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Akademisk avhandling

som för avläggande av filosofie doktorsexamen i matematik/tillämpad matematik vid Akademin för utbildning, kultur och kommunikation kommer att offentligen försvaras tisdagen den 29 november 2022, 13.15 i Kappa, Mälardalens universitet, Västerås.

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Abstract

We consider a market model of financial engineering with three factors represented by three correlated Brownian motions. The volatility of the risky asset in this model is the sum of two stochastic volatilities. The dynamic of each volatility is governed by a mean-reverting process. The first stochastic volatility of mean-reversion process reverts to the second volatility at a fast rate, while the second volatility moves slowly to a constant level over time with the state of the economy.

The double mean-reverting model by Gatheral (2008) is motivated by empirical dynamics of the variance of the stock price. This model can be consistently calibrated to both the SPX options and the VIX options. However due to the lack of an explicit formula for both the European option price and the implied volatility, the calibration is usually done using time consuming methods like Monte Carlo simulation or the finite difference method.

To solve the above issue, we use the method of asymptotic expansion developed by Pagliarani and Pascucci (2017). In paper **A**, we study the behaviour of the implied volatility with respect to the logarithmic strike price and maturity near expiry and at-the-money. We calculate explicitly the asymptotic expansions of implied volatility within a parabolic region up the second order. In paper **B** we improve the results obtain in paper A by calculating the asymptotic expansion of implied volatility under the Gatheral model up to order three. In paper **C**, we perform numerical studies on the asymptotic expansion up to the second order. The Monte-Carlo simulation is used as the benchmark value to check the accuracy of the expansions. We also proposed a partial calibration procedure using the expansions. The calibration procedure is implemented on real market data of daily implied volatility surfaces for an underlying market index and an underlying equity stock for periods both before and during the COVID-19 crisis. Finally, in paper **D** we check the performance of the third order expansion and compare it with the previous results.

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This thesis is dedicated to the memory of my beloved grandfather
Mohammed ibn Hamdiyah

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Västerås, October, 20, 2022
Mohammed Albuhayri

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Populärvetenskaplig sammanfattning

Finansiell matematik är en gren av sannolikhetsteorin som fokuserar på matematisk modellering på finansiella marknader. Varje gång när beteendet hos finansiella instrument på finansmarknaden förändras, utvecklas också motsvarande matematiska modeller som beskriver priserna. Till exempel var finansmarknadskraschen i oktober 1987 en vändpunkt för den berömda Black–Scholes-modellen från 1973. I efterdyningarna av finanskraschen var Black–Scholes modellantaganden starkt i konflikt med empiriska rön.

Spridningen av avkastningen för ett givet värdepapper eller marknadsindex mäts vanligtvis med en statistisk egenskap som kallas volatiliteten eller standardavvikelsen. Ju högre volatilitet, desto mer riskfyllt är värdepappret. Genom att jämföra marknadspriset för ett givet finansiellt instrument med det teoretiska priset som ges av Black–Scholes modellen får vi ett slags mått som kallas implicit volatilitet. Efter kraschen visade optionsmarknaderna ett volatilitetsleende: genom att rita den implicita volatiliteten för en grupp optioner mot lösenpriserna så fås en graf som ser ut som en leende mun. Black-Scholes-modellen förutsäger inte ett sådant beteende, eftersom den är baserad på antagandet om konstant volatilitet.

Att modellera volatilitet som en stokastisk process är lösningen! År 2009 genomfördes empiriska studier av Christoffersen et al. som visade att införandet av två tidsskalor i volatilitet, en snabb och en långsam, är effektiv för att fånga huvuddragen i de observerade termstrukturerna för implicit volatilitet. En populär teori som används inom finans antyder att både tillgångsprisvolatiliteten och den empiriska avkastningen så småningom återgår till det långa medelvärdet eller genomsnittsnivån. Följaktligen överväger vi Gatheral modellen där slumpmässiga faktorer beskrivs av en stokastisk volatilitetsmodell av medelåtergångstyp. Ingen lösning i sluten form för vare sig europeiskt optionspris eller implicit volatilitet finns i Gatheral modellen. Således, med hjälp av metoden av Pagliarani och Pascucci (2017), beräknar vi den så kallade asymptotiska expansionen av den implicita volatiliteten under Gatheral modellen upp till tredje ordningen. Med andra ord får vi en serie funktioner som har följande egenskap: trunkering av serien efter ett ändligt antal termer ger en approximation till en given funktion, i vårt fall till den implicita volatiliteten. Expansionen utförs med hänsyn till de två oberoende små parametrarna: det logaritmiska lösenpriset och optionens löptid. Den tredje ordningen innebär att approximationsfelet blir obetydligt med avseende på kuben för den första parametern och 1,5 potensen för den andra när både parameter går mot 0. Vi hittar den ekonomiska meningen med expansionen, utför numeriska studier och validerar vår resultat genom att jämföra dem med Monte Carlo-simuleringen.

Resultaten av vår forskning kan användas av både finansinstitutioner och enskilda handlare för att optimera sina inkomster.

Popular science summary

Financial mathematics is a branch of probability theory that focuses on mathematical modelling in financial markets. Each time when the behaviour of financial instruments in financial market evolves, the corresponding mathematical models describing the prices evolve as well. For instance, the financial market's crash of October 1987 was a turning point for the famous 1973 Black–Scholes model. In the aftermath of the financial crash, the assumptions of the Black–Scholes model were strongly conflicting with empirical findings.

The dispersion of returns for a given security or market index is usually measured by a statistical property called the volatility, or the standard deviation. The higher the volatility, the riskier the security. By comparing the market price of a given financial instrument with the theoretical price given by the Black–Scholes model, we obtain a kind of measure called the implied volatility. After the crash, option markets displayed a volatility smile: while plotting the implied volatility of a group of options against the strike prices, one obtains a graph looking as a smiling mouth. The Black–Scholes model does not predict such a behaviour, because it is based on the assumption of constant volatility.

Modeling volatility as a stochastic process is the solution! In 2009, empirical studies by Christoffersen et al. show that the introduction of two timescales in volatility, a fast and a slow, is efficient to capture the main features of the observed term structures of implied volatility. A popular theory used in finance suggests that both the asset price volatility and the empirical returns eventually return to the long-time mean value or average level. Accordingly, we consider the Gatheral model where random factors are described by a stochastic volatility model of mean-reversion type. No closed-form solution for either European option price or implied volatility exists in the Gatheral model. Thus, using the method by Pagliarani and Pascucci (2017), we calculate the so called asymptotic expansion of the implied volatility under the Gatheral model up to the third order. In other words, we obtain a series of functions which has the following property: truncating the series after a finite number of terms provides an approximation to a given function, in our case, to the implied volatility. The expansion is performed with respect to the two independent small parameters: the logarithmic strike price and time to maturity of the option. The third order means that the error of the approximation becomes insignificant with respect to the cube of the first parameter and the 1.5 power of the second one as both tend to 0. We found the financial sense of the expansion, performed numerical studies and validated our results by comparing them to the Monte Carlo simulation.

The results of our research can be used by both financial institutions and individual traders for optimisation of their incomes.

Contents

List of Papers

Paper A Mohammed Albuhayri, Anatoliy Malyarenko, Sergei Silvestrov, Ying Ni, Christopher Engström, Finnan Tewolde, and Jiahui Zhang. "Asymptotics of Implied Volatility in the Gatheral Double Stochastic Volatility Model." Applied Modeling Techniques and Data Analysis 2: Financial, Demographic, Stochastic and Statistical Models and Methods 8 (2021): 27–38.

Paper B Mohammed Albuhayri, Cristopher Engström, Anatoliy Malyarenko, Ying Ni, Sergei Silvestrov. An improved asymptotic of implied volatility in the Gatheral model. In Stochastic Processes, Statistical Methods, and Engineering Mathematics. SPAS 2019, volume 408 of Spring Proceedings, in Mathematics & Statistics, pages 3–14. Springer, 2022.

Paper C Marko Dimitrov, Mohammed Albuhayri, Ying Ni, Anatoliy Malyarenko. "Numerical Studies of Implied Volatility Expansions Under the Gatheral Model." In Data Analysis and Related Applications 1. Computational, Algorithmic and Applied Economic Data Analysis, vol. 9 of Big data, artificial intelligence, and data analysis set, K. N. Zafeiris, C. H. Skiadas, Y. Dimotikalis, A. Karagrigoriu, C. Karagrigoriou-Vonta, (eds.), Wiley, 2022, Chapter 10, pp. 135–148.

Paper D Mohammed Albuhayri, Sergei Silvestrov, Marko Dimitrov, Ying Ni, Anatoliy Malyarenko. Numerical Studies of the Implied Volatility Expansions up to Third Order under the Gatheral Model. Accepted for publication. Presented in the 7th Stochastic Modeling Techniques and Data Analysis International Conference (SMTDA2022) and Demographics 2022 Workshop, (hybrid) Athens, Greece, June 7–10, 2022.

Parts of the thesis have been presented at the following international conferences:

- The 18th Conference of the Applied Stochastic Models and Data Analysis International Society (ASMDA2019) with Demographics Workshop, Florence, Italy, June 11–14, 2019.
- The 2nd International Conference "Stochastic Processes and Algebraic Structures – From Theory Towards Applications" SPAS 2019, Västerås, Sweden, September 30–October 2, 2019.
- The 19th Conference of the Applied Stochastic Models and Data Analysis International Society (ASMDA2021) and Demographics 2021 Workshop, (online) Athens, Greece, June 1–4, 2021.
- The 7th Stochastic Modeling Techniques and Data Analysis International Conference (SMTDA2022) and Demographics 2022 Workshop, (hybrid) Athens, Greece, June 7–10, 2022.

Part I

The Thesis

Chapter 1

Introduction

Financial Engineering is a part of applied mathematics that studies market models. We do not give here an exact definition of a market model, but rather consider several examples.

Historically, the first market model has been created in the PhD thesis by Bachelier (1900), reprinted in Bachelier (1995, 2012). To explain his model, we introduce the Brownian motion in Section 1.2. The Bachelier model is described in Section 1.3.

1.1 A Brief History

- Brown (1828) observes Brownian motion.
- Fick (1855) derives the diffusion equation.
- Einstein (1905) explains Brownian motion.
- Langevin (1908) derives his equation (the second Newton law) offering a Newtonian explanation of Brownian motion.
- Fokker (1914) and Planck (1917) derive a general forward equation (Einstein's 1905 derivation treats only a free Brownian particle)

- Wiener (1923) proves the existence of Brownian motion i.g. the mathematical existence of stochastic process satisfying the Einstein's 1905 postulates. This process we call today the *standard* Brownian motion or the Wiener process.

- Uhlenbeck and Ornstein (1930) develop the theory of Brownian motion based on the 1908 Langevin equation. It leads to a process which we call today the Ornstien-Uhlenbeck process.

- Kolmogoroff (1931) derives his forward and backward equations (the forward equation is the same as the one derived by Fokker (1914) and Planck (1917)).

- Itô (1944) introduces the Ito integral which leads to the development of *stochastic differential equation* as a mathematical mean of constructing and representing diffusion processes.

1.2 The Brownian Motion

Economy is uncertain. To construct a mathematical model of uncertainty in economy, we suppose that all possible states of economy form a set. Call it Ω. Certain subsets of the set Ω are called *events*. We suppose that the set of events is a σ*-field*.

Definition 1 (σ-field)**.** A set of events $\mathfrak{F}$ is called a σ*-field* if and only if

F1 The *impossible event*, $\varnothing$, is an event.

F2 If $A \subseteq \Omega$ is an event, then the set "not A",

$$\Omega \setminus A = \{\, \omega \in \Omega \colon \omega \notin A \,\}$$

is an event.

F3 If $\{\, A_n \colon n \geq 1 \,\}$ is a sequence of events, then its union is an event.

A pair $(\Omega, \mathfrak{F})$ has its own name.

Definition 2 (sample space)**.** A *sample space* is the set of all possible outcomes (Ω).

Definition 3 (measurable space)**.** A *measurable space* is a pair $(\Omega, \mathfrak{F})$, where Ω is a set, and $\mathfrak{F}$ is a σ-field of subsets of Ω.

Next, we equip the measurable space with a filtration and a probability measure.

Definition 4 (filtration)**.** A *filtration* is a non-decreasing family $\{ \mathfrak{F}_t : t \geq 0 \}$ of sub-σ-fields of $\mathfrak{F}$: if $0 \leq s < t < \infty$, then $\mathfrak{F}_s \subseteq \mathfrak{F}_t \subseteq \mathfrak{F}$.

Definition 5 (probability measure)**.** A *probability measure* on a measurable space $(\Omega, \mathfrak{F})$ is a function $\mathsf{P} \colon \mathfrak{F} \to [0, 1]$ satisfying

P1 $\mathsf{P}(\varnothing) = 0$, $\mathsf{P}(\Omega) = 1$.

P2 If $\{ A_n : n \geq 1 \}$ is a sequence of pairwise nonintersecting events, then

$$\mathsf{P}\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathsf{P}(A_n).$$

The triple $(\Omega, \mathfrak{F}, \mathsf{P})$ has a special name.

Definition 6 (probability space)**.** A triple $(\Omega, \mathfrak{F}, \mathsf{P})$ is called a *probability space* if ad only if $(\Omega, \mathfrak{F})$ is a sample space and P is a probability measure on $\mathfrak{F}$.

The following definition is a technical requirement.

Definition 7 (usual conditions)**.** A filtration $\{ \mathfrak{F}_t : t \geq 0 \}$ satisfies the *usual conditions* if and only if

U1 The filtration is right-continuous: for every $t \geq 0$

$$\mathfrak{F}_{t+} = \cap_{\varepsilon > 0} \mathfrak{F}_{t+\varepsilon} = \mathfrak{F}_t.$$

U2 The σ-field $\mathfrak{F}_0$ contains all subsets of all *negligible events* $A \in \mathfrak{F}$, that is, the events A with $\mathsf{P}(A) = 0$.

Definition 8 (filtered probability space)**.** A *filtered probability space* is a quadruple $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$, where $(\Omega, \mathfrak{F}, \mathsf{P})$ is a probability space and $\{ \mathfrak{F}_t : t \geq 0 \}$ is a filtration satisfying the usual conditions.

Let T be a topological space.

Definition 9 (Borel set)**.** The σ-field of Borel sets, $\mathfrak{B}(T)$, is the smallest σ-field containing all open sets of the space T. A *Borel set* is an element of $\mathfrak{B}(T)$.

Let d be a positive integer. Next, we consider some important mappings from Ω to $\mathbb{R}^d$.

Definition 10 (random vector)**.** A mapping $\mathbf{X} \colon \Omega \to \mathbb{R}^d$ is called a *random vector* if and only if it is *measurable*, that is, for any $B \in \mathfrak{B}(\mathbb{R}^d)$, its inverse image

$$\mathbf{X}^{-1}(B) = \{ \omega \in \Omega \colon \mathbf{X}(\omega) \in B \}$$

is an event. A random vector $\mathbf{X}$ is called a *random variable* if and only if $d = 1$.

In what follows, we denote random variables by capital Roman Latin letters $X, Y, \dots$. Now we introduce stochastic processes.

Definition 11 (stochastic process)**.** A *stochastic process* is a collection of random variables $\{ X(t) \colon 0 \leq t < \infty \}$.

The argument $t \in [0, \infty)$ of the random variables $X(t)$ has an interpretation as *time*.

The most important example of a stochastic process is as follows.

Definition 12 (Brownian motion)**.** A *Brownian motion* is a stochastic process $\{ B(t) \colon 0 \leq t < \infty \}$ defined on a filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ such that

B1 $B(t)$ is *adapted*: for all $t \geq 0$ the random variable $B(t)$ is $\mathfrak{F}_t$-measurable.

B2 $B(t)$ is continuous: for P-almost all $\omega_0 \in \Omega$ the function $B(t, \omega_0)$ of the variable $t \in [0, \infty)$ is continuous.

B3 $\mathsf{P}(B(0) = 0) = 1$.

B4 For $0 \leq s < t$, the *increment* $B(t) - B(s)$ is independent of $\mathfrak{F}_s$ and is normally distributed with mean 0 and variance $t - s$.

Does such a process exist? The answer is positive, three different *explicit constructions* of a filtered probability space and a Brownian motions on it can be found in Karatzas and Shreve (1991, Chapter 2, Sections 2–4). The first one is based on the *Kolmogorov Extension Theorem*, see Daniell (1919), Kolmogoroff (1933) and subsequent editions and translations Kolmogorov (1950), Kolmogorov (1956), Kolmogoroff (1973), Kolmogorov (1974), Kolmogoroff (1977), and on the *Kolmogorov–Čentsov Continuity Theorem*, see Čentsov (1956). The second explicit construction is based on the original construction by Wiener (1923) with additions by Lévy (1948) and Ciesielski (1961). The third one is based on the ideas by Donsker (1951) and Prokhorov (1956).

In what follows, we replace the name "Brownian motion" with "Wiener process" and the notation $B(t)$ with $W(t)$ because of the historically first explicit construction given by Wiener (1923).

1.3 The Bachelier Model

The Bachelier model contains two securities: the stock and the bank account. The stock price $S(t)$ is given by

$$S(t) = S(0) + \mu t + \sigma W(t), \qquad t \in [0, \infty),$$

where $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ are two constants, see Musiela and Rutkowski (2005, Subsection 3.3). The bank account is given by

$$B(t) = 1.$$

1.3.1 Trading Strategies

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ be a filtered probability space on which the Wiener process $W(t)$ is defined. To discuss trading strategies, we need a technical definition.

Definition 13 (progressive measurability)**.** A stochastic process $X(t)$ defined on a filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ is called *progressively measurable* if and only if for every $t \in [0, \infty)$, the mapping $[0, t] \times \Omega \to \mathbb{R}$, $(u, \omega) \mapsto X(u, \omega)$ is measurable with respect to the σ-field $\mathfrak{B}([0, t]) \times \mathfrak{F}_t$, that is, for any $B \in \mathfrak{B}(\mathbb{R})$, the inverse image

$$X^{-1}(B) = \{ (u, \omega) \in [0, t] \times \Omega \colon X(u, \omega) \in B \}$$

belongs to the σ-field $\mathfrak{B}([0, t]) \times \mathfrak{F}_t$.

Definition 14 (trading strategy)**.** A *trading strategy* is a pair $(X_1(t), X_2(t))$ of progressively measurable stochastic processes.

At any time moment t, the random variable $X_1(t)$ is the number of units of the stock in the *trader's portfolio*, while $X_2(t)$ is the number of units of the bank account there. We have that the time t price of the portfolio is given by the *wealth process* or *portfolio process*

$$V(t) = X_1(t)S(t) + X_2(t)B(t). \tag{1.1}$$

The most important trading strategies or portfolio processes are self-financing ones. Intuitively, a trading strategy or the corresponding portfolio process (1.1) is self-financing on a finite trading interval $[0, T]$ if at any time moment $t \in (0, T]$ there are no cash flows inside or outside the portfolio. It turns out that this condition can be easily formulated mathematically.

Theorem 1. *A trading strategy $(X_1(t), X_2(t))$ is self-financing if and only if*

$$V(t) = V(0) + \int_0^t X_1(u)\,dS(u) + \int_0^t X_2(u)\,dB(u), \qquad t \in [0, T]. \tag{1.2}$$

In Equation (1.2), the second integral is the pathwise Lebesgue integral, while the first integral is the Itô integral constructed by Itô (1944). The construction of Itô integral can be found in many sources, see Karatzas and Shreve (1991), Musiela and Rutkowski (2005), among many others. In some sources, for example, Musiela and Rutkowski (2005, Definition 3.3.1), the conclusion of Theorem 1 serves as the *definition* of a self-financing trading strategy.

Some self-financing trading strategies can produce money from nothing. Mathematically, such a trading strategy is called an *arbitrage opportunity*.

Definition 15 (arbitrage opportunity)**.** A self-financing trading strategy is called an *arbitrage opportunity* if and only if its wealth process (1.1) satisfies the following conditions.

A1 $V(0) = 0$.

A2 $V(T) \geq 0$.

A3 $\mathsf{P}(V(T) > 0) > 0$.

The corresponding portfolio process (1.1) is often called a *free lunch portfolio*.

How to check if a given market model, in particular, the Bachelier one, contains an arbitrage opportunity? We need more theory.

1.3.2 Martingale Measures

Let P_1 and P_2 be two probability measures on a measurable space $(\Omega, \mathfrak{F})$.

Definition 16 (equivalent measures)**.** The probability measures P_1 and P_2 are called *equivalent* if and only if they have the same null events, that is, for any event A, we have $\mathsf{P}_1(A) = 0$ if and only if $\mathsf{P}_2(A) = 0$.

Definition 17 (martingale)**.** A stochastic process $X(t)$ defined on a filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ is called a *martingale* if and only if

M1 $X(t)$ is *adapted*, that is, the random variable $X(t)$ is $\mathfrak{F}_t$-measurable for all $t \in [0, T]$.

M2 $\mathsf{E}[|X(t)|] < \infty$, $t \in [0, T]$.

M3 $\mathsf{E}[X(t) \mid \mathfrak{F}_s] = X(s)$, $0 \leq s \leq t \leq T$.

Definition 18 (martingale measure)**.** A probability measure P^* defined on a measurable space $(\Omega, \mathfrak{F}_T)$ is called a *martingale measure* for the discounted price process $S^*(t) = \frac{S(t)}{B(t)}$ if and only if

MM1 The measure P^* is equivalent to the restriction of P to $\mathfrak{F}_T$.

MM2 The discounted price process $S^*(t)$ is a martingale on the filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P}^*)$.

Remark 1. It is possible to develop a more general theory, where the class of martingales is replaced by a much wider class of *local martingales*, see, e,g., Musiela and Rutkowski (2005). In this thesis, we do not develop and never use such a general theory.

The usefulness of the introduced notions is clarified by the following result.

Theorem 2 (The Fundamental Theorem of Financial Engineering, part 1)**.** *For a wide class of market models, including the Bachelier one, the following conditions are equivalent.*

- *The market model does not contain arbitrage opportunities.*
- *There exists a martingale measure for the discounted price process $S^*(t)$.*

Is the second condition of Theorem 2 hold true for the Bachelier model? To give an answer to this question, we need a technical tool.

