ | General fields |
| $V$ | Vector spaces over a field $F$ |
| $K$ | Compact sets in $\mathbb{R}^n$, defined in *Section 2* |
| $C^0(\Omega)$ | The space of continuous functions over $\Omega$, defined in *Section 2.1* |
| $C^k(\Omega)$ | The space of functions with $k$-times continuous derivatives over $\Omega$, defined in *Section 2.2* |
| $C^\infty(\Omega)$ | The space of smooth functions over $\Omega$, defined in *Section 2.3* |
| $L^p(\Omega)$ | The space of $L^p$ functions over $\Omega$, defined in *Section 2.4* |
| $\mathcal{H}(\Omega)$ | The space of holomorphic functions over $\Omega$, defined in *Section 2.5* |
| $\mathcal{H}^\infty(\Omega)$ | The space of bounded holomorphic functions over $\Omega$, defined in *Section 2.5* |
| $\mathcal{A}^k(\Omega)$ | The space of analytic functions with $k$ continuous derivatives over $\Omega$, defined in *Section 2.5* |

# Chapter 2

# Function Spaces

We will start by delving into the topic of function spaces. First we present some definitions and theorems that will be used to define certain function spaces along with some suitable norms. These function spaces will then be used in *Section 4* to prove the Gleason property for these spaces on specific domains. This section contains some insight from [4], [5], [8], [12], [13], and [14].

**Definition 1.** The *supremum* of a set, $\Omega \subseteq \mathbb{R}$, is the minimum upper bound of the set, and it is denoted $\sup\limits_{x\in\Omega} x$. If an upper bound for the set $\Omega$ does not exist, then $\sup\limits_{x\in\Omega} x = \infty$.

**Definition 2.** The *infimum* of a set, $\Omega \subseteq \mathbb{R}$, is the maximum lower bound of the set, denoted $\inf\limits_{x\in\Omega} x$. If a lower bound does not exist, we define $\inf\limits_{x\in\Omega} x = -\infty$.

**Definition 3.** A *metric space*, denoted $(M, d)$, is defined by a set $M$ as well as a metric $d : M \times M \to \mathbb{R}$ that is equipped on $M$ and it satisfies the following three conditions for all $x, y, z \in M$.

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

An example of a metric space is the Euclidean space $\mathbb{R}^n$ with the Euclidean distance as metric.

**Definition 4.** The distance between an element, $y$, and a set, $\Omega$, is given by

$$d(y, \Omega) = \inf_{x\in\Omega}(d(y, x)).$$

**Definition 5.** The *closure* of a set $\Omega$ is denoted as $\overline{\Omega}$ and it is the smallest closed set that contains $\Omega$.

**Definition 6.** The *boundary* of a set, $\Omega$, is denoted $\partial\Omega$ and it is defined as the intersection of the closure of $\Omega$ with the closure of its complement, that is

$$\partial\Omega = \overline{\Omega} \cap \overline{\Omega^c}.$$

**Definition 7.** A subset, $K \subset \Omega$, is called *compact* if for every collection, $C$, of open subsets of $\Omega$ such that

$$K = \bigcup_{x \in C} x$$

there exists a finite collection $Y \subseteq C$ such that

$$K = \bigcup_{x \in Y} x.$$

If $\overline{K}$ is a compact set in $\Omega$ then we denote it as $K \subset\subset \Omega$.

**Theorem 1.** *For a subset K of Euclidean space $\mathbb{R}^n$, the following statements are equivalent:*

1. *K is closed and bounded.*

2. *K is compact.*

This theorem is called the Heine-Borel theorem and the proof for it can be found in [12].

**Theorem 2.** *Suppose that $f$ is a continuous real function on a compact metric space M, and*

$$S = \sup_{x \in M} f(x), \qquad s = \inf_{x \in M} f(x).$$

*Then there exists points $p, q \in M$ such that $f(p) = S$ and $f(q) = s$.*

For proof of this Theorem, see [12].

**Definition 8.** A *normed space* is a vector space $V$ over a field $F \in \{\mathbb{R}, \mathbb{C}\}$ with a norm function $||\cdot|| : V \longrightarrow \mathbb{R}$ that satisfies the following three axioms.

1. $||v|| \geq 0$ for all $v \in V$ and $||v|| = 0$ if and only if $v = 0$.

2. $||av|| = |a|||v||$ for all $a \in F$ and $v \in V$.

3. $||v + u|| \leq ||v|| + ||u||$ for all $v, u \in V$.

The Euclidean space $\mathbb{R}^n$ with the Euclidean distance as metric is also a normed space. In fact, all normed spaces are metric spaces and the metric is generated by $d(v, u) = ||v - u||$. The proof of this can be found in [8].

**Definition 9.** A *Cauchy sequence* is a sequence of elements that as the sequence progresses they become arbitrarily close to each other. For a metric space, $(M, d)$, a sequence, $\{a_n\}$, is a Cauchy sequence if for every real $\varepsilon > 0$, there exists an integer $N > 0$ such that the metric distance

$$d(a_j, a_i) < \varepsilon \text{ for all integers } j, i > N.$$

For $a_i$ and $a$ elements in the metric space $(M, d)$, we say that if

$$\lim_{i\to\infty} d(a, a_i) = 0$$

then $\lim_{i\to\infty} a_i = a$ or equivalently $a_i \to a$. An example of a Cauchy sequence is the sequence $\{1/n\}$. One can easily prove that every convergent sequence is a Cauchy sequence using the triangular inequality, thus as $n \to \infty$ the sequence converges to 0, hence it is a Cauchy sequence in the space of real numbers.

**Definition 10.** A metric space in which every Cauchy sequence in the space converges to an element in the space is called *complete*.

**Definition 11.** A *Banach space* is a complete normed space.

**Definition 12.** A *function space* is a space of functions that are defined on a domain and that share a certain property. Some function spaces are equipped with a norm, which makes them a Banach space if the norm generates a complete space. These spaces are linear which means that they contain the zero function and they are closed under linear combinations of the functions. A well defined norm preserves the properties of a function space. In this thesis, we only consider normed function spaces $(V, \|\cdot\|)$ in which the elements have a finite norm, that is $\|f\| < \infty$ for all $f \in V$.

## 2.1 Continuous functions

The space of continuous functions over $\Omega$ is denoted $C^0(\Omega)$ and it consists of functions that are continuous on the domain $\Omega$. In this thesis we will look at two cases of continuous functions that have different norms. These function spaces are the continuous functions on a closed and bounded set as well as the general continuous functions.

### 2.1.1 Closed and bounded

For the space of continuous functions on a closed and bounded set $\Omega$ there exists a maximum value. Therefore, we can use the maximum value as a norm in this function space. An example of a continuous function on a closed and bounded interval is $f(x) = \sin(x)$ on the interval $[0, \pi]$. The norm of $f$ is

$$\|f\|_{C^0} = \max_{x\in[0,\pi]} |f(x)| = \max_{x\in[0,\pi]} |\sin(x)| = 1 < \infty.$$

### 2.1.2 General

For general continuous functions we cannot assume that we have a closed and bounded domain. Therefore a maximum value does not always exist, an example of this is the function $f(x) = x$ on the open interval $(0, 1)$. Thus, the typical norm for this function space is the supremum norm. The supremum norm of the function $f(x) = x$ on the interval $(0, 1)$ is

$$\|f\|_{L^\infty} = \sup_{x\in(0,1)} |x| = \sup_{f(x)\in(0,1)} |x| = 1 < \infty.$$

The notation of $L^\infty$ in the norm will be made clear in *Section 2.4*, where $L^p$ spaces will be defined. Later in this thesis we will denote the supremum norm taken over the set $\Omega$ as $\|f\|_{L^\infty(\Omega)}$.

