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On Hom-associative Ore Extensions

Per Bäck

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Abstract

In this thesis, we introduce and study *hom-associative Ore extensions*. These are non-unital, non-associative, non-commutative polynomial rings in which the associativity condition is "twisted" by an additive group *hom*omorphism. In particular, they are examples of *hom-associative algebras*, and they generalize the classical non-commutative polynomial rings introduced by Ore known as *Ore extensions* to the non-unital, hom-associative setting. At the same time, when the twisted associativity condition is null, they also generalize to the general non-unital, non-associative setting. We deduce necessary and sufficient conditions for hom-associative Ore extensions to exist, and construct concrete examples thereof. These include hom-associative generalizations of the quantum plane, the universal enveloping algebra of the two-dimensional non-abelian Lie algebra, and the first Weyl algebra, to name a few. The aforementioned algebras turn out to be formal hom-associative deformations of their associative counterparts, the latter two which cannot be formally deformed in the associative setting. Moreover, these are all *weakly unital* algebras, and we provide a way of embedding any multiplicative, non-unital hom-associative algebra into a multiplicative, weakly unital hom-associative algebra. This generalizes the classical unitalization of non-unital, associative algebras. We then study the hom-associative Weyl algebras in arbitrary characteristic, classify them up to isomorphism, and in the zero characteristic case, we prove that an analogue of the Dixmier conjecture is true. We also study hom-modules over hom-associative rings, and by doing so, we are able to prove a Hilbert's basis theorem for hom-associative Ore extensions. Our theorem includes as special cases both the classical Hilbert's basis theorem for Ore extensions and a Hilbert's basis theorem for unital, non-associative Ore extensions. Last, we construct examples of both hom-associative and non-associative Ore extensions which are all Noetherian by our theorem.

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Populärvetenskaplig sammanfattning

Ore-utvidgningar eller *algebror av icke-kommutativa polynom* dyker upp på många ställen i både matematik och fysik. I exempelvis kvantfysik är två icke-kommutativa polynom ensamma skyldiga för den berömda Schrödingers katts död (och födelse). I den här avhandlingen introducerar vi Ore-utvidgningar i en vidare bemärkelse än den vanliga. Vi vet inte om det finns fysik bortom kvantfysiken, men vi vet att det finns intressant matematik där.

Att gå från klassisk fysik till kvantfysik kräver att man ersätter *funktioner* med *operatorer*. När man multiplicerar två funktioner x och y spelar ordningen ingen roll eftersom den *kommutativa lagen*, $x \cdot y = y \cdot x$, gäller. Med vanlig multiplikation bildar funktioner en *kommutativ algebra*. Det gäller däremot inte för operatorer; operationen att först ta på sig strumporna och sedan skorna är inte samma operation som att först ta på sig skorna och sedan strumporna. Med denna multiplikation bildar operatorerna en *icke-kommutativ algebra*. Att gå från den kommutativa algebran i klassisk fysik till den icke-kommutativa algebran i kvantfysik, *Weylalgebran*, kräver att man *deformerar* den förra algebran till den senare. Det finns däremot inget sätt att "gå bortom" kvantfysiken genom att deformera Weylalgebran utan att också behöva ge upp den *associativa lagen*, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

I den här avhandlingen introducerar och undersöker vi algebror av icke-kommutativa polynom för vilka den *hom-associativa lagen* gäller, $\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z)$. Här är α en funktion känd som en *hom*omorfi, därav namnet. Algebrorna vi introducerar kallas *hom-associativa Ore-utvidgningar* då de generaliserar de klassiska algebrorna av icke-kommutativa polynom som Ore introducerade 1933 till hom-associativa. Vi konstruerar exempel av hom-associativa Ore-utvidgningar som generaliserar kända Ore-utvidgningar, som exempelvis de *hom-associativa Weylalgebrorna*. Dessa generaliserar och är en deformation av Weylalgebran. Vi undersöker sedan de hom-associativa Weylalgebrorna närmare, och visar bland annat att en motsvarighet till den berömda och ännu olösta *Dixmiers förmodan* från sextiotalet är sann för dessa.

Sist utvidgar vi en känd sats för Ore-utvidgningar vid namn Hilberts bassats till en hom-associativ motsvarighet. Enligt denna sats kan man "lyfta" egenskapen av att vara såkallat *noethersk* från en algebra till dess Ore-utvidgning. Vi konstruerar sedan hom-associativa Ore-utvidgningar som enligt vår sats är noetherska.

Popular science summary

Ore extensions or *algebras of non-commutative polynomials* show up in many places in both mathematics and physics. For instance, in quantum physics, two non-commutative polynomials alone are held responsible for the death (and birth) of the famous Schrödinger's cat. In this thesis, we introduce Ore extensions in a setting that goes beyond the classical one. We do not know if there is physics beyond quantum physics, but we do know there is interesting mathematics there.

Transitioning from classical physics to quantum physics requires replacing *functions* by *operators*. When multiplying any two functions x and y, the order does not matter since the *commutative law*, $x \cdot y = y \cdot x$, holds. Functions form a *commutative algebra* under ordinary multiplication. This is not true for operators, however; the operation of first putting on your socks and then your shoes is not the same as the operation of first putting on your shoes and then your socks. The operators form a *non-commutative algebra* under this multiplication. Transitioning from the commutative algebra in classical physics to the non-commutative algebra in quantum physics, the *Weyl algebra*, requires *deforming* the former algebra to the latter. However, there is no way to "go beyond" quantum physics by deforming the Weyl algebra without also giving up the *associative law*, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

In this thesis, we introduce and study algebras of non-commutative polynomials for which the *hom-associative law*, $\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z)$, hold. Here, α is a function known as a *hom*omorphism, hence the name. The algebras we introduce are called *hom-associative Ore extensions* as they generalize the classical algebras of non-commutative polynomials introduced by Ore in 1933 to the hom-associative setting. We construct examples of hom-associative Ore extensions generalizing classical Ore extensions, such as the *hom-associative Weyl algebras*. These generalize and are a deformation of the Weyl algebra. We then study the hom-associative Weyl algebras in greater detail, and show, among other things, that an analogue of the famous and still unresolved *Dixmier conjecture* from the 1960s is true in the hom-associative setting.

Last, we extend a classical theorem for Ore extensions known as Hilbert's basis theorem to the hom-associative setting. By this theorem, one can "lift" a property of being so-called *Noetherian* from an algebra to its Ore extension. We then construct hom-associative Ore extensions which are Noetherian by our theorem.

Till M, Pi & Viggo

Preface

This thesis is based on five scientific papers (**A**–**E**). A summary of each paper can be found in Section 1.4.

A P. Bäck, J. Richter, and S. Silvestrov,
Hom-associative Ore extensions and weak unitalizations,
Int. Electron. J. Algebra **24** (2018), pp. 174–194, `arXiv:1710.04190`.

B P. Bäck,
Notes on formal deformations of quantum planes and universal enveloping algebras,
J. Phys.: Conf. Ser. **1194**(1) (2019), `arXiv:1812.00083`.

C P. Bäck and J. Richter,
On the hom-associative Weyl algebras,
J. Pure Appl. Algebra **224**(9) (2020), `arXiv:1902.05412`.

D P. Bäck and J. Richter,
The hom-associative Weyl algebras in prime characteristic,
Int. Electron. J. Algebra **31** (2022), pp. 203–229, `arXiv:2012.11659`.

E P. Bäck and J. Richter,
Hilbert's basis theorem for non-associative and hom-associative Ore extensions,
Algebr. Represent. Theory (2022), `arXiv:1804.11304`.

By the same author, but not included in this thesis:

- P. Bäck,
  Multi-parameter formal deformations of ternary hom-Nambu-Lie algebras,
  Lie Theory and Its Applications in Physics, Springer Proceedings in Mathematics & Statistics **335**, Springer, Singapore (2020),
  `arXiv:1911.07051`.

- P. Bäck and G. Karaali,
  The algebra detective: if Snape is a snake, then $p = k!$,
  Front. Young Minds **8**:524026 (2020).

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Dö skâ hô tack no,
Per

Contents

Chapter 1

Chapter 1

Introduction and summary

"The mysterious character that entered through Fimle's window was a man of many names. Fimle knew him as X, and for simplicity, let us call him that as well."

In *The spies of Oreborg*,
by Jakob Wegelius

The mysterious man with the elegant mustache hiding in the bushes on the front cover, is no other than the spy X. X, depicted by Jakob Wegelius in his book *The spies of Oreborg*, turns out to be not only a main character in the story that takes place in Oreborg in Oreland, but also in that of *hom-associative Ore extensions*. This thesis is about the latter.

Hom-associative Ore extensions are examples of *hom-associative algebras*, which historically originate with the development of *hom-Lie algebras*. The latter constitute a class of algebras introduced by Hartwig, Larsson, and Silvestrov [35] to describe deformed Lie algebras obeying a generalized Jacobi identity twisted by a vector space *hom*omorphism; hence the name. Any Lie algebra can in fact be seen as a hom-Lie algebra in which this homomorphism is the identity map. Hartwig, Larsson, and Silvestrov were mainly motivated by studying so-called q-deformations of the Witt and Virasoro Lie algebra, which have been of interest in e.g. mathematical physics [1, 13–16, 19, 20, 38, 42, 49–51]. These two algebras cannot be formally deformed as Lie algebras [25–27, 72], meaning there is no way to 'bend' these Lie algebras into some other Lie algebras by using a formal power series expansion to deform the bracket. Moreover, the Witt and Virasoro algebra are far from the only Lie algebras that cannot be formally deformed. For example, by the so-called second Whitehead Lemma, this is also the case for any finite-dimensional semi-simple Lie algebra over a field of characteristic zero. Hence the urge to look for a more general setting in which e.g. these kinds of q-deformed Lie algebras fit into arose, and the class of hom-Lie algebras seemed to provide a context in which both deformed and various other generalizations of Lie algebras could now be described. In this setting, hom-associative algebras were later introduced by Makhlouf and Silvestrov [56] as the natural counterparts to associative algebras; taking a hom-associative algebra and defining the commutator as a new multiplication gives a hom-Lie algebra, this in the exact same way as an associative algebra can be made into a Lie algebra. In more detail, a hom-associative algebra is an algebra in which the associativity condition is twisted by a linear map. Both associative and non-associative algebras can in fact be seen as hom-associative algebras; in the first case by taking this map equal to the identity map, and in the latter case by taking it equal to the zero map. It did not take long until it was discovered that there were formally rigid associative algebras that could now be formally deformed as hom-associative algebras [55], this indicating that hom-associative algebras could be useful in studying deformations as well. Since then, many papers have been written in the subject, and many other algebraic structures have been discovered to have natural counterparts in the "hom-world" as well, such as e.g. hom-coalgebras, hom-bialgebras, and hom-Hopf algebras [57, 58], just to name a few. Let us also mention that Larsson and Silvestrov [45, 47, 48] have considered generalizations of hom-algebras which naturally arise in connection with *quasi-deformations*, such as *quasi-hom-Lie algebras*, generalizing the notion of hom-Lie algebras. However, so far, the field has not gained nearly as much attention as that of hom-algebras.

Another class of algebras that is associated with deformations is that of *Ore extensions*. Ore extensions were first introduced under the name *non-commutative polynomial rings* by the Norwegian mathematician Øystein Ore [68], who investigated how one could generalize in a natural way ordinary polynomial rings into non-commutative analogues. As it turns out, many classical Ore extensions that appear in the literature (see e.g. [32, 36, 60]) are deformations of associative algebras, such as e.g. the *Weyl algebra*, the *q-Weyl algebra*, and the *quantum plane*. In particular, the deformation of the ordinary polynomial algebra to the Weyl algebra is precisely what describes the transition from ordinary physics to quantum physics, replacing classical functions by non-commutative operators [18]. Moreover, many of these algebras, such as the aforementioned Weyl algebra, are in turn formally rigid as associative algebras. Hence, at least from a deformation point of view, it seems motivated to study Ore extensions in a wider – hom-associative – context. By doing so, the hope would also be to contribute to the understanding of hom-associative algebras in general, which is still a quite new field of research. Moreover, not long ago, Nystedt, Öinert, and Richter [66] introduced *non-associative Ore extensions* in the unital case, generalizing Ore's work to the non-associative setting (see also [67] for a further extension to monoid Ore extensions, [65] for a generalization of the closely related skew groupoid rings, and [64] for results on crossed products in the non-associative setting). They were able to generalize some classical examples and results on the ideals and on simplicity of a type of Ore extension known as *differential polynomial rings* to the non-associative setting. Are there more examples and classical results that can be generalized to the non-associative setting? What can be said in the slightly more general non-unital, hom-associative setting? In this thesis, we try to answer these questions. We introduce and study *hom-associative Ore extensions* as non-unital, non-commutative, hom-associative polynomial rings. These slightly generalize the non-associative Ore extensions introduced in [66] to the non-unital, hom-associative setting.

Here is an outline of the thesis:

– Chapter 1 gives preliminaries on non-associative algebras (Section 1.1), hom-associative algebras and hom-Lie algebras (Section 1.2), and on associative Ore extensions (Section 1.3).

– Chapter 2 introduces hom-associative Ore extensions together with necessary and sufficient conditions for such to exist (Section 2.1 – Section 2.4). Concrete examples of hom-associative Ore extensions are then constructed in Section 2.5, including hom-associative generalizations of the universal enveloping algebra of the non-abelian two-dimensional Lie algebra, the quantum plane, and the first Weyl algebra. It is then shown that these three are all formal deformations of their associative counterparts. Moreover, all the hom-associative Ore extensions constructed turn out to be *weakly unital*, and in Section 2.6 it is shown that there is a way to embed any multiplicative hom-associative algebra into a weakly unital ditto. In particular, this generalizes the procedure of embedding an associative algebra into a unital ditto, something that cannot be done for hom-associative algebras in general.

– Chapter 3 studies the hom-associative Weyl algebras in the zero characteristic case (Section 3.1) as introduced in Chapter 2, as well as in the prime characteristic case (Section 3.2). In the zero characteristic case, they are shown to be simple, and the center, commuter, and set of derivations are determined. They are also classified up to isomorphism, and an analogue of the Dixmier conjecture is proven true (Corollary 8). Last, they are shown to be a one-parameter formal deformation of the first Weyl algebra. In the prime characteristic case, some general assertions about hom-associative algebras coming from Yau twisted associative algebras are first proved (Subsection 3.2.2). Then, in Subsection 3.2.3, by using these results, the center, commuter, nuclei, and set of derivations are determined by slightly more general methods than used in the zero characteristic case. Moreover, the algebras are classified up to isomorphism, and it is seen that all non-zero endomorphisms on them are injective, but not surjective. Last, in Subsection 3.2.4, multi-parameter formal deformations of hom-associative algebras are introduced, and the hom-associative Weyl algebras are then shown to be a multi-parameter formal deformation of the first Weyl algebra.