Let $W(t)$ be a Wiener process on a filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$. Let γ be a real number. The random variable $Y = \exp(\gamma W(T))$ is positive. Moreover, we have $\mathsf{E}[Y] = \exp(\gamma^2 T/2)$. The random variable $Z = \exp(\gamma W(T) - \gamma^2 T/2)$ is positive with $\mathsf{E}[Z] = 1$. Let A be an event in $\mathfrak{F}_T$, and let $\mathbb{1}_A$ be the indicator of A:

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that the probability measure P^* given by

$$\mathsf{P}^*(A) = \mathsf{E}[Z\mathbb{1}_A], \qquad A \in \mathfrak{F}_T$$

is equivalent to P.

Theorem 3 (The Girsanov Theorem, Girsanov (1960))**.** *The stochastic process $W^*(t) = W(t) - \gamma t$, $0 \le t \le T$, is a Wiener process on the filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P}^*)$.*

In particular, consider the case of $\gamma = -\mu/\sigma$. The stochastic process $W^*(t) = W(t) - \gamma t$ is a Wiener process under the measure P^* with Radon–Nikodym density

$$\frac{d\mathsf{P}^*}{d\mathsf{P}} = \exp\left(\gamma W(T) - \frac{1}{2}\gamma^2 T\right).$$

Moreover, under P^*,

$$S(t) = S(0) + \sigma W^*(t)$$

which is a martingale.

1.4 The Black–Scholes Model

According to Samuelson (2002),

> As I expected, the 1914 popular Bachelier exposition was not in the limited MIT library. But a greater treasure was there: the 1900 Paris thesis and the 1912 item.

In the above citation, "the 1914 popular Bachelier exposition" is Bachelier (1914), reprinted in Bachelier (1929), Bachelier (1993), Bachelier (2018). "The 1912 item" is Bachelier (1912), reprinted in Bachelier (1992).

After rediscovering the 1900 Bachelier thesis, Samuelson published a paper Samuelson (1965), reprinted in Samuelson (2015). His ideas were extended by his PhD student Merton (1973) and independently and simultaneously by Black and Scholes (1973), reprinted in Black and Scholes (2012). Like the Bachelier model, the Black–Scholes one contains two securities, the bank account, $B(t)$, and the stock, $S(t)$. Their dynamics is given by the following system of equations.

$$\begin{aligned} dB(t) &= rB(t)\,dt, & B(0) &= 1, \\ dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dW(t), & S(0) &= S_0. \end{aligned} \tag{1.3}$$

The first equation is an ordinary differential equation, where the *spot interest rate* r is constant. The second equation is a stochastic differential equation, just a shortcut which is customary to write instead of a longer but more correct stochastic integral equation

$$S(t) = S_0 + \int_0^t \mu S(u)\, du + \int_0^t \sigma S(u)\, dW(u), \tag{1.4}$$

where, as usual, the first integral is a pathwise Lebesgue integral, while the second one is an Itô stochastic integral.

The number $\mu \in \mathbb{R}$ is called the *appreciation rate* of the stock price, while the number $\sigma > 0$ is called *volatility*.

Theorem 4. *The Black–Scholes model has the following properties.*

- *The stochastic process*

  $$S(t) = S_0 \exp(\sigma W(t) + (\mu - \sigma^2/2)t), \qquad t \in [0, T]$$

  is the unique solution of the second equation in (1.3) *or equivalently, Equation* (1.4).

- *The discounted stock price*

  $$S^*(t) = \frac{S(t)}{B(t)} = e^{-rt} S(t)$$

  is a martingale if and only if $\mu = r$.

- *The unique martingale measure* P^* *for the process* $S^*(t)$ *is given by the Radon–Nikodym derivative*

  $$\frac{d\mathsf{P}^*}{d\mathsf{P}} = \exp\left(\frac{r-\mu}{\sigma} W(T) - \frac{1}{2}\frac{(r-\mu)^2}{\sigma^2} T\right).$$

  Moreover, under P^*,

  $$dS^*(t) = \sigma S^*(t)\, dW^*(t),$$

  where the stochastic process

  $$W^*(t) = W(t) - \frac{r-\mu}{\sigma} t, \qquad t \in [0, T]$$

  is a Wiener process under P^*.

This result is proved in Musiela and Rutkowski (2005). The first item in Theorem 4 is their Proposition 3.1.1, the second one is Corollary 3.1.2, the third one is Lemma 3.1.3.

1.4.1 Risk-Neutral Valuation

One of the properties given in Theorem 4 is very important.

Definition 19. A *contingent claim* is a random variable X. A contingent claim X is called *European* if it is $\mathfrak{F}_T$-measurable.

Intuitively, a contingent claim X represents the payoff of a financial instrument. The payoff of a European contingent claim becomes known at *maturity*, or expiry date T. For example, a *European call option* with *strike price* K has payoff

$$X = \max\{S(T) - K, 0\}.$$

Economically, at time T, the owner of X has right, but not obligation, to buy one unit of stock S and pay K money units. If $S(T) \leq K$, then he/she will not use this right, and we say that the option is *out-of-the-money*. Otherwise, the owner pays K money units for the one unit of the stock and immediately sells it. His/her gain is the difference in prices, $S(T) - K$, the option is *in-the-money*.

Definition 20. A self-financing portfolio process $V(t)$ given by Equation (1.1) is called *replicating* for a European contingent claim X if

$$V(T) = X.$$

A European contingent claim X is called *attainable* if there exists a replicating portfolio for X.

Part 1 of the Fundamental Theorem of Financial Engineering (Theorem 2) holds true for the Black–Scholes model. It is easy to see that the time t no-arbitrage price of an attainable contingent claim X is equal to $V(t)$, the time t price of the replicating portfolio. If not, the reader can easily construct an arbitrage opportunity, that is, a portfolio satisfying Definition 15.

Is a European call option attainable in the Black–Scholes model? The answer is positive.

Definition 21. A no-arbitrage market model is called *complete* if and only if every European contingent claim is attainable.

Theorem 5 (The Fundamental Theorem of Financial Engineering, part 2)**.** *For a wide class of market models, including the Bachelier and the Black–Scholes one, the following conditions are equivalent.*

- *The market model is complete.*
- *There exists a unique martingale measure* P^* *for the discounted price process* $S^*(t)$.

Under these conditions, the time t no-arbitrage price of any European contingent claim X that settles at time T is

$$\pi_t(X) = B(t)\mathsf{E}^*[B^{-1}(T)X \mid \mathfrak{F}_t].$$

As we see, the discounted stock price process may be considered as a fair game in a risk-neutral economy, where the probabilities of future stock fluctuations are determined by the martingale measure P^*. By this reason, the above measure is also called *risk-neutral*.

Note that different variants of the Fundamental Theorems of Financial Engineering were proved in the papers by Harrison and Pliska (1981, 1983), Dalang et al. (1990), Delbaen and Schachermayer (1994, 1995a,b, 1998), Schachermayer (1994), Levental and Skorohod (1995), Yan (1998), Barski and Zabczyk (2010), Kardaras (2010), Wong and Heyde (2010), Bouchard et al. (2014), Takaoka and Schweizer (2014), Cuchiero et al. (2016), and in the book by Delbaen and Schachermayer (2006).

1.4.2 The Black–Scholes Formula

For the case of a European call option in the Black–Scholes model, the time t no-arbitrage price was explicitly calculated independently by Black and Scholes (1973), reprinted in Black and Scholes (2012), and Merton (1973). From now on, denote by $\tau = T - t > 0$ time to maturity. Let $d_+(s,t)$ and $d_-(s,t)$ be two functions of two positive real variables s and t defined by

$$d_\pm(s,t) = \frac{\ln(s/K) + (r \pm \sigma^2/2)t}{\sigma\sqrt{t}},$$

(recall that K, r, and σ are constants). Finally, let

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)\, dy$$

be the cumulative distribution function of the standard normal distribution.

Theorem 6 (Black and Scholes (1973); Merton (1973))**.** *The time t no-arbitrage price of a European call option in the Black–Scholes model is given by*

$$C(t) = S(t)N(d_+(S(t), \tau)) - Ke^{-r\tau}N(d_-(S(t), \tau)). \tag{1.5}$$

Following Pagliarani and Pascucci (2017), in what follows we put $r = 0$. From now on, we introduce the logarithmic stock price and the logarithmic strike price by

$$x = \ln(S(t)), \qquad k = \ln K.$$

With this convention, and in new variables, the Black–Scholes price becomes

$$u^{\mathrm{BS}}(\sigma, \tau, x, k) = e^x N(d_+(e^x, \tau)) - e^k N(d_-(e^x, \tau)). \tag{1.6}$$

1.5 Local Volatility Models

In practice, market prices of options cannot be explained by the Black–Scholes model. One possible solution to this problem is as follows. Following Dupire (1997), we *postulate* that under a martingale measure the stock price satisfies the following equation:

$$dS(t) = \eta(t, S(t))S(t)\, dW^*(t), \qquad S(0) = S_0.$$

The coefficient $\eta(t, S(t))$ is called a *local volatility function*.

We assume that the local volatility function satisfies another stochastic equation. We need to introduce a new concept here.

1.5.1 Correlated Wiener Processes

We follow (Musiela and Rutkowski, 2005, Subsection 7.1.9). Let d and n be two positive integers. Let $\tilde{W}_1(t), \ldots, \tilde{W}_d(t)$ be d independent copies of a Wiener process. Let $\mathbf{B}^i(t)$ be n $\mathbb{R}^d$-valued progressively measurable processes such that $B^i(t) \neq \mathbf{0}$ for every $t \in [0, \infty)$ and $i = 1, \ldots, n$. It is possible to prove that the n stochastic processes given by

$$W_i(t) = \sum_{j=1}^{d} \int_0^t \frac{B^i_j(t)}{\|\mathbf{B}^i(t)\|} \, d\tilde{W}_j(u)$$

are Wiener processes. Moreover, the time t correlation matrix of the $\mathbb{R}^n$-valued stochastic process with components $W_i(t)$ has entries

$$\rho_{ij}(t) = \frac{\sum_{k=1}^{d} B^i_k(t) B^j_k(t)}{\|\mathbf{B}^i(t)\| \cdot \|\mathbf{B}^j(t)\|} \in [-1, 1].$$

1.5.2 Local Stochastic Volatility Models

Following Pagliarani and Pascucci (2017), we assume that under a martingale probability measure, the market model is described by a $\mathbb{R}^d$-valued stochastic process $(S(t), Y_2(t), \ldots, Y_d(t))^\top$ that satisfies the following system of stochastic differential equations:

$$\begin{aligned} dS(t) &= \eta_1(t, S(t), \mathbf{Y}(t)) S(t) \, dW^*_1(t), & S(0) &= s, \\ dY_i(t) &= \mu_i(t, S(t), \mathbf{Y}(t)) \, dt + \eta_i(t, S(t), \mathbf{Y}(t)) \, dW^*_i(t), & \mathbf{Y}(0) &= \mathbf{y}, \end{aligned} \tag{1.7}$$

where $2 \leq i \leq d$, $S(t)$ is the price of a risky security, s is a deterministic positive number, $\mathbf{Y}(t)$ is a vector with components $Y_i(t)$, $\mathbf{y} \in \mathbb{R}^{d-1}$ is a deterministic vector, and the time t correlation matrix of the $\mathbb{R}^d$-valued stochastic process with components $W^*_i(t)$ has entries

$$\rho_{ij}(t, S(t), \mathbf{Y}(t)) \in [-1, 1].$$

In what follows, we refer to this model as a *local stochastic volatility model*.

1.5.3 The Gatheral Model

As an example, we consider the double-mean-reverting market model proposed by Gatheral (2008). In a subsequent publication Bayer et al. (2013), it is given as follows.

$$\begin{aligned} dS(t) &= \sqrt{v(t)}S(t)\,dW_1^*(t), \\ dv(t) &= \kappa_1(v'(t)-v(t))\,dt+\xi_1 v^{\alpha_1}(t)\,dW_2^*(t), \\ dv'(t) &= \kappa_2(\theta-v'(t))\,dt+\xi_2 v'^{\alpha_2}(t)\,dW_3^*(t), \end{aligned} \tag{1.8}$$

and the time t correlation matrix of the $\mathbb{R}^3$-valued stochastic process with components $W_i^*(t)$ has entries

$$\rho_{ij}(t, S(t), \mathbf{Y}(t)) = \rho_{ij} \in [-1, 1].$$

Compare this system with (1.7). We see that $d = 3$, $Y_2(t) = v(t)$, $Y_3(t) = v'(t)$,

$$\begin{aligned} \eta_1(v(t)) &= \sqrt{v(t)}, & \mu_1 &= 0, \\ \eta_2(v(t)) &= \xi_1 v^{\alpha_1}(t), & \mu_2(v(t), v'(t)) &= \kappa_1(v'(t)-v(t)), \\ \eta_3(v'(t)) &= \xi_2 v'^{\alpha_2}(t), & \mu_3(v'(t)) &= \kappa_2(\theta-v'(t)). \end{aligned} \tag{1.9}$$

The reason why we choose this model, was described by Bayer et al. (2013) as follows

> Thus variance mean-reverts to a level that itself moves slowly over time with the state of the economy.

1.6 Implied Volatility

The European call options are traded on the market, however the stock's volatility, σ, is not directly observable. A possible solution to this problem is as follows. It is well known that the Black–Sholes price (1.5) with $r = 0$ satisfies the following boundary value problem for the *Black–Scholes partial* *differential equation*

$$\begin{aligned} \frac{\partial C(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C(S,t)}{\partial S^2} &= 0, \\ \lim_{t \uparrow T} C(S,t) &= \max\{0, S-K\} \end{aligned} \tag{1.10}$$

with $(S,t) \in (0,\infty) \times (0,T)$. Berestycki et al. (2002) describe a possible solution as follows

> . . . it is common practice to start from the observed prices and invert the closed-form solution to (2) in order to find that constant σ — called *implied volatility* — for which the solution to (2) agrees with the market price at today's value of the stock.

Note that their Equation (2) is our Equation (1.10).

The literature on implied volatility for the options is huge and exploits several range of mathematical methods. The first appearance of implied volatility in history goes back to 1976 when it was first mentioned under the name "implied standard deviation" by Latane and Rendleman (1976). In their paper "standard deviations of stock price ratios implied in option prices", they suggest for the first time that implied volatility should be derived from traded options in the market:

> "If the assumptions underlying the Black and Scholes model were completely valid and the option market were completely efficient, then at any point in time all options on a particular stock would be priced with the same monthly standard deviation. As a practical matter, however, it is not likely that this will be the case, even in a market which is highly efficient. This is due to the fact that some options are more dependent upon a precise specification of the standard deviation than other".

Focusing to address the unrealistic assumption of constant volatility for pricing options in Black–Scholes model, the interest to investigate and identify the market implied volatility has grown substantially in both academia and industry. Many well-known models have been obtained to capture empirical features of financial prices. Among these models, models proposed by Cox and Ross (1976), Vasicek (1977), Hull and White (1990), Heston (1993), Hagan et al. (2002), Henry-Labordere (2008), Benhamou et al. (2010a), and Fouque et al. (2011). For an extensive comparison and overview of volatility models, we refer the reader to Gatheral (2011).

As volatility modelling evolves and takes shapes over the years, more studies emerged. Christoffersen et al. (2009) propose using a two-factor stochastic volatility model in response to the deficiencies of the single-factor regarding the ability to demonstrate large independent fluctuations in the level and the slope of volatility smirk. They conclude that the addition of volatility factors to an existing framework improves the model's flexibility to capture the volatility term structure. Merville and Pieptea (1989) find that market volatility follows a mixed mean-reverting diffusion with noise process and suggested that volatility is strong mean reverting. In recent studies, Lorig et al. (2017b) consider a general class of multi-factor local-stochastic volatility models with hybrid dynamics.

Another aspect of the evolution of implied volatility focused on the versatility of its modelling. A robust model of implied volatility has to capture the dynamics and the characteristics of the implied volatility surface along various strike prices and expiration dates. Moreover, calibrating models to market data and the stability of parameters and the quality of the fit are essentials.The calibration in models which lack closed-form formula for implied volatility is often time consuming or might lead to numerical instabilities. To overcome this dilemma, several results have been obtained to translate models formulations to explicit implied volatility expansions via techniques from perturbation theory, heat kernel, PDEs, large deviation and Malliavin calculus. We review some of these approaches.

Hagan and Woodward (1999) use singular perturbation methods to obtain an expansions formula for implied volatility for general local volatility models. In addition, regular perturbation methods as well as Fourier analysis were used by Jacquier and Lorig (2013) to derive an expansion for implied volatility for the same class of models. Both methods were combined and extended to general stochastic volatility models by Fouque et al. (2016). Large deviation methods have been employed by Forde and Jacquier (2009) to obtain small-time asymptotic behaviour of implied volatility under Heston model. Henry-Labordère (2005) derive a general asymptotic implied volatility at the first-order for any local stochastic volatility model using the heat kernel expansion on a Riemann manifold. Another contribution is due to Watanabe (1987) as well as the recent work by Benhamou et al. (2010b) who obtain closed-form approximation for implied volatility in local stochastic volatility sitting using Malliavin calculus.

Finally, we mention the recent results obtained by Lorig et al. (2017b) whose approach combine technique from perturbation theory with Dyson series to obtain approximations for partial differential equations. These authors calculated asymptotic expansions for implied volatilities and their derivatives which asymptotically converge to the exact values within a parabolic region in the space of log-strike and time to maturity. Their methodology builds upon a series of papers including Pagliarani and Pascucci (2012) in which they obtain asymptotic formulas for implied volatility for scalar diffusions. The approach was extended later for scalar Levy-type processes in Pagliarani et al. (2013) and Lorig et al. (2015).

1.6.1 Motivation

Beginning from the idea of Bachelier presented in his doctoral thesis at the Ecole Polytechnique in Paris, 1900 of using Brownian motion to model uncertainty in price behaviour, Black and Scholes (1973) assume that the asset price follows a geometric Brownian motion with constant drift and volatility. According to Black and Scholes model, the asset price $S(t), t > 0$, is governed by the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)\, dW(t),$$

where μ and σ are assumed to be constant and denote the drift and volatility receptively, $W(t), t > 0$ is a standard Brownian motion. It was the first model that produces a simple closed-form formula which computes the prices of European call and put options. In 1973 Merton contributed to the result of Black and Scholes and enhanced the option formula. Scholes and Merton received the Nobel prize for their work in economics in 1997, unfortunately Black passed away in 1995. On the 19th of October 1987 the world woke up to witness a sudden, sever and largely unexpected stock market crash, it was known as the **Black Monday**. Worldwide losses were estimated at 1.71$ trillion.

The crash causes unprecedented patterns for the implied volatility, the new patterns that arise in options pricing resemble smiles (skews) shapes. Governments, financial markets and banks impose new regulations and restrictions to limit the impact of the crash. The crash also open the door for new researches and ideas in the field of mathematical finance.

The ability of the Black–Scholes model to provide adequate prices for options was questionable. Moreover, experimental studies that follow the 1987 crash suggest that the assumption of the Black–Scholes model that underlying returns being normally distributed with constant volatility does not hold. The option prices observed in the financial markets show that volatility is non-constant quantity and the behaviour of the volatility can be described by a stochastic mean-reverting process.

Trying to overcome the inadequacy of the Black and Scholes model by assuming that the volatility is modelled as a stochastic process, stochastic volatility model were born. Hull and White are the first authors to introduce the stochastic volatility model in mathematical finance in Hull and White (1987). The dynamics of asset price in Hull and White model follows:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{v(t)}S(t)\, dW_1^*(t), \\ dv(t) &= \theta v(t)dt + \xi v(t)\, dW_2^*(t), \end{aligned}$$

where $v(t), t > 0$ and μ is the stochastic variance and the drift of the asset price $S(t), t > 0$ at time t, and θ and ξ are the drift and the volatility coefficients of the variance respectively, $dW_1^*(t)$ and $dW_2^*(t)$ are two Brownian motions with correlation $\rho = [-1, 1]$. The parameters θ, ξ and μ are real constants and $\theta < 0$. The issue of this model is that it is inaccurate when the variance is stochastic. Also, the assumption that $\rho = 0$ is not supported by observations of the prices in the financial market.

In response to the weakness in Hull and White model, the Stein and Stein model was proposed Stein and Stein (1991). It assumes that the underlying asset follows:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma(t)S(t)\, dW_1^*(t), \\ d\sigma(t) &= \eta(\theta - \sigma(t))dt + \xi\, dW_2^*(t), \end{aligned}$$

where μ, η, θ, and ζ are fixed constants and when $\eta > 0$, the stochastic volatility $\sigma(t), t > 0$ is governed by Ornstein-Uhlenbeck process with a tendency to revert back to a long-run average level θ. The model assumes that the volatility is uncorrelated with the spot asset and that makes it unable to capture important skewness effects that appear in such correlation.

Since its introduction in 1993, the Heston model has become one of the most popular stochastic volatility models. In Heston (1993) the model assumes that the asset price $S(t), t > 0$ follows the following process:

$$
\begin{aligned}
dS(t) &= \mu S(t)dt + \sqrt{v(t)}S(t)\, dW_1^*(t), \\
dv(t) &= \kappa(\theta - v(t))dt + \xi\sqrt{v(t)}\, dW_2^*(t),
\end{aligned}
$$

where the variance process $v(t)$ reverts to long-run level of θ, κ is the rate of mean-reversion, and ξ is the volatility of the volatility coefficient. The model depends on real constants μ, θ, κ and ξ and κ is positive. The variance in Heston model follows a Cox–Ingersoll–Ross process. The introduction of the model in mathematical finance was revolutionary due to the fact that the model admit a closed form solution for European call (put) option. However, the Heston model suffers when it comes to predicting the option prices for short term as the model fails to capture the high implied volatility, see Jäckel (2004).

Since the pioneering work of the previous authors, many other stochastic volatility models have been introduced and it is not possible to mention all of them. However, it is important to mention the sophisticated work by Christoffersen et al. 2009. In Christoffersen et al. (2009) they presented an empirical studies "The shape and term structure of the index option smirk: why multi-factor stochastic volatility models work so well". Christoffersen et al. (2009) propose using a two-factor stochastic volatility model in response to the deficiencies of the single-factor regarding the ability to demonstrate large independent fluctuations in the level and the slope of volatility smirk. They conclude that the addition of volatility factors are needed to improve the model's flexibility to capture the volatility term structure.