**Theorem 3.** *The space $C^0(\Omega)$ where $\Omega$ is a compact set is a Banach space with the supremum norm.*

The proof of this theorem can be found in [12].

## 2.2 Functions with $k$ continuous derivatives

The space of functions with $k$ continuous derivatives is denoted as $C^k(\Omega)$ and it consists of functions defined on the domain $\Omega$ that are $k < \infty$ times differentiable where every derivative is a continuous function. The norm typically used for this space is

$$\|f\|_{C^k} = \sum_{i=0}^{k} \sup_{x\in\Omega}|f^{(i)}(x)| = \sup_{x\in\Omega}|f(x)| + \sup_{x\in\Omega}|f^1(x)| + \ldots + \sup_{x\in\Omega}|f^k(x)|.$$

To read more about this norm, see [3]. An example of a function in $C^2(-1, 1)$ is

$$f(x) = \begin{cases} -x^3 & \text{for } -1 < x < 0 \\ x^3 & \text{for } 0 \leq x < 1. \end{cases} \tag{2.1}$$

The second derivative of $f(x)$ is continuous but not differentiable.

## 2.3 Smooth functions

The space of smooth functions is denoted as $C^\infty(\Omega)$ with the metric

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{\|f^{(k)} - g^{(k)}\|_{L^\infty}}{1 + \|f^{(k)} - g^{(k)}\|_{L^\infty}}.$$

This space consists of functions defined on the domain $\Omega$ that are infinitely differentiable where every derivative is a continuous function. There are norms that can be used for this space but the space of smooth functions is not a Banach space, see [13]. *Equation 2.1* in the section above is an example of a function that is 2 times differentiable on the interval $(-1, 1)$ with continuous derivatives but not 3 times differentiable, therefore it is not smooth. Any polynomial on a bounded domain is a smooth function. The function $\sin(x)$ is a function that is smooth on any domain, with both the supremum norm and the $L^p$-norm that is introduced in *Section 2.4*. The function $\arctan(x)$ is a smooth function on any domain with the supremum norm, since the function has the upper limit $y = \pi/2$ for all $x$. Moreover, in *Section 3* the smooth functions $\varphi$ and $\eta$ are introduced. For $\varphi$ we have the unique quality that on the entire negative real axis the function is equal to 0 and then it accelerates closer to 1 while still being smooth. The function $\eta$ is similar to $\varphi$ but it is equal to 0 on the region where $|x| < 1$ for $x \in \mathbb{R}^n$ and then it accelerates up to $e^{-1}$ elsewhere.

## 2.4 $L^p$ functions

The space of $L^p$ functions defined over $\Omega$ is denoted $L^p(\Omega)$. A function, $f$, is an $L^p$ function if $\|f\|_{L^p}$ is integrable over the entire domain of $f$. The norm used for $L^p$ functions with $1 \leq p < \infty$ is

$$\|f\|_{L^p} = \sqrt[p]{\int_\Omega |f(x)|^p dx}.$$

In this thesis, we will only consider the Riemann integral even though the integral above is usually Lebegue's integral, in which case the $L^p$ spaces are Banach spaces. If one would like to read about $L^p$ spaces with regards to Lebegue's integral, see [11]. One can also define the norm for the $L^\infty$ space, with $f \in L^\infty(\Omega)$, as

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{\Omega} |f(x)|.$$

We will study $L^\infty(\Omega)$ where $\Omega$ is a bounded domain and the functions are continuous on $\Omega$. Hence, the supremum norm will suffice as the norm of $L^\infty$ spaces in this thesis. To read more about the ess sup norm, see [13].

**Theorem 4.** *The space $L^\infty$ is a Banach space.*

The proof of this theorem is follows from the fact that all Cauchy sequences are bounded. An example of an $L^p(0, 1)$ function, with $1 \leq p < \infty$, is $f(x) = x^{-1/2p}$, on the interval $x \in (0, 1)$. The norm of $f$ is

$$\|f\|_{L^p} = \sqrt[p]{\int_0^1 |x^{-\frac{1}{2p}}|^p dx} = \sqrt[p]{\left[2x^{\frac{1}{2}}\right]_0^1} = \sqrt[p]{2} < \infty.$$

An example of a similar function that is not an $L^p(0, 1)$ function is $g(x) = x^{-1/p}$, on the interval $x \in (0, 1)$. The norm of $g$ is

$$\|g\|_{L^p} = \sqrt[p]{\int_0^1 |x^{-\frac{1}{p}}|^p dx} = \sqrt[p]{[\ln(x)]_0^1} = \lim_{s\to 0^+} \sqrt[p]{\ln(1) - \ln(s)} = \infty.$$

In fact one can show that $L^p(\Omega) \subset L^r(\Omega)$ if $1 \leq r < p \leq \infty$ for $\Omega$ bounded domain. The proof of this can be found in [4].

## 2.5 Holomorphic functions

The space of holomorphic functions defined over a subset of the complex plane $\Omega$ is denoted $\mathcal{H}(\Omega)$ and it is the space of functions that are complex differentiable on $\Omega$. A function is holomorphic if it satisfies the Cauchy-Reimann equations. Let's introduce two subspaces of the holomorphic functions, $\mathcal{H}^\infty(\Omega) = \mathcal{H}(\Omega) \cap L^\infty(\Omega)$ and $\mathcal{A}^k(\Omega) = \mathcal{H}(\Omega) \cap C^k(\overline{\Omega})$. The following two theorems are standard results in complex analysis and they show that $\mathcal{H}^\infty(\Omega)$ and $\mathcal{A}^k(\Omega)$ are Banach spaces with the supremum norm.

**Theorem 5.** *Suppose that $f_n : \Omega \to \mathbb{C}$ and that $f_n \to f$ uniformly on $\Omega$ with the supremum norm. If all $f_n$ are continuous on $\Omega$, then $f$ is also continuous on $\Omega$.*

**Theorem 6.** *Suppose that $\Omega \subset \mathbb{C}$ is a domain and that $f_n$ is a sequence of holomorphic functions on $\Omega$ that converge uniformly to $f$ with the supremum norm, then $f$ is also holomorphic. Meaning that $f_n$ is a Cauchy sequence on the space of holomorpic functions. In addition, the derivatives $f_n^{(1)}$ converge locally uniformly to $f^{(1)}$.*

The proof of these two theorems can be found in [14].

# Chapter 3

# Partition of Unity

In this chapter we will construct a smooth partition of unity that can later be used to solve a case of the Gleason problem. A partition of unity is a method used to construct a function by puzzling together local functions defined on overlapping domains. First we will list some definitions and theorems that will be used to construct our smooth partition of unity. Then, in *Section 3.1*, we will construct a bump function with unit mass. Afterward, in *Section 3.4*, we will use this bump function to construct a smooth partition of unity. The development of our smooth partition of unity utilized ideas from [5], [9] and [10].

**Definition 13.** The *support* of a function $f$ is written as $\operatorname{supp} f$ and it is the closure of the subsection of the domain that leads to $f \neq 0$. That is

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}$$

where $\Omega$ is the domain.

**Definition 14.** If a function has a support that is a compact subset of the domain $\Omega$, then we say that the function has a *compact support* on the space $\Omega$. A function space with a compact support is denoted by a lowered $c$.

**Theorem 7.** *The composition of two smooth functions is also smooth.*

The proof of this theorem can be found in [9].