– Chapter 4 first introduces hom-module theory, and with the help of this a Hilbert's basis theorem for hom-associative Ore extensions (Theorem 8) is proved. Both the classical Hilbert's basis theorem for associative Ore extensions (Theorem 3) and a Hilbert's basis theorem for unital, non-associative Ore extensions (Corollary 21) then follow immediately from this result. Several new examples of both non-associative and hom-associative algebras that are Noetherian by the above theorem and its corollary are then provided in Section 4.3.

1.1 Non-associative algebras

Throughout this thesis, we denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{N}_{>0}$ the set of positive integers. $\mathbb{Z}$ is the ring of integers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, $\mathbb{H}$ the division ring of real quaternions, and $\mathbb{O}$ the division ring of real octonions. $\mathbb{F}_{p^n}$ is the finite field of characteristic p and cardinality p^n for some prime p and $n \in \mathbb{N}_{>0}$. In general, if K is a field, we will denote by $\operatorname{char}(K)$ its characteristic, and by $K^\times$ its multiplicative group of non-zero elements. By an *algebra* over a unital, associative, commutative ring R, we mean an R-algebra in the broadest sense, not requiring associativity, commutativity, or unitality. To emphasize the fact that we do not require our algebras to be associative, we shall often use the term *non-associative algebra*, however. Moreover, when the context requires, we will also use the term *non-unital* to emphasize that an algebra is not necessarily unital. Hence unital and associative algebras are both non-unital, non-associative algebras by definition (this in accordance with the red herring principle; a red herring need not, in general, be either red or a herring). Now, in particular, a *non-associative ring* is a non-associative algebra over $\mathbb{Z}$. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are all unital, non-associative algebras over $\mathbb{R}$ where $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, and $\mathbb{O}$ is the only of the four that is not associative. Moreover, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are (up to isomorphism) the only normed, unital division algebras over $\mathbb{R}$ [77].

For a non-associative algebra A, we denote by $D_l(A)$ the set of left zero divisors of A, and by $D_r(A)$ the set of right zero divisors of A. The *commutator* $[\cdot,\cdot]\colon A \times A \to A$ is defined by $[a,b] := a\cdot b - b\cdot a$ for arbitrary $a, b \in A$, and the *commuter* of A, written $C(A)$, is defined as $\{a \in A\colon [a,b] = 0,\ b \in A\}$. The *associator* $(\cdot,\cdot,\cdot)\colon A \times A \times A \to A$ is defined by $(a,b,c) = (a\cdot b)\cdot c - a\cdot(b\cdot c)$ for arbitrary elements $a, b, c \in A$, and the *left*, *middle*, and *right nuclei* of A, denoted by $N_l(A), N_m(A)$, and $N_r(A)$, respectively, are the sets $\{a \in A\colon (a,b,c) = 0,\ b,c \in A\}$, $\{b \in A\colon (a,b,c) = 0,\ a,c \in A\}$, and $\{c \in A\colon (a,b,c) = 0,\ a,b \in A\}$, respectively. The *nucleus* of A, written $N(A)$, is defined as the set $N_l(A) \cap N_m(A) \cap N_r(A)$. By the *associator identity* $a\cdot(b,c,d) + (a,b,c)\cdot d + (a, b\cdot c, d) = (a\cdot b, c, d) + (a, b, c\cdot d)$, holding for all $a, b, c, d \in A$, $N_l(A), N_m(A), N_r(A)$, and hence also $N(A)$, are all associative R-subalgebras of A. The *center* of A, denoted by $Z(A)$, is the intersection of the commuter and the nucleus, $C(A) \cap N(A)$. A way to measure the non-associativity of A can be done by using the associator: A is called *power associative* if $(a,a,a) = 0$, *left alternative* if $(a,a,b) = 0$, *right alternative* if $(b,a,a) = 0$, *flexible* if $(a,b,a) = 0$, and *associative* if $(a,b,c) = 0$ for all $a, b, c \in A$. Hence, if A is not power associative, then A is not left alternative, right alternative, flexible, or associative.

An R-linear map $\delta\colon A \to A$ is called a *derivation* if for any $a, b \in A$, $\delta(a\cdot b) = \delta(a)\cdot b + a\cdot\delta(b)$, and the set of derivations of A is denoted by $\mathrm{Der}_R(A)$. If A is associative, then all maps of the form $\mathrm{ad}_a := [a,\cdot]\colon A \to A$ for an arbitrary $a \in A$ are derivations, called *inner derivations*, and the set of all inner derivations of A are denoted by $\mathrm{InnDer}_R(A)$. If A is not associative, such a map need not be a derivation, however. All derivations that are not inner are *outer*. Last, recall that A is called *simple* if the only two-sided ideals of A are the zero ideal and A itself, and that A *embeds* into a non-associative algebra B if there is an injective homomorphism from A to B, so that A may be seen as a subalgebra of B.

For a general introduction to non-associative algebras, we refer the reader to Schafer's book [71].

1.2 Hom-associative algebras and hom-Lie algebras

In this section, we introduce basic definitions and results concerning hom-associative algebras and hom-Lie algebras. Hom-associative algebras were first introduced in [56] and hom-Lie algebras in [35], in both cases by starting from vector spaces. Here, we take a slightly more general approach by starting from modules. As it turns out, most of the basic theory still hold in this latter case.

1.2.1 Hom-associative algebras

Definition 1 (Hom-associative algebra). A *hom-associative algebra* over a unital, associative, and commutative ring R is a triple $(M, \cdot, \alpha)$, consisting of an R-module M, an R-bilinear map $\cdot\colon M \times M \to M$, and an R-linear map $\alpha\colon M \to M$, satisfying, for all $a, b, c \in M$,

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c), \quad \text{(hom-associativity)}.$$

In the above definition, α is in a sense "twisting" the usual associativity condition, and hence it is referred to as a *twisting map*. A hom-associative algebra is called *multiplicative* if the twisting map is multiplicative, i.e. if it is an R-algebra homomorphism.

Remark 1. A hom-associative algebra $A := (M, \cdot, \alpha)$ over some unital, associative, commutative ring R is in particular a non-associative R-algebra, and if α is (any non-zero multiple of) the identity map id_A on A, then A is an associative R-algebra. If $\alpha = 0$, then the hom-associativity condition becomes null, and hence hom-associative algebras can be considered as generalizations of both associative and non-associative algebras.

Any product on a one-dimensional vector space V over a field K is necessarily associative, and hence so is any one-dimensional hom-associative algebra over K. In this case, the twisting map may be defined arbitrarily on V. The following is an example, though a rather trivial one, of a two-dimensional hom-associative algebra.

Example 1 ([29]). Let $\{v_1, v_2\}$ be a basis of a two-dimensional vector space V over a field K. The multiplication $\cdot$ and linear map α defined here below makes V a hom-associative K-algebra $(V, \cdot, \alpha)$:

$$\begin{aligned} v_1 \cdot v_1 &= v_2, & v_2 \cdot v_1 &= 0, \\ v_1 \cdot v_2 &= 0, & v_2 \cdot v_2 &= 0, \\ \alpha(v_1) &= v_1, & \alpha(v_2) &= v_1 + v_2. \end{aligned}$$

Even though α is not a multiple of the identity map, this is still an associative algebra. Moreover, it is commutative, but not multiplicative. We will return to this example later in this section and in Section 2.6.

Makhlouf and Zahari [59] have classified all multiplicative hom-associative algebras up to dimension three over an algebraically closed field of characteristic zero. The following example can be found in their paper.

Example 2 ([59]). Let $\{v_1, v_2\}$ be a basis of a two-dimensional vector space V over an algebraically closed field K of characteristic zero. The multiplication $\cdot$ and linear map α defined here below makes V a hom-associative K-algebra $(V, \cdot, \alpha)$:

$$\begin{aligned} v_1 \cdot v_1 &= 0, & v_2 \cdot v_1 &= v_1, \\ v_1 \cdot v_2 &= 0, & v_2 \cdot v_2 &= v_1 + v_2, \\ \alpha(v_1) &= v_1, & \alpha(v_2) &= v_1 + v_2. \end{aligned}$$

This algebra is, however, not even power associative, since e.g. $(v_2 \cdot v_2) \cdot v_2 = (v_1 + v_2) \cdot v_2 = v_1 + v_2$ while $v_2 \cdot (v_2 \cdot v_2) = v_2 \cdot (v_1 + v_2) = 2v_1 + v_2$. By a straightforward calculation, α is multiplicative. We will return to this example later in this section and in Section 2.6.

Example 3 ([55]). Let $\{v_1, v_2, v_3\}$ be a basis of a three-dimensional vector space V over an algebraically closed field K of characteristic zero. The multiplication $\cdot$ and linear map α defined here below makes V a hom-associative K-algebra $(V, \cdot, \alpha)$:

$$\begin{aligned} v_1 \cdot v_1 &= k_1 v_1, & v_2 \cdot v_1 &= k_1 v_2, & v_3 \cdot v_1 &= k_2 v_3, \\ v_1 \cdot v_2 &= k_1 v_2, & v_2 \cdot v_2 &= k_1 v_2, & v_3 \cdot v_2 &= 0, \\ v_1 \cdot v_3 &= k_2 v_3, & v_2 \cdot v_3 &= k_2 v_3, & v_3 \cdot v_3 &= 0, \\ \alpha(v_1) &= k_1 v_1, & \alpha(v_2) &= k_1 v_2, & \alpha(v_3) &= k_2 v_3, \end{aligned}$$

for any $k_1, k_2 \in K$. In other words, we have a two-parameter family of hom-associative algebras, and when e.g. $k_1 \neq k_2$ and $k_2 \neq 0$, they are not associative since $(v_1, v_1, v_3) = (k_1 - k_2)k_2v_3$. Moreover, α is in general not multiplicative, since e.g. $\alpha(v_1)\cdot\alpha(v_1) = k_1v_1\cdot k_1v_1 = k_1^3v_1$ while $\alpha(v_1\cdot v_1) = \alpha(k_1v_1) = k_1^2v_1$. We will return to this example in Subsection 1.2.3.

Definition 2 (Hom-associative ring). A *hom-associative ring* is a hom-associative algebra over $\mathbb{Z}$.

There is a natural generalization of associative matrix rings to *hom-associative matrix rings*. The following proposition is well known and straightforward to prove.

Proposition 1. *Let R be a hom-associative ring and $M_n(R)$ the non-associative matrix ring over R for some $n \in \mathbb{N}_{>0}$. Then $M_n(R)$ can be made hom-associative by extending the twisting map α of R to $M_n(R)$ by letting α act on all entries of a matrix.*

Definition 3 (Opposite hom-associative ring). Let $S := (R, \cdot, \alpha)$ be a hom-associative ring. The *opposite hom-associative ring* of S, written S^{op}, is the hom-associative ring $(R, \cdot_{\text{op}}, \alpha)$ where $r \cdot_{\text{op}} s := s \cdot r$ for any $r, s \in R$.

Definition 4 (Homomorphism). Let A and B be two hom-associative R-algebras with twisting maps α and β, respectively. A *homomorphism* from A to B is a multiplicative, R-linear map $f\colon A \to B$, satisfying $f \circ \alpha = \beta \circ f$. If $A = B$, then f is an *endomorphism*. If f is a bijective homomorphism, then f is an *isomorphism*, and A and B are *isomorphic*, written $A \cong B$.

For any two hom-associative R-algebras A and B, we denote by $\operatorname{Hom}_R(A, B)$ the set of homomorphisms from A to B, and put $\operatorname{End}_R(A) := \operatorname{Hom}_R(A, A)$ for the set of endomorphisms. We also introduce $C_{\operatorname{End}_R(A)}(\alpha, \beta) := \{f \in \operatorname{End}_R(A)\colon f \circ \alpha = \beta \circ f\}$, $C_{\operatorname{End}_R(A)}(\alpha) := C_{\operatorname{End}_R(A)}(\alpha, \alpha)$, and moreover $C_{\operatorname{Der}_R(A)}(\alpha) := \{\delta \in \operatorname{Der}_R(A)\colon \delta \circ \alpha = \alpha \circ \delta\}$.

Take any hom-associative algebra with twisting map α not equal to the zero map, such as e.g. the one in Example 2 or in Example 3. Certainly the very same algebra can be made hom-associative by instead considering the zero map as twisting map. However, since the two twisting maps are not equal, the two hom-associative algebras are by definition not the same. Moreover, we also see that the two hom-associative algebras cannot be isomorphic, since then $\alpha = f^{-1} \circ 0 \circ f = 0$ for any bijection f.

Definition 5 (Hom-associative subalgebra). Let $A := (M, \cdot, \alpha)$ be a hom-associative algebra and N a submodule of M that is closed under the multiplication $\cdot$ and invariant under α. The hom-associative algebra $(N, \cdot, \alpha|_N)$ is said to be a *hom-associative subalgebra* of A.

Definition 6 (Hom-ideal). A *right (left) hom-ideal* of a hom-associative R-algebra A is an R-submodule I of A such that $\alpha(I) \subseteq I$, and for all $a \in A, i \in I, i \cdot a \in I$ $(a \cdot i \in I)$. If I is both a left and a right hom-ideal, I is simply called a *hom-ideal*.

Remark 2. In case the twisting map is equal to the identity map or the zero map, a right (left) hom-ideal is simply a right (left) ideal. Also note that a hom-ideal is in particular a hom-subalgebra.

Definition 7 (Hom-simplicity). We say that a hom-associative algebra A is *hom-simple* provided its only hom-ideals are 0 and A.

Any simple hom-associative algebra is also hom-simple. The converse need not be true, however, since there may exist ideals that are not invariant under the twisting map.

Definition 8 (Weakly unital hom-associative algebra). Let A be a hom-associative algebra with twisting map α. If for all $a \in A$, $e_l \cdot a = \alpha(a)$ for some $e_l \in A$, we say that A is *weakly left unital* with *weak left identity element* e_l. In case $a \cdot e_r = \alpha(a)$ for some $e_r \in A$, A is called *weakly right unital* with *weak right identity element* e_r. If there is an $e \in A$ which is both a weak left and a weak right identity element, e is called a *weak identity element*, and A is called *weakly unital*.

Remark 3. First, any weak identity element, when it exists, need not be unique. Now, any unital hom-associative algebra A with twisting map α is weakly unital with weak identity element $\alpha(1_A)$, since by hom-associativity $\alpha(1_A) \cdot a = \alpha(1_A) \cdot (1_A \cdot a) = (1_A \cdot 1_A) \cdot \alpha(a) = \alpha(a)$, and similarly for the right case (in this case, α is thus completely determined by $\alpha(1_A)$). We see that the notion of a weak identity element is, as the name suggests, a weakening of that of an identity element.

Example 4. Consider the two-dimensional hom-associative algebra in Example 2. Let $e := k_1 v_1 + k_2 v_2$ for some $k_1, k_2 \in K$. Then $v_1 \cdot e = 0 \neq v_1 = \alpha(v_1)$, so there is no weak right identity element in this algebra (and hence, by the discussion prior to this example, no identity element). However, $e \cdot v_1 = k_2 v_1$, so if $k_2 = 1_K$, then $e \cdot v_1 = \alpha(v_1)$. Moreover, we then have $e \cdot v_2 = (k_1 v_1 + v_2) \cdot v_2 = v_1 + v_2 =$

$\alpha(v_2)$, so we can conclude that e is a weak left identity element for any $k_1 \in K$. By similar calculations, the two-dimensional hom-associative algebra in Example 1 is not weakly unital.