Christoffersen et al. (2009) suggest that the price process is determined by two factors stochastic volatility process:

$$
\begin{aligned}
dS(t) &= \mu S(t)dt + \sqrt{v_1(t)}S(t)\,dW_1^*(t) + \sqrt{v_2(t)}S(t)\,dW_2^*(t),\\
dv_1(t) &= \kappa_1(\theta_1 - v_1(t))dt + \xi_1\sqrt{v_1(t)}\,dW_3^*(t)\\
dv_2(t) &= \kappa_2(\theta_2 - v_2(t))dt + \xi_2\sqrt{v_2(t)}\,dW_4^*(t),
\end{aligned}
$$

where the variance of the stock return is the sum of v_1 and v_2. Moreover, it can be the sum of two uncorrelated factors that might be correlated individually with the stock returns. For instance, $W_1^*(t)$ and $W_3^*(t)$ has correlation ρ_1, and $W_2^*(t)$ and $W_4^*(t)$ has correlation ρ_2, but all other correlations are zero. More specifically, correlations are zero for the following pairs: $(W_1^*(t),\ W_4^*(t))$, $(W_2^*(t),\ W_3^*(t))$, $(W_1^*(t),\ W_2^*(t))$ and $(W_3^*(t),\ W_4^*(t))$.

Thus, pricing options under double stochastic volatility model gains some popularity in the recent years. We choose the Gatheral double stochastic volatility model which was first proposed in Gatheral (2008). The model is given by system of stochastic differential equations 1.8. There is no analytical solution for European options nor an implied volatility under this model. Moreover, the model proves to be versatile in term of pricing both options on SPX and VIX with the market in Gatheral (2008) and in a subsequent publication Bayer et al. (2013) but since there was no closed-form solution for European options, the calibration was not easy. Lastly, we have the result by Pagliarani and Pascucci (2017) to calculate the implied volatility for general class of double stochastic volatility model, and the Gatheral model can be a special case of their model. Thus, equipped with the result of Pagliarani and Pascucci (2017), we conduct studies to calculate the asymptotic expansions of implied volatility for European options under Gatheral model up to third order. We also check the accuracy of the expansions against the Monte-Carlo benchmark and calibrate the model parameters to real market data.

1.7 Asymptotic Expansions

There are many problems in mathematics which are hard to solve, where the governing equations contain many variables that are nonlinear and do not have exact analytical solutions. Therefore, we are forced to resort to either approximations or numerical solutions or combinations of both. Among approximation methods, one method stands out; *asymptotic expansions*.

The modern theory of asymptotic expansion was proposed by Stieltjes (1886) and Poincaré (1886). According to this theory, it is usually sufficient to represent the solution and give a very accurate approximation to a *complicated* function by the first few terms of an asymptotic expansion, no more than two or three terms in most cases.

1.7.1 Ordering Symbols O, $\sim$, and o

The order of magnitude of functions is fundamental in asymptotic analysis. The exact mathematical meaning of the order of magnitude is provided by the so called **Bachmann–Landau notations**.

Following Erdélyi (1956), let the variable x ranges over a set R and x_0 is the base point of R not necessarily belonging to R. $f(x)$, $g(x)$ stand for real or complex valued functions defined when x is in R.

One writes

$$f(x) = O(g(x)),$$

in R as $x \longrightarrow x_0$ if $g(x) \neq 0$ and $f(x)/g(x)$ is bounded as $x \longrightarrow x_0$ in R.

It means that $f(x)$ is order big O of $g(x)$ or $f(x)$ is asymptotically bounded by $g(x)$, as $x \longrightarrow x_0$.

If the $\lim_{x\longrightarrow +x_0} f(x)/g(x)$ exists and is equal to 1 as we approaching a point of interest, then

$$f(x) \sim g(x).$$

It means that $f(x)$ is asymptotically equal to $g(x)$ as $x \longrightarrow x_0$.

For the "little o" one writes

$$f(x) = o(g(x)),$$

in R as $x \longrightarrow x_0$ if $f(x) \neq 0$ and $f(x)/g(x) \longrightarrow 0$ as $x \longrightarrow x_0$.

It means that $f(x)$ is asymptotically smaller than $g(x)$ or $f(x)$ tends to 0 faster than $g(x)$ does as $x \longrightarrow x_0$.

1.7.2 Asymptotic Sequences

Definition 22. Let $\phi_1(x), \phi_2(x)$,....., denote a finite or infinite sequence of functions abbreviated as $\{\phi_j(x)\}$. The sequence of functions $\{\phi_j(x)\}$ is called an *asymptotic sequence* for $x \to x_0$ in R if for each j, $\{\phi_j(x)\}$ is defined in R and

$$\phi_{j+1}(x) = o(\phi_j(x)) \quad \text{as} \quad x \to x_0$$

in R.

If the sequence is infinite and $\phi_{j+1}(x) = o(\phi_j(x))$ uniformly in j, then $\{\phi_j(x)\}$ is considered to be an *asymptotic sequence uniformly* in j. If the $\{\phi_j(x)\}$ depends on parameters and $\phi_{j+1}(x) = o(\phi_j(x))$, then $\{\phi_j(x)\}$ is considered to be an *asymptotic sequence uniformly* in the parameters.

1.7.3 Asymptotic Expansion

Definition 23. Let $\{\phi_j(x)\}$ be an asymptotic sequence for $x \to x_0$ in R; $f(x)$ is a numerical function of x defined on R; and a is a constant (i.e., independent of x). The formal series $\sum a_j\phi_j(x)$ not necessarily convergent, is said to be an *asymptotic expansion* to N terms of $f(x)$ in the form of Poincaré as $x \to x_0$ if :

$$f(x) = \sum_{j=1}^{N} a_j\phi_j(x) + o(\phi_N) \text{ as } x \to x_0. \tag{1.11}$$

It means the partial sum $\sum_{j=1}^{N} a_j\phi_j(x)$ is an approximation to $f(x)$ with an error $o(\phi_N)$ as $x \to x_0$. Sometimes this is written as

$$f(x) - \sum_{j=1}^{N} a_j\phi_j(x) \ll (\phi_N) \; as \; x \to x_0,$$

which is read as the error or the reminder is much less the last term in the sum.

If such an asymptotic expansion exists, it is unique and It follows that the coefficients in asymptotic expansion to N terms can be determined using the following recurrence formula

$$a_m = \lim_{x \to +x_0} \{[f(x) - \sum_{n=1}^{m-1} a_j\phi_j(x)]/\phi_m(x)\} \; m = 1, ..., N$$

We write

$$f(x) \sim \sum a_j\phi_j(x),$$

to N terms as $x \to x_0$ in R, if the function $f(x)$ possesses an asymptotic expansion. It is frequently written as $f(x) \sim a_j\phi_j(x)$ which means that $f(x)/\phi_j(x)$ tends to a_j as $x \to x_0$

Theorem 7. *If we have $N + 1$ functions, $f(x), \phi_1(x), ..., \phi_N(x)$ defined in R. It follows that $\{\phi_j\}$ is an asymptotic sequence for $x \to x_0$ and $\sum a_j\phi_j$ is an asymptotic expansion for N terms of $f(x)$ as $x \to x_0$* provided (1.11) *holds and $a_m \neq 0$ for $m = 1, ..., N$.*

Remark 2. The choice of the asymptotic sequence obviously effects the form of an asymptotic expansion. Furthermore, the same function might have different asymptotic expansions. For example, as $x \to \infty$,

1. $$\frac{1}{x-1} \sim \sum_{1}^{\infty} \frac{1}{x^n}.$$

2. $$\frac{1}{x-1} \sim \sum_{1}^{\infty} \frac{x+1}{x^{2n}},$$

see Copson (2004, p. 7).

In addition, two different functions might have the same asymptotic expansion.

1.8 A Survey of the Rest of the Thesis

This thesis is divided into two main parts: part one is devoted to the thesis and part two contains four articles. In the first part of the thesis we have four chapters. The current chapter, **Chapter 1** is about a survey of literature, we describes some definitions and theorems and introduce generally some major concepts to be used in the later chapters. We also include several enlightening examples and description of the previous work that related to stochastic volatility models to motivate the work done in latter chapters.

Chapter 2 is the base of the Paper A and B and investigates the theoretical part of the thesis where we first have to prove that the Gatheral model satisfies the assumptions by Pagliarani and Pascucci (2017). Then, based on the results by Pagliarani and Pascucci (2017), we use asymptotic expansions method to calculate option price under the Gatheral model. Using the results in Chapter 2, we go through heavy calculations in **Chapter 3** to compute the asymptotic expansions of the implied volatility up to third order. This chapter is based on Paper A and B. **Chapter 4** is based on paper C and paper D and it briefly describes the practical part of the work and summarises the results.

Chapter 2

The Asymptotic Expansion of the Option Price

2.1 Introduction

In order to derive explicit expansions for that the implied volatility, it is essential to expand the call prices around a Black-Scholes price. This chapter is dedicated to the derivation of option prices under the Gatheral model. However, one need to first check that the Gatheral model satisfies the assumptions given in Pagliarani and Pascucci (2017), then expand operator $\mathcal{A}_t(\mathbf{z})$ by replacing the coefficients $a_{ij}(t, s, y_2, y_3)$ and $a_i(t, s, y_2, y_3)$ with their Taylor series around $\bar{\mathbf{z}} = (s_0, v_0, v_0')$.

2.2 The Backward Kolmogorov Equation

Following Pagliarani and Pascucci (2017, Equation (3.1)), introduce the following function:

$$\begin{aligned} &V(t, x, y_2, y_3, T, K) \\ &\quad = \mathsf{E}^*[\max\{S(T) - K, 0\} \mid \mathfrak{F}_t, S(t) = x, v(t) = y_2, v'(t) = y_3]. \end{aligned} \tag{2.1}$$

In what follows, we use the symbols x, y_2, y_3 as formal variables during calculations. In the formulation of results, we substitute the values from

Equation 2.1: $x = S(t)$, $y_2 = v(t)$, $y_3 = v'(t)$.

On the one hand, by the Fundamental Theorem of Financial Engineering, the time t no-arbitrage price of a European call option with strike price K and maturity T is equal to $V(t, S(t), v(t), v'(t), T, K)$.

Remark 3. If the spot risk-free interest rate is a non-zero, but deterministic function of time, say $r(t)$, then the function (2.1) is replaced with another one:

$$\begin{aligned}\tilde{V}(t, \tilde{s}, y_2, y_3, T, K) = \mathsf{E}^*[\exp(-\int_t^T r(u)\,du) \\ \times \max\{\tilde{S}(T) - K, 0\} \mid \mathfrak{F}_t, \tilde{S}(t) = \tilde{s}, v(t) = y_2, v'(t) = y_3],\end{aligned}$$

where

$$d\tilde{S}(t) = dS(t) + r(t)\,dt.$$

In this case, we just change a variable

$$V(t, \tilde{s}, y_2, y_3, T, K) = \exp\left(\int_t^T r(u)\,du\right)\tilde{V}(t, \tilde{s}, y_2, y_3, T, K).$$

On the other hand, by (Pagliarani and Pascucci, 2017, Theorem 2.6, Remark 2.9), the function (2.1) satisfies the backward Kolmogorov equation, which has the form

$$\left(\frac{\partial}{\partial t} + \overline{\mathcal{A}}_t(s, y_2, y_3)\right) v(t, s, y_2, y_3, T, K) = 0. \tag{2.2}$$

By Pagliarani and Pascucci (2017, Equation (2.8)), the linear differential operator $\overline{\mathcal{A}}_t$ has the form

$$\overline{\mathcal{A}}_t(\mathbf{z}) = \frac{1}{2}\sum_{i,j=1}^{3} \overline{a}_{ij}(t, \mathbf{z})\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^{3} \overline{a}_i(t, \mathbf{z})\frac{\partial}{\partial z_i}, \tag{2.3}$$

where $\mathbf{z} = (s, y_2, y_3)^\top \in \mathbb{R}^3$, and where the entries $\overline{a}_{ij}(t, \mathbf{z})$ of a symmetric matrix and the components $\overline{a}_i(t, \mathbf{z})$ of a vector are given by Pagliarani and Pascucci (2017, p. 671):

$$\begin{aligned} &\overline{a}_1(t, \mathbf{z}) = 0, && \overline{a}_i(t, \mathbf{z}) = \mu_i(t, \mathbf{z}), \\ &\overline{a}_{11}(t, \mathbf{z}) = \eta_1^2(t, \mathbf{z})s^2, && \overline{a}_{1i}(t, \mathbf{z}) = \rho_{1i}\eta_1(t, \mathbf{z})\eta_i(t, \mathbf{z})s, \\ &\overline{a}_{ij}(t, \mathbf{z}) = \rho_{ij}\eta_i(t, \mathbf{z})\eta_j(t, \mathbf{z}), && \end{aligned}$$

for any $i, j = 2, 3$. Equation (1.9) gives the following nonzero coefficients:

$$
\begin{aligned}
\overline{a}_2(y_2, y_3) &= \kappa_1(y_3 - y_2), & \overline{a}_3(y_3) &= \kappa_2(\theta - y_3), \\
\overline{a}_{11}(s, y_2) &= s^2 y_2, & \overline{a}_{12}(s, y_2) &= \rho_{12}\xi_1 s y_2^{\alpha_1+1/2}, \\
\overline{a}_{13}(s, y_2, y_3) &= \rho_{13}\xi_2 s y_2^{1/2} y_3^{\alpha_2}, & \overline{a}_{22}(y_2) &= \xi_1^2 y_2^{2\alpha_1}, \\
\overline{a}_{23}(y_2, y_3) &= \rho_{23}\xi_1\xi_2 y_2^{\alpha_1} y_3^{\alpha_2}, & \overline{a}_{33}(y_3) &= \xi_2^2 y_3^{2\alpha_2}.
\end{aligned}
$$

The operator $\overline{\mathcal{A}}_t(\mathbf{z})$ in the backward Kolmogorov equation (2.2) takes the form

$$
\begin{aligned}
\overline{\mathcal{A}}_t(\mathbf{z}) &= \frac{1}{2} s^2 y_2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}\xi_1^2 y_2^{2\alpha_1} \frac{\partial^2}{\partial y_2^2} + \frac{1}{2}\xi_2^2 y_3^{2\alpha_2} \frac{\partial^2}{\partial y_3^2} \\
&\quad + \rho_{12}\xi_1 s y_2^{\alpha_1+1/2} \frac{\partial^2}{\partial s \partial y_2} + \rho_{13}\xi_2 s y_2^{1/2} y_3^{\alpha_2} \frac{\partial^2}{\partial s \partial y_3} \\
&\quad + \rho_{23}\xi_1\xi_2 y_2^{\alpha_1} y_3^{\alpha_2} \frac{\partial^2}{\partial y_2 \partial y_3} + \kappa_1(y_3 - y_2)\frac{\partial}{\partial y_2} + \kappa_2(\theta - y_3)\frac{\partial}{\partial y_3}.
\end{aligned}
$$

Now we check that the the Gatheral model satisfies the assumptions given in Pagliarani and Pascucci (2017). We start from their Assumption 2.4, part (i): the coefficients $\overline{a}_{ij}$ and $\overline{a}_i$ belong to a parabolic Hölder space $C_P^{N,1}([0, T_0) \times D)$, where D is a domain in $\mathbb{R}^3$, and $N \geq 2$ is an integer, see Chou and Zhu (2001). We do that for the coefficient $\overline{a}_{11}(s, y) = s^2 y$, a proof for the remaining cases is similar.

First, we need to prove that there is a domain $D \subseteq \mathbb{R}^3$ such that for any $s, t \in (0, T_0)$ and $\mathbf{x}$, $\mathbf{y} \in D$ we have

$$
[\overline{a}_{11}] = \sup_{(s,\mathbf{x}),(t,\mathbf{y}) \in [0,T_0)\times D} \frac{|\overline{a}_{11}(s, \mathbf{x}) - \overline{a}_{11}(t, \mathbf{y})|}{(|s - t| + \|\mathbf{x} - \mathbf{y}\|^2)^{1/2}} < \infty.
$$

Define

$$
D = \{\, \mathbf{x} \in \mathbb{R}^3 : r < \|\mathbf{x}\| < R \,\}.
$$

From the point of view of financial applications, this means that the asset price and both volatilities are bounded and separated from 0 on a time interval

$[0, T_0)$. Estimate the middle term:

$$\frac{|\overline{a}_{11}(s,\mathbf{x}) - \overline{a}_{11}(t,\mathbf{y})|}{(|s-t| + \|\mathbf{x}-\mathbf{y}\|^2)^{1/2}} = \frac{|s^2 x_2 - t^2 y_2|}{(|s-t| + |x_2 - y_2|^2)^{1/2}}$$
$$\leq \frac{T_0^2}{|x_2 - y_2|}|x_2 - y_2| = T_0^2 < \infty.$$

Second, we need to prove that

$$\left[\frac{\partial \overline{a}_{11}(s,\mathbf{x})}{\partial s}\right] < \infty.$$

Indeed, we have $\frac{\partial \overline{a}_{11}(s,\mathbf{x})}{\partial s} = 2sy_2$, and the above middle term takes the form

$$\frac{|2sx_2 - 2ty_2|}{(|s-t| + |x_2 - y_2|^2)^{1/2}} \leq \frac{2T_0|x_2 - y_2|}{|x_2 - y_2|} < \infty.$$

Last, we have to prove that

$$\sum_{i+2j \leq 2} \max_{(s,\mathbf{x}) \in [0,T_0) \times D} \left|\frac{\partial^{i+j} \overline{a}_{11}(s,\mathbf{x})}{\partial x^i \partial s^j}\right| + \sum_{i+2j=2} \left[\frac{\partial^{i+j} \overline{a}_{11}(s,\mathbf{x})}{\partial x^i \partial s^j}\right] < \infty.$$

Estimating each remaining term separately, we have

$$\max_{(s,\mathbf{x}) \in [0,T_0) \times D} |\overline{a}_{11}(s,\mathbf{x})| \leq T_0^2 R < \infty,$$
$$\max_{(s,\mathbf{x}) \in [0,T_0) \times D} \left|\frac{\partial \overline{a}_{11}(s,\mathbf{x})}{\partial x_2}\right| \leq 2T_0^2 < \infty,$$
$$\max_{(s,\mathbf{x}) \in [0,T_0) \times D} \left|\frac{\partial^2 \overline{a}_{11}(s,\mathbf{x})}{\partial x_2^2}\right| = 0 < \infty,$$
$$\max_{(s,\mathbf{x}) \in [0,T_0) \times D} \left|\frac{\partial \overline{a}_{11}(s,\mathbf{x})}{\partial s}\right| \leq 2RT_0 < \infty.$$

Part (ii) of Assumption 2.4 is as follows: the operator $\overline{\mathcal{A}}_t(\mathbf{z})$ is strongly elliptic in the sense that there exist $M > 0$ and $\varepsilon \in (0,1)$ such that for all $t \in [0, T_0)$, $\mathbf{z} \in D$, and $\boldsymbol{\zeta} \in \mathbb{R}^3$ we have

$$\varepsilon M \|\boldsymbol{\zeta}\|^2 \leq \sum_{i,j=1}^{3} \overline{a}_{ij}(t,\mathbf{z}) \zeta_i \zeta_j \leq M \|\boldsymbol{\zeta}\|^2.$$

In other words: the norm of the positive-definite matrix $\overline{a}_{ij}(t, \mathbf{z})$ is separated from 0 and finite. This is checked by direct calculations.

By (Pagliarani and Pascucci, 2017, Lemma 2.3), Assumption 2.1 is satisfied if Assumption 2.4 is satisfied *and* the coefficients of the system (1.8) are continuous and bounded on $[0, T_0] \times D$. The latter is obvious.

Finally, we need to check Assumption 2.5: the solution to the system (1.8) is a *Feller process* on D, that is, for any $T \in (0, T_0)$, and for any continuous function $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ with compact support, the function $(0, T) \times D \to \mathbb{R}$, $(t, \mathbf{z}) \mapsto (\mathbf{T}_{t,T}\varphi)(\mathbf{z})$ is continuous, where

$$(\mathbf{T}_{t,T}\varphi)(\mathbf{z}) = \int_{\mathbb{R}^3} \varphi(\boldsymbol{\zeta})\overline{p}(t, \mathbf{z}; T, \mathrm{d}\boldsymbol{\zeta}),$$

and $\overline{p}(t, \mathbf{z}; T, \mathrm{d}\boldsymbol{\zeta})$ is the transition probability function for the solution to the system (1.8).

We use (Ethier and Kurtz, 1986, Chapter 8, Theorem 1.4). This result says that under some conditions, the solution to a system of stochastic differential equations is a Feller process. We check the above conditions one by one.

The condition $d \geq 2$ is trivially satisfied. By construction, our set D is bounded, connected, and open. To check the next condition, we need a notation.

Let $\mathbf{x}_0$ be a point on the boundary ∂D of the set D. According to Ethier and Kurtz (1986), a map $\mathbb{R}^d \to \mathbb{R}^d$, $\mathbf{x} \mapsto \mathbf{y} = U(\mathbf{x} - \mathbf{x}_0)$, is called a *local Cartesian coordinate system with origin at* $\mathbf{x}_0$ if U is an orthogonal matrix and the outer normal to ∂D at $\mathbf{x}_0$ is mapped to the nonnegative y_d axis. For a positive integer m and a real number $\mu \in [0, 1]$, the surface ∂D is of class $C^{m,\mu}$ if there exists $\rho > 0$ such that for every $\mathbf{x}_0 \in \partial D$ the intersection of ∂D with an open ball $B(\mathbf{x}_0, \rho)$ of radius ρ centred at $\mathbf{x}_0$ is a connected surface of the form $y_d = v(y_1, \dots, y_{d-1})$, the function v is m times continuously differentiable on the closure $\overline{P}$ of the projection P of the set $\partial D \cap B(\mathbf{x}_0, \rho)$ onto the hyperplane $y_d = 0$, and their partial derivatives of orders $0, 1, \dots, m$ satisfy a Hölder condition with exponent μ with some fixed u_0:

$$\sup_{0<u\leq u_0} \{\sup_{\mathbf{y}\in\overline{P}} v(\mathbf{y}) - \inf_{\mathbf{y}\in\overline{P}} v(\mathbf{y})\} < \infty.$$

In the latter case we write $v \in C^{m,\mu}(\overline{P})$.

One may check by direct calculations that ∂D is of class $C^{2,1}$ and the coefficients of the system (1.8) belong to $C^{0,1}$. The last remaining condition of (Pagliarani and Pascucci, 2017, Lemma 2.3) is as follows:

$$\inf_{\mathbf{x}\in D}\inf_{\|\theta\|=1}\sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\theta_i\theta_j > 0,$$

and is trivial to check. Thus, the solution to the system (1.8) is indeed a Feller process.