**Definition 15.** A subset $D$ of a space $\Omega$ is said to be a *dense subset* if any of the following equivalent conditions are satisfied

1. The smallest closed subset of $\Omega$ containing $D$ is $\Omega$ itself.

2. The closure of $D$ in $\Omega$ is equal to $\Omega$.

3. Every point in $\Omega$ is either in $D$ or is a limit point of $D$.

## 3.1 The Bump Function

A bump function is smooth function with a compact support, particularly the function is non-zero on the compact support and zero elsewhere. In this chapter we will construct a bump function such that the mass of it will be equal to one.

### 3.1.1 Bump function with support on the positive real line

In this section we want to prove that the function $\varphi$ is smooth on $\mathbb{R}$ where

$$\varphi(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0. \end{cases} \tag{3.1}$$

For $t \leq 0$ we have that $\varphi = 0$ which is clearly infinitely differentiable. For $t > 0$ the function $\varphi$ is a composition of smooth functions and hence from *Theorem 7*, it follows that $\varphi$ is itself a smooth function. To investigate the smoothness at the origin, we present a form of the derivatives of $\varphi$ when $t > 0$ by using the principle of mathematical induction.

**Theorem 8.** *The n-th derivative of $\varphi(t)$ for $t > 0$ can be described by*

$$\varphi^{(n)}(t) = P\left(\frac{1}{t}\right)e^{-\frac{1}{t}},$$

*where P is a polynomial.*

*Proof.* Induction base: The base case $n = 0$ is trivial since $\varphi(t) = e^{-\frac{1}{t}}$. For $n = 1$ one gets

$$\varphi^{(1)}(t) = \frac{1}{t^2}e^{-\frac{1}{t}} = \left(\frac{1}{t}\right)^2 e^{-\frac{1}{t}} = P_1\left(\frac{1}{t}\right)e^{-\frac{1}{t}},$$

where $P_1(x) = x^2$ is a polynomial, so *Theorem 8* holds for $n = 1$.

For $n = 2$ one gets

$$\varphi^{(2)}(t) = \frac{1}{t^4}e^{-\frac{1}{t}} - \frac{2}{t^3}e^{-\frac{1}{t}} = \left(\left(\frac{1}{t}\right)^4 - 2\left(\frac{1}{t}\right)^3\right)e^{-\frac{1}{t}} = P_2\left(\frac{1}{t}\right)e^{-\frac{1}{t}},$$

where $P_2(x) = x^4 - 2x^3$ is a polynomial, so *Theorem 8* also holds for $n = 2$.

Induction assumption: Assume that *Theorem 8* holds for some $p > 0$. That is

$$\varphi^{(p)}(t) = P\left(\frac{1}{t}\right)e^{-\frac{1}{t}}$$

for some polynomial $P$.

Induction step: For $n = p + 1$ we get

$$\begin{aligned}
\varphi^{(p+1)}(t) =& \frac{d}{dt}\left(P\left(\frac{1}{t}\right)e^{-\frac{1}{t}}\right) \\
=& P^{(1)}\left(\frac{1}{t}\right)e^{-\frac{1}{t}} + P\left(\frac{1}{t}\right)\left(\frac{1}{t}\right)^2 e^{-\frac{1}{t}} \\
=& \underbrace{\left(P^{(1)}\left(\frac{1}{t}\right) + P\left(\frac{1}{t}\right)\left(\frac{1}{t}\right)^2\right)}_{\tilde{P}(1/t)} e^{-\frac{1}{t}}.
\end{aligned}$$

The expression of polynomials, $\tilde{P}$, is a polynomial itself, therefore

$$\varphi^{(p+1)}(t) = \tilde{P}\left(\frac{1}{t}\right)e^{-\frac{1}{t}}.$$

Conclusion: From the steps taken above and the principle of induction it follows that *Theorem 8* holds for $n \geq 0$.

$\square$

Now we only need to show that $\varphi$ is smooth on the origin. We do this by proving that all derivatives of $\varphi(t)$, for $t > 0$, converge to 0 as $t$ tends to the origin. We have

$$\lim_{t \to 0^+} \varphi^{(n)}(t) = \lim_{t \to 0^+} P\left(\frac{1}{t}\right)e^{-\frac{1}{t}}.$$

To show that the limit tends to 0, one can now do a variable change $s = 1/t$ such that $s \to \infty$ when $t = 1/s \to 0^+$. Thus

$$\lim_{t \to 0^+} \varphi^{(n)}(t) = \lim_{s \to \infty} P(s)e^{-s}.$$

Since exponential functions grow faster than polynomial functions, the expression above has the limit 0. Therefore $\varphi$ is smooth on $\mathbb{R}$. In *Figure 3.1* one can see the graph of $\varphi$ over the origin.

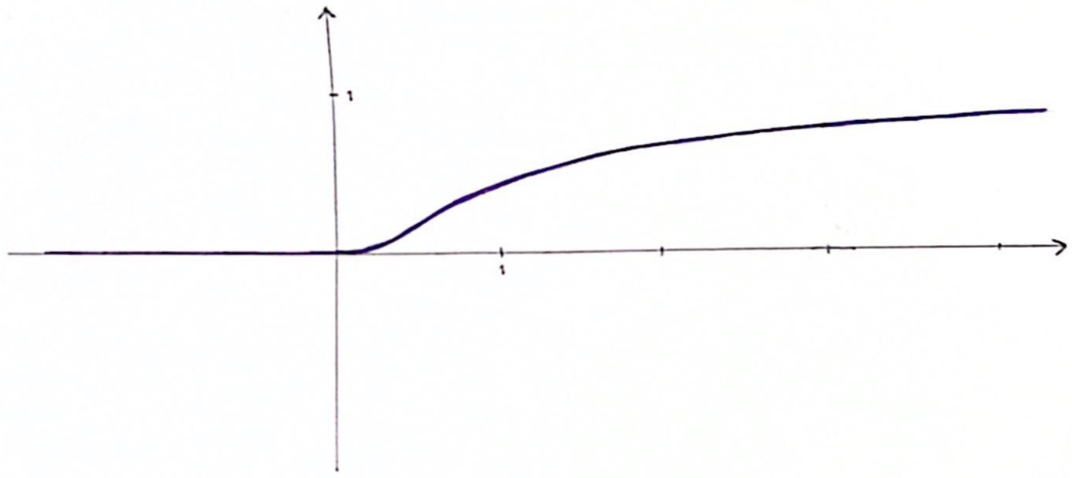

Figure 3.1: The graph of the smooth function $\varphi$, defined by equation 3.1.

### 3.1.2 Bump function with compact support on $\overline{B(0,1)}$

Now we build on the function defined in *Section 3.1.1* such that our new function is smooth on $\mathbb{R}^n$ and has compact support on $\overline{B(0,1)}$. Let

$$\eta(x) = \varphi(1 - |x|^2).$$

This generates a smooth function that has compact support on $\overline{B(0,1)}$.

*Proof.* It follows directly from *Theorem 7* that $\eta$ is a smooth function. If we expand $\eta$, we get

$$\begin{aligned} \eta(x) =& \varphi(1 - |x|^2) \\ =& \begin{cases} e^{-\frac{1}{(1-|x|^2)}}, & 1 - |x|^2 > 0 \\ 0, & 1 - |x|^2 \leq 0. \end{cases} \\ =& \begin{cases} e^{-\frac{1}{(1-|x|^2)}}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases} \end{aligned} \tag{3.2}$$

Since $\eta = 0$ when $|x| \geq 1$ and $\eta$ is positive for $|x| < 1$ one gets

$$\operatorname{supp} \eta = \overline{\{x \in \mathbb{R}^n : |x| < 1\}}.$$

Which is the closure of $B(0,1)$, thus $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \eta = \overline{B(0,1)}$. $\square$

In *Figure 3.2* one can see the graph of $\eta$ on $\mathbb{R}$.