Proposition 2 ([78]). *Let A be an associative R-algebra, and $\alpha \in \operatorname{End}_R(A)$. Define a product $*\colon A \times A \to A$ by $a*b := \alpha(a \cdot b)$ for all $a, b \in A$. Then $A^\alpha := (A, *, \alpha)$ is a hom-associative R-algebra.*

Proof. That $*$ really is a product and that A^α is an R-algebra is immediate. Regarding hom-associativity, for all $a, b, c \in A$, we have

$$\alpha(a) * (b * c) = \alpha(a) * (\alpha(b \cdot c)) = \alpha(\alpha(a) \cdot \alpha(b \cdot c)) = \alpha(\alpha(a \cdot b \cdot c)),$$
$$(a * b) * \alpha(c) = \alpha(a \cdot b) * \alpha(c) = \alpha(\alpha(a \cdot b) \cdot \alpha(c)) = \alpha(\alpha(a \cdot b \cdot c)). \quad \square$$

The construction of A^α from A in Proposition 2 was introduced by Yau [78], and is often referred to as the *Yau twist of A*. We will use the subscript $*$ whenever the multiplication is that in A^α; hence $[\cdot, \cdot]_*$ denotes the commutator in A^α and $(\cdot, \cdot, \cdot)_*$ denotes the associator in A^α.

Example 5 ([59]). The two-dimensional hom-associative algebra in Example 2 is the Yau twist of the following two-dimensional, non-unital, associative algebra A defined on the same underlying vector space, but whose multiplication $\cdot$ is given by

$$v_1 \cdot v_1 = 0, \qquad v_2 \cdot v_1 = v_1,$$
$$v_1 \cdot v_2 = 0, \qquad v_2 \cdot v_2 = v_2.$$

In [59] in which this example can be found, the products $v_1 \cdot v_2$ and $v_2 \cdot v_1$ seem to have been mixed up, however. We therefore provide the detailed calculations here below. First, denote the multiplication in Example 2 by $*$ instead. Then,

$$v_1 * v_1 = 0 = \alpha(0) = \alpha(v_1 \cdot v_1), \quad v_2 * v_1 = v_1 = \alpha(v_1) = \alpha(v_2 \cdot v_1),$$
$$v_1 * v_2 = 0 = \alpha(0) = \alpha(v_1 \cdot v_2), \quad v_2 * v_2 = v_1 + v_2 = \alpha(v_2) = \alpha(v_2 \cdot v_2).$$

Moreover, we see that α is multiplicative, since

$$\alpha(v_1) \cdot \alpha(v_1) = v_1 \cdot v_1 = 0 = \alpha(v_1 \cdot v_1),$$
$$\alpha(v_1) \cdot \alpha(v_2) = v_1 \cdot (v_1 + v_2) = 0 = \alpha(v_1 \cdot v_2),$$
$$\alpha(v_2) \cdot \alpha(v_1) = (v_1 + v_2) \cdot v_1 = v_1 = \alpha(v_2 \cdot v_1),$$
$$\alpha(v_2) \cdot \alpha(v_2) = (v_1 + v_2) \cdot (v_1 + v_2) = v_1 + v_2 = \alpha(v_2 \cdot v_2).$$

What is left to check is that the product $\cdot$ is associative. By linearity, it suffices to check that $v_i \cdot (v_j \cdot v_k) = (v_i \cdot v_j) \cdot v_k$ for any i, j, k; hence there are eight cases to check. However, the product is defined in such a way that $v_1 \cdot (v_j \cdot v_k) = 0 = (v_1 \cdot v_j) \cdot v_k$ for any j, k, so it suffices to check the following four products:

$$\begin{aligned} v_2 \cdot (v_1 \cdot v_1) &= 0 = (v_2 \cdot v_1) \cdot v_1, \\ v_2 \cdot (v_1 \cdot v_2) &= 0 = (v_2 \cdot v_1) \cdot v_2, \\ v_2 \cdot (v_2 \cdot v_1) &= v_1 = (v_2 \cdot v_2) \cdot v_1, \\ v_2 \cdot (v_2 \cdot v_2) &= v_2 = (v_2 \cdot v_2) \cdot v_2. \end{aligned}$$

Hence A is associative, $\alpha \in \mathrm{End}_K(A)$, and $a * b = \alpha(a \cdot b)$ for any $a, b \in A$. By Proposition 7, A^α in Example 2 is indeed hom-associative.

Corollary 1 ([29]). *Let A be a unital, associative R-algebra with identity element 1_A and $\alpha \in \mathrm{End}_R(A)$. Then A^α is weakly unital with weak identity element 1_A.*

Proof. For all $a \in A$, $1_A * a = \alpha(1_A \cdot a) = \alpha(a) = \alpha(a \cdot 1_A) = a * 1_A$. $\square$

The following example is perhaps the simplest example of a non-trivial hom-associative algebra arising in a natural way, and therefore it should be well known. However, we have not been able to find it in the literature.

Example 6. Denote by $\overline{u}$ the complex conjugate of $u \in \mathbb{C}$. We can equip $\mathbb{C}$ with a new product $*$ defined by $u * v := \overline{u} \cdot \overline{v}$ for any $u, v \in \mathbb{C}$. This turns $\mathbb{C}$ into a non-associative division algebra over $\mathbb{R}$, where $u * (v * w) = \overline{u} \cdot \overline{\overline{v} \cdot \overline{w}} = \overline{u} \cdot v \cdot w$ and $(u * v) * w = \overline{\overline{u} \cdot \overline{v}} \cdot \overline{w} = u \cdot v \cdot \overline{w}$. Complex conjugation can be seen as a map $\alpha \colon \mathbb{C} \to \mathbb{C}$, and as such, it is an $\mathbb{R}$-algebra automorphism (in fact an involution; $\alpha^2 = \mathrm{id}_{\mathbb{C}}$). Hence $\alpha(u) * (v * w) = (u * v) * \alpha(w)$, so the algebra we have found is hom-associative with twisting map α. Moreover, it is of course no coincidence we decided to denote the multiplication by $*$: the hom-associative algebra is the Yau twist $\mathbb{C}^\alpha$ of $\mathbb{C}$. Therefore $\mathbb{C}^\alpha$ is weakly unital with weak identity element $1_{\mathbb{C}}$, and complex conjugation can thus be expressed as $*$-multiplication by $1_{\mathbb{C}}$, where $1_{\mathbb{C}} * u = u * 1_{\mathbb{C}} = \alpha(u) = \overline{u}$.

1.2.2 Hom-Lie algebras

Definition 9 (Hom-Lie algebra). A *hom-Lie algebra* over a unital, associative, commutative ring R is a triple $(M, [\cdot,\cdot], \alpha)$, consisting of an R-module M, an R-bilinear map $[\cdot,\cdot] \colon M \times M \to M$, and an R-linear map $\alpha \colon M \to M$, satisfying, for all $a, b, c \in M$, the following two axioms:

$$[a,a] = 0, \quad \text{(alternativity)},$$
$$[\alpha(a),[b,c]] + [\alpha(c),[a,b]] + [\alpha(b),[c,a]] = 0, \quad \text{(hom-Jacobi identity)}.$$

In the above definition, $[\cdot,\cdot]$ is called a *hom-Lie bracket*, and just like in the definition of hom-associative algebras, α is called a twisting map. Note that we immediately also get anti-commutativity of the bracket from the bilinearity and alternativity, since $0 = [a+b, a+b] = [a,a] + [a,b] + [b,a] + [b,b] = [a,b] + [b,a]$, so $[a,b] = -[b,a]$ holds for all a and b in any hom-Lie algebra. Unless the characteristic of R is two, anti-commutativity also implies alternativity, since $[a,a] = -[a,a]$ for all a. A hom-Lie algebra in which the bracket is the zero bracket is called *abelian*, and a hom-Lie algebra in which that is not the case is called *non-abelian*.

Remark 4. If $\alpha = \mathrm{id}_M$ in Definition 9, we get the definition of a *Lie algebra*. Hence the notion of a hom-Lie algebra can be seen as a generalization of that of a Lie algebra.

Example 7 ([55]). Let $\{v_1, v_2, v_3\}$ be a basis of a three-dimensional vector space V over an algebraically closed field K of characteristic zero. The bracket $[\cdot,\cdot]$ and linear map α defined here below makes V a hom-Lie algebra $(V, [\cdot,\cdot], \alpha)$ over K:

$$\begin{aligned}
&[v_1,v_1] = 0, && [v_2,v_1] = -k_1v_1 - k_2v_3, && [v_3,v_1] = -k_3v_3,\\
&[v_1,v_2] = k_1v_1 + k_2v_3, && [v_2,v_2] = 0, && [v_3,v_2] = -k_4v_1 - 2k_1v_3,\\
&[v_1,v_3] = k_3v_2, && [v_2,v_3] = k_4v_1 + 2k_1v_3, && [v_3,v_3] = 0,\\
&\alpha(v_1) = v_1, && \alpha(v_2) = 2v_2, && \alpha(v_3) = 2v_3,
\end{aligned}$$

for any $k_1, k_2, k_3, k_4 \in K$, and we thus have a four-parameter family of hom-Lie algebras. To verify that the hom-Jacobi identity is satisfied for all $k_1, k_2, k_3, k_4 \in K$, it is, due to alternativity of the bracket, sufficient to check that $[\alpha(v_1),[v_2,v_3]] + [\alpha(v_3),[v_1,v_2]] + [\alpha(v_2),[v_3,v_1]] = 0$ holds. To check when $(V,[\cdot,\cdot],\alpha)$ is a Lie algebra, it is sufficient to determine when $[v_1,[v_2,v_3]] + [v_3,[v_1,v_2]] + [v_2,[v_3,v_1]] = 0$ holds. By a direct computation, $[v_1,[v_2,v_3]] + [v_3,[v_1,v_2]] + [v_2,[v_3,v_1]] = k_1k_3v_2$, so $(V,[\cdot,\cdot],\alpha)$ is a Lie algebra if and only if $k_1 = 0$ or $k_3 = 0$.

Example 8 ([46,56])**.** Let $\{v_1, v_2, v_3\}$ be a basis of a three-dimensional vector space V over an algebraically closed field K of characteristic zero. The bracket $[\cdot,\cdot]$ and linear map α defined here below makes V a hom-Lie algebra $(V, [\cdot,\cdot], \alpha)$ over K:

$$\begin{aligned}
&[v_1, v_1] = 0, && [v_2, v_1] = -2v_2, && [v_3, v_1] = 2v_3 + 2kv_3,\\
&[v_1, v_2] = 2v_2, && [v_2, v_2] = 0, && [v_3, v_2] = -v_1 - \frac{k}{2}v_1,\\
&[v_1, v_3] = -2v_3 - 2kv_3, && [v_2, v_3] = v_1 + \frac{k}{2}v_1, && [v_3, v_3] = 0,\\
&\alpha(v_1) = v_1, && \alpha(v_2) = \frac{2+k}{2(1+k)}v_2, && \alpha(v_3) = v_3 + \frac{k}{2}v_3,
\end{aligned}$$

for any $k \in K\backslash\{-1\}$, and hence we have a one-parameter family of hom-Lie algebras. To verify that the hom-Jacobi identity is satisfied for all $k \in K$, it is, just as in Example 7, sufficient to check that $[\alpha(v_1), [v_2, v_3]] + [\alpha(v_3), [v_1, v_2]] + [\alpha(v_2), [v_3, v_1]] = 0$ holds. To check when $(V, [\cdot,\cdot], \alpha)$ is a Lie algebra, it is, again as in Example 7, sufficient to determine when $[v_1, [v_2, v_3]] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = 0$ holds. We see that $[v_1, [v_2, v_3]] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = k(2 + k)v_1$, so $(V, [\cdot,\cdot], \alpha)$ is a Lie algebra if and only if $k = 0$ or $k = -2$. When $k = 0$, this Lie algebra is known as $\mathfrak{sl}_2$, and the hom-Lie algebra $(V, [\cdot,\cdot], \alpha)$ for an arbitrary $k \in K\backslash\{-1\}$ is known as the *Jackson* $\mathfrak{sl}_2$. The name comes from the fact that it was originally constructed in [46] by using the so-called *Jackson derivative* (see Example 17 in Section 1.3). We will return to this example in Subsection 1.2.3.

The following example is what originally motivated the introduction of hom-Lie algebras.

Example 9 ([35])**.** Let K be an algebraically closed field of characteristic zero. The *Witt algebra* W over K is the Lie algebra with basis $\{w_m\}_{m\in\mathbb{Z}}$ and Lie bracket $[w_m, w_n] = (m-n)w_{m+n}$. The set of derivations on a K-algebra A, $\mathrm{Der}_K(A)$, is an associative algebra with multiplication composition, and hence it can be made a Lie algebra over K with Lie bracket the commutator. It turns out that W is isomorphic to $\mathrm{Der}_K(K[x^{\pm}])$ where $K[x^{\pm}]$ is the Laurent polynomial ring over K and w_m is identified with $-x^{m+1}\frac{\mathrm{d}}{\mathrm{d}x}$ for any $m \in \mathbb{Z}$. In [35], Hartwig, Larsson, and Silvestrov replaced these derivations by so-called *σ-derivations* (see Definition 12 in Section 1.3). In doing so, they got a family of algebras which resembled – but in general were not – Lie algebras. These algebras satisfied a "twisted" Jacobi identity, something we now know as the hom-Jacobi identity. Now, going back to the first definition of W, they ended up with a *q-deformation* of W in which the bracket was given by $[w_m, w_n]_q = (\{m\}_q - \{n\}_q)x_{m+n}$ where $\{m\}_q := (1-q^m)/(1-q)$ for $q \in K\backslash\{1\}$ and $\{m\}_1 := m$ (see also [1, 13–16, 19, 38] for earlier works on similar q-deformations of the Witt and Virasoro algebra. Moreover, the books [23, 40] give general introductions to the world of *q-calculus*). The q-deformed Jacobi identity was then given by

$$(1+q^l)[x_l, [x_m, x_n]_q]_q + (1+q^m)[x_m[x_n, x_l]_q]_q + (1+q^n)[x_n[x_l, x_m]_q]_q = 0.$$

The bracket $[\cdot,\cdot]_q$ is alternative, and so by defining a linear map α on the underlying vector space of W by $\alpha(x_m) = (1+q^m)x_m$ for all $m \in \mathbb{Z}$, we get a hom-Lie algebra $(W, [\cdot,\cdot]_q, \alpha)$. The term q-deformation here refers to the fact that when we put $q = 1$, we recover the Witt algebra W, and in some sense $(W, [\cdot,\cdot]_q, \alpha)$ is then a deformation of W that depends on the "quantum" parameter q. Moreover, in [35], the authors also showed that this construction generalizes to the so-called central extension of the Witt algebra known as the *Virasoro algebra*, resulting in a q-deformed Virasoro algebra. We will come back to the q-deformed Witt algebra in Subsection 1.2.3.