The next step is switching to logarithmic variables. The differential operator $\overline{\mathcal{A}}_t(\mathbf{z})$ given by Equation (2.3) becomes the operator

$$\mathcal{A}_t(\mathbf{z}) = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}(t,\mathbf{z})\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^{3} a_i(t,\mathbf{z})\frac{\partial}{\partial z_i},$$

where this time $\mathbf{z} = (x, \mathbf{y}^\top)^\top = (x, y_2, y_3)^\top \in \mathbb{R}^3$, the functions $a_{ij}(t,\mathbf{z})$ and $a_i(t,\mathbf{z})$ are given by Pagliarani and Pascucci (2017, p. 680):

$$\begin{aligned}
a_{11}(t,x,\mathbf{y}) &= e^{-2x}\overline{a}_{11}(t,e^x,\mathbf{y}), \quad & a_1(t,x,\mathbf{y}) &= -\frac{1}{2}e^{-2x}\overline{a}_{11}(t,e^x,\mathbf{y}),\\
a_{1i}(t,x,\mathbf{y}) &= e^{-x}\overline{a}_{1i}(t,e^x,\mathbf{y}), \quad & a_{ij}(t,x,\mathbf{y}) &= \overline{a}_{ij}(t,e^x,\mathbf{y}),\\
a_i(t,x,\mathbf{y}) &= \overline{a}_i(t,e^x,\mathbf{y}),
\end{aligned}$$

for any i, $j = 2, 3$. This gives

$$\begin{aligned}
a_1(y_2) &= -\frac{1}{2}y_2, & a_2(y_2,y_3) &= \kappa_1(y_3 - y_2),\\
a_3(y_3) &= \kappa_2(\theta - y_3), & a_{11}(y_2) &= y_2,\\
a_{12}(y_2) &= \rho_{12}\xi_1 y_2^{\alpha_1+1/2}, & a_{13}(y_2,y_3) &= \rho_{13}\xi_2 y_2^{1/2}y_3^{\alpha_2},\\
a_{22}(y_2) &= \xi_1^2 y_2^{2\alpha_1}, & a_{23}(y_2,y_3) &= \rho_{23}\xi_1\xi_2 y_2^{\alpha_1}y_3^{\alpha_2},\\
a_{33}(y_3) &= \xi_2^2 y_3^{2\alpha_2},
\end{aligned} \tag{2.4}$$

and the operator $\mathcal{A}_t(\mathbf{z})$ becomes

$$\begin{aligned}\mathcal{A}_t(\mathbf{z}) = {} & \frac{1}{2}y_2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\xi_1^2 y_2^{2\alpha_1}\frac{\partial^2}{\partial y_2^2} + \frac{1}{2}\xi_2^2 y_3^{2\alpha_2}\frac{\partial^2}{\partial y_3^2} \\ & + \rho_{12}\xi_1 y_2^{\alpha_1+1/2}\frac{\partial^2}{\partial x\partial y_2} + \rho_{13}\xi_2 y_2^{1/2} y_3^{\alpha_2}\frac{\partial^2}{\partial x\partial y_3} \\ & + \rho_{23}\xi_1\xi_2 y_2^{\alpha_1} y_3^{\alpha_2}\frac{\partial^2}{\partial y_2\partial y_3} - \frac{1}{2}y_2\frac{\partial}{\partial x} + \kappa_1(y_3 - y_2)\frac{\partial}{\partial y_2} \\ & + \kappa_2(\theta - y_3)\frac{\partial}{\partial y_3}.\end{aligned} \tag{2.5}$$

2.3 The Formal Expansion of the Operator $\mathcal{A}_t(\mathbf{z})$

Fix a point $\overline{\mathbf{z}} = (\overline{z}_1, \overline{z}_2, \overline{z}_3)^\top \in \mathbb{R}^3$. For example, we may choose:

$$\overline{z}_1 = \ln S(t), \qquad \overline{z}_2 = v(t), \qquad \overline{z}_3 = v'(t),$$

where t is current time, close to maturity T. Expand the operator $\mathcal{A}_t(\mathbf{z})$ by replacing the coefficients $a_{ij}(t, x, y_2, y_3)$ and $a_i(t, x, y_2, y_3)$ with their Taylor series around $\overline{\mathbf{z}}$. Formally,

$$\mathcal{A}_t(\mathbf{z}) \sim \sum_{n=0}^{\infty} \mathcal{A}_{t,n}^{(\overline{\mathbf{z}})}(\mathbf{z}).$$

We call this expansion *formal* because we do not discuss the questions of its convergence. In this Section, we explicitly calculate the first four terms of the above formal expansion.

2.3.1 The Zeroth Term

By (Pagliarani and Pascucci, 2017, Equation (3.4)), the zeroth term of the expansion is just the value of the operator $\mathcal{A}_t(\mathbf{z})$ when $\mathbf{z} = \overline{\mathbf{z}}$. Using Equation (2.5), we obtain

$$
\begin{aligned}
\mathcal{A}_{t,0}^{(\bar{\mathbf{z}})}(x, y_2, y_3) &= \frac{1}{2}v(t)\frac{\partial^2}{\partial x^2} + \frac{1}{2}\xi_1^2 v^{2\alpha_1}(t)\frac{\partial^2}{\partial y_2^2} + \frac{1}{2}\xi_2^2 v'^{2\alpha_2}(t)\frac{\partial^2}{\partial y_3^2} \\
&+ \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\frac{\partial^2}{\partial x\partial y_2} + \rho_{13}\xi_2 v^{1/2}(t)v'^{\alpha_2}(t)\frac{\partial^2}{\partial x\partial y_3} \\
&+ \rho_{23}\xi_1\xi_2 v^{\alpha_1}(t)v'^{\alpha_2}(t)\frac{\partial^2}{\partial y_2\partial y_3} - \frac{1}{2}v(t)\frac{\partial}{\partial x} + \kappa_1(v'(t) - v(t))\frac{\partial}{\partial y_2} \\
&+ \kappa_2(\theta - v'(t))\frac{\partial}{\partial y_3}.
\end{aligned}
$$

2.3.2 The First Term

According to (Pagliarani and Pascucci, 2017, Equation (3.4)), the first term can be obtained by the following replacements in Equation (2.5):

$$
a_{ij}(t, \mathbf{z}) \to \sum_{\beta=1}^{3} \frac{\partial a_{ij}}{\partial z_\beta}(t, \bar{\mathbf{z}})(z_\beta - \bar{z}_\beta), \qquad a_i(t, \mathbf{z}) \to \sum_{\beta=1}^{3} \frac{\partial a_i}{\partial z_\beta}(t, \bar{\mathbf{z}})(z_\beta - \bar{z}_\beta).
$$

Calculating partial derivatives, we obtain

$$
\begin{aligned}
\mathcal{A}_{t,1}^{(\bar{\mathbf{z}})}(x, y_2, y_3) &= \frac{1}{2}(y_2 - v(t))\frac{\partial^2}{\partial x^2} + \xi_1^2\alpha_1 v^{2\alpha_1-1}(t)(y_2 - v(t))\frac{\partial^2}{\partial y_2^2} \\
&+ \xi_2^2\alpha_2 v'^{2\alpha_2-1}(t)(y_3 - v'(t))\frac{\partial^2}{\partial y_3^2} \\
&+ \rho_{12}\xi_1(\alpha_1 + 1/2)v^{\alpha_1-1/2}(t)(y_2 - v(t))\frac{\partial^2}{\partial x\partial y_2} \\
&+ \rho_{13}\xi_2[(1/2)v^{-1/2}(t)v'^{\alpha_2}(t)(y_2 - v(t)) \\
&+ \alpha_2 v^{1/2}(t)v'^{\alpha_2-1}(y_3 - v'(t))]\frac{\partial^2}{\partial x\partial y_3} \\
&+ \rho_{23}\xi_1\xi_2[\alpha_1 v^{\alpha_1-1}(t)v'^{\alpha_2}(t)(y_2 - v(t)) \\
&+ \alpha_2 v^{\alpha_1}(t)v'^{\alpha_2-1}(t)(y_3 - v'(t))]\frac{\partial^2}{\partial y_2\partial y_3}
\end{aligned} \tag{2.6}
$$

$$+\frac{1}{2}(v(t)-y_2)\frac{\partial}{\partial x}+\kappa_1[-y_2+v(t)+y_3-v'(t)]\frac{\partial}{\partial y_2}+\kappa_2(v'(t)-y_3)\frac{\partial}{\partial y_3}.$$

2.3.3 The Second Term

This time, the replacements have the form

$$\begin{aligned} a_{ij}(t,\mathbf{z}) &\to \sum_{\|\boldsymbol{\beta}\|=n}\frac{1}{\boldsymbol{\beta}!}\frac{\partial^n a_{ij}}{\partial \mathbf{z}^{\boldsymbol{\beta}}}(t,\overline{\mathbf{z}})(\mathbf{z}_{\boldsymbol{\beta}}-\overline{\mathbf{z}}_{\boldsymbol{\beta}}),\\ a_i(t,\mathbf{z}) &\to \sum_{\|\boldsymbol{\beta}\|=n}\frac{1}{\boldsymbol{\beta}!}\frac{\partial^n a_i}{\partial \mathbf{z}^{\boldsymbol{\beta}}}(t,\overline{\mathbf{z}})(\mathbf{z}_{\boldsymbol{\beta}}-\overline{\mathbf{z}}_{\boldsymbol{\beta}}), \end{aligned} \tag{2.7}$$

where $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)^\top$ is a multi-index with nonnegative integer coordinates, $\|\boldsymbol{\beta}\|=\beta_1+\beta_2+\beta_3$, $n=2$, and

$$\partial\mathbf{z}^{\boldsymbol{\beta}}=\prod_{k=1}^{3}\partial z_k^{\beta_k},\qquad \mathbf{z}_{\boldsymbol{\beta}}-\overline{\mathbf{z}}_{\boldsymbol{\beta}}=\prod_{k=1}^{3}(z_k-\overline{z}_k)^{\beta_k},\qquad \boldsymbol{\beta}!=\prod_{k=1}^{3}\beta_k!.$$

Again, by calculating partial derivatives, we obtain

$$\begin{aligned} \mathcal{A}^{(\overline{\mathbf{z}})}_{t,2}(x,y_2,y_3) = {} & \xi_1^2\alpha_1(\alpha_1-1/2)v^{2(\alpha_1-1)}(t)(y_2-v(t))^2\frac{\partial^2}{\partial y_2^2}\\ & +\xi_2^2\alpha_2(\alpha_2-1/2)v'^{2(\alpha_2-1)}(t)(y_3-v'(t))^2\frac{\partial^2}{\partial y_3^2}\\ & +\frac{1}{2}\rho_{12}\xi_1(\alpha_1^2-1/4)v^{\alpha_1-3/2}(t)(y_2-v(t))^2\frac{\partial^2}{\partial x\partial y_2}\\ & +\rho_{13}\xi_2[(-1/8)v^{-3/2}(t)v'^{\alpha_2}(t)(y_2-v(t))^2\\ & +(1/2)\alpha_2 v^{-1/2}(t)v'^{\alpha_2-1}(t)(y_2-v(t))(y_3-v'(t))\\ & +(1/2)\alpha_2(\alpha_2-1)v^{1/2}(t)v'^{\alpha_2-2}(t)(y_3-v'(t))^2]\frac{\partial^2}{\partial x\partial y_3}\\ & +\rho_{23}\xi_1\xi_2[(1/2)\alpha_1(\alpha_1-1)v^{\alpha_1-2}(t)v'^{\alpha_2}(t)(y_2-v(t))^2\\ & +\alpha_1\alpha_2 v^{\alpha_1-1}(t)v'^{\alpha_2-1}(t)(y_2-v(t))(y_3-v'(t))\\ & +(1/2)\alpha_2(\alpha_2-1)v^{\alpha_1}(t)v'^{\alpha_2-2}(t)(y_3-v'(t))^2]\frac{\partial^2}{\partial y_2\partial y_3}. \end{aligned}$$

2.3.4 The Third Term

The replacements have the form (2.7), but with $n = 3$. The differential operator takes the form

$$\begin{aligned}
\mathcal{A}_{t,3}^{(\bar{\mathbf{z}})}(x, y_2, y_3) &= \frac{1}{3}\xi_1^2\alpha_1(2\alpha_1 - 1)(\alpha_1 - 1)v^{2\alpha_1-3}(t)(y_2 - v(t))^3\frac{\partial^2}{\partial y_2^2} \\
&+ \frac{1}{3}\xi_2^2\alpha_2(2\alpha_2 - 1)(\alpha_2 - 1)v'^{(2\alpha_2-3)}(t)(y_3 - v'(t))^3\frac{\partial^2}{\partial y_3^2} \\
&+ \frac{1}{6}\rho_{12}\xi_1(\alpha_1^2 - 1/4)(\alpha_1 - 3/2)v^{\alpha_1-5/2}(t)(y_2 - v(t))^3\frac{\partial^2}{\partial x\partial y_2} \\
&+ \rho_{13}\xi_2[(1/16)v^{-5/2}(t)v'^{\alpha_2}(t)(y_2 - v(t))^3 \\
&- (1/8)\alpha_2 v^{-3/2}(t)v'^{\alpha_2-1}(t)(y_2 - v(t))^2(y_3 - v'(t)) \\
&+ (1/4)\alpha_2(\alpha_2 - 1)v^{-1/2}(t)v'^{\alpha_2-2}(t)(y_2 - v(t))(y_3 - v'(t))^2 \\
&+ (1/6)\alpha_2(\alpha_2 - 1)(\alpha_2 - 2)v^{1/2}(t)v'^{\alpha_2-3}(t)(y_3 - v'(t))^3]\frac{\partial^2}{\partial x\partial y_3} \\
&+ \rho_{23}\xi_1\xi_2[(1/6)\alpha_1(\alpha_1 - 1)(\alpha_1 - 2)v^{\alpha_1-3}(t)v'^{\alpha_2}(t)(y_2 - v(t))^3 \\
&+ (1/2)\alpha_1\alpha_2(\alpha_1 - 1)v^{\alpha_1-2}(t)v'^{\alpha_2-1}(t)(y_2 - v(t))^2(y_3 - v'(t)) \\
&+ (1/2)\alpha_1\alpha_2(\alpha_2 - 1)v^{\alpha_1-1}(t)v'^{\alpha_2-2}(t)(y_2 - v(t))(y_3 - v'(t))^2 \\
&+ (1/6)\alpha_1(\alpha_2 - 1)(\alpha_2 - 2)v^{\alpha_1}(t)v'^{\alpha_2-3}(t)(y_3 - v'(t))^3]\frac{\partial^2}{\partial y_2\partial y_3}.
\end{aligned}$$

We explain why we need to write down the above operators. It is proved in Pagliarani and Pascucci (2017) that, in logarithmic variables, the pricing function

$$u(t, x, y_2, y_3, T, k) = V(t, e^x, y_2, y_3, T, e^k),$$

where the function V is given by (2.1), can be expanded into a series

$$u(t, x, y_2, y_3, T, k) \sim \sum_{n=0}^{\infty} u_n^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k). \tag{2.8}$$

The zeroth term is a solution to the boundary value problem

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathcal{A}^{(\bar{\mathbf{z}})}_{t,0}(x,y_2,y_3)\right)u^{(\bar{\mathbf{z}})}_0(t,x,y_2,y_3,T,k)&=0,\\ u^{(\bar{\mathbf{z}})}_0(T,x,y_2,y_3,T,k)&=\max\{0,e^x-e^k\},\end{aligned}\tag{2.9}$$

where the variables t, x, y_2, and y_3 in the partial differential equation run over the region $[0,T)\times\mathbb{R}^3$, and the variables x, y_2, and y_3 in the boundary condition run over $\mathbb{R}^3$. The subsequent terms are solutions to the boundary value problems

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathcal{A}^{(\bar{\mathbf{z}})}_{t,0}\right)u^{(\bar{\mathbf{z}})}_n(t,x,y_2,y_3,T,k)&=-\sum_{h=1}^{n}\mathcal{A}^{(\bar{\mathbf{z}})}_{t,h}u^{(\bar{\mathbf{z}})}_{n-h}(t,x,y_2,y_3,T,k),\\ u^{(\bar{\mathbf{z}})}_n(T,x,y_2,y_3,T,k)&=0,\end{aligned}$$

where the variables run over the same regions.

2.4 The Expansion

2.4.1 The Zeroth Term

The solution to the boundary value problem (2.9) has the form

$$u^{(\bar{\mathbf{z}})}_0(t,x,y_2,y_3,T,k)=u^{\mathrm{BS}}(\sigma^{(\bar{\mathbf{z}})}_0,\tau,x,k),$$

where the function u^{BS} is given by Equation (1.6), and the parameter $\sigma^{(\bar{\mathbf{z}})}_0$ is given by

$$\sigma^{(\bar{\mathbf{z}})}_0=\left(\frac{1}{\tau}\int_t^T a_{11}(u,\bar{\mathbf{z}})\,du\right)^{1/2}.$$

Using equation (2.4), we obtain

$$\sigma^{(\bar{\mathbf{z}})}_0=\sqrt{v_0}.\tag{2.10}$$

To obtain the time t approximation of the option's price, we put $x=\ln S(t)$.

Theorem 8. *The approximation of order* 0 *of the European call option's price has the form*

$$u(t,S(t),T,k)\approx u^{\mathrm{BS}}(\sqrt{v(t)},\tau,\ln S(t),k).\tag{2.11}$$

2.4.2 Auxiliary calculations

First, we need to calculate the vector $\mathbf{m}^{(\overline{\mathbf{z}})}(t,s)$ whose components are given by

$$m_i^{(\overline{\mathbf{z}})}(t,s) = \int_t^s a_i(u,\overline{\mathbf{z}})\,du,$$

see Pagliarani and Pascucci (2017, p. 713). Equation (2.4) gives

$$m_1^{(\overline{\mathbf{z}})}(t,s) = -\frac{1}{2}v(t)(s-t), \qquad m_2^{(\overline{\mathbf{z}})}(t,s) = \kappa_1(v'(t)-v(t))(s-t),$$
$$m_3^{(\overline{\mathbf{z}})}(t,s) = \kappa_2(\theta - v'(t))(s-t).$$

The next task is to calculate the entries of the matrix

$$C_{ij}^{(\overline{\mathbf{z}})}(t,s) = \int_t^s a_{ij}(u,\overline{\mathbf{z}})\,du.$$

This gives

$$\begin{aligned}
C_{12}^{(\overline{\mathbf{z}})}(t,s) &= \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(s-t), & C_{11}^{(\overline{\mathbf{z}})}(t,s) &= v(t)(s-t),\\
C_{13}^{(\overline{\mathbf{z}})}(t,s) &= \rho_{13}\xi_2 v^{1/2}(t)v'^{\alpha_2}(t)(s-t), & C_{22}^{(\overline{\mathbf{z}})}(t,s) &= \xi_1^2 v^{2\alpha_1}(t)(s-t),\\
C_{23}^{(\overline{\mathbf{z}})}(t,s) &= \rho_{23}\xi_1\xi_2 v^{\alpha_1}(t)v'^{\alpha_2}(t)(s-t), & C_{33}^{(\overline{\mathbf{z}})}(t,s) &= \xi_2^2 v'^{2\alpha_2}(t)(s-t).
\end{aligned}$$

Introduce the following notation for the coefficients of the linear differential operator $\mathcal{A}_{t,n}^{(\overline{\mathbf{z}})}(\mathbf{z})$:

$$\mathcal{A}_{t,n}^{(\overline{\mathbf{z}})}(\mathbf{z}) = \sum_{1\le i\le j\le 3} a_{ij,n}(t,\mathbf{z})\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^{3} a_{i,n}(t,\mathbf{z})\frac{\partial}{\partial z_i}.$$

In particular, when $n=1$, Equation (2.6) gives

$$\begin{aligned}
a_{11,1}(t,y_2) &= \frac{1}{2}(y_2 - v(t)),\\
a_{12,1}(t,y_2) &= \rho_{12}\xi_1(\alpha_1+1/2)v^{\alpha_1-1/2}(t)(y_2-v(t)),
\end{aligned}$$

$$
\begin{aligned}
a_{13,1}(t,y_2,y_3) &= \rho_{13}\xi_2[(1/2)v^{-1/2}(t)v'^{\alpha_2}(t)(y_2-v(t)) \\
&\quad + \alpha_2 v^{1/2}(t)v'^{\alpha_2-1}(y_3-v'(t))], \\
a_{22,1}(t,y_2) &= \xi_1^2\alpha_1 v^{2\alpha_1-1}(t)(y_2-v(t)), \\
a_{23,1}(t,y_2,y_3) &= \rho_{23}\xi_1\xi_2[\alpha_1 v^{\alpha_1-1}(t)v'^{\alpha_2}(t)(y_2-v(t)) \\
&\quad + \alpha_2 v^{\alpha_1}(t)v'^{\alpha_2-1}(t)(y_3-v'(t))], \\
a_{33,1}(t,y_3) &= \xi_2^2\alpha_2 v'^{2\alpha_2-1}(t)(y_3-v'(t)), \\
a_{1,1}(t,y_2) &= \frac{1}{2}(v(t)-y_2), \\
a_{2,1}(t,y_2,y_3) &= \kappa_1[-y_2+v(t)+y_3-v'(t)], \\
a_{3,1}(t,y_3) &= \kappa_2(v'(t)-y_3).
\end{aligned}
$$

Following Pagliarani and Pascucci (2017), define the symbols $\mathcal{M}_i(t,s,z_i)$, $1 \le i \le 3$, by the equation:

$$
\mathcal{M}_i(t,s,z_i) = z_i + m_i(t,s) + \sum_{j=1}^{3} C_{ij}(t,s)\frac{\partial}{\partial z_j}.
$$

Using the values of $m_i(t,s)$ and $C_{ij}(t,s)$ calculated above, we obtain the following equalities:

$$
\begin{aligned}
\mathcal{M}_1(t,s,x) &= x - \frac{1}{2}v(t)(s-t) + v(t)(s-t)\frac{\partial}{\partial x} + \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(s-t)\frac{\partial}{\partial y_2} \\
&\quad + \rho_{13}\xi_2 v^{1/2}(t)v'^{\alpha_2}(t)(s-t)\frac{\partial}{\partial y_3}, \\
\mathcal{M}_2(t,s,y_2) &= y_2 + \kappa_1(v'(t)-v(t))(s-t) + \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(s-t)\frac{\partial}{\partial x} \\
&\quad + \xi_1^2 v^{2\alpha_1}(t)(s-t)\frac{\partial}{\partial y_2} + \rho_{23}\xi_1\xi_2 v^{\alpha_1}(t)v'^{\alpha_2}(t)(s-t)\frac{\partial}{\partial y_3}, \\
\mathcal{M}_3(t,s,y_3) &= y_3 + \kappa_2(\theta - v'(t))(s-t) + \rho_{13}\xi_2 v^{1/2}(t)v'^{\alpha_2}(t)(s-t)\frac{\partial}{\partial x} \\
&\quad + \rho_{23}\xi_1\xi_2 v^{\alpha_1}(t)v'^{\alpha_2}(t)(s-t)\frac{\partial}{\partial y_2} + \xi_2^2 v'^{2\alpha_2}(t)(s-t)\frac{\partial}{\partial y_3}.
\end{aligned}
\tag{2.12}
$$

Now, we calculate the operators

$$
\begin{aligned}
&\mathcal{G}_n^{(\bar{\mathbf{z}})}(t,s,x,y_2,y_3)\\
&\quad=\sum_{1\le i\le j\le 3} a_{ij,n}(t,\mathcal{M}_1(t,s,x),\mathcal{M}_2(t,s,y_2),\mathcal{M}_3(t,s,y_3))\frac{\partial^2}{\partial z_i\partial z_j}\\
&\quad+\sum_{i=1}^{3} a_{i,n}(t,\mathcal{M}_1(t,s,x),\mathcal{M}_2(t,s,y_2),\mathcal{M}_3(t,s,y_3))\frac{\partial}{\partial z_i}
\end{aligned}
\tag{2.13}
$$

for $n=1,2,3$. They have the form

$$
\mathcal{G}_n^{(\bar{\mathbf{z}})}(t,s,x,y_2,y_3)=\sum_{\gamma} g_{\gamma,n}^{(\bar{\mathbf{z}})}(t,s,x,y_2,y_3)\frac{\partial^{\|\gamma\|}}{\partial\mathbf{z}^{\gamma}},
$$

where the sum is taken over a finite set of multi-indices.