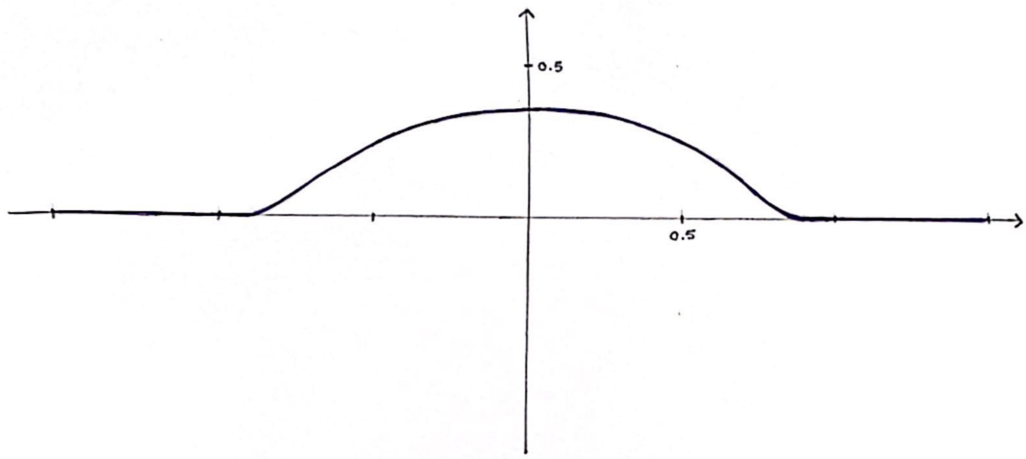


Figure 3.2: The graph of the smooth function $\eta$, defined by equation 3.2, on $\mathbb{R}$.

### 3.1.3 Bump function with compact support on $\overline{B(0, \varepsilon)}$

In this section we expand upon the function defined in *Section 3.1.2* such that our new function is smooth on $\mathbb{R}^n$ and has for a constant $\varepsilon > 0$ compact support on $\overline{B(0, \varepsilon)}$. Let

$$\eta_\varepsilon(x) = \varepsilon^{-n} \eta \left( \frac{x}{\varepsilon} \right).$$

We will now prove that this generates a smooth function that has compact support on $\overline{B(0, \varepsilon)}$.

*Proof.* It follows directly from *Theorem 7* that $\eta(x/\varepsilon)$ is a smooth function. Therefore, since $\varepsilon^{-n}$ is a constant, one gets that $\eta_\varepsilon$ is smooth. If we expand $\eta_\varepsilon$, we get

$$\begin{aligned} \eta_\varepsilon(x) =& \varepsilon^{-n} \varphi(1 - \left| \frac{x}{\varepsilon} \right|^2) \\ =& \varepsilon^{-n} \begin{cases} e^{-\frac{1}{(1-|\frac{x}{\varepsilon}|^2)}}, & (1 - |\frac{x}{\varepsilon}|^2) > 0 \\ 0, & (1 - |\frac{x}{\varepsilon}|^2) \le 0 \end{cases} \\ =& \varepsilon^{-n} \begin{cases} e^{-\frac{1}{(1-|\frac{x}{\varepsilon}|^2)}}, & |x| < |\varepsilon| \\ 0, & |x| \ge |\varepsilon|. \end{cases} \end{aligned} \tag{3.3}$$

Since $\eta_\varepsilon = 0$ when $|x| \ge |\varepsilon|$ and $\eta_\varepsilon$ is positive for $|x| < |\varepsilon|$ the support is

$$\operatorname{supp} \eta_\varepsilon = \overline{\{x \in \mathbb{R}^n : |x| < |\varepsilon|\}}.$$

This is the closure of $B(0,\varepsilon)$, thus $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\eta_\varepsilon = \overline{B(0,\varepsilon)}$. ☐

In *Figure 3.3* one can see the graph of $\eta_\varepsilon$ on $\mathbb{R}$.

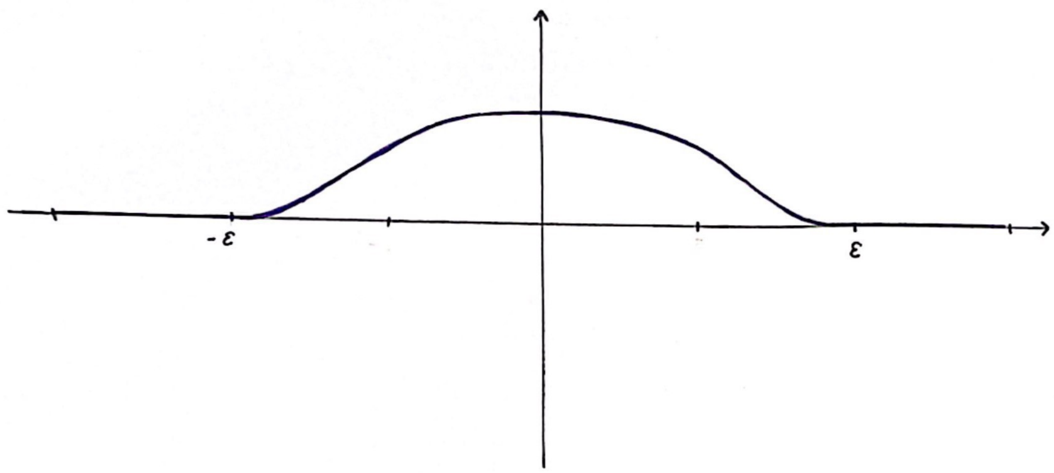


Figure 3.3: The graph of the smooth function $\eta_\varepsilon$, defined by equation 3.3, on $\mathbb{R}$.

### 3.1.4 Bump function with the mass one

Now we will prove that for some variable $c_n$, we have

$$c_n \int_{\mathbb{R}^n} \eta_\varepsilon dx = 1.$$

*Proof.* Since $\eta_\varepsilon(x) = 0$ when $x \notin B(0,\varepsilon)$ we have

$$\int_{\mathbb{R}^n} \eta_\varepsilon dx = \int_{B(0,\varepsilon)} \eta_\varepsilon dx.$$

From *Theorem 2*, since $\eta_\varepsilon$ is continuous on the closed and bounded set $\overline{B(0,\varepsilon)}$, we have that $\eta_\varepsilon$ has a maximum and minimum value on that set. We also know that $\eta_\varepsilon \geq 0$, thus

$$0 \leq \int_{B(0,\varepsilon)} \eta_\varepsilon dx \leq \int_{B(0,\varepsilon)} M dx$$

where

$$M = \max_{\overline{B(0,\varepsilon)}} (\eta_\varepsilon) < \infty$$

which follows directly from *Theorem 2*. Furthermore, because $\eta_\varepsilon(x) > 0$ for all $x \in B(0,\varepsilon/2)$ we have

$$0 < \int_{B(0,\varepsilon/2)} \eta_\varepsilon(x)dx < \int_{B(0,\varepsilon)} \eta_\varepsilon(x)dx < \infty.$$

Therefore the integral will take a positive value, let's say $N > 0$, such that

$$c_n \int_{\mathbb{R}^n} \eta_\varepsilon dx = c_n N.$$

Thus we pick $c_n = 1/N$ with the aim that the integral will take the value of 1. □

From here on, we assume that $\eta_\varepsilon$ has the mass one.