Proposition 3 ([56]). *Let $(M, \cdot, \alpha)$ be a hom-associative algebra over a unital, associative, commutative ring R, with commutator $[\cdot,\cdot]$. Then $(M, [\cdot,\cdot], \alpha)$ is a hom-Lie algebra over R.*

Note that when α in Proposition 3 is the identity map, we recover the classical construction of a Lie algebra from an associative algebra (this construction was for instance used in Example 9 here above to construct a Lie algebra of derivations). We refer to the above construction as the *commutator construction*.

Example 10. Let, for example, $(M, \cdot, \alpha)$ be any of the hom-associative algebras in Example 1, Example 2, Example 3, Example 6, or even the corresponding hom-associative matrix rings over them (cf. Proposition 1). Using the commutator construction, $(M, [\cdot,\cdot], \alpha)$ is then a hom-Lie algebra.

1.2.3 One-parameter formal deformations

In the seminal paper [30], Gerstenhaber introduced formal deformation theory for associative algebras. Later, Nijenhuis and Richardson [61, 62] extended Gerstenhaber's work to Lie algebras (see e.g. [28] for a gentle introduction to algebraic deformation theory). Makhlouf and Silvestrov [55] have introduced formal deformation theory for hom-associative algebras and for hom-Lie algebras together with an attempt at describing a so-called compatible cohomology theory in lower degrees. In the multiplicative case, this was later expanded on by Ammar, Ejbehi and Makhlouf [2], and then by Hurle and Makhlouf [39]. Only in this latter paper, treating the multiplicative, hom-associative case, did the cohomology theory include the twisting map α in a natural way. This is indeed essential, as the idea behind these kinds of deformations is to deform not only the multiplication map, or the Lie bracket, but also the twisting map α, resulting also in a deformation of the twisted associativity condition and the twisted Jacobi identity, respectively. In the special case when the deformations start from α being the identity map and the multiplication being associative, or the bracket being the Lie bracket, one gets a deformation of an associative algebra into a hom-associative algebra, and in the latter case a deformation of a Lie algebra into a hom-Lie algebra. As mentioned in the introduction, many Lie algebras cannot be formally deformed in a non-trivial way; they are *formally rigid*. We have already mentioned that this is the case for the Witt and Virasoro Lie algebra, but we could also add to that list $\{0\}$, K, the two-dimensional non-abelian Lie algebra $\mathfrak{r}_2$, $\mathfrak{sl}_2$, $\mathfrak{gl}_2$, $\mathfrak{sl}_2\times\mathfrak{r}_2$, $\mathfrak{sl}_2\times\mathfrak{sl}_2$, and $\mathfrak{gl}_2\times K^2$, all formally rigid Lie algebras over a field K of characteristic zero. In fact, the eight Lie algebras just mentioned are, up to isomorphism, all the *strongly rigid* Lie algebras of dimension at most six [12]. Here, strongly rigid means that the algebras are formally rigid as Lie algebras, and that their corresponding universal enveloping algebras are formally rigid as associative algebras (a rigid Lie algebra need not be strongly rigid, however [12]). Moreover, any finite-dimensional semi-simple Lie algebra over a field of characteristic zero is not only rigid, but also strongly rigid [12]. The fact that there seem to be both many interesting Lie algebras as well as associative algebras that are formally rigid is, at present, perhaps the main motivation for studying their hom-counterparts; in the framework of the latter algebras, many algebras can now be deformed, which otherwise could not when considered as objects of the former categories.

Definition 10 (One-parameter formal hom-associative deformation). A *one-parameter formal hom-associative deformation* of a hom-associative algebra $(M,\cdot_0,\alpha_0)$ over R, is a hom-associative algebra $(M[\![t]\!],\cdot_t,\alpha_t)$ over $R[\![t]\!]$, where

$$\cdot_t=\sum_{i\in\mathbb{N}}\cdot_i t^i,\quad \alpha_t=\sum_{i\in\mathbb{N}}\alpha_i t^i,$$

and for each $i \in \mathbb{N}$, $\cdot_i\colon M \times M \to M$ is an R-bilinear map, and $\alpha_i\colon M \to M$ is an R-linear map. We further extend $\cdot_i$ homogeneously to a binary operation linear over $R[\![t]\!]$ in both arguments, $\cdot_i\colon M[\![t]\!] \times M[\![t]\!] \to M[\![t]\!]$, and α_i to an $R[\![t]\!]$-linear map $\alpha_i\colon M[\![t]\!] \to M[\![t]\!]$.

Here, a homogeneous extension is defined on arbitrary monomials rat^j where $r \in R$, $a \in M$, and $j \in \mathbb{N}$ by $\alpha_i(rat^j) = r\alpha_i(a)t^j$ for any $i \in \mathbb{N}$, and then extended linearly to polynomials. A homogeneous extension of the product $\cdot_i$ is defined analogously.

Example 11 ([55])**.** The hom-associative algebra in Example 3 may be viewed as a one-parameter formal hom-associative deformation of the associative algebra corresponding to $k_1 = k_2 = 1_K$. This by considering $k_1 = 1_K$ and $k_2 = 1_K + t$, or $k_1 = 1_K + t$ and $k_2 = 1_K$ where t is a formal parameter. The algebra corresponding to $k_1 = k_2 = 1_K$ is formally rigid as an associative algebra [34], but can now be deformed non-trivially as a hom-associative algebra.

Definition 11 (One-parameter formal hom-Lie deformation)**.** A *one-parameter formal hom-Lie deformation* of a hom-Lie algebra $(M, [\cdot,\cdot]_0, \alpha_0)$ over R is a hom-Lie algebra $(M[\![t]\!], [\cdot,\cdot]_t, \alpha_t)$ over $R[\![t]\!]$, where

$$[\cdot,\cdot]_t = \sum_{i\in\mathbb{N}} [\cdot,\cdot]_i t^i, \quad \alpha_t = \sum_{i\in\mathbb{N}} \alpha_i t^i,$$

and for each $i \in \mathbb{N}$, $[\cdot,\cdot]_i\colon M \times M \to M$ is an R-bilinear map, and $\alpha_i\colon M \to M$ is an R-linear map. We further extend $[\cdot,\cdot]_i$ homogeneously to a binary operation linear over $R[\![t]\!]$ in both arguments, $[\cdot,\cdot]_i\colon M[\![t]\!] \times M[\![t]\!] \to M[\![t]\!]$, and α_i to an $R[\![t]\!]$-linear map $\alpha_i\colon M[\![t]\!] \to M[\![t]\!]$.

Remark 5. Alternativity of $[\cdot,\cdot]_t$ is equivalent to alternativity of $[\cdot,\cdot]_i$ for all $i \in \mathbb{N}$.

Example 12 ([55])**.** The hom-Lie algebra Jackson $\mathfrak{sl}_2$ in Example 8 may be viewed as a one-parameter formal hom-Lie deformation of the Lie algebra $\mathfrak{sl}_2$ with deformation parameter $t = k$ and $\frac{2+t}{2(1+t)} = 1 + \sum_{i=0}^{\infty} \frac{(-1)^i}{2} t^i \in K[\![t]\!]$. As mentioned in the introduction of this subsection, the Lie algebra $\mathfrak{sl}_2$ is strongly rigid (and hence also formally rigid), but can now be deformed non-trivially as a hom-Lie algebra.

Example 13 ([55])**.** Recall from the introduction that the Witt algebra W defined in Example 9 is formally rigid as a Lie algebra. However, in the same example, we saw that W could, in some sense, be deformed into a hom-Lie algebra. The resulting hom-Lie algebra depended on a parameter $q \in K$, and when we put $q = 1$, we recovered the Lie algebra W. This was then coined a q-deformation of W. For the non-negative Witt algebra $W_{\geq 0}$, i.e. the Lie subalgebra of W obtained from W by restricting the basis of W to $\{w_m\}_{m\in\mathbb{N}}$, the corresponding q-deformation of $W_{\geq 0}$ is also a formal hom-Lie deformation. To show this, let us put $t := q - 1$. Then, for any $m \in \mathbb{N}$ and $q \in K$, $\{m\}_1 = m$,

$$\begin{aligned}\{m\}_q = \frac{1-q^m}{1-q} = \sum_{j=0}^{m-1} q^j = \sum_{j=0}^{m-1}(1+t)^j = \sum_{j=0}^{m-1}\sum_{i=0}^{j}\binom{j}{i}t^i\\ = \sum_{i=0}^{m-1}\left(\sum_{j=i}^{m-1}\binom{j}{i}\right)t^i, \quad q \in K\backslash\{1\},\\ q^m = (1+t)^m = \sum_{i=0}^{m}\binom{m}{i}t^i.\end{aligned}$$

Here, we define $\sum_{i=0}^{m-1}\left(\sum_{j=i}^{m-1}\binom{j}{i}\right)t^i := 0$ whenever $m = 0$, so that $\{0\}_q = 0$ for all $q \in K$. Hence, if we define $\alpha_t \colon W_{\geq 0}[\![t]\!] \to W_{\geq 0}[\![t]\!]$ by

$$\begin{aligned}\alpha_t(x_m) := (1+q^m)x_m = (1+(1+t)^m)x_m = \left(1+\sum_{i=0}^{m}\binom{m}{i}t^i\right)x_m\\ = 2x_m + \sum_{i=1}^{m}\binom{m}{i}x_m t^i,\end{aligned}$$

then $\alpha_t(x_m) = \sum_{i\in\mathbb{N}}\alpha_i(x_m)t^i$ where $\alpha_0(x_m) = 2x_m$ and $\alpha_i(x_m) = \binom{m}{i}x_m$ for $i \in \mathbb{N}_{>0}, m \in \mathbb{N}$. Moreover, if we also define $[\cdot,\cdot]_t \colon W_{\geq 0}[\![t]\!] \times W_{\geq 0}[\![t]\!] \to W_{\geq 0}[\![t]\!]$ by

$$[x_m, x_n]_t := \sum_{i=0}^{\max(m,n)-1}\left(\left(\sum_{j=i}^{m-1}\binom{j}{i} - \sum_{j=i}^{n-1}\binom{j}{i}\right)x_{m+n}\right)t^i,$$

then $[x_m, x_n]_t = \sum_{i\in\mathbb{N}}[x_m, x_n]_i t^i$ where

$$[x_m, x_n]_i = \left(\sum_{j=i}^{m-1}\binom{j}{i} - \sum_{j=i}^{n-1}\binom{j}{i}\right)x_{m+n},\ i, m, n \in \mathbb{N}.$$

In particular, when $t = 0$, we have for the bracket $[\cdot,\cdot]_t$

$$[x_m, x_n]_0 := \left(\left(\sum_{j=0}^{m-1} \binom{j}{0} - \sum_{j=0}^{n-1} \binom{j}{0} \right) x_{m+n} \right) t^0 = (m-n)x_{m+n},$$

which is the bracket in $W_{\geq 0}$. If we consider $2 \cdot \mathrm{id}_{W_{\geq 0}} = \alpha_0$ as the twisting map of $W_{\geq 0}$, the twisting map α_t and hom-Lie bracket $[\cdot,\cdot]_t$ defined here above define a one-parameter formal hom-Lie deformation of $W_{\geq 0}$.

Later, in Chapter 2, we will see more examples of formal deformations of both Lie algebras and associative algebras into hom-Lie algebras and hom-associative algebras, respectively.

1.3 Associative Ore extensions

Let R be a unital, associative ring. Even though R may fail to be commutative, it is still fully possible to consider a polynomial ring $R[x]$ over R in the indeterminate x with the usual addition and multiplication of polynomials. Of course, one need to be a bit careful as elements of R, and hence elements of $R[x]$, in general do not commute. However, all elements of $R[x]$ still commute not only with x, but also with all powers of x. In this sense, $R[x]$ is still some kind of "commutative" polynomial ring, even though defined over a non-commutative ring R. A natural question then arise: would it be possible to define, in a natural way, a polynomial ring over R in which the elements of R not necessarily commute with x? Would it be possible to define a *non-commutative polynomial ring* over R? The Norwegian mathematician Øystein Ore considered this question already in the 1930s, though originally regarding the case when R was a "non-commutative field" (nowadays more often referred to as a division ring). In his paper *Theory of non-commutative polynomials* [68] from 1933, Ore investigated this question and what properties be natural to consider defining in this new context:

> *"In the present paper I have tried to give the principal results of a general non-commutative polynomial theory. The polynomials considered have coefficients in an arbitrary commutative or non-commutative field, while the multiplication of polynomials is so restricted that the degree of a product is equal to the sum of the degrees of the factors."*

Let us follow Ore's train of thought and see where it takes us. First, we would like to study the slightly more general case when R is an arbitrary unital, associative ring. Even for commutative such, Ore's [68] assumption

> *"The degree of a product shall be equal to the sum of the degrees of the factors."*

does in general not hold even on $R[x]$, unless R is an integral domain. However, the degree of a product is always *less than or equal to* the sum of the degrees of the factors. Let us use this as an assumption when defining a new multiplication on $R[x]$, where R is a unital, associative ring. To this end, forget all about the usual multiplication of polynomials defined on $R[x]$ and consider $R[x]$ only as a free left R-module with coefficients written on the left. As a set, we are considering that of formal sums $\sum_{i\in\mathbb{N}} r_i x^i$ where only finitely many $r_i \in R$ are non-zero, and when we write x, what we really mean is the formal sum where $r_1 = 1_R$ and $r_i = 0$ for $i \neq 1$. Moreover, terms for which the coefficients are zero are naturally not written out. The left R-module multiplication is then defined by $r \cdot sx^m = (r \cdot s)x^m$ for any $r, s \in R$ and $m \in \mathbb{N}$, and we consider R as a subring of $R[x]$ by identifying any $r \in R$ with $rx^0 \in R[x]$. It also seems natural to still require that $x^m \cdot x^n = x^{m+n}$ for any $m, n \in \mathbb{N}$. Now, we have that the new product $x \cdot r$ must be a polynomial of degree (at most) $1 + 0 = 1$. Moreover, the coefficients of this degree one polynomial must, in some way, depend on the choice of $r \in R$. In other words,

$$x \cdot r = \sigma(r)x + \delta(r) \tag{1.1}$$

for some coefficients $\sigma(r)$ and $\delta(r)$. Also, $\sigma(r)$ and $\delta(r)$, being coefficients of an expression in a free left R-module, must be unique. Hence we get maps $\sigma\colon R \to R$ and $\delta\colon R \to R$. We want our new ring to be unital with identity element inherited from R, so $x = 1_R \cdot x = x \cdot 1_R = \sigma(1_R)x + \delta(1_R)$. By comparing coefficients, we must have $\sigma(1_R) = 1_R$ and $\delta(1_R) = 0$. Now, the multiplication should be distributive over addition, so in particular $x \cdot (r+s) = x \cdot r + x \cdot s$ for any $r, s \in R$. Using (1.1) to compute the left- and right-hand side,

$$\begin{aligned} x \cdot (r + s) &= \sigma(r + s)x + \delta(r + s), \\ x \cdot r + x \cdot s &= \sigma(r)x + \delta(r) + \sigma(s)x + \delta(s). \end{aligned}$$

By comparing coefficients, we see that σ and δ must be additive maps. In addition, we would like our new ring to be associative. Hence $x \cdot (r \cdot s) = (x \cdot r) \cdot s$ should hold. Expanding both sides gives us

$$\begin{aligned} x \cdot (r \cdot s) &= \sigma(r \cdot s)x + \delta(r \cdot s), \\ (x \cdot r) \cdot s &= (\sigma(r)x + \delta(r)) \cdot s = \sigma(r)x \cdot s + \delta(r) \cdot s \\ &= (\sigma(r) \cdot x) \cdot s + \delta(r) \cdot s = \sigma(r) \cdot (x \cdot s) + \delta(r) \cdot s \\ &= \sigma(r) \cdot (\sigma(s)x + \delta(s)) + \delta(r) \cdot s \\ &= \sigma(r) \cdot (\sigma(s) \cdot x) + \sigma(r) \cdot \delta(s) + \delta(r) \cdot s \\ &= (\sigma(r) \cdot \sigma(s)) \cdot x + \sigma(r) \cdot \delta(s) + \delta(r) \cdot s \\ &= (\sigma(r) \cdot \sigma(s))x + \sigma(r) \cdot \delta(s) + \delta(r) \cdot s, \end{aligned}$$

which in turn imply that σ is an endomorphism (respecting 1_R) and δ a so-called *σ-derivation*. The exact definition is as follows:

Definition 12 (σ-derivation)**.** Let R be a ring, and let σ be an endomorphism and δ an additive map on R. Then δ is called a *σ-derivation* if for all $r, s \in R$, $\delta(r \cdot s) = \sigma(r) \cdot \delta(s) + \delta(r) \cdot s$. If $\sigma = \mathrm{id}_R$, then δ is a derivation.