We calculate in details the coefficient of the differential operator $\frac{\partial}{\partial x}$, which is $g_{(1,0,0)^\top,1}^{(\bar{\mathbf{z}})}(t,s,y_2)$. By Equation (2.13), the above coefficient is that part of the expression $a_{1,1}(t,\mathcal{M}_1(t,s,x),\mathcal{M}_2(t,s,y_2),\mathcal{M}_3(t,s,y_3))$ which does not contain any partial derivatives. We have

$$
a_{1,1}(t,\mathcal{M}_2(t,s,y_2))=\frac{1}{2}(v(t)-\mathcal{M}_2(t,s,y_2)).
$$

Substituting the value of $\mathcal{M}_2(t,s,y_2)$ from Equation (2.12) and ignoring the terms containing partial derivatives, we obtain

$$
g_{(1,0,0)^\top,1}^{(\bar{\mathbf{z}})}(t,s,y_2)=\frac{1}{2}[v(t)-y_2-\kappa_1(v'(t)-v(t))(s-t)]. \tag{2.14}
$$

Similarly, the coefficient of $\frac{\partial^2}{\partial x^2}$ is

$$
g_{(2,0,0)^\top,1}^{(\bar{\mathbf{z}})}(t,s,y_2)=\frac{1}{2}[y_2-v(t)-\rho_{12}\xi_1 v^{\alpha_1+1/2}(s-t)+\kappa_1(v'(t)-v(t))(s-t)]. \tag{2.15}
$$

The coefficient of $\frac{\partial^3}{\partial x^3}$ is

$$
g_{(3,0,0)^\top,1}^{(\bar{\mathbf{z}})}(t,s)=\frac{1}{2}\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(s-t). \tag{2.16}
$$

As we will see later, we do not need to know the values of the remaining coefficients.

2.4.3 The First Term

Our next task is to calculate the differential operator $\mathcal{L}_n^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3)$ with $n = 1$. It is given by Pagliarani and Pascucci (2017, Equation (D.2)). In the case of $n = 1$ this equation takes the form

$$\mathcal{L}_1^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3) = \int_t^T \mathcal{G}_1^{(\bar{\mathbf{z}})}(t, s_1, x, y_2, y_3)\, ds_1.$$

The above differential operator is used to calculate the nth term in the approximation (2.8). This approximation is given by Pagliarani and Pascucci (2017, Equation (D.1)):

$$u_n^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k) = \mathcal{L}_n^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3) u_0^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k),$$

The expression $u_0^{(\bar{\mathbf{z}})}(t, x, y_1, y_2, T, k)$ is given by the right hand side of Equation (2.11) and does not depend on y_2 and y_3. The approximation of the nth term takes the form

$$u_n^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k) = \hat{\mathcal{L}}_n^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3) u_0^{(\bar{\mathbf{z}})}(t, x, T, k),$$

where the differential operator $\hat{\mathcal{L}}_n^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3)$ is obtained from the operator $\mathcal{L}_n^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3)$ by deleting all terms containing partial derivatives with respect to the variables y_2 and y_3. To calculate this operator when $n = 1$, we have to integrate the coefficient of the differential operator $\frac{\partial^k}{\partial x^k}$, $1 \leq k \leq 3$, from t to T with respect to s. We calculate the integral

$$\int_t^T (t - s)\, ds = \frac{1}{2} t^2 + \tau t - \frac{1}{2} T^2 = -\frac{1}{2} \tau^2. \tag{2.17}$$

The operator $\hat{\mathcal{L}}_1^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3)$ takes the form

$$\begin{aligned}
\hat{\mathcal{L}}_1^{(\bar{\mathbf{z}})}(t, T, y_2) &= \frac{\tau}{4} [2(v(t) - y_2) - \kappa_1 (v'(t) - v(t))\tau] \frac{\partial}{\partial x} \\
&+ \frac{\tau}{4} [2(v(t) - y_2) - \rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t)\tau + \kappa_1 (v'(t) - v(t))\tau] \frac{\partial^2}{\partial x^2} \\
&+ \frac{\tau}{4} [\rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t)\tau] \frac{\partial^3}{\partial x^3}.
\end{aligned}$$

To calculate the approximation $u_1^{(\overline{\mathbf{z}})}(t, x, y_2, y_3, T, k)$, we have to apply the above differential operator to $u_0^{(\overline{\mathbf{z}})}(t, x, T, k)$ and put $y_2 = v(t)$. We obtain

$$\begin{aligned} u_1^{(\overline{\mathbf{z}})}(t, T, x, k) = \frac{\tau^2}{4} \bigg[& (-\kappa_1(v'(t) - v(t))) \frac{\partial}{\partial x} \\ & + (-\rho_{12}\xi_1 v^{\alpha_1 + 1/2}(t) + \kappa_1(v'(t) - v(t))) \frac{\partial^2}{\partial x^2} \\ & + \rho_{12}\xi_1 v^{\alpha_1 + 1/2}(t) \frac{\partial^3}{\partial x^3} \bigg] u^{\mathrm{BS}}(\sqrt{v(t)}, \tau, x, k). \end{aligned}$$

To formulate the approximation of order 1 of a European call option's price, we introduce the following notation

$$u^{\mathrm{BS},n}(\sqrt{v(t)}, \tau, S(t), k) = \frac{\partial^n}{\partial x^n} u^{\mathrm{BS}}(\sqrt{v(t)}, \tau, \ln S(t), k).$$

Theorem 9. *The approximation of order* 1 *of the European call option's price has the form*

$$\begin{aligned} & u(t, S(t), T, k) \approx u^{\mathrm{BS}}(\sqrt{v(t)}, \tau, \ln S(t), k) \\ & + \frac{\tau^2}{4}(-\kappa_1(v'(t) - v(t))) u^{\mathrm{BS},1}(\sqrt{v(t)}, \tau, S(t), k) \\ & + \frac{\tau^2}{4}(-\rho_{12}\xi_1 v^{\alpha_1+1/2}(t) + \kappa_1(v'(t) - v(t))) u^{\mathrm{BS},2}(\sqrt{v(t)}, \tau, S(t), k) \\ & + \frac{\tau^2}{4}\rho_{12}\xi_1 v^{\alpha_1+1/2}(t) u^{\mathrm{BS},3}(\sqrt{v(t)}, \tau, S(t), k). \end{aligned}$$

2.4.4 The Second Term

Pagliarani and Pascucci (2017, Equation (D.2)) gives

$$\begin{aligned} & \mathcal{L}_2^{(\overline{\mathbf{z}})}(t, T, x, y_2, y_3) \\ & \quad = \int_t^T \int_{s_1}^T \mathcal{G}_1^{(\overline{\mathbf{z}})}(t, s_1, x, y_2, y_3) \mathcal{G}_1^{(\overline{\mathbf{z}})}(t, s_2, x, y_2, y_3)\, ds_2\, ds_1. \end{aligned} \tag{2.18}$$

We write down the coefficients of the partial derivatives $\frac{\partial^2}{\partial x^2}, \dots, \frac{\partial^6}{\partial x^6}$ of the linear differential operator under the integral sign.

The coefficient of the partial derivative $\frac{\partial^2}{\partial x^2}$ is

$$g^{(\bar{\mathbf{z}})}_{(2,0,0)^\top,2}(t,s_1,s_2,y_2) = g^{(\bar{\mathbf{z}})}_{(1,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(1,0,0)^\top,1}(t,s_2,y_2).$$

The coefficient of $\frac{\partial^3}{\partial x^3}$ is

$$\begin{aligned} g^{(\bar{\mathbf{z}})}_{(3,0,0)^\top,2}(t,s_1,s_2,y_2) = {} & g^{(\bar{\mathbf{z}})}_{(1,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(2,0,0)^\top,1}(t,s_2,y_2) \\ & + g^{(\bar{\mathbf{z}})}_{(2,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(1,0,0)^\top,1}(t,s_2,y_2). \end{aligned}$$

The coefficient of $\frac{\partial^4}{\partial x^4}$ is

$$\begin{aligned} g^{(\bar{\mathbf{z}})}_{(4,0,0)^\top,2}(t,s_1,s_2,y_2) = {} & g^{(\bar{\mathbf{z}})}_{(1,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(3,0,0)^\top,1}(t,s_2,y_2) \\ & + g^{(\bar{\mathbf{z}})}_{(2,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(2,0,0)^\top,1}(t,s_2,y_2) \\ & + g^{(\bar{\mathbf{z}})}_{(3,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(1,0,0)^\top,1}(t,s_2,y_2). \end{aligned}$$

The coefficient of $\frac{\partial^5}{\partial x^5}$ is

$$\begin{aligned} g^{(\bar{\mathbf{z}})}_{(5,0,0)^\top,2}(t,s_1,s_2,y_2) = {} & g^{(\bar{\mathbf{z}})}_{(1,0,0)^\top,2}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(3,0,0)^\top,1}(t,s_2,y_2) \\ & + g^{(\bar{\mathbf{z}})}_{(3,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(2,0,0)^\top,1}(t,s_2,y_2). \end{aligned}$$

Finally, the coefficient of $\frac{\partial^6}{\partial x^6}$ is

$$g^{(\bar{\mathbf{z}})}_{(6,0,0)^\top,2}(t,s_1,s_2,y_2) = g^{(\bar{\mathbf{z}})}_{(3,0,0)^\top,1}(t,s_1,y_2))g^{(\bar{\mathbf{z}})}_{(3,0,0)^\top,1}(t,s_2,y_2).$$

We substitute the values (2.14), (2.15), and (2.16) to the above equations and integrate the result according to Equation (2.18). The integral (2.17) is now replaced with the following integrals:

$$\begin{aligned} \int_t^T \int_{s_1}^T \mathrm{d}s_2\,\mathrm{d}s_1 &= \frac{1}{2}\tau^2, & \int_t^T \int_{s_1}^T (s_2-t)\,\mathrm{d}s_2\,\mathrm{d}s_1 &= \frac{1}{3}\tau^3, \\ \int_t^T \int_{s_1}^T (s_1-t)\,\mathrm{d}s_2\,\mathrm{d}s_1 &= \frac{1}{6}\tau^3, & \int_t^T \int_{s_1}^T (s_1-t)(s_2-t)\,\mathrm{d}s_2\,\mathrm{d}s_1 &= \frac{1}{8}\tau^4. \end{aligned}$$

The operator $\hat{\mathcal{L}}_2^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3)$ takes the form

$$
\begin{aligned}
\hat{\mathcal{L}}_2^{(\bar{\mathbf{z}})}(t, T, x, y_2, y_3) &= \left[\frac{\tau^4}{32} \kappa_1^2 (v'(t) - v(t))^2 - \frac{\tau^3}{24} (v(t) - y_2) \kappa_1 (v'(t) - v(t)) \right. \\
&\quad \left. + \frac{\tau^2}{8} (v(t) - y_2)^2 \right] \frac{\partial^2}{\partial x^2} \\
&\quad + \left[\frac{\tau^4}{32} (\rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t) - \kappa_1 (v'(t) - v(t))) \kappa_1 (v'(t) - v(t)) \right. \\
&\quad \left. - \frac{\tau^3}{12} (v(t) - y_2) \rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t) - \frac{\tau^2}{4} (v(t) - y_2)^2 \right] \frac{\partial^3}{\partial x^3} \\
&\quad + \left[\frac{\tau^4}{32} (\rho_{12}^2 \xi_1^2 v^{2\alpha_1 + 1}(t) - 4 \rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t) \kappa_1 (v'(t) - v(t)) \right. \\
&\quad + \kappa_1^2 (v'(t) - v(t))^2) + \frac{\tau^3}{8} (v(t) - y_2(t)) \kappa_1 (v'(t) - v(t)) \\
&\quad \left. + \frac{\tau^2}{8} (v(t) - y_2(t))^2 \right] \frac{\partial^4}{\partial x^4} \\
&\quad + \left[\frac{\tau^4}{16} (\rho_{12}^2 \xi_1^2 v^{2\alpha_1 + 1}(t) - \rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t) \kappa_1 (v'(t) - v(t))) \right. \\
&\quad \left. + \frac{\tau^3}{4} (v(t) - y_2(t)) \rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t) \right] \frac{\partial^5}{\partial x^5} \\
&\quad + \frac{\tau^4}{32} \rho_{12}^2 \xi_1^2 v^{2\alpha_1 + 1}(t) \frac{\partial^6}{\partial x^6}.
\end{aligned}
$$

Again, to calculate the approximation $u_2^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k)$, we have to apply the above differential operator to $u_0^{(\bar{\mathbf{z}})}(t, x, T, k)$ and put $y_2 = v(t)$. We obtain

$$
\begin{aligned}
u_2^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k) &= \frac{\tau^4}{32} \kappa_1^2 (v'(t) - v(t))^2 u^{\mathrm{BS},2}(\sqrt{v(t)}, \tau, S(t), k) \\
&\quad + \frac{\tau^4}{32} (\rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t) - \kappa_1 (v'(t) - v(t))) \kappa_1 (v'(t) - v(t)) \\
&\qquad \times u^{\mathrm{BS},3}(\sqrt{v(t)}, \tau, S(t), k) \\
&\quad + \frac{\tau^4}{32} (\rho_{12}^2 \xi_1^2 v^{2\alpha_1 + 1}(t) - 4 \rho_{12} \xi_1 v^{\alpha_1 + 1/2}(t) \kappa_1 (v'(t) - v(t))
\end{aligned}
$$

$$
\begin{aligned}
&+ \kappa_1^2(v'(t) - v(t))^2) u^{\mathrm{BS},4}(\sqrt{v(t)}, \tau, S(t), k) \\
&+ \frac{\tau^4}{16}(\rho_{12}^2 \xi_1^2 v^{2\alpha_1+1}(t) - \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\kappa_1(v'(t) - v(t))) \\
&\quad \times u^{\mathrm{BS},5}(\sqrt{v(t)}, \tau, S(t), k) \\
&+ \frac{\tau^4}{32}\rho_{12}^2 \xi_1^2 v^{2\alpha_1+1}(t) u^{\mathrm{BS},5}(\sqrt{v(t)}, \tau, S(t), k).
\end{aligned}
$$

The subsequent terms can be calculated similarly, but we do not need them for calculations of implied volatility.

Chapter 3

The Asymptotic Expansion of Implied Volatility

This Chapter is based on Papers **A** and **B** and adds more details of calculations.

Asymptotics of implied volatility in the Gatheral double stochastic volatility model
M. Albuhayri, A. Malyarenko, S. Silvestrov, Y. Ni, C. Engström, F. Tewolde, J. Zhang

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An improved asymptotics of implied volatility in the Gatheral model
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3.1 Introduction

Recall Equation (2.10): $\sigma_0^{(\bar{\mathbf{z}})} = \sqrt{v_0}$. Let $u_n^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k)$ be the terms in the expansion (2.8) of the European call option's price, $u(t, x, y_2, y_3, T, k)$. Following Pagliarani and Pascucci (2017, Equation (3.12)), introduce a family of approximation to the above price given by

$$
\begin{aligned}
u^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, \delta, N) &= u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})}) + \sum_{n=1}^{N} \delta^N u_n^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k) \\
&+ \delta^{N+1}\left(u(t, x, y_2, y_3, T, k) - \sum_{n=0}^{N} u_n^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k)\right),
\end{aligned}
$$

where $0 \leq \delta \leq 1$, and where N is a positive integer. Note that

$$
u^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, 1, N) = u(t, x, y_2, y_3, T, k).
$$

The function $g^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, \delta, N)$ is defined by

$$
g^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, \delta, N) = (u^{\mathrm{BS}})^{-1}(u^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, \delta, N)).
$$

We see that the implied volatility is given by

$$
\sigma(\bar{\mathbf{z}}, t, x, y_2, y_3, T, k) = g^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, 1, N).
$$

(Pagliarani and Pascucci, 2017, Lemma 5.8) states the following. There exist two positive constants τ_0 and M, such that if $T - t \leq \tau_0$ and $|x_0 - k| \leq \lambda\sqrt{M\tau}$ for some positive λ, then the function g is well-defined.

The Taylor series expansion gives

$$
\begin{aligned}
\sigma(\bar{\mathbf{z}}, t, x, y_2, y_3, T, k) &= \sigma_0^{(\bar{\mathbf{z}})} \\
&+ \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n g^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, \delta, N)}{\partial \delta^n}(t, x, y_2, y_3, T, k, 0, N).
\end{aligned}
$$

It remains to calculate the partial derivatives in the right hand side. Denote

$$
\begin{aligned}
&\sigma_n^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k) \\
&= \frac{1}{n!} \frac{\partial^n g^{(\bar{\mathbf{z}})}(t, x, y_2, y_3, T, k, \delta, N)}{\partial \delta^n}(t, x, y_2, y_3, T, k, 0, N).
\end{aligned}
$$

Apply Faá di Bruno's formula, see Section A.2. We obtain

$$\begin{aligned}\sigma_n^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k) &= \frac{u_n^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}\\ &\quad - \frac{1}{n!}\sum_{h=2}^{n} B_{n,h}(1!\sigma_1^{(\bar{\mathbf{z}})}, 2!\sigma_2^{(\bar{\mathbf{z}})}, \dots, (n-h+1)!\sigma_{n-h+1}^{(\bar{\mathbf{z}})})\frac{\frac{\partial^h}{\partial\sigma^h}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}.\end{aligned}$$

The function $\sigma_0^{(\bar{\mathbf{z}})}$ is already calculated. We will calculate the functions $\sigma_n^{(\bar{\mathbf{z}})}(t,x,y_1,y_2,T,k)$ for $1 \le n \le 3$. In this case, using Equation (A.1), we obtain

$$\begin{aligned}\sigma_1^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k) &= \frac{u_1^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})},\\ \sigma_2^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k) &= \frac{u_2^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} - \frac{1}{2}(\sigma_1^{(\bar{\mathbf{z}})})^2\frac{\frac{\partial^2}{\partial\sigma^2}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})},\\ \sigma_3^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k) &= \frac{u_3^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} - \sigma_1^{(\bar{\mathbf{z}})}\sigma_2^{(\bar{\mathbf{z}})}\frac{\frac{\partial^2}{\partial\sigma^2}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}\\ &\quad - \frac{1}{6}(\sigma_1^{(\bar{\mathbf{z}})})^3\frac{\frac{\partial^3}{\partial\sigma^3}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}.\end{aligned} \tag{3.1}$$

We need to calculate the terms

$$\frac{u_n^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}, \qquad 1 \le n \le 3$$

and

$$\frac{\frac{\partial^n}{\partial\sigma^n}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}, \qquad 2 \le n \le 3.$$

The second one is easier to calculate. Denote $c_{2,2} = c_{2,0} = c_{3,3} = 1$, $c_{3,1} = 3$ and

$$\zeta = \frac{x-k-\sigma^2\tau/2}{\sigma\sqrt{2\tau}}.$$

By Pagliarani and Pascucci (2017, Proposition C.4),

$$\frac{\frac{\partial^n}{\partial\sigma^n}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)} = \sum_{q=0}^{\lfloor n/2\rfloor} c_{n,n-2q}\sigma^{n-2q-1}\tau^{n-q-1}\sum_{p=0}^{n-q-1}\binom{n-q-1}{p} \times\left(\frac{1}{\sigma\sqrt{2\tau}}\right)^{p+n-q-1} H_{p+n-q-1}(\zeta), \tag{3.2}$$

where $H_{p+n-q-1}(\zeta)$ are the "physicists" Hermite polynomials, see Appendix, Section A.3.

By Pagliarani and Pascucci (2017, Proposition D.3), we have

$$\frac{u_n^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} = \sum_m \left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^m \chi_{m,n}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k)H_m(\zeta), \tag{3.3}$$

but the exact form of the functions $\chi_{m,n}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k)$ is omitted. The authors refer to Lorig et al. (2017a, Proposition 3.6) instead.