## 3.2 Convolution

In this section we will define the convolution between two functions and then show how our bump function $\eta_\varepsilon$, defined by equation 3.3, can be used to construct a smooth function with compact support from a function that is compactly supported and integrable.

**Definition 16.** The *convolution* between two functions $f$ and $g$ is written as $f * g$ and it is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Let's look at the convolution, $f^\varepsilon(x)$, between a function $f \in L^1(\mathbb{R})$ and the bump function $\eta_\varepsilon$.

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{-\infty}^{\infty} f(x-y)\eta_\varepsilon(y)dy.$$

One can now construct a variable change

$$\left\{ \begin{array}{c} s = x - y \\ dy = -ds \\ y \longrightarrow -\infty \Rightarrow s \longrightarrow \infty \\ y \longrightarrow \infty \Rightarrow s \longrightarrow -\infty \end{array} \right\}.$$

Hence, we get the following

$$f^\varepsilon(x) = -\int_{\infty}^{-\infty} f(s)\eta_\varepsilon(x-s)ds = \int_{-\infty}^{\infty} f(s)\eta_\varepsilon(x-s)ds = \int_{-R}^{R} f(s)\eta_\varepsilon(x-s)ds$$

for $R$ large enough. This function is now differentiable for $x$ since $f$ is well defined and $\eta_\varepsilon$ is smooth. Thus $f^\varepsilon$ is smooth on $\mathbb{R}^n$ with compact support for any $n > 0$.

Let's look at the convolution between a simple not continuous function and the bump function $\eta_\varepsilon$. Suppose we have the function, $g$, defined as

$$g(x) = \begin{cases} 1, & |x| < 0.5 \\ 0, & |x| \geq 0.5. \end{cases} \tag{3.4}$$

This function is compactly supported and integrable. The convolution, $g^\varepsilon$, of $g$ and $\eta_\varepsilon$ is then

$$g^\varepsilon(x) = \int_{-\infty}^{\infty} g(s)\eta_\varepsilon(x-s)ds = \int_{-0.5}^{0.5} g(s)\eta_\varepsilon(x-s)ds. \tag{3.5}$$

Thus $g^\varepsilon$ is smooth on $\mathbb{R}$ with compact support, $\operatorname{supp} g^\varepsilon = \overline{\{y \in \mathbb{R} : |y| \leq 0.5 + \varepsilon\}}$. In *Figure 3.4* one can see the graph of $g$ as well as the graph of $g^\varepsilon$.

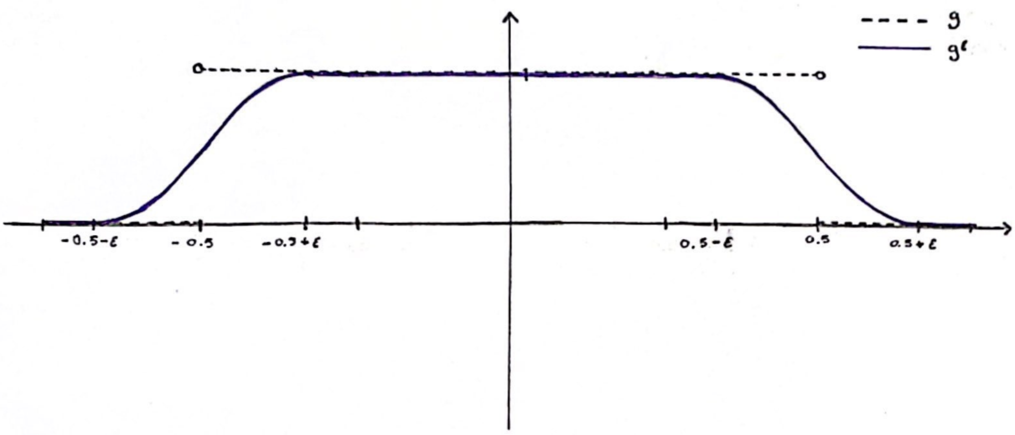


Figure 3.4: The graph of the function $g$, defined by equation 3.4, as well as $g^\varepsilon$, defined by equation 3.5.

## 3.3 Smooth functions are dense in the set of continuous functions

Smooth functions are easy to work with. If one does not have prior knowledge about if a function is smooth, but one knows that it is continuous, then one may use the fact that smooth functions are dense in the set of continuous functions. In general, one could prove a statement that holds for smooth functions and then if it is possible to construct a sequence of smooth functions that converge to a continuous function, the statement might hold for continuous functions as well. This idea is often used when proving properties of solutions on a varied form of a PDE, see for example [2]. In this section we show that one can prove that smooth functions are dense in the set of continuous functions using our bump function $\eta_\varepsilon$ that was introduced in *Section 3.1*.

**Theorem 9.** *Smooth functions are dense in the set of continuous functions.*

*Proof.* For $f, g \in C^0(\Omega)$, where $\Omega$ is a subset of $\mathbb{R}^n$, the distance between $f$ and $g$ is given by

$$d(f, g) = \sup |f(x) - g(x)|, \forall x \in \Omega.$$

We want to show that the space $C^\infty(\Omega)$ is locally dense in $C^0(\Omega)$. Let $f \in C^0(\Omega)$, given a $\delta > 0$, we want to find a $g \in C^\infty(\Omega)$ such that $d(f, g) < \delta$. So, let $g(x) = (f * \eta_\varepsilon)(x)$ and take $x_0 \in K$, where $K$ is any compact subset of $\Omega$. Then

$$|f(x_0) - g(x_0)| = |f(x_0) - (f * \eta_\varepsilon)(x_0)| = |f(x_0) - \int_{B(0,\varepsilon)} f(x_0 - y)\eta_\varepsilon(y)dy|.$$

Since the mass of $\eta_\varepsilon = 1$ we can multiply $f(x_0)$ with the integral of $\eta_\varepsilon$ over the domain $B(0, \varepsilon)$, which will give us

$$|f(x_0) \int_{B(0,\varepsilon)} \eta_\varepsilon(y)dy - \int_{B(0,\varepsilon)} f(x_0 - y)\eta_\varepsilon(y)dy|.$$

Since $x_0$ is a fixed point in $K$, then $f(x_0)$ is a constant that can be moved into the integral. That is

$$\begin{aligned}
|\int_{B(0,\varepsilon)} f(x_0)\eta_\varepsilon(y)dy - \int_{B(0,\varepsilon)} f(x_0 - y)\eta_\varepsilon(y)dy| &= |\int_{B(0,\varepsilon)} (f(x_0)\eta_\varepsilon(y) - f(x_0 - y)\eta_\varepsilon(y))dy| \\
&= |\int_{B(0,\varepsilon)} (f(x_0) - f(x_0 - y))\eta_\varepsilon(y)dy| \\
&\leq \int_{B(0,\varepsilon)} |(f(x_0) - f(x_0 - y))\eta_\varepsilon(y)|dy \\
&= \int_{B(0,\varepsilon)} |f(x_0) - f(x_0 - y)||\eta_\varepsilon(y)|dy.
\end{aligned}$$

For an $\varepsilon > 0$ that is small enough we have

$$\max_{y \in B(0,\varepsilon)} |f(x_0) - f(x_0 - y)| < \delta$$

since $x_0 \in K$ where $K$ is a compact set, there exists an $\varepsilon$ so that the inequality above holds. Therefore, we can determine that

$$\int_{B(0,\varepsilon)} |f(x_0) - f(x_0 - y)||\eta_\varepsilon(y)|dy < \int_{B(0,\varepsilon)} \delta \cdot |\eta_\varepsilon(y)|dy = \delta \int_{B(0,\varepsilon)} |\eta_\varepsilon(y)| = \delta.$$