Lemma 1. *Let R be an associative ring, and let σ be an endomorphism on R. For any $r \in R$, the rule $\delta_r(s) := r \cdot s - \sigma(s) \cdot r$ defines a σ-derivation δ_r on R called an* inner σ-derivation *(all others are called* outer*).*

Proof. For any $r \in R$, the map δ_r is is clearly additive. For any $s, t \in R$, $\sigma(s) \cdot \delta_r(t) + \delta_r(s) \cdot t = \sigma(s) \cdot (r \cdot t - \sigma(t) \cdot r) + (r \cdot s - \sigma(s) \cdot r) \cdot t = r \cdot s \cdot t - \sigma(s) \cdot \sigma(t) \cdot r = r \cdot s \cdot t - \sigma(s \cdot t) \cdot r = \delta_r(s \cdot t)$. □

Note in particular that when $\sigma = \mathrm{id}_R$, then an inner σ-derivation δ_r is an inner derivation $\mathrm{ad}_r \in \mathrm{InnDer}_{\mathbb{Z}}(R)$.

Remark 6. Let R be a unital ring with identity element 1_R, and let σ be a unital endomorphism and δ a σ-derivation on R. Then $\delta(1_R) = 0$ since $\delta(1_R) = \delta(1_R \cdot 1_R) = \sigma(1_R) \cdot \delta(1_R) + \delta(1_R) \cdot 1_R = 2\delta(1_R) \iff \delta(1_R) = 0$.

At this stage, we know that given a unital, associative ring R, it is necessary to find a unital endomorphism σ and a σ-derivation δ on R in order to have even the slightest chance of finding a unital, associative non-commutative polynomial ring defined by (1.1). However, using these as assumptions, it is not clear at all how many such rings there are: zero, one, or maybe forty-two? As it turns out, there is (up to isomorphism) *precisely* one such ring (see e.g. [32] for proofs of both existence and uniqueness), and it is called the *Ore extension* of R.

Definition 13 (Ore extension). Let R be a unital, associative ring, σ a unital endomorphism and δ a σ-derivation on R. The *Ore extension* of R, $R[x;\sigma,\delta]$, is the polynomial ring $R[x]$ as a free left R-module, equipped with the unital, associative multiplication induced by (1.1).

One can of course now multiply arbitrary polynomials in $R[x;\sigma,\delta]$ by iterating the product rule (1.1). However, the computations soon get rather involved. For instance, already for the seemingly simple product $x^2\cdot r$, we have $x^2\cdot r = (x\cdot x)\cdot r = x\cdot(x\cdot r) = x\cdot(\sigma(r)x+\delta(r)) = x\cdot\sigma(r)x+x\cdot\delta(r) = x\cdot(\sigma(r)\cdot x)+x\cdot\delta(r) = (x\cdot\sigma(r))\cdot x + x\cdot\delta(r) = (\sigma(\sigma(r))x+\delta(\sigma(r)))\cdot x+\sigma(\delta(r))x+\delta(\delta(r)) = \sigma(\sigma(r))x^2 + (\delta(\sigma(r))+\sigma(\delta(r)))x+\delta(\delta(r))$. Fortunately, one can deduce a general formula for the product of two arbitrary monomials, which by extension to polynomials defines the multiplication uniquely. The product of two arbitrary monomials rx^m and sx^n in $R[x;\sigma,\delta]$ where $r,s\in R$ and $m,n\in\mathbb{N}$, is given by

$$rx^m\cdot sx^n = \sum_{i\in\mathbb{N}}(r\cdot\pi_i^m(s))x^{i+n}. \tag{1.2}$$

Here, the functions $\pi_i^m\colon R\to R$, called π *functions*, are defined as the sum of all $\binom{m}{i}$ compositions of i instances of σ and $m-i$ instances of δ. For instance, $\pi_2^3 = \sigma\circ\sigma\circ\delta+\sigma\circ\delta\circ\sigma+\delta\circ\sigma\circ\sigma$, while π_0^0 is defined as id_R. Whenever $i>m$, we set $\pi_i^m = 0$.

Now, it is finally time to see some examples of Ore extensions. First, note that for any endomorphism σ, the zero map is a σ-derivation. The first example of an Ore extension is one we have already seen.

Example 14. Let R be a unital, associative ring. Then $R[x]$ is the Ore extension $R[x;\mathrm{id}_R,0]$.

By definition, the previous example is a *skew polynomial ring*:

Definition 14. Let R be a unital, associative ring, and σ a unital endomorphism on R. Then $R[x;\sigma,0]$ is called a *skew polynomial ring*.

The multiplication in a skew polynomial ring $R[x;\sigma,0]$ is thus induced by the relation $x \cdot r = \sigma(r)x$ for any $r \in R$. Goodearl has shown (Lemma 1.5 in [33]) that if δ_r is an inner σ-derivation for some $r \in R$, then there is an isomorphism of the Ore extension $R[x;\sigma,\delta_r]$ to the skew polynomial ring $R[x';\sigma,0]$ where $x-r$ is mapped to x'.

Example 15. Let $\sigma\colon \mathbb{C} \to \mathbb{C}$ be complex conjugation. Then σ is a ring automorphism (or even an $\mathbb{R}$-algebra automorphism) of $\mathbb{C}$, and so we can build a skew polynomial ring (or even $\mathbb{R}$-algebra) $\mathbb{C}[x;\sigma,0]$. In $\mathbb{C}[x;\sigma,0]$, we have $x \cdot i = \sigma(i)x = -ix$ while $x^2 \cdot i = (x \cdot x) \cdot i = x \cdot (x \cdot i) = x \cdot (-ix) = ix^2$. By using (1.1), we see that x^2 commutes with all elements of $\mathbb{C}[x;\sigma,0]$, and so $x^2 \in C(\mathbb{C}[x;\sigma,0])$. In fact, one can show that $C(\mathbb{C}[x;\sigma,0]) = \mathbb{R}[x^2]$.

Example 16. Let K be a field and $q \in K^\times$. The *quantized coordinate ring of* K^2, $\mathcal{O}_q(K^2)$ is the free, unital, associative algebra over K on two letters x and y, modulo the ideal generated by the q-commutation relation $x \cdot y = qy \cdot x$. In algebraic geometry, K^2 is known as the affine plane over K, and therefore $\mathcal{O}_q(K^2)$ is often called the *coordinate ring of the quantum plane*, or just the *quantum plane*. $\mathcal{O}_q(K^2)$ is (isomorphic to) the iterated skew polynomial ring $K[y][x;\sigma,0]$ where σ is the K-algebra automorphism on $K[y]$ defined by $\sigma(y) = qy$ (the elements x and y in $K[y][x;\sigma,0]$ are represented by the cosets of x and y in $\mathcal{O}_q(K^2)$). $\mathcal{O}_q(K^2)$, considered as the skew polynomial ring $K[y][x;\sigma,0]$, is a vector space over K with a basis consisting of the monomials $y^m x^n$ where $m,n \in \mathbb{N}$.

Now, we will have a first look at an Ore extension in which both σ and δ are non-trivial.

Example 17. Let K be a field and $q \in K^\times$. The *quantized Weyl algebra* over K, or just the *q-Weyl algebra*, is the free, unital, associative algebra $K\langle x,y\rangle$ modulo the ideal generated by the q-commutation relation $x \cdot y - qy \cdot x = 1_{K\langle x,y\rangle}$. Now, assume $q \neq 1_K$, and let σ be the K-algebra automorphism of $K[y]$ defined by $\sigma(y) = qy$. The rule

$$\delta(p(y)) := \frac{p(qy)-p(y)}{qy-y} = \frac{\sigma(p(y))-p(y)}{\sigma(y)-y}$$

where $p(y) \in K[y]$ is an arbitrary polynomial defines a σ-derivation on $K[y]$ known as the *Eulerian derivative*, the *Jackson derivative*, the *q-difference operator*, or simply the *q-derivative*. The q-Weyl algebra with $q \neq 1_K$ is then (isomorphic to) the Ore extension $K[y][x; \sigma, \delta]$.

Definition 15 (Differential polynomial ring)**.** Let R be a unital, associative ring, and δ a derivation on R. Then $R[x; \mathrm{id}_R, \delta]$ is called a *differential polynomial ring* or a *differential operator ring*.

The multiplication in a differential polynomial ring $R[x; \mathrm{id}_R, \delta]$ is thus induced by the relation $x \cdot r = rx + \delta(r)$ for any $r \in R$.

Example 18. Let K be a field. The simplest non-abelian Lie algebra over K, is two-dimensional. Moreover, there is (up to isomorphism) precisely one two-dimensional non-abelian Lie algebra $\mathfrak{r}_2$ over K. In characteristic zero, this Lie algebra was mentioned in Subsection 1.2.3 as an example of a strongly rigid Lie algebra. If $\{x, y\}$ is a basis of $\mathfrak{r}_2$ as a K-vector space, the Lie bracket $[\cdot, \cdot]_{\mathfrak{r}_2}$ is given by $[x, y]_{\mathfrak{r}_2} = y$. The *universal enveloping algebra* of $\mathfrak{r}_2$, $U(\mathfrak{r}_2)$, is the free, unital, associative algebra on the letters x and y modulo the ideal generated by the commutation $[x, y] = [x, y]_{\mathfrak{r}_2}$, where $[\cdot, \cdot]$ denotes the usual commutator. The defining relation $[x, y] = y$ of $U(\mathfrak{r}_2)$ can be written as $x \cdot y = y \cdot x + y \cdot \frac{\mathrm{d}}{\mathrm{d}y} y$, and from this one can deduce that $U(\mathfrak{r}_2)$ is (isomorphic to) the differential polynomial ring $K[y][x; \mathrm{id}_{K[y]}, y \cdot \mathrm{d}/\mathrm{d}y]$.

Example 19. Let K be a field. The *first Weyl algebra* A_1 over K is the free, unital, associative algebra $K\langle x, y\rangle$ on two letters x and y, modulo the ideal generated by the commutation relation $[x, y] = 1_{K\langle x,y\rangle}$. A_1 is (isomorphic to) the differential polynomial ring $K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$. In particular, A_1 is the q-Weyl algebra with $q = 1_K$. The name indicates that there are more Weyl algebras, and indeed, there are. The nth Weyl algebra A_n for $n \in \mathbb{N}_{>0}$ is the n-fold tensor product of the first Weyl algebra, and it is in turn (isomorphic to) a certain iterated differential polynomial ring over K. In the subsection here below, we will have a closer look at A_1 and its properties.

1.3.1 The first Weyl algebra

To quote S. C. Coutinho [18] in his excellent survey on A_1, this is an algebra that turns up in many different contexts and under many different guises. Perhaps the most famous role is that in which it also made its first appearance, in the 1920s; that as an algebra of quantum mechanical operators over $\mathbb{C}$. In this setting, x plays the role of a momentum operator, and y that of a position operator. The non-commutative nature of these two operators is what actually leads to the famous Heisenberg uncertainty relation, and ultimately to the death (and birth) of the famous Schrödinger's cat. (Disclaimer: no cats were harmed during the writing of this thesis, however.) As a vector space over K, A_1 has a basis $\{y^m x^n : m, n \in \mathbb{N}\}$. Apart from this, the case when $\operatorname{char} K = 0$ and that when $\operatorname{char} K > 0$ are quite different. For instance, if $\operatorname{char} K = 0$, there are no finite-dimensional representations of A_1 (as opposed to the case when $\operatorname{char} K > 0$, in which case there are many). A_1 is a non-commutative domain, but it should be mentioned that there is an alternative definition of A_1 as an algebra of differential operators, and as such, it is not a domain when $\operatorname{char} K > 0$ (see e.g. Chapter 2.3 in [17]). Throughout this thesis, we will stick to the former definition of A_1, though. The fact that A_1 contains no zero divisors can be used to derive another useful fact, namely that any non-zero endomorphism f on A_1 is unital: $f(1_{A_1}) = f(1_{A_1}) \cdot f(1_{A_1}) \iff f(1_{A_1}) \cdot (1_{A_1} - f(1_{A_1})) = 0 \iff f(1_{A_1}) = 1_{A_1}$. Another important fact is that $C(A_1) = K$ when $\operatorname{char} K = 0$, and, as first shown by Revoy [70], $C(A_1) = K[x^p, y^p]$ when $\operatorname{char} K = p > 0$. Littlewood [52] has proved that A_1 is simple when $K = \mathbb{R}$ and $K = \mathbb{C}$, and Hirsch [37] then generalized this to when K is an arbitrary field of characteristic zero, as well as for higher order Weyl algebras. A_1 contains non-trivial ideals when $\operatorname{char} K > 0$, however. Sridharan [74] has shown (cf. Remark 6.2 and Theorem 6.1) that the cohomology of A_1 is zero in all positive degrees when $\operatorname{char} K = 0$ (see also Theorem 5 in [31]). In particular, the vanishing of the cohomology in the first and second degree imply that all derivations are inner and that A_1 is formally rigid in the classical sense of Gerstenhaber [30]. It should be mentioned that there exists however a non-trivial so-called non-commutative deformation, which is due to Pinczon [69]. In this deformation, the deformation parameter no longer commutes with the original algebra, making it possible to deform A_1 into $U(\mathfrak{osp}(1,2))$, the universal enveloping algebra of the orthosymplectic Lie superalgebra (cf. Proposition 4.5 in [69]). A_1 is not rigid, and not all derivations are inner when $\operatorname{char} K > 0$, however (contrary to what is stated in [70]). In this latter case, Benkart, Lopes, and Ondrus [11] have found two non-inner derivations, and also been able to describe all derivations of A_1:

Theorem 1 ([11]). *Let* $\operatorname{char} K = p > 0$. *Then* $\operatorname{Der}_K(A_1) = C(A_1)E_x \oplus C(A_1)E_y \oplus \operatorname{InnDer}_K(A_1)$ *where* $E_x, E_y \in \operatorname{Der}_K(A_1)$ *are defined by* $E_x(x) = y^{p-1}$, $E_x(y) = E_y(x) = 0$, $E_y(y) = x^{p-1}$.