The above proposition tells the following. Consider the linear differential operator

$$\begin{aligned}\tilde{\mathcal{L}}_n(t,T) = \sum_{k=1}^{n}\int_t^T\int_{t_1}^T\cdots\int_{t_{k-1}}^T\sum_{\mathbf{i}\in I_{n,k}}\mathcal{G}_{i_1}(t,t_1)\cdots\mathcal{G}_{i_{k-1}}(t,t_{k-1})\\ \times a_{11,i_k}(t_k,\mathcal{M}_1(t,t_k,x),\mathcal{M}_2(t,t_k,y_2),\mathcal{M}_3(t,t_k,y_3))\,dt_k\cdots dt_1,\end{aligned} \tag{3.4}$$

where $I_{n,k}$ is the set of all multi-indices $\mathbf{i}=(i_1,\dots,i_k)$ with positive integer entries satisfying $i_1+\cdot+i_k=n$. Then we have

$$\frac{u_n^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} = \frac{\tilde{\mathcal{L}}_n(t,T)\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}{\tau\sigma_0^{(\bar{\mathbf{z}})}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}. \tag{3.5}$$

On the other hand, Lorig et al. (2017a, Lemma 3.4) gives

$$\frac{\frac{\partial^m}{\partial x^m}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0)}{\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0)} = \left(-\frac{1}{\sigma_0\sqrt{2\tau}}\right)^m H_m(\zeta). \tag{3.6}$$

We obtain: to calculate $\chi^{(\bar{\mathbf{z}})}_{m,n}(t,\mathbf{z},T,k)$, we divide the coefficient of the partial derivative $\frac{\partial^m}{\partial x^m}$ in the differential operator $\tilde{\mathcal{L}}_n(t,T)$ by $\tau\sigma_0^{(\bar{\mathbf{z}})}$.

Following Pagliarani and Pascucci (2017, Definition 3.4), define

$$\overline{\sigma}_N(t,x,y_2,y_3,T,k) = \sum_{n=0}^{N} \sigma^{(x,y_2,y_3)}(t,x,y_2,y_3,T,k) \tag{3.7}$$

and call this expression the *Nth order approximation of the implied volatility*. By Pagliarani and Pascucci (2017, Corollary 5.2), for all nonnegative integers q and m and for a fixed point (x_0,y_0,z_0) with $2q+m \le N$ the limit

$$\lim_{\substack{(T,k)\to(t,x_0)\\ |x_0-k|\le\lambda\sqrt{T-t}}} \frac{\partial^q}{\partial T^q}\frac{\partial^m}{\partial k^m}\overline{\sigma}_N(t,x_0,y_0,z_0,T,k)$$

exists and is finite. Moreover, it is also equal to another limit:

$$\lim_{\substack{(T,k)\to(t,x_0)\\ |x_0-k|\le\lambda\sqrt{T-t}}} \frac{\partial^q}{\partial T^q}\frac{\partial^m}{\partial k^m}\sigma(t,x_0,y_0,z_0,T,k).$$

When the last limit exists, its value can be used for an asymptotic expansion of implied volatility:

$$\begin{aligned}\sigma(t,x_0,y_0,z_0,T,k) = \sum_{2q+m\le N} &\lim_{\substack{(T,k)\to(t,x_0)\\ |x_0-k|\le\lambda\sqrt{T-t}}} \frac{\partial^q}{\partial T^q}\frac{\partial^m}{\partial k^m}\overline{\sigma}_N(t,x_0,y_0,z_0,T,k)\\ &\times\frac{(T-t)^q(k-x_0)^m}{q!m!} + o(|T-t|^{N/2}+|k-x_0|^n)\end{aligned} \tag{3.8}$$

as $(T,k)\to(t,x_0)$ with $|x_0-k|\le\lambda\sqrt{T-t}$.

In the following Sections, we realise these calculations for $0 \le N \le 3$. To simplify formulas, we omit the arguments of σ.

3.2 The Asymptotic Expansion of Order 0

Theorem 10. *The asymptotic expansion of implied volatility of order* 0 *in the Gatheral model has the form*

$$\sigma = \sqrt{v_0} + o(1). \tag{3.9}$$

Proof. Using Equations (2.10) and (3.7), we obtain, that the 0th order approximation of the implied volatility takes the form

$$\overline{\sigma}_0(y_2) = \sqrt{y_2}.$$

Equation (3.8) and the equality $y_0 = v_0$ give (3.9). □

3.3 The Asymptotic Expansion of Order 1

Theorem 11. *The asymptotic expansion of implied volatility of order* 1 *in the Gatheral model has the form*

$$\sigma = \sqrt{v_0} + \frac{1}{4}\rho_{12}\xi_1 v^{\alpha_1 - 1}(t)(k - x_0) + o(\sqrt{T-t} + |k - x_0|).$$

Proof. Recall that the first equation in (3.1) reads

$$\sigma_1^{(\overline{\mathbf{z}})}(t, x, y_1, y_2, T, k) = \frac{u_1^{(\overline{\mathbf{z}})}(t, x, y_2, y_3, T, k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\overline{\mathbf{z}})})}.$$

To calculate the right hand side, we use Equation (3.5) for $n = 1$. First, we calculate the linear differential operator $\tilde{\mathcal{L}}_1(t, T)$. In this case the index k takes only one value $k = 1$, the set $I_{1,1}$ is a singleton containing the multi-index $\mathbf{i} = (1)$. Equation (3.4) takes the form

$$\tilde{\mathcal{L}}_1(t, T) = \int_t^T a_{11,1}(t_1, \mathcal{M}_1(t, t_1, x), \mathcal{M}_2(t, t_1, y_2), \mathcal{M}_3(t, t_1, y_3))\, dt_1.$$

Equation (2.6) gives

$$a_{11,1}(t_1, x, y_2, y_3) = \frac{1}{2}(y_2 - v(t)).$$

The linear differential operator $\tilde{\mathcal{L}}_1(t, T)$ takes the form

$$\tilde{\mathcal{L}}_1(t, T) = \int_t^T \frac{1}{2}(\mathcal{M}_2(t, t_1, y_2) - v(t))\, dt_1.$$

We use the second equation in (2.12) and obtain

$$\begin{aligned}\tilde{\mathcal{L}}_1(t,T) = \int_t^T & \frac{1}{2}(y_2 + \kappa_1(v'(t) - v(t))(t_1 - t) + \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_1 - t)\frac{\partial}{\partial x} \\ & + \xi_1^2 v^{2\alpha_1}(t)(t_1 - t)\frac{\partial}{\partial y_2} + \rho_{23}\xi_1\xi_2 v^{\alpha_1}(t)v'^{\alpha_2}(t)(t_1 - t)\frac{\partial}{\partial y_3} \\ & - v(t))\, dt_1.\end{aligned}$$

Note that the Black–Scholes price u^{BS} depends on x, but does not depend on y_2 and y_3. By this reason, in what follows, we do not calculate the terms of the operator $\tilde{\mathcal{L}}_n(t,T)$ that include the partial derivatives with respect to y_2 and y_3, but replace them with dots $\cdots$. Using the integral (2.17), we obtain

$$\begin{aligned}\tilde{\mathcal{L}}_1(t,T) = {} & \frac{1}{2}(y_2 - v(t))\tau + \frac{1}{4}\kappa_1(v'(t) - v(t))\tau^2 \\ & + \frac{1}{4}\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^2\frac{\partial}{\partial x} + \cdots .\end{aligned}$$

Substitute the obtained operator to Equation (3.5) with $n = 1$. We have

$$\begin{aligned}\frac{u_1^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} = {} & \frac{y_2 - v(t)}{2\sqrt{v(t)}} + \frac{\kappa_1(v'(t) - v(t))\tau}{4\sqrt{v(t)}} \\ & + \frac{1}{4}\rho_{12}\xi_1 v^{\alpha_1}(t)\tau\frac{\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}{\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})}.\end{aligned}$$

To calculate the last term, we use Equation (3.6) and the first equation in (A.2), which gives

$$\frac{\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0)}{\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)u^{\mathrm{BS}}(\sigma_0)} = -\frac{\sqrt{2}}{\sigma_0\sqrt{\tau}}\zeta.$$

The term $\sigma_1^{(\bar{\mathbf{z}})}(t,x,y_1,y_2,T,k)$ becomes

$$\begin{aligned}\sigma_1^{(\bar{\mathbf{z}})}(t,x,y_1,y_2,T,k) = {} & \frac{y_2 - v(t)}{2\sqrt{v(t)}} + \frac{\kappa_1(v'(t) - v(t))\tau}{4\sqrt{v(t)}} \\ & - \frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1-1}(t)(2x - 2k - v(t)\tau).\end{aligned} \tag{3.10}$$

Note that the first term is equal to 0 because at time t we have $y_2 = v_0$.

Equation (3.7) with $N = 1$ gives

$$\overline{\sigma}_1(t, x, y_2, y_3, T, k) = \sqrt{v_0} + \frac{\kappa_1(y_3 - y_2)(T - t)}{4\sqrt{y_2}} - \frac{1}{8}\rho_{12}\xi_1 y_2^{\alpha_1 - 1}(2x - 2k - y_2(T - t)).$$

In particular,

$$\overline{\sigma}_1(t, x_0, y_0, z_0, T, k) = \sqrt{v_0} + \frac{\kappa_1(v'(t) - v(t))(T - t)}{4\sqrt{v(t)}} - \frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1 - 1}(t)(2x_0 - 2k - v(t)(T - t)).$$

The only possible values for q and m are $q = m = 0$ and $q = 0$, $m = 1$. In the first case, as $(T, k) \to (t, x_0)$ with $|x_0 - k| \leq \lambda\sqrt{T - t}$, the second and the third term go to 0, as it should be. In the second case, we calculate the partial derivative

$$\frac{\partial}{\partial k}\overline{\sigma}_1(t, x_0, y_0, z_0, T, k) = \frac{1}{4}\rho_{12}\xi_1 v^{\alpha_1 - 1}(t).$$

□

3.4 The Asymptotic Expansion of Order 2

Theorem 12. *The asymptotic expansion of implied volatility of order* 2 *in the Gatheral model has the form*

$$\begin{aligned}\sigma(t) = {} & \sqrt{v_0} + \frac{1}{4}\rho_{12}\xi_1 v_0^{\alpha_1 - 1}(t)(k - x_0) \\ & - \frac{3}{16}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 5/2}(t)(k - x_0)^2 \\ & + \left[\frac{\kappa_1(v_0'(t) - v_0)}{4\sqrt{v(t)}} + \frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1}(t) + \frac{3}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 3/2}(t)\right](T - t) \\ & + o(T - t + (k - x_0)^2).\end{aligned} \tag{3.11}$$

Proof. To calculate $\sigma_2^{(\bar{\mathbf{z}})}(t, x, y_1, y_2, T, k)$, we use the second equation in (3.1). First, we calculate the linear differential operator $\tilde{\mathcal{L}}_2(t, T)$. This time, the index k takes two values: 1 and 2. The set $I_{2,1}$ is a singleton containing the multi-index $\mathbf{i} = (2)$. The corresponding part of the operator $\tilde{\mathcal{L}}_2(t, T)$ is

$$\int_t^T a_{11,2}(t_1, \mathcal{M}_1(t, t_1, x), \mathcal{M}_2(t, t_1, y_2), \mathcal{M}_3(t, t_1, y_3))\, dt_1 = 0.$$

The set $I_{2,2}$ is also a singleton containing the multi-index $\mathbf{i} = (1, 1)$. The operator $\tilde{\mathcal{L}}_2(t, T)$ becomes

$$\begin{aligned}\tilde{\mathcal{L}}_2(t, T) = \int_t^T \int_{t_1}^T & \mathcal{G}_1(t, t_1) \\ & \times a_{11,1}(t_2, \mathcal{M}_1(t, t_2, x), \mathcal{M}_2(t, t_2, y_2), \mathcal{M}_3(t, t_2, y_3))\, dt_2\, dt_1.\end{aligned}$$

Using Equations (2.14), (2.15), and (2.16), we obtain

$$\begin{aligned}\mathcal{G}_1(t, t_1) &= \frac{1}{2}[v(t) - y_2 - \kappa_1(v'(t) - v(t))(t_1 - t)]\frac{\partial}{\partial x} \\ &+ \frac{1}{2}[y_2 - v(t) - \rho_{12}\xi_1 v^{\alpha_1+1/2}(t_1 - t) + \kappa_1(v'(t) - v(t))(t_1 - t)]\frac{\partial^2}{\partial x^2} \\ &+ \frac{1}{2}\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_1 - t)\frac{\partial^3}{\partial x^3} + \cdots .\end{aligned}$$

The operator $\tilde{\mathcal{L}}_2(t, T)$ takes the form

$$\begin{aligned}\tilde{\mathcal{L}}_2(t, T) = \frac{1}{4}\int_t^T \int_{t_1}^T & \Big\{[v(t) - y_2 - \kappa_1(v'(t) - v(t))(t_1 - t)]\frac{\partial}{\partial x} \\ & + [y_2 - v(t) - \rho_{12}\xi_1 v^{\alpha_1+1/2}(t_1 - t) + \kappa_1(v'(t) - v(t))(t_1 - t)]\frac{\partial^2}{\partial x^2} \\ & + \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_1 - t)\frac{\partial^3}{\partial x^3}\Big\}\, [y_2 + \kappa_1(v'(t) - v(t))(t_2 - t) \\ & + \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_2 - t)\frac{\partial}{\partial x} - v(t)\Big]\, dt_2\, dt_1 + \cdots .\end{aligned}$$

We group all terms with the same partial derivatives together:

$$
\begin{aligned}
\tilde{\mathcal{L}}_2(t,T) &= \frac{1}{4}\int_t^T \int_{t_1}^T [v(t) - y_2 - \kappa_1(v'(t) - v(t))(t_1 - t)] \\
&\times [y_2 - v(t) + \kappa_1(v'(t) - v(t))(t_2 - t)]\, dt_2\, dt_1 \frac{\partial}{\partial x} \\
&+ \frac{1}{4}\int_t^T \int_{t_1}^T \Big\{[v(t) - y_2 - \kappa_1(v'(t) - v(t))(t_1 - t)]\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_2 - t) \\
&+ [y_2 - v(t) - \rho_{12}\xi_1 v^{\alpha_1+1/2}(t_1 - t) + \kappa_1(v'(t) - v(t))(t_1 - t)] \\
&\times [y_2 - v(t) + \kappa_1(v'(t) - v(t))(t_2 - t)]\Big\}\, dt_2\, dt_1 \frac{\partial^2}{\partial x^2} \\
&+ \frac{1}{4}\int_t^T \int_{t_1}^T \Big\{[y_2 - v(t) - \rho_{12}\xi_1 v^{\alpha_1+1/2}(t_1 - t) + \kappa_1(v'(t) - v(t))(t_1 - t)] \\
&\times \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_2 - t) + \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_1 - t) \\
&\times [y_2 - v(t) + \kappa_1(v'(t) - v(t))(t_2 - t)]\Big\}\, dt_2\, dt_1 \frac{\partial^3}{\partial x^3} \\
&+ \frac{1}{4}\int_t^T \int_{t_1}^T \rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_1 - t)\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)(t_2 - t)\, dt_2\, dt_1 \frac{\partial^4}{\partial x^4},
\end{aligned}
$$

and use the values of the following integrals:

$$
\begin{aligned}
\int_t^T \int_{t_1}^T dt_2\, dt_1 &= \frac{1}{2}\tau^2, \\
\int_t^T \int_{t_1}^T (t_1 - t)\, dt_2\, dt_1 &= \frac{1}{6}\tau^3, \\
\int_t^T \int_{t_1}^T (t_2 - t)\, dt_2\, dt_1 &= \frac{1}{3}\tau^3, \\
\int_t^T \int_{t_1}^T (t_1 - t)(t_2 - t)\, dt_2\, dt_1 &= \frac{1}{8}\tau^4.
\end{aligned}
$$

The functions $\chi^{(\bar{\mathbf{z}})}_{m,1}(t,\mathbf{z},T,k)$ become:

$$
\chi^{(\bar{\mathbf{z}})}_{1,1}(t,\mathbf{z},T,k) = -\frac{1}{96\sqrt{v(t)}}[12(y_2 - v(t))^2\tau + 12(y_2 - v(t))\kappa_1(v'(t) - v(t))\tau^2
$$

$$
\begin{aligned}
&+ 3\kappa_1^2(v'(t) - v(t))^2\tau^3], \\
\chi_{2,1}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k) &= \frac{1}{96\sqrt{v(t)}}[-12(y_2 - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^2 \\
&- 6\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^3 + 12(y_2 - v(t))^2\tau \\
&+ 12(y_2 - v(t))\kappa_1(v'(t) - v(t))\tau^2 \\
&+ 3\kappa_1^2(v'(t) - v(t))^2\tau^3], \\
\chi_{3,1}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k) &= \frac{1}{96\sqrt{v(t)}}[12(y_2 - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^2 \\
&- 3\rho_{12}^2\xi_1^2 v^{2\alpha_1+1}(t)\tau^3 \\
&+ 6\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^3], \\
\chi_{4,1}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k) &= \frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1+1/2}(t)\tau^3.
\end{aligned}
$$

Equation (3.3) gives

$$
\begin{aligned}
&\frac{u_2^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} = \frac{1}{48\sqrt{2}v(t)}[12(y_2 - v(t))^2\tau^{1/2} \\
&+ 12(y_2 - v(t))\kappa_1(v'(t) - v(t))\tau^{3/2} + 3\kappa_1^2(v'(t) - v(t))^2\tau^{5/2}]\zeta \\
&+ \frac{1}{96v^{3/2}(t)}[-12(y_2 - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau \\
&- 6\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^2 + 12(y_2 - v(t))^2 \\
&+ 12(y_2 - v(t))\kappa_1(v'(t) - v(t))\tau + 3\kappa_1^2(v'(t) - v(t))^2\tau^2](2\zeta^2 - 1) \\
&- \frac{1}{48\sqrt{2}v^2(t)}[12(y_2 - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^{1/2} - 3\rho_{12}^2\xi_1^2 v^{2\alpha_1+1}(t)\tau^{3/2} \\
&+ 6\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)\tau^{3/2}](2\zeta^3 - 3\zeta) \\
&+ \frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-3/2}(t)\tau(4\zeta^4 - 12\zeta^2 + 3).
\end{aligned}
$$

We use Equation (3.2) with $n = 2$:

$$
\frac{\frac{\partial^2}{\partial\sigma^2}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)} = \frac{2\zeta^2 - \sqrt{2y_2\tau}\zeta}{\sqrt{y_2}}.
$$

The second equation in (3.1) gives

$$\sigma_2^{(\bar{\mathbf{z}})}(t,x,y_1,y_2,T,k) = \frac{u_2^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} - \frac{1}{2}\left[\frac{2\zeta^2-\sqrt{2y_2\tau}\zeta}{\sqrt{y_2}}\right]$$
$$\times\left[\frac{y_2-v(t)}{2\sqrt{v(t)}} + \frac{\kappa_1(v(t)-v'(t))\tau}{4\sqrt{v(t)}} - \frac{1}{2\sqrt{2}}\rho_{12}\xi_1 v^{\alpha_1-1/2}(t)\sqrt{\tau}\zeta\right]^2.$$

We do not calculate this expression, but calculate $\overline{\sigma}_2(t,x,y_2,y_3,T,k)$ by Equation (3.7) with $N=2$:

$$\begin{aligned}
\overline{\sigma}_2(t,x,y_2,y_3,T,k) &= \sqrt{y_2} + \frac{\kappa_1(y_3-y_2)(T-t)}{4\sqrt{y_2}} \\
&- \frac{1}{8}\rho_{12}\xi_1 y_2^{\alpha_1-1}(2x-2k-y_2(T-t)) \\
&+ \frac{1}{16\sqrt{2}y_2}\kappa_1^2(y_3-y_2)^2\tau^{5/2}\zeta \\
&- \frac{\kappa_1(y_3-y_2)\tau^2}{32y_2^{3/2}}[2\rho_{12}\xi_1 y_2^{\alpha_1+1/2} - \kappa_1(y_3-y_2)](2\zeta^2-1) \\
&+ \frac{1}{16\sqrt{2}}[\rho_{12}^2\xi_1^2 y_2^{2\alpha_1-1}\tau^{3/2} \\
&- 2\kappa_1(y_3-y_2)\rho_{12}\xi_1 y_2^{\alpha_1-3/2}\tau^{3/2}](2\zeta^3-3\zeta) \\
&+ \frac{1}{32}\rho_{12}^2\xi_1^2 y_2^{2\alpha_1-3/2}\tau(4\zeta^4-12\zeta^2+3) \\
&- \frac{2\zeta^2-\sqrt{2y_2\tau}\zeta}{2\sqrt{y_2}}\left[\frac{\kappa_1(y_3-y_2)\tau}{4\sqrt{y_2}} - \frac{1}{2\sqrt{2}}\rho_{12}\xi_1 y_2^{\alpha_1-1/2}\sqrt{\tau}\zeta\right]^2.
\end{aligned}$$

In particular,

$$\begin{aligned}
\overline{\sigma}_2(t,x_0,y_0,z_0,T,k) &= \sqrt{y_2} + \frac{\kappa_1(v(t)-v'(t))(T-t)}{4\sqrt{v(t)}} \\
&- \frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1-1}(t)(2x_0-2k-v(t)(T-t)) \\
&+ \frac{1}{16\sqrt{2}v(t)}\kappa_1^2(v'(t)-v(t))^2(T-t)^{5/2}\zeta
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa_1(v'(t)-v(t))(T-t)^2}{32v^{3/2}(t)}[2\rho_{12}\xi_1 v^{\alpha_1+1/2}(t)-\kappa_1(v'(t)-v(t))](2\zeta^2-1)\\
&+\frac{1}{16\sqrt{2}}\rho_{12}^2\xi_1^2 v^{2\alpha_1-1}(t)(T-t)^{3/2}(2\zeta^3-3\zeta)\\
&+\frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-3/2}(t)(T-t)(4\zeta^4-12\zeta^2+3)\\
&-\frac{2\zeta^2-\sqrt{2y_2\tau}\zeta}{2\sqrt{y_2}}\left[\frac{\kappa_1(v'(t)-v(t))(T-t)}{4\sqrt{v(t)}}\right.\\
&\left.-\frac{1}{2\sqrt{2}}\rho_{12}\xi_1 v^{\alpha_1-1/2}(t)\sqrt{T-t}\zeta\right]^2.
\end{aligned}
$$