Consequently, since the space $C^\infty(\Omega)$ is locally dense in $C^0(\Omega)$ for an $x_0 \in K$ where $K$ is any compact subset of $\Omega$, it must also hold for all $x \in \Omega$. $\square$

## 3.4 Proof of smooth partition of unity subordinate a finite open covering of finite domain

**Theorem 10.** *Given an open bounded set $\Omega$ in $\mathbb{R}^n$ and finite open sets $\{V_n\}$ in $\mathbb{R}^n$ such that*

$$\overline{\Omega} \subset \bigcup_{n=1}^{N} V_n,$$

*then there exists functions $\psi_n$ such that*

1. *$\psi_n$ is smooth on $\mathbb{R}^n$.*
2. *$\operatorname{supp} \psi_n \subset V_n$.*
3. *$0 \leq \psi_n \leq 1$.*
4. *$\sum_{n=1}^{N} \psi_n(x) = 1 \ \forall x \in \Omega$.*

*Proof.* For each sub covering $V_n$ we define the function

$$\varphi_n(x) = \begin{cases} 1 \text{ for all } x \text{ on } V_n \\ 0 \text{ for all } x \text{ outside of } V_n. \end{cases}$$

If we take the convolution of $\varphi_n$ with $\eta_\varepsilon$, then we will get a smooth function with compact support on the closed $\varepsilon$-neighborhood of $V_n$. This is not a subset of $V_n$, so property 2 would not hold. Therefore we define the function

$$\tilde{\varphi}_n(x) = \begin{cases} 1 \text{ for all } x \text{ on } W_n \\ 0 \text{ for all } x \text{ outside of } W_n \end{cases}$$

where $W_n \subset V_n$ is defined by $W_n = \{y \in V_n : d(y, \partial V_n) > 2\varepsilon\}$. We pick the biggest $\varepsilon$ such that the closure of $\Omega$ is compact on the covering of all $W_n$, that is

$$\overline{\Omega} \subset \bigcup_{n=1}^{N} W_n.$$

Now we define the convolution $\chi_\varepsilon^n(x) = (\tilde{\varphi}_n * \eta_\varepsilon)(x)$. This function is smooth and has compact support with these conditions

$$W_n \subset \operatorname{supp}(\chi_\varepsilon^n) \subset V_n.$$

So $\chi_\varepsilon^n(x)$ upholds properties 1, 2 and 3. However, this function does not hold for property 4 since if we take a point, $x_1$, in an area where two sets, $V_1$ and $V_2$, overlap, then the sum in property 4 would be $\chi_\varepsilon^1(x_1) + \chi_\varepsilon^2(x_1) = 2$. To solve this we simply define a new function, $\psi_n$, that takes $\chi_\varepsilon^n(x)$ and divides it by the sum of each overlapping functions on the point $x$. That is

$$\psi_n(x) = \frac{\chi_\varepsilon^n(x)}{\sum\limits_{k=1}^{n} \chi_\varepsilon^k(x)}.$$

Since $\{W_n\}$ is an open finite covering of $\Omega$, then $\sum \chi_\varepsilon^k(x) > 0 \ \forall x \in \Omega$, and so all four properties hold for $\psi_n(x)$. $\square$

# Chapter 4

# The Gleason Problem

Partition of unity can be used when dealing with the Gleason problem by constructing local solutions to the problem on smaller areas. This section utilizes ideas from [1] as well as [6]. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ or $\mathbb{R}^n$ and let $R(\Omega)$ be a ring (with addition and multiplication defined pointwise in $\Omega$) of complex or real valued functions on $\Omega$. We say that a domain $\Omega$ has Gleason $R$-property at $\varepsilon \in \Omega$ if for all functions $f \in R(\Omega)$, such that $f(\varepsilon) = 0$, there exists a set of functions $\{f_i\}_{i=1}^n \subset R(\Omega)$ such that

$$f(z) = \sum_{i=1}^{n} f_i(z)(z_i - \varepsilon_i), \text{ for all } z \in \Omega.$$

The Gleason problem is the issue of determining if a domain has a certain Gleason property and it can also be formulated with the use of maximal ideals.

**Definition 17.** A subset, $I$, of a ring $R$ is called an *ideal* of $R$ if the following three properties hold

1. $I$ is non empty.
2. If $a, b \in I$, then $a + b \in I$.
3. If $a \in I$ and $r \in R$ then $ra \in I$ and $ar \in I$.

**Definition 18.** An ideal $I \subset R$ is called a *proper ideal* if $I$ is not the set $R$ itself. That is, $I$ is a proper subset of $R$.

**Definition 19.** A proper ideal $I$ is called a *maximal ideal* if there does not exist any other proper ideals $J$ such that $I$ is a proper subset of $J$.

If the maximal ideal of $R(\Omega)$ consisting of functions that are vanishing at a point $\varepsilon \in \Omega$ is algebraically finitely generated by the coordinate functions $(z_1 - \varepsilon_1), (z_2 - \varepsilon_2), \ldots, (z_n - \varepsilon_n)$, then $\Omega$ has Gleason $R$-property at $\varepsilon$.

## 4.1 Implementation

In the following sections we will investigate the Gleason problem for some of the spaces that were defined in *Section 2*. Afterwards, an implementation of how smooth partition of unity can be used to solve the Gleason problem will be shown in *Section 4.1.5*.

The following theorem is called the maximum modulus principle.

**Theorem 11.** *Let $\Omega \subset \mathbb{C}$ be a bounded set and let $f \in \mathcal{A}(\Omega)$, then the maximum value of $|f|$ on $\overline{\Omega}$ exists and*

$$\max_{\overline{\Omega}} |f| = \max_{\partial\Omega} |f|.$$

The proof of this can be found in [12].

### 4.1.1 Complex analytic functions on the unit disk

In this section we will show that the unit disk has Gleason $\mathcal{H}$-property at a point $z = 0$. If $f \in \mathcal{H}(\Omega)$, where $\Omega$ is the open unit disk, and $f(0) = 0$ then the Taylor expansion of $f(z)$ around $z = 0$ is

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)z^k}{k!} = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)z^k}{k!}.$$

This sum converges to $f(z)$ for all $z \in \Omega$. Now we can factor out $z$ from the sum, since $f(0) = 0$, thus

$$f(z) = z \cdot \sum_{k=0}^{\infty} \frac{f^{(k+1)}(0)z^k}{(k+1)!} = z \cdot \sum_{k=0}^{\infty} \frac{1}{(k+1)} \cdot \frac{f^{(k+1)}(0)z^k}{k!}. \tag{4.1}$$

Let

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)} \cdot \frac{f^{(k+1)}(0)z^k}{k!}. \tag{4.2}$$

To see if $g(z) \in \mathcal{H}(\Omega)$ we take the Taylor expansion of $f^{(1)}(z)$ around $z = 0$

$$f^{(1)}(z) = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(0)z^k}{k!}$$

which converges to $f^{(1)}(z)$ for all $z \in \Omega$ so $g(z)$ clearly converges to some holomorphic function for all $z \in \Omega$. Therefore, the unit disk has Gleason $\mathcal{H}$-property at $z = 0$.