Dixmier [21] first described the automorphism group of A_1, $\mathrm{Aut}_K(A_1)$, in characteristic zero, and later Makar-Limanov [54] generalized Dixmier's result to arbitrary characteristic:

Theorem 2 ([54]). $\mathrm{Aut}_K(A_1)$ *is generated by* linear automorphisms,

$$x \mapsto k_1 x + k_2 y, \quad y \mapsto k_3 x + k_4 y, \quad \begin{vmatrix} k_1 & k_3 \\ k_2 & k_4 \end{vmatrix} = 1_K, \quad k_1, k_2, k_3, k_4 \in K,$$

and triangular automorphisms, $x \mapsto x, \quad y \mapsto y + q(x), \quad q(x) \in K[x]$.

In characteristic zero, any non-zero endomorphism on A_1 is injective since A_1 is simple (the kernel of any endomorphism is an ideal, and since there are only two ideals in a simple algebra, the kernel of a non-zero endomorphism must be the zero ideal). In [21], Dixmier further asked (cf. 11. Problèmes) if all non-zero endomorphisms on A_1 are also automorphisms? The question is still open, and the statement that all non-zero endomorphisms are automorphisms is now known as the *Dixmier conjecture*. As we will return to this conjecture a couple of times, let us write down the formal statement.

Conjecture 1 (Dixmier [21]). *Over a field of characteristic zero, all non-zero endomorphisms on A_1 are automorphisms.*

In prime characteristic, it is also known that all non-zero endomorphisms on A_1 are injective, but that there are non-trivial endomorphisms that are not automorphisms.

Tsuchimoto [75] and Kanel-Belov and Kontsevich [41] have proven, independently, that the Dixmier conjecture is stably equivalent to the more famous *Jacobian conjecture*, a generalization of the classical Rolle's theorem in calculus (see e.g. [10, 24] for nice introductions to the conjecture). Originally, the Jacobian conjecture was formulated as a problem for polynomials with integer coefficients by Keller [43], and is therefore also known as *Keller's problem*.

Conjecture 2 (Keller [43]). *Let K be a field of characteristic zero, and $f\colon K^2 \to K^2$ a polynomial map such that* $\det J(f) \in K^\times$. *Then f has an inverse which is also a polynomial map.*

A *polynomial map* $f\colon K^2 \to K^2$ is a map $(x_1, x_2) \mapsto (f_1(x_1, x_2), f_2(x_1, x_2))$ where $f_1, f_2 \in K[x_1, x_2]$, and $J(f) := (\partial f_i / \partial x_j)_{1 \le i,j \le 2}$, the *Jacobian matrix*.

1.3.2 Hilbert's basis theorem

Recall that a family $\mathcal{F}$ of subsets of a set S satisfies the *ascending chain condition* if there is no properly ascending infinite chain $S_1 \subset S_2 \subset \ldots$ of subsets from $\mathcal{F}$. Furthermore, an element in $\mathcal{F}$ is called a *maximal element* of $\mathcal{F}$ provided there is no element in $\mathcal{F}$ that properly contains that element.

Proposition 4. *Let R be an associative ring. Then the following conditions are equivalent:*

(NR1) *R satisfies the ascending chain condition on its right (left) ideals.*

(NR2) *Any non-empty family of right (left) ideals of R has a maximal element.*

(NR3) *Any right (left) ideal of R is finitely generated.*

Proof. See e.g. the proof of Proposition 1.1 in [32]. □

Definition 16 (Noetherian ring)**.** An associative ring R is called *right (left) Noetherian* if it satisfies the three equivalent conditions of Proposition 4 on its right (left) ideals. If R satisfies the conditions on both its right and left ideals, it is called *Noetherian*.

Theorem 3 (Hilbert's basis theorem)**.** *Let R be a unital, associative ring, σ an automorphism and δ a σ-derivation on R. If R is right (left) Noetherian, then so is $R[x;\sigma,\delta]$.*

Proof. See e.g. the proof of Theorem 2.6 in [32]. □

Remark 7. If $\sigma = \mathrm{id}_R$ and $\delta = 0$, we recover the classical Hilbert's basis theorem for the ordinary polynomial ring $R[x]$.

Example 20. If K is a field, I a non-zero ideal of K, and $i \in I$ a non-zero element, then $1_K = i^{-1} \cdot i \in I$ so that for any $k \in K$, $k = k \cdot 1_K \in I$. Therefore, there are only two ideals of K, the zero ideal and K itself, and in particular, they are both singly generated (by 0 and 1_K, respectively). Hence K is Noetherian, and by Remark 7, so is $K[y]$. From Theorem 3, it then follows that the skew polynomial ring made from complex conjugation in Example 15, the quantum plane in Example 16, the q-Weyl algebra in Example 17, the universal enveloping algebra in Example 18, and the first Weyl algebra in Example 19 are all Noetherian.

If R is right (left) Noetherian, but σ is not an automorphism, then $R[x;\sigma,\delta]$ may fail to be right (left) Noetherian. For instance, if σ is an endomorphism that is not an automorphism on a field K, then $K[x;\sigma,0]$ is left Noetherian, but not right Noetherian (see Example 1.25 in [44]). If σ is the K-algebra endomorphism on $K[y]$ given by $\sigma(p(y)) = p(y^2)$ on an arbitrary polynomial $p(y) \in K[y]$, then $K[y][x;\sigma,0]$ is neither right, nor left Noetherian (see Exercise 2P in [32]).

There is of course much more to say about associative Ore extensions and their properties, but which is beyond the scope of this thesis (and even if one insisted on using the margins, they would at least be too narrow to contain any proofs). We refer the interested reader to the books by Goodearl and Warfield [32] and McConnell and Robson [60], which both give nice introductions to the subject. Moreover, Lam's book [44] contains some illuminating examples and counterexamples.

1.4 Summaries of papers

1.4.1 Summary of Paper A [9]

In this paper, we introduce hom-associative Ore extensions as non-unital, non-associative Ore extensions with a hom-associative multiplication, and give some necessary and sufficient conditions for such to exist. Within this framework we then construct families of hom-associative quantum planes, hom-associative universal enveloping algebras of the two-dimensional non-abelian Lie algebra, and hom-associative Weyl algebras. All these families contain their associative counterparts, and hence, as algebras, they can be seen as hom-associative generalizations thereof. In this paper, we also prove that the hom-associative Weyl algebras are simple. Moreover, we provide a way of embedding any non-unital, multiplicative hom-associative algebra into a weakly unital, multiplicative hom-associative algebra, which we call a weak unitalization. The thesis author is the main author of this paper.

1.4.2 Summary of Paper B [4]

In this paper, we show that the hom-associative quantum planes and the hom-associative universal enveloping algebras of the two-dimensional non-abelian Lie algebra are formal deformations of their associative counterparts. We also show that these deformations induce formal deformations of the corresponding Lie algebras into hom-Lie algebras, when using this commutator as a bracket. Here, it should be mentioned that the two-dimensional non-abelian Lie algebra cannot be formally deformed as a Lie algebra, and that its universal enveloping algebra cannot be formally deformed as an associative algebra. The thesis author is the single author of this paper.

1.4.3 Summary of Paper C [7]

In this paper, we show that over a field of characteristic zero, the hom-associative Weyl algebras are a formal deformation of the first associative Weyl algebra, the latter which cannot be formally deformed as an associative algebra. We then show that some properties are preserved by the deformation, such as the commuter, while others are deformed, such as the center, the set of derivations, and power associativity. We also show that the deformation induces a formal deformation of the corresponding Lie algebra into a hom-Lie algebra, when using the commutator as bracket. Moreover, we prove that all homomorphisms between any two hom-associative Weyl algebras of which none is associative, are in fact isomorphisms. In particular, all endomorphisms are automorphisms in this case, and hence we prove a hom-associative analogue of the Dixmier conjecture to be true. The thesis author is the main author of this paper.

1.4.4 Summary of Paper D [8]

In this paper, we introduce hom-associative Weyl algebras over a field of prime characteristic as a generalization of the first associative Weyl algebra in prime characteristic. First, we study properties of hom-associative algebras constructed from associative algebras by a general "twisting" procedure, called the Yau twist. Then, with the help of these results, we determine the commuter, center, nuclei, and set of derivations of the hom-associative Weyl algebras. We also classify them up to isomorphism, and show, among other things, that all non-zero endomorphisms on them are injective, but not surjective. Last, we show that they can be described as a multi-parameter formal hom-associative deformation of the first associative Weyl algebra, and that this deformation induces a multi-parameter formal hom-Lie deformation of the corresponding Lie algebra, when using the commutator as bracket. The thesis author is the main author of this paper.

1.4.5 Summary of Paper E [6]

In this paper, we prove a hom-associative version of Hilbert's basis theorem, which includes as special cases both a non-associative version and the classical Hilbert's basis theorem for associative Ore extensions. Along the way, we develop hom-module theory, including the introduction of corresponding isomorphism theorems and a notion of being hom-Noetherian. We conclude with several new examples, including both non-associative and hom-associative Ore extensions which are all Noetherian by our theorem. The thesis author is the main author of this paper.

Chapter 2

Chapter 2

Hom-associative Ore extensions and weak unitalizations

"Aha, said X."

In *The spies of Oreborg*,
by Jakob Wegelius

This chapter is based on Papers **A**, **B**, and **C**.

A P. Bäck, J. Richter, and S. Silvestrov,
Hom-associative Ore extensions and weak unitalizations,
Int. Electron. J. Algebra **24** (2018), pp. 174–194, `arXiv:1710.04190`.

B P. Bäck,
Notes on formal deformations of quantum planes and universal enveloping algebras,
J. Phys.: Conf. Ser. **1194**(1) (2019), `arXiv:1812.00083`.

C P. Bäck and J. Richter,
On the hom-associative Weyl algebras,
J. Pure Appl. Algebra **224**(9) (2020), `arXiv:1902.05412`.

2.1 Non-unital, non-associative Ore extensions

In this section, we define non-unital, non-associative Ore extensions. Let R be a non-unital, non-associative ring, and σ and δ two maps on R. If R is unital, we further require that $\sigma(1_R) = 1_R$ and $\delta(1_R) = 0$. As a set, a *non-unital, non-associative Ore extension* of R, written $R[x;\sigma,\delta]$, consists of all formal sums $\sum_{i\in\mathbb{N}} r_i x^i$, called *polynomials*, where only finitely many $r_i \in R$ are non-zero. To simplify the notation a bit, we will in most cases only write out the terms with non-zero coefficients. $R[x;\sigma,\delta]$ is then equipped with the same termwise addition as an ordinary polynomial ring,

$$\sum_{i\in\mathbb{N}} r_i x^i + \sum_{i\in\mathbb{N}} s_i x^i = \sum_{i\in\mathbb{N}} (r_i + s_i) x^i, \quad r_i, s_i \in R.$$

We then use the multiplication (1.2) as a rule on $R[x;\sigma,\delta]$ for R non-unital and non-associative. In other words, our rule is first defined on *monomials* rx^m and sx^n by

$$rx^m \cdot sx^n = \sum_{i\in\mathbb{N}} (r \cdot \pi_i^m(s)) x^{i+n}, \quad m, n \in \mathbb{N},\ r, s, \in R,$$

and then extend biadditively to arbitrary polynomials. Here, the π functions are, just as in the associative case, defined as the sum of all $\binom{m}{i}$ compositions of i instances of σ and $m-i$ instances of δ. Whenever $i > m$ or $i < 0$, we put $\pi_i^m = 0$. The following two results are new.

Proposition 5. *Let R be a non-unital, non-associative ring. The rule* (1.2) *on $R[x;\sigma,\delta]$ is distributive over addition if and only if for all $r_j, s_j, t_j \in R$ and $j \in \mathbb{N}$,*

$$\sum_{i\in\mathbb{N}} \sum_{k=0}^{j} t_i \cdot \pi_k^i (r_{j-k} + s_{j-k}) = \sum_{i\in\mathbb{N}} \sum_{k=0}^{j} t_i \cdot (\pi_k^i (r_{j-k}) + \pi_k^i (s_{j-k})). \tag{2.1}$$

Proof. We begin by showing that right-distributivity follows immediately, without any conditions on σ and δ. To this end, let $r_i, s_i, t_i \in R$ and $i \in \mathbb{N}$ be arbitrary.

Then,

$$\left(\sum_{i\in\mathbb{N}} r_i x^i + \sum_{i\in\mathbb{N}} s_i x^i\right) \cdot \sum_{j\in\mathbb{N}} t_j x^j = \sum_{i\in\mathbb{N}} (r_i + s_i) x^i \cdot \sum_{j\in\mathbb{N}} t_j x^j$$
$$= \sum_{i,j,k\in\mathbb{N}} \left((r_i + s_i) \cdot \pi_k^i(t_j)\right) x^{j+k},$$
$$\sum_{i\in\mathbb{N}} r_i x^i \cdot \sum_{j\in\mathbb{N}} t_j x^j + \sum_{i\in\mathbb{N}} s_i x^i \cdot \sum_{j\in\mathbb{N}} t_j x^j$$
$$= \sum_{i,j,k\in\mathbb{N}} (r_i \cdot \pi_k^i(t_j)) x^{j+k} + \sum_{i,j,k\in\mathbb{N}} (s_i \cdot \pi_k^i(t_j)) x^{j+k}$$
$$= \sum_{i,j,k\in\mathbb{N}} ((r_i + s_i) \cdot \pi_k^i(t_j)) x^{j+k}.$$

We now proceed to investigate left-distributivity.

$$\sum_{i\in\mathbb{N}} t_i x^i \cdot \left(\sum_{l\in\mathbb{N}} r_l x^l + \sum_{l\in\mathbb{N}} s_l x^l\right) = \sum_{i\in\mathbb{N}} t_i x^i \cdot \sum_{l\in\mathbb{N}} (r_l + s_l) x^l$$
$$= \sum_{i,k,l\in\mathbb{N}} (t_i \cdot \pi_k^i(r_l + s_l)) x^{l+k} = \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \sum_{k=0}^{j} (t_i \cdot \pi_k^i(r_{j-k} + s_{j-k})) x^j,$$
$$\sum_{i\in\mathbb{N}} t_i x^i \cdot \sum_{l\in\mathbb{N}} r_l x^l + \sum_{i\in\mathbb{N}} t_i x^i \cdot \sum_{l\in\mathbb{N}} s_l x^l$$
$$= \sum_{i,k,l\in\mathbb{N}} t_i \cdot \pi_k^i(r_l) x^{l+k} + \sum_{i,k,l\in\mathbb{N}} t_i \cdot \pi_k^i(s_l) x^{l+k}$$
$$= \sum_{i,k,l\in\mathbb{N}} (t_i \cdot (\pi_k^i(r_l) + \pi_k^i(s_l))) x^{l+k}$$
$$= \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \sum_{k=0}^{j} (t_i \cdot (\pi_k^i(r_{j-k}) + \pi_k^i(s_{j-k}))) x^j.$$

By comparing coefficients, the result now follows. □

By the above proposition, if R is a non-unital, non-associative ring with two maps σ and δ, then $R[x;\sigma,\delta]$ is a non-unital, non-associative ring if and only if σ

and δ satisfy (2.1). Now, recall that if R is unital, then we also assume that $\sigma(1_R) = 1_R$ and $\delta(1_R) = 0$. From this it follows that if R is a unital, non-associative ring with two maps σ and δ that satisfy (2.1), then $R[x;\sigma,\delta]$ is a unital, non-associative ring with identity element $1_R x^0$ (recall that this is the formal sum $\sum_{i\in\mathbb{N}} r_i x^i$ in which $r_0 = 1_R$ is the only (possible) non-zero coefficient). Moreover, if R is unital, we will often write x to denote the monomial $1_R x^1$. If R is not unital, it does not necessarily make sense to think of x as an element of $R[x;\sigma,\delta]$. From now on, when we write $R[x;\sigma,\delta]$, we shall implicitly assume that σ and δ satisfy (2.1) in Proposition 5, so that $R[x;\sigma,\delta]$ is a non-unital, non-associative ring.