When we substitute ζ we get

$$
\begin{aligned}
\overline{\sigma}_2(t,x_0,y_0,z_0,T,k) &= \sqrt{y_2}+\frac{\kappa_1(v'(t)-v(t))(T-t)}{4\sqrt{v(t)}}\\
&-\frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1-1}(t)(2x_0-2k-v(t)(T-t))\\
&+\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^2(x_0-k-v(t)(T-t)/2)}{64v^{3/2}(t)}\\
&-\frac{1}{32}\kappa_1(v'(t)-v(t))(T-t)\rho_{12}\xi_1 v^{\alpha_1-2}(t)(x_0-k-v(t)(T-t)/2)^2\\
&+\frac{1}{16}\kappa_1(v'(t)-v(t))(T-t)^2\rho_{12}\xi_1 v^{\alpha_1-1}(t)\\
&+\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)(x_0-k-v(t)(T-t)/2)^2}{32v^{5/2}(t)}\\
&-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^2}{32v^{3/2}(t)}+\frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-5/2}(t)(x_0-k-v(t)(T-t)/2)^3\\
&-\frac{3}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-3/2}(t)(T-t)(x_0-k-v(t)(T-t)/2)\\
&-\frac{1}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-3}(x_0-k-v(t)(T-t)/2)^3\\
&+\frac{3}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-2}(T-t)(x_0-k-v(t)(T-t)/2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\rho_{12}^2\xi_1^2 v^{2\alpha_1-3}(t)(x_0-k-v(t)(T-t)/2)^4}{32(T-t)}\\
&-\frac{3}{16}\rho_{12}^2\xi_1^2 v^{2\alpha_1-3/2}(t)(x_0-k-v(t)(T-t)/2)^2+\frac{3}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-1}(t)(T-t)\\
&-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^2(x_0-k-v(t)(T-t)/2)}{32v^{3/2}(t)}\\
&-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)(x_0-k-v(t)(T-t)/2)^2}{32v^{5/2}(t)}\\
&+\frac{1}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-2}(t)(x_0-k-v(t)(T-t)/2)^2\\
&+\frac{1}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-3}(t)(x_0-k-v(t)(T-t)/2)^3\\
&-\frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-5/2}(t)(x_0-k-v(t)(T-t)/2)^3\\
&-\frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-7/2}(t)(x_0-k-v(t)(T-t)/2)^4.
\end{aligned}
$$

We do the multiplication to get:

$$
\begin{aligned}
&\overline{\sigma}_2(t,x_0,y_0,z_0,T,k)=\sqrt{v_0}+\frac{\kappa_1(v'(t)-v(t))(T-t)}{4\sqrt{v(t)}}\\
&-\frac{1}{4}\rho_{12}\xi_1 v^{\alpha_1-1}(t)(x_0-k)+\frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1}(t)(T-t)\\
&+\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^2(x_0-k)}{32v^{3/2}(t)}-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^3}{64\sqrt{v(t)}}\\
&-\frac{1}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-2}(t)(T-t)(x_0-k)^2\\
&+\frac{1}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-1}(t)(T-t)^2(x_0-k)\\
&-\frac{1}{64}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1}(t)(T-t)^3\\
&+\frac{1}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-1}(t)(T-t)^2\\
&+\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)(x_0-k)^2}{32v^{5/2}(t)}-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^2(x_0-k)}{32v^{3/2}(t)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^3}{128\sqrt{v(t)}}-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^2}{32v^{3/2}(t)}\\
&+\frac{1}{32}\rho_{12}^2\xi_1^2v^{2\alpha_1-5/2}(t)(x_0-k)^3-\frac{3}{64}\rho_{12}^2\xi_1^2v^{2\alpha_1-3/2}(t)(x_0-k)^2(T-t)\\
&+\frac{3}{128}\rho_{12}^2\xi_1^2v^{2\alpha_1-1/2}(t)(x_0-k)(T-t)^2-\frac{1}{256}\rho_{12}^2\xi_1^2v^{2\alpha_1+1/2}(t)(T-t)^3\\
&-\frac{3}{32}\rho_{12}^2\xi_1^2v^{2\alpha_1-3/2}(t)(x_0-k)(T-t)+\frac{3}{64}\rho_{12}^2\xi_1^2v^{2\alpha_1-1/2}(t)(T-t)^2\\
&-\frac{1}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1v^{\alpha_1-3}(t)(x_0-k)^3\\
&+\frac{3}{32}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1v^{\alpha_1-2}(t)(T-t)(x_0-k)^2\\
&-\frac{3}{64}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1v^{\alpha_1-1}(t)(T-t)^2(x_0-k)\\
&+\frac{1}{128}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1v^{\alpha_1}(t)(T-t)^3\\
&+\frac{3}{16}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1v^{\alpha_1-2}(t)(T-t)(x_0-k)\\
&-\frac{3}{32}\kappa_1(v'(t)-v(t))\rho_{12}\xi_1v^{\alpha_1-1}(t)(T-t)^2\\
&+\frac{\rho_{12}^2\xi_1^2v^{2\alpha_1-7/2}(t)(x_0-k)^4}{32(T-t)}-\frac{1}{16}\rho_{12}^2\xi_1^2v^{2\alpha_1-5/2}(t)(x_0-k)^3\\
&+\frac{3}{64}\rho_{12}^2\xi_1^2v^{2\alpha_1-3/2}(t)(x_0-k)^2(T-t)\\
&-\frac{1}{64}\rho_{12}^2\xi_1^2v^{2\alpha_1-1/2}(t)(x_0-k)(T-t)^2\\
&+\frac{1}{512}\rho_{12}^2\xi_1^2v^{2\alpha_1+1/2}(t)(T-t)^3-\frac{3}{16}\rho_{12}^2\xi_1^2v^{2\alpha_1-5/2}(t)(x_0-k)^2\\
&+\frac{3}{16}\rho_{12}^2\xi_1^2v^{2\alpha_1-3/2}(t)(x_0-k)(T-t)-\frac{3}{64}\rho_{12}^2\xi_1^2v^{2\alpha_1-1/2}(t)(T-t)^2\\
&+\frac{3}{32}\rho_{12}^2\xi_1^2v^{2\alpha_1-3/2}(t)(T-t)-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^2(x_0-k)}{32v^{3/2}(t)}\\
&+\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)^3}{64v^{1/2}(t)}-\frac{\kappa_1^2(v'(t)-v(t))^2(T-t)(x_0-k)^2}{32\sqrt{v^{5/2}(t)}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{32}\kappa_1^2(v'(t) - v(t))^2 v^{-3/2}(t)(x_0 - k)(T - t)^2 \\
&- \frac{1}{128}\kappa_1^2(v'(t) - v(t))^2 v^{-1/2}(t)(T - t)^3 \\
&+ \frac{1}{16}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1 - 2}(t)(T - t)(x_0 - k)^2 \\
&+ \frac{1}{16}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1 - 1}(t)(T - t)^2(x_0 - k) \\
&+ \frac{1}{64}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1}(t)(T - t)^3 \\
&+ \frac{1}{16}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1 - 3}(t)(x_0 - k)^3 \\
&- \frac{3}{32}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1 - 2}(t)(x_0 - k)^2(T - t) \\
&+ \frac{3}{64}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1 - 1}(t)(x_0 - k)(T - t)^2 \\
&- \frac{1}{128}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1}(t)(T - t)^3 \\
&- \frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 5/2}(t)(x_0 - k)^3 + \frac{3}{64}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 3/2}(t)(x_0 - k)^2(T - t) \\
&- \frac{3}{128}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 1/2}(t)(T - t)^2(x_0 - k) \\
&+ \frac{1}{256}\kappa_1(v'(t) - v(t))\rho_{12}\xi_1 v^{\alpha_1 + 1/2}(t)(T - t)^3 \\
&- \frac{1}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 7/2}(t)(T - t)^{-1}(x_0 - k)^4 \\
&+ \frac{1}{16}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 5/2}(t)(x_0 - k)^3 \\
&- \frac{3}{64}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 3/2}(t)(x_0 - k)^2(T - t) \\
&+ \frac{1}{64}\rho_{12}^2\xi_1^2 v^{2\alpha_1 - 1/2}(t)(x_0 - k)(T - t)^2 \\
&- \frac{1}{512}\rho_{12}^2\xi_1^2 v^{2\alpha_1 + 1/2}(t)(T - t)^3.
\end{aligned}
$$

We write the expression in the following form $\sum C_i(T-t)^{\alpha_i}|x_0-k|^{\beta_i}$:

$$
\begin{aligned}
\overline{\sigma}_2 &= \sqrt{v_0} \\
&+ \left[\frac{1}{4}\kappa_1(v_0' - v(t))v^{-1/2} + \frac{1}{8}\rho_{12}\xi_1 v_0^{\alpha_1} + \frac{3}{32}\rho_{12}^2\xi_1^2 v_0^{2\alpha_1-3/2}\right](T-t) \\
&- \frac{1}{4}\rho_{12}\xi_1 v_0^{\alpha_1-1}(x_0-k) - \frac{3}{16}\rho_{12}^2\xi_1 v_0^{2\alpha_1-5/2}(x_0-k)^2 \\
&+ \frac{3}{32}[2\kappa_1(v_0'-v_0)\rho_{12}\xi_1 v_0^{\alpha_1-2} + \rho_{12}^2\xi_1^2 v_0^{2\alpha_1-3/2}](T-t)(x_0-k) \quad (3.12) \\
&- \frac{1}{32}\left[\kappa_1(v_0'-v_0)\rho_{12}\xi_1 v_0^{\alpha_1-1} + \kappa_1^2(v_0'-v_0)^2 v_0^{-3/2}\right](T-t)^2 \\
&+ \frac{1}{128}\Big[22\kappa_1(v_0'-v_0)\rho_{12}\xi_1 v_0^{\alpha_1-1} + 4\kappa_1^2(v_0'-v_0)^2 v_0^{-3/2} \\
&+ 3\rho_{12}^2\xi_1^2 v_0^{2\alpha_1-1/2}\Big](T-t)^2(x_0-k).
\end{aligned}
$$

We group together all terms that include $(T-t)^q|x_0-k|^m$ with $2q+m=2$. The term with $q=0$ and $m=2$ is

$$-\frac{3}{16}\rho_{12}^2\xi_1^2 v^{2\alpha_1-5/2}(t)(k-x_0)^2.$$

The term with $q=1$ and $m=0$ is

$$\left[\frac{\kappa_1(v(t)-v'(t))}{4\sqrt{v(t)}} + \frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1}(t) + \frac{3}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-3/2}(t)\right](T-t).$$

These terms give the contribution of order 2 to Equation (3.11). □

3.5 The Asymptotic Expansion of Order 3

Theorem 13. *The asymptotic expansion of implied volatility of order* 3 *in the Gatheral model has the form*

$$
\begin{aligned}
\sigma &= \sqrt{v_0} + \frac{1}{4}\rho_{12}\xi_1 v^{\alpha_1-1}(t)(k-x_0) \\
&- \frac{3}{16}\rho_{12}^2\xi_1^2 v^{2\alpha_1-5/2}(t)(k-x_0)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{\kappa_1(v'(t)-v(t))}{4\sqrt{v(t)}}+\frac{1}{8}\rho_{12}\xi_1 v^{\alpha_1}(t)+\frac{3}{32}\rho_{12}^2\xi_1^2 v^{2\alpha_1-3/2}(t)\right](T-t)\\
&+\frac{1}{128}[16\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-3}+8\rho_{12}^2\xi_1^2 v_0^{2\alpha_1-5/2}\\
&-32\rho_{12}^3\xi_1^3 v_0^{3\alpha_1-4}](k-x_0)^3\\
&+\frac{1}{128}[24\kappa_1(v'(t)-v(t))\rho_{12}\xi_1 v^{\alpha_1-2}+12\rho_{12}^2\xi_1^2 v_0^{2\alpha_1-3/2}\\
&+35\rho_{12}^3\xi_1^3 v_0^{3\alpha_1-3}-8\rho_{12}\xi_1 v_0^{\alpha_1}](T-t)(k-x_0)\\
&+o((T-t)^{3/2}+|k-x_0|^3).
\end{aligned}
$$

Proof. Note that the first two coefficients of $(T-t)(k-x_0)$ go from (3.12).

First, we calculate the linear differential operator $\tilde{\mathcal{L}}_3(t,T)$. The sets $I_{3,k}$ have the form

$$
I_{3,1}=\{(3)\},\qquad I_{3,2}=\{(1,2),(2,1)\},\qquad I_{3,3}=\{(1,1,1)\}.
$$

Note that $a_{(2,0,0)^\top,2}=a_{(2,0,0)^\top,3}=0$, which makes the terms with $k=1$ and $(k=2,\mathbf{i}=(1,2))$ in (3.4) equal to 0. Equation (2.13) gives $\mathcal{G}_2(t,t_m)=0$, which excludes the term with $(k=2,\mathbf{i}=(2,1))$. We obtain

$$
\begin{aligned}
\tilde{\mathcal{L}}_3(t,T)=\int_t^T \mathrm{d}t_1\int_{t_1}^T \mathrm{d}t_2\int_{t_2}^T \mathrm{d}t_3\,&\mathcal{G}_1(t,t_1)\mathcal{G}_1(t,t_2)\\
&\times a_{(2,0,0)^\top,1}(t,\mathcal{M}_1(t,t_3),\mathcal{M}_2(t,t_3),\mathcal{M}_3(t,t_3)).
\end{aligned}
$$

Observe that the differential operator $\tilde{\mathcal{L}}_3(t,T)$ will be applied to the Black–Scholes price, which does not depend on v and v'. We denote by $\cdots$ all terms that contain partial derivatives with respect to the above two variables. In particular, we have

$$
\mathcal{G}_1(t,t_m)=a_{(2,0,0)^\top,1}(t,\mathcal{M}_1(t,t_m),\mathcal{M}_2(t,t_m),\mathcal{M}_3(t,t_m))\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right) \tag{3.13}
$$

with

$$
\begin{aligned}
&a_{(2,0,0)^\top,1}(t,\mathcal{M}_1(t,t_m),\mathcal{M}_2(t,t_m),\mathcal{M}_3(t,t_m))\\
&=\frac{1}{2}\left(v-v_0+(t_m-t)\kappa_1(v'-v)+(t_m-t)\rho_{12}\xi_1 v_0^{\alpha_1+1/2}\frac{\partial}{\partial x}+\cdots\right).
\end{aligned} \tag{3.14}
$$

The differential operator $\tilde{\mathcal{L}}_3(t,T)$ takes the form

$$\tilde{\mathcal{L}}_3(t,T) = \int_t^T \mathrm{d}t_1 \int_{t_1}^T \mathrm{d}t_2 \int_{t_2}^T \mathrm{d}t_3$$
$$\times a_{(2,0,0)^\top,1}(t, \mathcal{M}_1(t,t_1), \mathcal{M}_2(t,t_1), \mathcal{M}_3(t,t_1))$$
$$\times \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) a_{(2,0,0)^\top,1}(t, \mathcal{M}_1(t,t_2), \mathcal{M}_2(t,t_2), \mathcal{M}_3(t,t_2)) \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)$$
$$\times a_{(2,0,0)^\top,1}(t, \mathcal{M}_1(t,t_3), \mathcal{M}_2(t,t_3), \mathcal{M}_3(t,t_3)).$$

Substitute Equations (3.13) and (3.14) to the last display formula, group like terms, and introduce the following notation:

$$A(s,t) = v - v_0 + (s-t)\kappa_1(v' - v), \qquad B(s,t) = (s-t)\rho_{12}\xi_1 v_0^{\alpha_1+1/2}.$$

Equation (3.4) takes the form

$$\tilde{\mathcal{L}}_3(t,T) = \frac{1}{8}\int_t^T \int_{t_1}^T \int_{t_2}^T \Bigg[B(t_1,t)B(t_2,t)B(t_3,t)\frac{\partial^7}{\partial x^7}$$
$$+ [B(t_1,t)B(t_2,t)A(t_3,t) - 2B(t_1,t)B(t_2,t)B(t_3,t)$$
$$+ B(t_1,t)A(t_2,t)B(t_3,t)$$
$$+ A(t_1,t)B(t_2,t)B(t_3,t)]\frac{\partial^6}{\partial x^6} + [-2B(t_1,t)B(t_2,t)A(t_3,t)$$
$$+ B(t_1,t)A(t_2,t)A(t_3,t) + A(t_1,t)B(t_2,t)A(t_3,t)$$
$$- 2B(t_1,t)A(t_2,t)B(t_3,t) - 2A(t_1,t)B(t_2,t)B(t_3,t)$$
$$+ B(t_1,t)B(t_2,t)B(t_3,t) + A(t_1,t)A(t_2,t)B(t_3,t)]\frac{\partial^5}{\partial x^5}$$
$$+ [-2B(t_1,t)A(t_2,t)A(t_3,t) - 2A(t_1,t)B(t_2,t)A(t_3,t)$$
$$+ B(t_1,t)B(t_2,t)A(t_3,t) + A(t_1,t)A(t_2,t)A(t_3,t)$$
$$- 2A(t_1,t)A(t_2,t)B(t_3,t) + B(t_1,t)A(t_2,t)B(t_3,t)$$
$$+ A(t_1,t)B(t_2,t)B(t_3,t)]\frac{\partial^4}{\partial x^4} + [-2A(t_1,t)A(t_2,t)A(t_3,t)$$
$$+ B(t_1,t)A(t_2,t)A(t_3,t) + A(t_1,t)B(t_2,t)A(t_3,t)$$
$$+ A(t_1,t)A(t_2,t)B(t_3,t)]\frac{\partial^3}{\partial x^3} + A(t_1,t)A(t_2,t)A(t_3,t)\frac{\partial}{\partial x^2}\Bigg] \mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1.$$

To simplify this expression, the following integrals are important:

$$\int_t^T \int_{t_1}^T \int_{t_2}^T \mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{6}(T-t)^3,$$

$$\int_t^T \int_{t_1}^T \int_{t_2}^T (t_1-t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{8}(T-t)^4,$$

$$\int_t^T \int_{t_1}^T \int_{t_2}^T (t_2-t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{12}(T-t)^4,$$

$$\int_t^T \int_{t_1}^T \int_{t_2}^T (t_3-t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{24}(T-t)^4,$$

$$\int_t^T \int_{t_1}^T \int_{t_2}^T (t_1-t)(t_2-t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{15}(T-t)^5,$$

$$\int_t^T \int_{t_1}^T \int_{t_2}^T (t_1-t)(t_3-t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{30}(T-t)^5,$$

$$\int_t^T \int_{t_1}^T \int_{t_2}^T (t_2-t)(t_3-t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{40}(T-t)^5,$$

$$\int_t^T \int_{t_1}^T \int_{t_2}^T (t_1-t)(t_2-t)(t_3-t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{1}{48}(T-t)^6.$$

Equations (3.5), (3.6), and (A.2) give

$$\frac{u_3^{(\bar{\mathbf{z}})}(t,x,y_2,y_3,T,k)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma_0^{(\bar{\mathbf{z}})})} = \sum_{m=2}^{7}\left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^m \chi_{m,3}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k)H_m(\zeta),$$

where

$$\chi_{2,3}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k) = \frac{1}{48}\tau^2(v-v_0)^3 v_0^{-1/2} + \frac{1}{32}\tau^3(v-v_0)^2\kappa_1(v'-v)v_0^{-1/2}$$
$$+ \frac{1}{64}\tau^4(v-v_0)\kappa_1^2(v'-v)^2 v_0^{-1/2} + \frac{1}{384}\tau^5\kappa_1^3(v'-v)^3 v_0^{-1/2},$$

$$\chi_{3,3}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k) = -\frac{1}{24}\tau^2(v-v_0)^3 v_0^{-1/2} - \frac{1}{16}\tau^3(v-v_0)^2\kappa_1(v'-v)v_0^{-1/2}$$
$$- \frac{1}{32}\tau^4(v-v_0)\kappa_1^2(v'-v)^2 v_0^{-1/2} - \frac{1}{192}\tau^5\kappa_1^3(v'-v)^3 v_0^{-1/2}$$

$$
\begin{aligned}
&+\frac{1}{32}\tau^3\rho_{12}\xi_1 v_0^{\alpha_1}(v-v_0)+\frac{1}{128}\tau^5\rho_{12}\xi_1 v_0^{\alpha_1}\kappa_1^2(v'-v)^2\\
&+\frac{1}{32}\tau^4\rho_{12}\xi_1 v_0^{\alpha_1}(v-v_0)(v'-v),\\
\chi^{(\bar{\mathbf{z}})}_{4,3}(t,\mathbf{z},T,k)&=-\frac{1}{16}\tau^3\rho_{12}\xi_1 v_0^{\alpha_1}(v-v_0)^2\\
&-\frac{1}{16}\tau^4\rho_{12}\xi_1 v_0^{\alpha_1}(v-v_0)\kappa_1(v'-v)-\frac{1}{64}\tau^5\rho_{12}\xi_1 v_0^{\alpha_1}\kappa_1^2(v'-v)^2\\
&+\frac{1}{64}\tau^4\rho_{12}^2\xi_1^2 v_0^{2\alpha_1+1/2}(v-v_0)+\frac{1}{128}\tau^5\rho_{12}^2\xi_1^2 v_0^{2\alpha_1+1/2}\kappa_1(v'-v)\\
&+\frac{1}{48}\tau^2 v_0^{-1/2}(v-v_0)^3+\frac{1}{32}\tau^3 v_0^{-1/2}(v-v_0)^2\kappa_1(v'-v)\\
&+\frac{1}{64}\tau^4 v_0^{-1/2}(v-v_0)\kappa_1^2(v'-v)^2+\frac{1}{384}\tau^5 v_0^{-1/2}\kappa_1^3(v'-v)^3,\\
\chi^{(\bar{\mathbf{z}})}_{5,3}(t,\mathbf{z},T,k)&=-\frac{1}{32}\tau^4\rho_{12}^2\xi_1^2 v_0^{2\alpha_1+1/2}(v-v_0)+\frac{1}{128}\tau^5\rho_{12}\xi_1 v_0^{\alpha_1}\kappa_1^2(v'-v)^2\\
&+\frac{1}{32}\tau^4\rho_{12}\xi_1 v_0^{\alpha_1}(v-v_0)(v'-v)-\frac{1}{64}\tau^5\rho_{12}^2\xi_1^2 v_0^{2\alpha_1+1/2}\kappa_1(v'-v)\\
&+\frac{1}{32}\tau^3\rho_{12}\xi_1 v_0^{\alpha_1}(v-v_0)^2+\frac{1}{384}\tau^5\rho_{12}^3\xi_1^3 v_0^{3\alpha_1+1},\\
\chi^{(\bar{\mathbf{z}})}_{6,3}(t,\mathbf{z},T,k)&=\frac{1}{64}\tau^4\rho_{12}^2\xi_1^2 v_0^{2\alpha_1+1/2}(v-v_0)-\frac{1}{192}\tau^5\rho_{12}^3\xi_1^3 v_0^{3\alpha_1+1}\\
&+\frac{1}{128}\tau^5\rho_{12}^2\xi_1^2 v_0^{2\alpha_1+1/2}\kappa_1(v'-v),\\
\chi^{(\bar{\mathbf{z}})}_{7,3}(t,\mathbf{z},T,k)&=\frac{1}{384}\tau^5\rho_{12}^3\xi_1^3 v_0^{3\alpha_1+1}.
\end{aligned}
$$

Observe that the right hand side of the equation for $\chi^{(\bar{\mathbf{z}})}_{2,3}(t,\mathbf{z},T,k)$ has 4 terms. Each of them is multiplied by $4\zeta^2-2$ that also contains 4 terms. Overall, we have 285 terms here. Which of them give a contribution to the asymptotic expansion? We explain this issue using $\chi^{(\bar{\mathbf{z}})}_{2,3}(t,\mathbf{z},T,k)$ as an example.