### 4.1.2 Bounded holomorphic functions on the unit disk

In this section we will show that the unit disk has Gleason $\mathcal{H}^\infty$-property at a point $z = 0$. For $f \in \mathcal{H}^\infty(\Omega)$, where $\Omega$ is again the unit disk and $f(0) = 0$ we do the same Taylor expansion as in *Equation 4.1* and define $g(z)$ as in *Equation 4.2*. In *Section 4.1.1*, we showed that $g(z)$ is holomorphic on $|z| < 1$ which means that it is continuous on any closed subset of the unit disk, let's say $|z| \leq 1/2$, so from *Theorem 2* we have that a continuous function on a compact set is bounded. Thus *Theorem 4* provides that $g(z)$ is bounded on $|z| \leq 1/2$. For $|z| > 1/2$ we have

$$\|g(z)\|_{L^\infty} = \left\|\frac{f(z)}{z}\right\|_{L^\infty} < 2\|f(z)\|_{L^\infty} < \infty.$$

Therefore $g(z)$ converges to an element in $\mathcal{H}^\infty(\Omega)$, so the unit disk also has Gleason $\mathcal{H}^\infty$-property at $z = 0$.

### 4.1.3 Analytic functions on the unit disk

In this section we will show that the unit disk has Gleason $\mathcal{A}^0$-property at a point $z = 0$. Let $f \in \mathcal{A}^0(\Omega)$ where $\Omega$ is the unit disk and $f(0) = 0$. We take the same Taylor expansion as in *Equation 4.1* and define $g(z)$ as in *Equation 4.2*. Let

$$g_n(z) = \sum_{k=0}^{n} \frac{1}{(k+1)} \cdot \frac{f^{(k+1)}(0)z^k}{k!}. \tag{4.3}$$

Now we have the limit

$$f(z) = z \cdot g(z) = z \cdot \lim_{n\to\infty} g_n(z).$$

Clearly $g_n \in \mathcal{H}(\mathbb{C})$, we need to show that $g_n$ converges to a continuous function on the boundary of $\Omega$. From *Theorem 11* we get

$$\|g_n\|_{L^\infty(\overline{\Omega})} = \|g_n\|_{L^\infty(\partial\Omega)}.$$

Since $\partial\Omega$ is given by $|z| = 1$ we have

$$\|g_l - g_k\|_{L^\infty(\partial\Omega)} = \|z\|_{L^\infty(\partial\Omega)} \cdot \|g_l - g_k\|_{L^\infty(\partial\Omega)} = \|zg_l - zg_k\|_{L^\infty(\partial\Omega)} = \|f_l - f_k\|_{L^\infty(\partial\Omega)}.$$

Considering that $f_n$ is a Cauchy sequence that converges to an element $f \in C^0(\overline{\Omega})$ with the supremum norm, there exists an $N \in \mathbb{N}$ such that for all $l, k > N$ there is an $\varepsilon > 0$ such that

$$\|g_l - g_k\|_{L^\infty(\overline{\Omega})} = \|g_l - g_k\|_{L^\infty(\partial\Omega)} = \|f_l - f_k\|_{L^\infty(\partial\Omega)} < \varepsilon.$$

Therefore $g_n$ is a Cauchy sequence under the supremum norm taken over the closed unit disk so from *Theorem 3* we have that $g_n$ converges to an element in $C^0(\overline{\Omega})$. Thus the unit disk has Gleason $\mathcal{A}^0$-property at $z = 0$.

### 4.1.4 Analytic functions with continuous derivatives on the unit disk

In this section we will show that the unit disk has Gleason $\mathcal{A}^1$-property at point $z = 0$. Let $f \in \mathcal{A}^1(\Omega)$ where $\Omega$ is the unit disk and $f(0) = 0$. We define $g_n(z)$ like in *Equation 4.3* and do the same calculations as in *Section 4.1.3* to prove that $g_n$ converges to an element in $C^0(\overline{\Omega})$. We also need to prove that $g_n^{(1)}$ converges to an element in $C^1(\overline{\Omega})$. Clearly $g_n^{(1)} \in \mathcal{H}(\mathbb{C})$ so from *Theorem 11* we get

$$\|g_n^{(1)}\|_{L^\infty(\overline{\Omega})} = \|g_n^{(1)}\|_{L^\infty(\partial\Omega)}.$$

Thus for two functions in the sequence $g_n^{(1)}$ we have

$$\|g_l^{(1)} - g_k^{(1)}\|_{L^\infty(\overline{\Omega})} = \|g_l^{(1)} - g_k^{(1)}\|_{L^\infty(\partial\Omega)}.$$

The boundary is given by $|z| = 1$, therefore

$$\begin{aligned}
\|g_l^{(1)} - g_k^{(1)}\|_{L^\infty(\partial\Omega)} =& \|z\|_{L^\infty(\partial\Omega)} \cdot \|g_l^{(1)} - g_k^{(1)}\|_{L^\infty(\partial\Omega)} \\
=& \|z \cdot g_l^{(1)} - z \cdot g_k^{(1)}\|_{L^\infty(\partial\Omega)} \\
=& \|f_l^{(1)} - f_k^{(1)}\|_{L^\infty(\partial\Omega)} < \varepsilon.
\end{aligned}$$

Using the same reasoning as in *Section 4.1.3* we can see that $\underline{g_n^{(1)}}$ converges to an element in $C^0(\overline{\Omega})$ now it is clear that $g_n$ converges to an element in $C^1(\overline{\Omega})$. Thus the unit disk has Gleason $\mathcal{A}^1$-property at $z = 0$. With the same reasoning, one can prove that the unit disk has Gleason $\mathcal{A}^k$-property at $z = 0$ for all $k \in \mathbb{N}$.

### 4.1.5 A subset of smooth functions

In this section we will use partition of unity to show that a bounded subset $\Omega \subset \mathbb{R}^2$ has Gleason $R$-property at a point $p \in \Omega$, where $R(\Omega) \subset C^\infty(\Omega)$ is the ring of functions that are real analytic in $p \in \Omega$. One can analyze an open covering, $\Omega_1$, over a point where $f \in C^\infty(\Omega)$ is real analytic, and then an open covering, $\Omega_2$, over the rest of the domain. Let $\Omega_1$ be an open disk centered at the origin, then we can do a Taylor expansion of $f(x, y)$ around $(x, y) = (0, 0)$. We let $f(0, 0) = 0$ which means that we can factor out the first terms of the sum as well as the variables $x$ and $y$ like in the following calculations.

$$
\begin{aligned}
f(x,y) =& \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!}\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0,0)x^m y^n \\
=& y \cdot \frac{\partial f}{\partial y}(0,0) + x \cdot \frac{\partial f}{\partial x}(0,0) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m!n!}\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0,0)x^m y^n \\
=& y \cdot \frac{\partial f}{\partial y}(0,0) + x \cdot \frac{\partial f}{\partial x}(0,0) + x \cdot \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m!n!}\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0,0)x^{(m-1)} y^n \\
=& y \cdot \frac{\partial f}{\partial y}(0,0) + x \cdot \left(\frac{\partial f}{\partial x}(0,0) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m!n!}\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0,0)x^{(m-1)} y^n\right).
\end{aligned}
$$

For simplicity we construct the following notations

$$
\begin{aligned}
f_1^x(x,y) =& \frac{\partial f}{\partial x}(0,0) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m!n!}\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0,0)x^{(m-1)} y^n, \\
f_1^y(x,y) =& \frac{\partial f}{\partial y}(0,0).
\end{aligned}
$$

The region of convergence for this Taylor expansion is given by the disks $|x| < r_x$ and $|y| < r_y$ where $r_x$ and $r_y$ are dependent on the function $f(x,y)$. Let our covering $\Omega_1 = D_r = \sqrt{x^2+y^2} < r$, where $r = \min(r_x, r_y)$. Now $f(x,y)$ can be expressed as a Taylor series centered at the origin, which converges to a smooth function on $\Omega_1$. We let $\Omega_2 = (\overline{D_{r/2}})^c$, so that the two coverings are open and overlapping. Since $(0,0) \notin \Omega_2$ we have for $f(x,y) \in \Omega_2$:

$$
f(x,y) = 1 \cdot f(x,y) = \frac{x^2+y^2}{x^2+y^2} \cdot f(x,y) = x \cdot \frac{x}{x^2+y^2} \cdot f(x,y) + y \cdot \frac{y}{x^2+y^2} \cdot f(x,y).
$$

For simplicity we construct the following functions

$$
\begin{aligned}
f_2^x(x,y) =& \frac{x}{x^2+y^2} \cdot f(x,y), \\
f_2^y(x,y) =& \frac{y}{x^2+y^2} \cdot f(x,y).
\end{aligned}
$$

In *Figure 4.1* one can see how the two domains, $\Omega_1$ and $\Omega_2$, overlap to cover the full domain of $\Omega$.