Corollary 2. *Let R be a non-unital, non-associative ring. The rule* (1.2) *on $R[x;\sigma,\delta]$ is distributive over addition if σ and δ are additive.*

Proof. If σ and δ are additive, then so are π_i^m for any $i, m \in \mathbb{N}$. Hence the result follows from Proposition 5. □

By identifying any $r \in R$ with $rx^0 \in R[x;\sigma,\delta]$, R can be identified with a subring of $R[x;\sigma,\delta]$ (we shall often say that R *is* a subring of $R[x;\sigma,\delta]$). Hence, as suggested by the name, a non-unital, non-associative Ore extension of a non-unital, non-associative ring R is an extension of the ring R. Now, in the special case when $\delta = 0$, $R[x;\sigma,0]$ is said to be a *non-unital, non-associative skew polynomial ring*, and in case $\sigma = \mathrm{id}_R$, we say that $R[x;\mathrm{id}_R,\delta]$ is a *non-unital, non-associative differential polynomial ring*. In this latter case, (1.2) simplifies to

$$rx^m \cdot sx^n = \sum_{i\in\mathbb{N}} \binom{m}{i} \left(r \cdot \delta^{m-i}(s)\right) x^{i+n}. \tag{2.2}$$

2.2 Non-associative Ore extensions of non-associative rings

We use this small section to present a couple of results that hold true for any non-unital, non-associative Ore extension of a non-unital, non-associative ring R. First, with some abuse of notation, we extend an additive map α on R *homogeneously* to an additive map on $R[x;\sigma,\delta]$ by putting $\alpha(rx^m) := \alpha(r)x^m$ for any $r \in R$ and $m \in \mathbb{N}$, and then extend this additively to polynomials. The extended map is then called a *homogenous extension* of α.

Lemma 2 (A [9])**.** *Let R be a non-unital, non-associative ring. If α is an endomorphism on R, then the homogenous extension of α is an endomorphism on $R[x;\sigma,\delta]$ if and only if for all $i,m\in\mathbb{N}$ and $r,s\in R$,*

$$\alpha(r)\cdot\pi_i^m(\alpha(s))=\alpha(r)\cdot\alpha(\pi_i^m(s)). \tag{2.3}$$

Proof. Additivity follows from the definition, while for any monomials rx^m and sx^n,

$$\alpha(rx^m\cdot sx^n)=\alpha\left(\sum_{i\in\mathbb{N}}\left(r\cdot\pi_i^m(s)\right)x^{i+n}\right)=\sum_{i\in\mathbb{N}}\big(\alpha(r)\cdot\alpha\big(\pi_i^m(s)\big)\big)\,x^{i+n},$$

$$\alpha(rx^m)\cdot\alpha(sx^n)=\alpha(r)x^m\cdot\alpha(s)x^n=\sum_{i\in\mathbb{N}}\left(\alpha(r)\cdot\pi_i^m(\alpha(s))\right)x^{i+n}.$$

Comparing coefficients completes the proof. □

Lemma 3 (A [9])**.** *Let R be a non-unital, non-associative ring. If α is an endomorphism on R that commutes with σ and δ, then the homogeneous extension of α is an endomorphism on $R[x;\sigma,\delta]$.*

Proof. This follows immediately from Lemma 2, since if α commutes with σ and δ, then α also commutes with π_i^m for any $i,m\in\mathbb{N}$. □

2.3 Hom-associative Ore extensions of non-associative rings

In this small section, we focus on the question when non-unital, non-associative Ore extensions of non-unital, non-associative rings are hom-associative? Reader discretion is advised, as the formulas presented here can be considered quite unappealing, and sometimes maybe even vulgar. However, these will for instance all be used in the next section where we construct concrete examples of hom-associative Ore extensions.

Lemma 4 (A [9])**.** *Let R be a non-unital, non-associative ring. Define an additive map α on $R[x;\sigma,\delta]$ by*

$$\alpha\left(rx^m\right):=\sum_{i\in\mathbb{N}}\alpha_{i,m}(r)x^i,\quad r,\alpha_{i,m}(r)\in R,\ m\in\mathbb{N}. \tag{2.4}$$

Then $R[x;\sigma,\delta]$ is hom-associative with twisting map α if and only if for all $r,s,t\in R$ and $k,l,m,n\in\mathbb{N}$,

$$\begin{aligned}&\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}\alpha_{i,l}(r)\cdot\pi_{k-j}^{i}\left(s\cdot\pi_{j-n}^{m}(t)\right)\\&=\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}\left(r\cdot\pi_{i}^{l}(s)\right)\cdot\pi_{k-j}^{i+m}\left(\alpha_{j,n}(t)\right).\end{aligned}\tag{2.5}$$

Proof. For any $r,s,t\in R$ and $l,m,n\in\mathbb{N}$,

$$\begin{aligned}&\alpha\left(rx^{l}\right)\cdot\left(sx^{m}\cdot tx^{n}\right)=\alpha\left(rx^{l}\right)\cdot\sum_{q\in\mathbb{N}}\left(s\cdot\pi_{q}^{m}(t)x^{q+n}\right)\\&=\sum_{q\in\mathbb{N}}\alpha\left(rx^{l}\right)\cdot\left(\left(s\cdot\pi_{q}^{m}(t)\right)x^{q+n}\right)=\sum_{q\in\mathbb{N}}\sum_{i\in\mathbb{N}}\alpha_{i,l}(r)x^{i}\cdot\left(\left(s\cdot\pi_{q}^{m}(t)\right)x^{q+n}\right)\\&=\sum_{q\in\mathbb{N}}\sum_{i\in\mathbb{N}}\sum_{p\in\mathbb{N}}\alpha_{i,l}(r)\cdot\pi_{p}^{i}\left(s\cdot\pi_{q}^{m}(t)\right)x^{n+q+p}\\&=\sum_{p\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}\alpha_{i,l}(r)\cdot\pi_{p}^{i}\left(s\cdot\pi_{j-n}^{m}(t)\right)x^{p+j}\\&=\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}\alpha_{i,l}(r)\cdot\pi_{k-j}^{i}\left(s\cdot\pi_{j-n}^{m}(t)\right)x^{k},\end{aligned}$$

$$\begin{aligned}&\left(rx^{l}\cdot sx^{m}\right)\cdot\alpha\left(tx^{n}\right)=\left(\sum_{i\in\mathbb{N}}\left(r\cdot\pi_{i}^{l}(s)\right)x^{i+m}\right)\cdot\alpha\left(tx^{n}\right)\\&=\sum_{i\in\mathbb{N}}\left(r\cdot\pi_{i}^{l}(s)\right)x^{i+m}\cdot\alpha\left(tx^{n}\right)=\sum_{i\in\mathbb{N}}\left(r\cdot\pi_{i}^{l}(s)\right)x^{i+m}\cdot\sum_{j\in\mathbb{N}}\alpha_{j,n}(t)x^{j}\\&=\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\left(r\cdot\pi_{i}^{l}(s)\right)x^{i+m}\cdot\alpha_{j,n}(t)x^{j}\\&=\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{p\in\mathbb{N}}\left(r\cdot\pi_{i}^{l}(s)\right)\cdot\pi_{p}^{i+m}\left(\alpha_{j,n}(t)\right)x^{p+j}\\&=\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}\left(r\cdot\pi_{i}^{l}(s)\right)\cdot\pi_{k-j}^{i+m}\left(\alpha_{j,n}(t)\right)x^{k}.\end{aligned}$$

Comparing coefficients then completes the proof. □

Lemma 5 (A [9]). *Let R be a non-unital, non-associative ring, and let $R[x;\sigma,\delta]$ be a non-unital, hom-associative Ore extension of R with twisting map defined by* (2.4). *Then the following assertions hold for all $r,s,t \in R$ and $k,n \in \mathbb{N}$:*

$$\sum_{i\in\mathbb{N}} \alpha_{i,0}(r)\cdot \pi^i_{k-n}(s\cdot t) = (r\cdot s)\cdot \alpha_{k,n}(t), \tag{2.6}$$

$$\begin{aligned}
&\sum_{i\in\mathbb{N}} \alpha_{i,0}(r)\cdot \left(\pi^i_{k-n-1}(s\cdot\sigma(t)) + \pi^i_{k-n}(s\cdot\delta(t))\right)\\
&= (r\cdot s)\cdot(\delta(\alpha_{k,n}(t)) + \sigma(\alpha_{k-1,n}(t)))\\
&= (r\cdot s)\cdot(\alpha_{k,n}(\delta(t)) + \alpha_{k-1,n}(\sigma(t)))\,,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\sum_{i\in\mathbb{N}} \alpha_{i,1}(r)\cdot \pi^i_{k-n}(s\cdot t)\\
&= (r\cdot\sigma(s))\cdot(\delta(\alpha_{k,n}(t)) + \sigma(\alpha_{k-1,n}(t))) + (r\cdot\delta(s))\cdot\alpha_{k,n}(t).
\end{aligned} \tag{2.8}$$

Here, $\alpha_{-1,n} := 0$.

Proof. We get (2.6), the first equality in (2.7), and (2.8) immediatly from the cases $l=m=0$, $l=0, m=1$, and $l=1, m=0$ in (2.5), respectively. The second equality in (2.7) follows from comparison with (2.6). □

2.4 Hom-associative Ore extensions of hom-associative rings

In this section, we continue our previous investigation, but narrowed down to hom-associative Ore extensions of hom-associative rings.

Lemma 6 (A [9]). *Let R be a non-unital, hom-associative ring, and extend the twisting map α of R to $R[x;\sigma,\delta]$ homogeneously. Then $R[x;\sigma,\delta]$ is hom-associative with twisting map α if and only if for all $r,s,t\in R$ and $l,m,n\in\mathbb{N}$,*

$$\sum_{i\in\mathbb{N}} \alpha(r)\cdot\pi^l_i\left(s\cdot\pi^m_{n-i}(t)\right) = \sum_{i\in\mathbb{N}}\left(r\cdot\pi^l_i(s)\right)\cdot\pi^{i+m}_n\left(\alpha(t)\right). \tag{2.9}$$

Proof. A homogeneous α corresponds to $\alpha_{i,m}(r) = \alpha(r)\cdot\delta_{i,m}$ in (2.4) in Lemma 4, where $\delta_{i,m}$ is the Kronecker delta. The left-hand side of (2.5) in Lemma 4 then reads

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} \alpha_{i,l}(r)\cdot\pi^{i}_{k-j}\left(s\cdot\pi^{m}_{j-n}(t)\right)\\
&=\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}} \alpha(r)\cdot\delta_{i,l}\cdot\pi^{i}_{k-j}\left(s\cdot\pi^{m}_{j-n}(t)\right)=\sum_{j\in\mathbb{N}}\alpha(r)\cdot\pi^{l}_{k-j}\left(s\cdot\pi^{m}_{j-n}(t)\right)\\
&=\sum_{i\in\mathbb{N}}\alpha(r)\cdot\pi^{l}_{i}\left(s\cdot\pi^{m}_{k-i-n}(t)\right)=\sum_{i\in\mathbb{N}}\alpha(r)\cdot\pi^{l}_{i}\left(s\cdot\pi^{m}_{n'-i}(t)\right),
\end{aligned}
$$

while the right-hand side reads

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}\left(r\cdot\pi^{l}_{i}(s)\right)\cdot\pi^{i+m}_{k-j}\left(\alpha_{j,n}(t)\right)\\
&=\sum_{j\in\mathbb{N}}\sum_{i\in\mathbb{N}}\left(r\cdot\pi^{l}_{i}(s)\right)\cdot\pi^{i+m}_{k-j}\left(\alpha(t)\cdot\delta_{j,n}\right)=\sum_{i\in\mathbb{N}}\left(r\cdot\pi^{l}_{i}(s)\right)\cdot\pi^{i+m}_{k-n}\left(\alpha(t)\right)\\
&=\sum_{i\in\mathbb{N}}\left(r\cdot\pi^{l}_{i}(s)\right)\cdot\pi^{i+m}_{n'}\left(\alpha(t)\right).
\end{aligned}
$$

By renaming n' to n, the result now follows. □

Lemma 7 (**A** [9]). *Let R be a non-unital, hom-associative ring, and extend the twisting map α of R to $R[x;\sigma,\delta]$ homogeneously. If $R[x;\sigma,\delta]$ is hom-associative with twisting map α, then for all $r,s,t\in R$,*

$$(r\cdot s)\cdot\delta(\alpha(t))=(r\cdot s)\cdot\alpha(\delta(t)), \tag{2.10}$$

$$(r\cdot s)\cdot\sigma(\alpha(t))=(r\cdot s)\cdot\alpha(\sigma(t)), \tag{2.11}$$

$$\alpha(r)\cdot\delta(s\cdot t)=\alpha(r)\cdot(\sigma(s)\cdot\delta(t)+\delta(s)\cdot t), \tag{2.12}$$

$$\alpha(r)\cdot\sigma(s\cdot t)=\alpha(r)\cdot(\sigma(s)\cdot\sigma(t)). \tag{2.13}$$

Proof. Using the same technique as in the proof of Lemma 6, this follows from Lemma 5 with a homogeneous α. □

Remark 8 (**A** [9]). For the last two equations, it is worth noting the resemblance to the associative case (see Definition 12 in Chapter 1 and the calculations made prior to it).