First, the terms with nonzero powers of $v-v_0$ give no contribution by the same reason as previously. The only term which may give contribution is

$$
\frac{1}{384}(T-t)^5\kappa_1^3(v'-v)^3 v_0^{-1/2}.
$$

After multiplying by $\left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^2$, it becomes $\frac{1}{768}(T-t)^4\kappa_1^3(v'-v)^3 v_0^{-3/2}$. Write ζ in the following form:

$$\zeta = \frac{x-k}{\sigma\sqrt{2(T-t)}} - \frac{\sigma\sqrt{T-t}}{2\sqrt{2}}.$$

Checking all 16 terms of the product

$$\frac{1}{768}(T-t)^4\kappa_1^3(v'-v)^3 v_0^{-3/2}(4\zeta^2-2),$$

we see that no term has either the form $C(k-x_0)^3$ or $C(k-x_0)(T-t)$. There are no contributions to the asymptotic expansion here.

We continue at the same way and find two contributions from the term $\left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^7 \chi_{7,3}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k)H_7(\zeta)$. The first one is

$$\left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^7 \chi_{7,3}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k)\times 3360\left(\frac{x-k}{\sigma\sqrt{2(T-t)}}\right)^3,$$

which gives the third coefficient in $(k-x_0)^3$. The second one is

$$\left(\frac{1}{\sigma_0\sqrt{2\tau}}\right)^7 \chi_{7,3}^{(\bar{\mathbf{z}})}(t,\mathbf{z},T,k)\times(-1680)\frac{x-k}{\sigma\sqrt{2(T-t)}},$$

which gives the third coefficient in $(T-t)(k-x_0)$.

So far we calculated the contribution of the first term in the right hand side of the third equation in (3.1). We proceed to the second term.

The terms $\sigma_1^{(\bar{\mathbf{z}})}(t,x,y_1,y_2,T,k)$ and $\sigma_2^{(\bar{\mathbf{z}})}(t,x,y_1,y_2,T,k)$ are given by Equations (3.10) and (3.12). For the last term Equation (3.2) gives

$$\begin{aligned}\frac{\frac{\partial^2}{\partial\sigma^2}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)} &= \sum_{q=0}^{1} c_{2,2-2q}v_0^{1-2q}(T-t)^{1-q}\sum_{p=0}^{1-q}\binom{1-q}{p}\\ &\quad\times\left(\frac{1}{v_0\sqrt{2(T-t)}}\right)^{p+1-q} H_{p+1-q}(\zeta)\\ &= v_0(T-t)\left[\frac{H_1(\zeta)}{v_0\sqrt{2(T-t)}}+\frac{H_2(\zeta)}{2v_0^2(T-t)}\right]+v_0^{-1}.\end{aligned}$$

Using Equation (A.2), we find

$$\frac{\frac{\partial^2}{\partial\sigma^2}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)} = -\frac{\sqrt{v_0}}{2} + v_0^{-3/2}(x-k)^2(T-t)^{-1}.$$

The product of 3 terms in Equation (3.10), 12 terms in Equation (3.12), and 3 terms in the last display contains 72 terms. Of that, three terms contribute to all coefficients in $(k-x_0)^3$ and one term contributes to the last coefficient in $(k-x_0)(T-t)$.

We proceed to calculate the term $\frac{\frac{\partial^3}{\partial\sigma^3}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)}$. Equation (3.2) gives

$$\begin{aligned}\frac{\frac{\partial^3}{\partial\sigma^3}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)} &= \sum_{q=0}^{1} c_{3,3-2q}\sigma^{2-2q}\tau^{2-q}\sum_{p=0}^{2-q}\binom{2-q}{p}\left(\frac{1}{\sigma\sqrt{2\tau}}\right)^{p+2-q} H_{p+2-q}(\zeta)\\ &= \frac{\tau}{2}H_2(\zeta) + \frac{\sqrt{\tau}}{\sqrt{2}\sigma}H_3(\zeta) + \frac{1}{4\sigma^2}H_4(\zeta) + \frac{3\sqrt{\tau}}{\sqrt{2}\sigma}H_1(\zeta)\\ &\quad + \frac{3}{2\sigma^2}H_2(\zeta).\end{aligned}$$

Using Equation (A.2), we obtain

$$\begin{aligned}\frac{\frac{\partial^3}{\partial\sigma^3}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)} &= \tau(2\zeta^2-1) + \frac{\sqrt{\tau}}{\sigma}(4\sqrt{2}\zeta^3 - 6\sqrt{2}\zeta) + \frac{1}{\sigma^2}(4\zeta^4 - 12\zeta^2 + 3)\\ &\quad + \frac{3\sqrt{2\tau}}{\sigma}\zeta + \frac{3}{\sigma^2}(2\zeta^2-1)\\ &= \frac{4}{\sigma^2}\zeta^4 + \frac{4\sqrt{2\tau}}{\sigma}\zeta^3 + \left(2\tau - \frac{6}{\sigma^2}\right)\zeta^2 - \frac{3\sqrt{2\tau}}{\sigma}\zeta - \tau,\end{aligned}$$

and finally,

$$\begin{aligned}\frac{\frac{\partial^3}{\partial\sigma^3}u^{\mathrm{BS}}(\sigma)}{\frac{\partial}{\partial\sigma}u^{\mathrm{BS}}(\sigma)} &= \frac{1}{\sigma^6}(x-k)^4\tau^{-2} - \frac{1}{2\sigma^2}(x-k)^2 + \frac{\sigma^2}{16}\tau^2 - \frac{3}{\sigma^4}(x-k)^2\tau^{-1}\\ &\quad - \frac{1}{4}\tau.\end{aligned}$$

The term $(\sigma_1^{(\bar{\mathbf{z}})}(t, x, y_1, y_2, T, k))^3$ contains 4 parts. Its product by the right hand side of the last display contains 24 terms and no of them give any contribution to the asymptotic expansion of order 3.

$\square$

Chapter 4

Model Calibration

This Chapter is based on Papers **C** and **D**.

Numerical studies of implied volatility expansions under the Gatheral model
M. Dimitrov, M. Albuhayri, Y. Ni, A Malyarenko

Applied Modeling Techniques and Data Analysis 2: Financial, Demographic, Stochastic and Statistical Models and Methods, vol. 9 of Big data, artificial intelligence, and data analysis set, K. N. Zafeiris, C. H. Skiadas, Y. Dimotikalis, A. Karagrigoriu, C. Karagrigoriou-Vonta, (eds.), Wiley, 2022, Chapter 10, pp. 135–148.

Numerical studies of the implied volatility expansions up to third order under the Gatheral model
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4.1 Introduction

The calibration of a financial model can be described as a reversal process of an optimisation problem. Thus, in calibration, we know the answer (the price of contracts) and want to find the problem (the parameters). Therefore, the price of a financial instrument is not the objective in the first place but rather to get a set of parameters for a specific model that generates values for financial instruments consistent with the market. It is a reversal process of the whole theoretical valuation.

In order to go through this process, one need to pay serious attention to some fundamental questions that arise.

4.1.1 Why We Do Calibration?

In the famous Black–Scholes formula which takes five variables as inputs: the price of the underlying, the strike price, time to maturity, interest rate and the volatility. If we put numerical values for the five variables, then the formula returns a value for the option. This called the theoretical value of the model and it is based on the assumptions that in an efficient market, one can observe anything in the market and eventually calculate the price of the security.

Nevertheless, if this is the case and the market is always right, what does a deviated model value from an observed market value mean?

One of the fundamental conditions of a financial market is to be complete, meaning that any given derivative can be replicated perfectly by trading in its underlying. In contrast, market incompleteness can be a hurdle when derivative assets have multiple prices. Calibration comes into play to solve this issue because when calibration returns the market-consistent values for the derivative product, it chooses the right risk-neutral probability measure.

4.1.2 What Error Function?

To judge the reliability of the calibration, one needs a tool to measure the calibration's performance, an error function to be minimised. Moreover, when choosing an error function for calibration purpose, the objective of calibration e.g. valuation, speculation and hedging should be taken into consideration

Christoffersen and Jacobs (2004). There are many error functions in the literature, for instance more often the performance of options pricing models is evaluated using the mean-square error loss function which is given by:

$$MSE(\theta) = \frac{1}{n}\sum_{i=1}^{n}(C_i - C_i(\theta))^2,$$

where θ is the set of model parameters to be calibrated.

Some researchers prefer the relative option price differences of the loss function due to the fact that the previous function assigns a lot of weight to options with high valuations and thus high error. The mean square error of the relative error price differences is defined as

$$RMSE(\theta) = \frac{1}{n}\sum_{i=1}^{n}\frac{(C_i - C_i(\theta))^2}{C_i}.$$

This type of error function emphasises out-of-money products. Another popular choices of the error function is the mean error of the implied volatility difference and it is given by:

$$IVMSE(\theta) = \frac{1}{n}\sum_{i=1}^{n}(\sigma_i - \sigma_i(\theta))^2,$$

where σ_i and $\sigma_i(\theta)$ are the market and model implied volatility respectively. Finally, in Pagliarani and Pascucci (2017) paper which has been used extensively in this work, the error function has the following form.

$$P\&P\ \ MSE(\theta) = \sum_{i=1}^{n}\frac{|\sigma_i - \sigma_i(\theta)|}{\sigma_i}.$$

It is important to note that relying on the specific error function, the resulting model parameters may vary.

4.1.3 What Market Data to Calibrate a Model?

There are relevant questions to the type of market data that one should try to calibrate a mode, for instance: what the associate index is when analysing stock index derivatives? For instance, S&P 500, Dow Jones, Nasdaq, and EURO STOXX 50 could be candidates. Another question is about the concrete options quotes to be used for the calibration e.g. option strikes and maturities. Moreover, there are different quotes per options, like ask, bid, last and settlement price. One also might need to modify the raw data since there is no guarantee that market prices are arbitrage-free.

4.2 The Results

Before conducting numerical analysis on asymptotic expansions under the Gatheral model, one needs to check the accuracy of the expansions against a benchmark value. In order to do that, we try to calculate the option price in Gatheral model using Fourier–based method and compare it with the expansions. Unfortunately, this was not possible due to the fact that the Gatheral model is not an affine process in the general case and therefore not compatible with the Fourier–based method. We explain below why the Gatheral model is not affine.

$$\begin{aligned}
\mathrm{d}S_t &= \sqrt{v_t} S_t \,\mathrm{d}W_t^1, \\
\mathrm{d}v_t &= \kappa_1 (v_t' - v_t)\,\mathrm{d}t + \xi_1 v_t^{\alpha_1}\,\mathrm{d}W_t^2, \\
\mathrm{d}v_t' &= \kappa_2 (\theta - v_t')\,\mathrm{d}t + \xi_2 v'^{\alpha_2}_t\,\mathrm{d}W_t^3,
\end{aligned}$$

where the Wiener processes W_t^i are correlated: $\mathsf{E}[W_s^i W_t^j] = \rho_{ij}\min\{s,t\}$. After transition to the logarithmic price $s_t = \ln S_t$, we write it in the form

$$\mathrm{d}\mathbf{X}_t = \boldsymbol{\mu}(\mathbf{X}_t)\,\mathrm{d}t + \Sigma(\mathbf{X}_t)\,\mathrm{d}\mathbf{W}_t,$$

where

$$\mathbf{X}_t = \begin{pmatrix} s_t \\ v_t \\ v_t' \end{pmatrix}, \qquad \boldsymbol{\mu}(\mathbf{X}_t) = \begin{pmatrix} -v_t/2 \\ \kappa_1(v_t' - v_t) \\ \kappa_2(\theta - v_t') \end{pmatrix},$$

$$\Sigma(\mathbf{X}_t)\Sigma^\top(\mathbf{X}_t) = \begin{pmatrix} v_t & \rho_{12}\xi_1 v_t^{\alpha_1+1/2} & \rho_{13}\xi_2 v_t^{1/2} v_t'^{\alpha_2} \\ \rho_{12}\xi_1 v_t^{\alpha_1+1/2} & \xi_1^2 v_t^{2\alpha_1} & \rho_{23}\xi_1\xi_2 v_t^{\alpha_1} v_t'^{\alpha_2} \\ \rho_{13}\xi_2 v_t^{1/2} v_t'^{\alpha_2} & \rho_{23}\xi_1\xi_2 v_t^{\alpha_1} v_t'^{\alpha_2} & \xi_2^2 v_t'^{2\alpha_2} \end{pmatrix}.$$

By definition, the system is affine if and only if both the vector $\boldsymbol{\mu}(\mathbf{X}_t)$ and the matrix $\Sigma(\mathbf{X}_t)\Sigma^\top(\mathbf{X}_t)$ may be written in the form

$$\begin{aligned}
\boldsymbol{\mu}(\mathbf{X}_t) &= \mathbf{K}_0 + \mathbf{K}_1 s_t + \mathbf{K}_2 v_t + \mathbf{K}_3 v_t', \\
\Sigma(\mathbf{X}_t)\Sigma^\top(\mathbf{X}_t) &= H_0 + H_1 s_t + H_2 v_t + H_3 v_t',
\end{aligned}$$

where all vectors and matrices are constants. It is easy to see that this may happen if and only if $\alpha_1 = \alpha_2 = 1/2$, $\rho_{13} = \rho_{23} = 0$. For Fourier-based option pricing method, we refer to Heston (1993) who introduces this method to mathematical finance and also to the important work of Carr and Madan (1999) and Lewis (2001).

Therefore, we are left with the Monte-Carlo method as the only mean of checking the accuracy of our asymptotic expansions. Thus, we use the Monte-Carlo simulation to generate the benchmark value for the implied volatilities. All errors are calculated by treating the benchmark values as the exact values. The parameters used for the simulation are given in Dimitrov et al. (2022, Table 1). The performance of the asymptotic expansions of the implied volatility of the first and second order under the Gatheral model is investigated. We focus on two special cases of the Gatheral model given in (1.8): the double Heston model and the double lognormal model.

We consider 130 options with ten maturities (30, 60, 91, 122, 152, 182, 273, 365, 547, and 730 calendar days) and with log-moneyness between -0.2 and 0.2 and report the proportion of options that can be approximated within a relative error of 5% using the second-order asymptotic expansion. For the Double Heston model, this proportion is 45% of all options. However, the accuracy becomes much higher for options with log-moneyness between -0.1 and 0.07, and maturities from 30 days to 1 year.

It can be seen in Dimitrov et al. (2022, Fig 1 and Fig 2), the asymptotic expansions of order 1 and 2 of the implied volatility and the benchmark values for two different maturities, 30 days and 1 year respectively. The asymptotic expansion of order 2 gives better result as expected and it is more accurate for maturities shorter than 1 year.

For the Double Lognormal model, with the second order expansion, the corresponding proportion of option that can be approximated within a relative error of 5% is around 55%. For options with log-moneyness between -0.07 and 0.096, and maturities from 30 days to 1 year options, the accuracy again becomes higher.

Similarly, we conducted extensive studies in paper **D** to check and compare the accuracy of the asymptotic expansions of the implied volatility up to order 3. The parameter choices come from Gatheral (2008). We consider 100 options with strike prices varying from 80 to 122 with maturities ranging from 30 days to 2 years and log-moneyness between −0.2 and 0.2 as well as for shorter range of log-moneyness between −0.1 and 0.1. Figures 1 in paper **D** gives examples of the relative errors of asymptotic expansions of orders 1, 2 and 3 of the implied volatility compared to the Monte–Carlo benchmark value for 4 different times to maturities, 30 days, 60 days , 1 year and 2 years, respectively. For these maturities, we use a larger number of 500 time steps. The accuracy of asymptotic expansion of order one can only give a decent approximation for a maturity of 30 day. In contrast, the asymptotic expansions of order two and three give good performances for maturities up to 2 years and are almost identical, though we hope that order three expansion gives much better result. Moreover, for the first order expansion with a range of log-moneyness between −0.1 and 0.1, the proportion of options that can be approximated within a relative error of 5% is around 67% in Fig. 1a in paper **D** for 30 days of maturities.

In contrast, the accuracy becomes higher for the second and third order expansions in Fig. 1 in paper **D** with almost 70% of options within relative error of 5% in Fig. 1a and Fig. 1b and Fig. 1c in paper **D** and a 100% of options within error of 5% for 2 years of options. Similar experiments have been done using long range of log-moneyness, the results are alike. Parts of the experiment is presented in Table 3 in paper **D**.

In Gatheral (2008) and Bayer et al. (2013) Gatheral et al. shown that the model given in (1.8) calibrated so well for the SPX and VIX indices using a multitude of long and complicated steps. Though, there is no need to repeat the calibration procedure for this model, we are motivated by two reasons:

1. We have analytical solutions under the Gatheral model; the asymptotic expansions of implied volatility for European option.

2. The model has not been calibrated to equity stock before.

We propose a partial calibration procedure for some model parameters or some group of parameters in Dimitrov et al. (2022) . The calibration procedure is implemented on real and synthetic market data of daily implied volatility surfaces for an underlying market index; EURO Stoxx 50 index and underlying equity stock; ABB stock for periods both before and during the pandemic crisis. The data set consists of 10 time to maturities 30, 60, 91, 122, 152, 182, 273, 365, 547, and 730 calendar days. There are also 30 strike prices obtained from the well-known Greek Deltas ($0.20+0.05n$, $n = 0, 1, 2, \ldots, 12$).

We exploit the simplicity of the asymptotic expansions of order one and two to perform daily calibration for the expansion parameters. As expected, the calibrated parameters give values with fairly small differences compare to the true values, see Dimitrov et al. (2022, Table 2). Also, when applying the calibration procedure to the real market data, we note clearly the effect of the pandemic on the model. For instance, we calibrated the daily value of the volatility process of the ABB stock and Eurostock 50 Index, respectively in Dimitrov et al. (2022, Fig 3 and Fig 4). This is the period when Covid-19 start to spread in Europe and as a result high volatility was expected. The effect of the pandemic was more clear on the ABB stock compare to Eurostock 50 Index. We also note stabilisation issue and some extreme values during the calibration, therefore, one should pay particular attention to the calibration when the market is undergoing a similar crisis.

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Appendix A

Miscellaneous formulae

In this Appendix, we collect various definitions and theorems necessary for understanding the main text of this thesis. Usually, they are not present in standard courses.

A.1 The Bell Polynomials

These polynomials have been introduced by Bell (1927/28). Let m be a positive integer, and let an integer h satisfies $1 \le h \le m$. Let $\mathbf{z} \in \mathbb{R}^{m-h+1}$.

Definition 24. The *Bell polynomial* $B_{m,h}(z_1, \dots, z_{m-h+1})$ is given by

$$B_{m,h}(z_1, \dots, z_{m-h+1}) = m! \sum_{\beta_1,\dots,\beta_{m-h+1}} \prod_{i=1}^{m-h+1} \frac{z_i^{\beta_i}}{\beta_i!(i!)^{\beta_i}},$$

where the sum is taken over all sequences $\beta_1, \dots, \beta_{m+h-1}$ of nonnegative integers satisfying

$$\begin{aligned} \beta_1 + \beta_2 + \cdots + \beta_{m+h-1} &= h, \\ \beta_1 + 2\beta_2 + \cdots + (m-h+1)\beta_{m-h+1} &= m. \end{aligned}$$

In the main text, we use the following three Bell polynomials:

$$\begin{aligned} B_{2,2}(z_1) &= z_1^2, \\ B_{3,2}(z_1, z_2) &= 3z_1 z_2, \\ B_{3,3}(z_1) &= z_1^3. \end{aligned} \tag{A.1}$$

Note that the Bell polynomial $B_{m,h}(z_1, \ldots, z_{m-h+1})$ is a homogeneous polynomial in $m - h + 1$ variables of degree h, it has the form

$$B_{m,h}(\mathbf{z}) = \sum_{\beta} b_{\beta} \mathbf{z}^{\beta}$$

with $b_{\beta} = m! \prod_{i=1}^{m-h+1} \frac{1}{\beta_i!(i!)^{\beta_i}}$.

A.2 The Faà di Bruno Formula

The above formula is just a general version of the Chain Rule. Let Let $\mathbf{G}\colon \mathbb{R} \to \mathbb{R}^n$ and $F\colon \mathbb{R}^n \to \mathbb{R}$ be two smooth functions. Let $\nabla^h F$ be the rank h tensor of partial derivatives of order h of the function F.

Theorem 14 (The Faà di Bruno formula)**.** *For any positive integer m, we have*

$$\begin{aligned} \frac{d^m F(\mathbf{G}(t))}{dt^m} = &\sum_{h=1}^{m} \partial^h F(G_1(x), \ldots, G_n(x)) \\ &\times B_{n,h}(G'(x), G''(x), \ldots, G^{(m-h+1)}(x)). \end{aligned}$$

This result was proved by Arbogast (1800), and only more than half century later by Faà di Bruno (1855). See also a historical survey in Frabetti and Manchon (2015).

A.3 The "Physicists" Hermite Polynomials

Definition 25. The "physicists" Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{\partial^n e^{-x^2}}{\partial x^n}.$$

These polynomials are orthogonal with respect to the measure $d\mu(x) = e^{-x^2}\,dx$, that is,

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)\,d\mu(x) = \sqrt{\pi}2^n n!\delta_{mn}.$$

The fist several physicists Hermite polynomials are

$$\begin{aligned}
H_0(x) &= 1,\\
H_1(x) &= 2x,\\
H_2(x) &= 4x^2 - 2,\\
H_3(x) &= 8x^3 - 12x,\\
H_4(x) &= 16x^4 - 48x^2 + 12,\\
H_5(x) &= 32x^5 - 160x^3 + 120x,\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120,\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x.
\end{aligned} \tag{A.2}$$

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