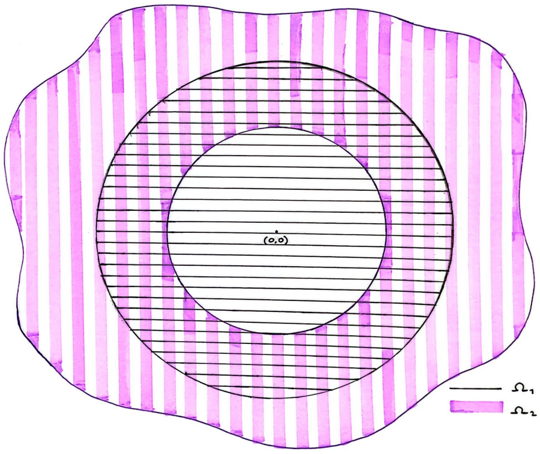


Figure 4.1: An illustration of the overlapping domains $\Omega_1$ and $\Omega_2$, defined in *Section 4.1.5*.

Now we can use our smooth partition of unity, where $\psi_1$ and $\psi_2$ are the partitions over $\Omega_1$ and $\Omega_2$ respectively. We pick our $\psi_1$ so that it has compact support in $D_r$ and we can pick our $\psi_2$ so that it has compact support in $(\overline{D_{r/2}})^c$.

$$\begin{aligned}
f(x,y) =& f(x,y)\psi_1(x,y) + f(x,y)\psi_2(x,y)\\
=&(x\cdot f_1^x(x,y) + y\cdot f_1^y(x,y))\psi_1(x,y) + (x\cdot f_2^x(x,y) + y\cdot f_2^y(x,y))\psi_2(x,y)\\
=&x\cdot(f_1^x(x,y)\psi_1(x,y) + f_2^x(x,y)\psi_2(x,y)) + y\cdot(f_1^y(x,y)\psi_1(x,y) + f_2^y(x,y)\psi_2(x,y)).
\end{aligned}$$

Since $\operatorname{supp}\psi_1 \subset D_r$, we have that $\psi_1(x,y) = 1$ for $(x,y) \in D_{r/2}$ and we also have that $\psi_2(x,y) = 0$ for $(x,y) \in D_{r/2}$ so the function $f(x,y)$ on $D_{r/2}$ is given by

$$f(x,y) = x\cdot f_1^x(x,y) + y\cdot f_1^y(x,y),$$

where the functions $f_1^x(x,y)$ and $f_1^y(x,y)$ are smooth functions that are analytic on the origin. Similarly $\operatorname{supp}\psi_2 \subset (\overline{D_{r/2}})^c$ so $\psi_2(x,y) = 1$ and $\psi_1(x,y) = 0$ for $(x,y) \in (\overline{D_{3r/2}})^c$ so $f(x,y)$ on $(\overline{D_{3r/2}})^c$ is

$$f(x,y) = x\cdot f_2^x(x,y) + y\cdot f_2^y(x,y),$$

where the functions $f_2^x(x,y)$ and $f_2^y(x,y)$ are smooth functions on $\Omega_2$. For points in between the two sets $D_{r/2}$ and $(\overline{D_{3r/2}})^c$ we still have that $\psi_1(x,y) + \psi_2(x,y) = 1$. Thus we can now prove the following theorem:

**Theorem 12.** *Let $\Omega \subset \mathbb{R}^2$ and let $R(\Omega) \subset C^\infty(\Omega)$ be the ring of functions that are real analytic in $p \in \Omega$, then $\Omega$ has the Gleason R-property at $p$.*

*Proof.* We want to show that $\tilde{f} \in \Omega$ has Gleason $R$-property at $p$. We construct a variable change such that the point $p = (p_w, p_z)$ is represented by the origin, that is $(w,z) = (x + p_w, y + p_z)$.

Now for $(x, y) = (0, 0)$ we have $(w, z) = (p_w, p_z) = p$. Then we compose the partition of unity above such that

$$f(x, y) = x \cdot f_1(x, y) + y \cdot f_2(x, y).$$

Now we execute the variable change $(x, y) = (w - p_w, y - p_z)$ so that we are back in our original domain. That is

$$f(w - p_w, z - p_z) = (w - p_w)f_1(w - p_w, z - p_z) + (z - p_z)f_2(w - p_w, z - p_z). \tag{4.4}$$

Since $p_w$ and $p_z$ are constants we can introduce functions $\tilde{f}$, $\tilde{f}_1$ and $\tilde{f}_2$ such that *Equation 4.4* becomes

$$\tilde{f}(w, z) = (w - p_w)\tilde{f}_1(w, z) + (z - p_z)\tilde{f}_2(w, z).$$

□

With similar calculations and reasoning one can also prove that *Theorem 12* holds for domains $\Omega \subset \mathbb{R}^n$.

# Chapter 5

# Conclusion

The Gleason problem can be tricky to solve, in order to tackle this complicated problem we initially delved into the topic of function spaces. By properly defining the function spaces and assigning appropriate norms to them we could later in *Section 4* prove that the unit disk has the Gleason property on the origin for these spaces. In addition to function spaces, we delved into the construction of partition of unity, which has proven to be a useful tool when solving the Gleason problem. Specifically, in this thesis we used smooth partition of unity to prove that a domain in $\mathbb{R}^2$ has Gleason $R$-property at any point $p$ in the domain, where $R$ is the ring of smooth functions that are real analytic in $p$. The proof centered around the fact that if a domain has Gleason $R$-property at the origin, one can study the Gleason problem on a region around the origin and an overlapping region on the rest of the domain. Afterwards, we combine these areas together using our smooth partition of unity such that the Gleason property holds for the entire domain. Lastly, we prove with a change of variable that the domain in fact has Gleason $R$-property at any point $p$ in the domain.

The results in this thesis are well known however, the final outcome in *Section 4.1.5* is according to my knowledge an entirely new result. Moreover, the proofs of *Theorem 8, 9, 10* and *12* as well as the proofs in *Section 3.1* are constructed by myself. In addition, I have produced the implementations and calculations in *Section 4.1* and in *Section 3.2*. Furthermore all figures in this thesis are personally illustrated.

The findings derived from this thesis touch on current work. For future research, one interesting topic to investigate is the dilemma of whether a Hartogs domain possesses the Gleason $\mathcal{H}^\infty$-property. Another subject one could investigate is the Gleason problem on other rings of real valued functions. One could also study whether the ring of smooth functions that are real analytic in $p$ is Noetherian.

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