Lemma 8 (A [9]). *Let R be a non-unital, hom-associative ring, and extend the twisting map α of R to $R[x;\sigma,\delta]$ homogeneously. If α commutes with σ and δ, then $R[x;\sigma,\delta]$ is hom-associative with twisting map α if and only if for all $r,s,t \in R$, $l,m,n \in \mathbb{N}$,*

$$\alpha(r)\cdot\sum_{i\in\mathbb{N}}\pi_i^l\left(s\cdot\pi_{n-i}^m(t)\right)=\alpha(r)\cdot\sum_{i\in\mathbb{N}}\left(\pi_i^l(s)\cdot\pi_n^{i+m}(t)\right). \tag{2.14}$$

Proof. Using Lemma 6, we know that $R[x;\sigma,\delta]$ is hom-associative if and only if for all $r,s,t\in R$ and $l,m,n\in\mathbb{N}$,

$$\sum_{i\in\mathbb{N}}\alpha(r)\cdot\pi_i^l\left(s\cdot\pi_{n-i}^m(t)\right)=\sum_{i\in\mathbb{N}}\left(r\cdot\pi_i^l(s)\right)\cdot\pi_n^{i+m}\left(\alpha(t)\right).$$

However, since R is hom-associative and α commutes with σ and δ, the right-hand side can be rewritten as

$$\sum_{i\in\mathbb{N}}\left(r\cdot\pi_i^l(s)\right)\cdot\alpha\left(\pi_n^{i+m}(t)\right)=\sum_{i\in\mathbb{N}}\alpha(r)\cdot\left(\pi_i^l(s)\cdot\pi_n^{i+m}(t)\right).$$

As a last step, we pull out $\alpha(r)$ from the two sums. □

Recall from Definition 12 in Chapter 1 that if σ is an endomorphism on a ring R, then an additive map δ on R is called a σ-derivation if for all $r,s\in R$, $\delta(r\cdot s)=\sigma(r)\cdot\delta(s)+\delta(r)\cdot s$. Now, when R is unital and associative, we also saw in Chapter 1 that it was both necessary and sufficient that σ is an endomorphism and δ a σ-derivation for the Ore extension $R[x;\sigma,\delta]$ to be associative. By Lemma 7 (and Proposition 5 for additivity), we see that in the non-unital, hom-associative setting, it is *almost* necessary that σ is an endomorphism and δ a σ-derivation that both commute with α on R for the Ore extension $R[x;\sigma,\delta]$ to be hom-associative. If we get rid of the "almost" in these necessary conditions, they are actually also sufficient. The next proposition demonstrates this fact.

Proposition 6 (A [9]). *Let R be a non-unital (unital), hom-associative ring with twisting map α, σ an (unital) endomorphism and δ a σ-derivation on R that both commute with α. If α is extended to $R[x;\sigma,\delta]$ homogeneously, then $R[x;\sigma,\delta]$ is a (unital) hom-associative Ore extension with twisting map α.*

Proof. We refer the reader to the proof in [63], where it is seen that neither associativity, nor unitality is used to prove that for all $s, t \in R$ and $l, m, n \in \mathbb{N}$,

$$\sum_{i \in \mathbb{N}} \pi_i^l \left(s \cdot \pi_{n-i}^m(t) \right) = \sum_{i \in \mathbb{N}} \pi_i^l(s) \cdot \pi_n^{i+m}(t) . \tag{2.15}$$

Hence (2.14) in Lemma 8 holds. $\square$

Note that in the above proposition, α need not be an endomorphism on R. Now, recall from Proposition 2 in Chapter 1 that if R is an associative ring and α is an endomorphism on R, then R^α denotes the hom-associative ring $(R, *, \alpha)$, known as the Yau twist of R. As additive groups, R and R^α are the same, but the multiplication $*$ in R^α is defined by $r * s = \alpha(r \cdot s)$ for any $r, s \in R$. In fact, this also makes α an endomorphism on R^α, since for any $r, s \in R$, $\alpha(r * s) = \alpha(\alpha(r \cdot s)) = \alpha(\alpha(r) \cdot \alpha(s)) = \alpha(r) * \alpha(s)$. Moreover, if R is unital with identity element 1_R, then from Corollary 1 in Chapter 1, R^α is weakly unital with weak identity element 1_R.

Lemma 9 (**A** [9]). *Let R be an associative ring, α and endomorphism, σ an endomorphism and δ a σ-derivation on R that both commute with α. Then σ is an endomorphism and δ is a σ-derivation on R^α.*

Proof. For any $r, s \in R$,

$$\begin{aligned}
\sigma(r * s) &= \sigma(\alpha(r \cdot s)) = \alpha(\sigma(r \cdot s)) = \alpha(\sigma(r) \cdot \sigma(s)) = \sigma(r) * \sigma(s), \\
\delta(r * s) &= \delta(\alpha(r \cdot s)) = \alpha(\delta(r \cdot s)) = \alpha(\sigma(r) \cdot \delta(s) + \delta(r) \cdot s) \\
&= \alpha(\sigma(r) \cdot \delta(s)) + \alpha(\delta(r) \cdot s) = \sigma(r) * \delta(s) + \delta(r) * s.
\end{aligned}$$

$\square$

If we start with a unital, associative ring R, an endomorphism α, a unital endomorphism σ and a σ-derivation δ that both commute with α, then we may form the non-unital, non-associative Ore extension $R^\alpha[x; \sigma, \delta]$. By Lemma 9, σ is then an endomorphism and δ a σ-derivation on R^α. Hence, if we extend α homogeneously to $R^\alpha[x; \sigma, \delta]$, by Proposition 6, we have a hom-associative Ore extension. In the next proposition, we will see that this hom-associative Ore extension is precisely the Yau twist of a unital, associative Ore extension.

Proposition 7 (**A** [9]). *Let R be a unital, associative ring, α an endomorphism, σ a unital endomorphism and δ a σ-derivation on R that both commute with α. If α is extended to $R[x; \sigma, \delta]$ (or $R^\alpha[x; \sigma, \delta]$) homogeneously, then $R[x; \sigma, \delta]^\alpha = R^\alpha[x; \sigma, \delta]$.*

Proof. If α is an endomorphism on R that commutes with σ and δ, by Lemma 3, the homogeneous extension of α is then an endomorphism on $R[x;\sigma,\delta]$. Hence, by Proposition 2 in Chapter 1, we may indeed define the Yau twist of $R[x;\sigma,\delta]$, written $R[x;\sigma,\delta]^\alpha$. On the other hand, as already mentioned, $R[x;\sigma,\delta]^\alpha = R[x;\sigma,\delta]$ and $R^\alpha = R$ as additive groups, and hence we also have $R[x;\sigma,\delta]^\alpha = R^\alpha[x;\sigma,\delta]$ as additive groups. Hence, extending α to $R[x;\sigma,\delta]$ homogeneously is equivalent to extending α to $R^\alpha[x;\sigma,\delta]$ homogeneously. What is left to check is that the multiplication $*$ in $R[x;\sigma,\delta]^\alpha$ is the same as the multiplication, temporarily denoted by $\bullet$, in $R^\alpha[x;\sigma,\delta]$. For any $r,s\in R$ and $m,n\in\mathbb{N}$, we have

$$rx^m * sx^n = \alpha\left(\sum_{i\in\mathbb{N}}\left(r\cdot\pi_i^m(s)\right)x^{i+n}\right) = \sum_{i\in\mathbb{N}}\alpha\left(r\cdot\pi_i^m(s)\right)x^{i+n}$$
$$= \sum_{i\in\mathbb{N}}\left(r*\pi_i^m(s)\right)x^{i+n} = rx^m \bullet sx^n. \qquad \square$$

Remark 9 (**A** [9]). Note in particular that $R^\alpha[x;\sigma,\delta]$ in Proposition 7 is weakly unital with weak identity element $1_R x^0$ (cf. Corollary 1 in Chapter 1).

Remark 10 (**A** [9]). Let R be a unital, associative ring and σ a unital endomorphism on R. By Proposition 7, we may always define a weakly unital, hom-associative skew polynomial ring $R^\sigma[x;\sigma,0]$ as the Yau twist of the unital, associative Ore extension $R[x;\sigma,0]$ by σ, i.e. $R[x;\sigma,0]^\sigma$.

Let us conclude this section with an observation. The equality $R[x;\sigma,\delta]^\alpha = R^\alpha[x;\sigma,\delta]$ in Proposition 7 can be interpreted that there are (at least) two different ways of constructing hom-associative Ore extensions. If we take the left-hand side as our starting point, we could start with a known unital, associative Ore extension $R[x;\sigma,\delta]$ of some unital, associative ring R. Next, we could try to find all endomorphisms α on R that commute with σ and δ, extend α homogeneously to an endomorphism on $R[x;\sigma,\delta]$, and then take the Yau twist of $R[x;\sigma,\delta]$, $R[x;\sigma,\delta]^\alpha$. The right-hand side of the equality then tells us that this is an Ore extension of the hom-associative ring R^α. If we take the right-hand side as our starting point, then we could start with a known Yau twist R^α of a unital, associative ring R. Next, we could try to find all unital endomorphisms σ and σ-derivations δ that commute with α to construct a non-associative Ore extension $R^\alpha[x;\sigma,\delta]$. If we then extend α homogeneously to an endomorphism on $R^\alpha[x;\sigma,\delta]$, the left-hand side of the equality tells us that this is a hom-associative ring which is the Yau twist of the unital, associative Ore extension $R[x;\sigma,\delta]$. To make a bold statement, the equality then provides, in a sense, a link between the hom-associative world and the world of associative Ore extensions, or at least between some parts of the two.

2.5 Examples

In this section, we use Proposition 7 to construct explicit examples of hom-associative Ore extensions. Here, we demonstrate both methods described at the end of the previous section. The first three examples are new.

Example 21. Let R be a unital, associative ring and α an endomorphism on R. Then $R[x]^\alpha$ is the weakly unital, hom-associative polynomial ring $R^\alpha[x]$. If $\alpha = \mathrm{id}_R$, then we have the unital, associative polynomial ring $R[x]$.

Example 22. Let us start with the, to our knowledge, simplest example of a hom-associative ring, or even $\mathbb{R}$-algebra, arising in a natural way. In other words, consider $\mathbb{C}^\alpha$ where α is complex conjugation (cf. Example 6 in Chapter 1). We wish to find weakly unital, hom-associative skew polynomial rings $\mathbb{C}^\alpha[x;\sigma,0]$ by using the construction in Proposition 7. In more detail, it seems natural to try to find all unital $\mathbb{R}$-algebra endomorphisms σ that commute with α. (In fact, any endomorphism f on $\mathbb{C}$ is automatically unital since e.g. $f(1_\mathbb{C}) = f(1_\mathbb{C}\cdot 1_\mathbb{C}) = f(1_\mathbb{C})\cdot f(1_\mathbb{C}) \iff f(1_\mathbb{C})\cdot(1_\mathbb{C} - f(1_\mathbb{C})) = 0 \iff f(1_\mathbb{C}) = 1_\mathbb{C}$.) Now, σ is assumed $\mathbb{R}$-linear and hence is completely determined by $\sigma(i)$ since for any $a,b\in\mathbb{R}$, $\sigma(a+bi) = a + b\cdot\sigma(i)$. Moreover, α is also $\mathbb{R}$-linear, so we see that σ and α commute if and only if $\alpha(\sigma(i)) = \sigma(\alpha(i))$. Hence, let $\sigma(i) = j + ki$ for some $j,k\in\mathbb{R}$. Then $\alpha(\sigma(i)) = \alpha(j+ki) = j - ki$ while $\sigma(\alpha(i)) = \sigma(-i) = -j - ki$. Hence $\alpha(\sigma(i)) = \sigma(\alpha(i)) \iff j = 0$. Last, let us determine what values of k that makes σ multiplicative. For any $a,b,c,d\in\mathbb{R}$, $\sigma((a+bi)\cdot(c+di)) = a\cdot c - b\cdot d + a\cdot d\cdot ki + b\cdot c\cdot ki$ while $\sigma(a+bi)\cdot\sigma(c+di) = (a+b\cdot ki)(c+d\cdot ki) = a\cdot c - b\cdot d\cdot k^2 + a\cdot d\cdot ki + b\cdot c\cdot ki$, and the two are equivalent if and only if $k^2 = 1_\mathbb{C} \iff k = \pm 1_\mathbb{C}$. The choice $k = 1_\mathbb{C}$ corresponds to $\sigma = \mathrm{id}_\mathbb{C}$ while $k = -1_\mathbb{C}$ corresponds to $\sigma = \alpha$. We thus have two hom-associative skew polynomial rings, $\mathbb{C}^\alpha[x]$ and $\mathbb{C}^\alpha[x;\alpha,0]$. By Proposition 7, $\mathbb{C}^\alpha[x] = \mathbb{C}[x]^\alpha$ and $\mathbb{C}^\alpha[x;\alpha,0] = \mathbb{C}[x;\alpha,0]^\alpha$; the former is thus the Yau twist of the commutative polynomial ring $\mathbb{C}[x]$, and the latter is the Yau twist of the unital, associative skew polynomial ring $\mathbb{C}[x;\alpha,0]$ in Example 15 in Chapter 1.

Example 23. To contrast the previous example, let us start with the skew polynomial ring $\mathbb{C}[x;\sigma,0]$ in Example 15 in Chapter 1, where σ is complex conjugation. We now wish to find all non-zero $\mathbb{R}$-algebra endomorphisms α on $\mathbb{C}$ that commute with σ. By the exact same calculations as in Example 22, but with the roles of α and σ changed, we see that $\alpha = \mathrm{id}_{\mathbb{C}}$ or $\alpha = \sigma$. We thus have two hom-associative skew polynomial rings, $\mathbb{C}[x;\sigma,0]$ and $\mathbb{C}[x;\sigma,0]^{\sigma} = \mathbb{C}^{\sigma}[x;\sigma,0]$. The latter is of course no other hom-associative skew polynomial ring than the one we ended up with in Example 22.

Example 24 (A [9])**.** Consider the quantum plane $\mathcal{O}_q(K^2)$ in Example 16 in Chapter 1 where K is a field of characteristic zero. In the very same example, we saw that we may exhibit $\mathcal{O}_q(K^2)$ as the iterated skew polynomial ring $K[y][x;\sigma,0]$ where σ is the K-algebra automorphism on $K[y]$ defined by $\sigma(y) = qy$. We now wish to find all non-zero K-algebra endomorphisms α on $K[y]$ that commute with σ. First, by the same argument as in e.g. Example 22, any endomorphism on $K[y]$ is necessarily unital. Now, put $\alpha(y) = k_0 + k_1 y + \ldots + k_m y^m$ for some $m \in \mathbb{N}$ and $k_0, \ldots, k_m \in K$. Then $\alpha(\sigma(y)) = q\alpha(y) = qk_0 + qk_1 y + \ldots qk_m y^m$ while $\sigma(\alpha(y)) = \sigma(k_0 + k_1 y + \ldots + k_m y^m) = k_0 + qk_1 y + \ldots + q^m k_m y^m$. By comparing coefficients, $k_0 = k_2 = ... = k_m = 0$ since $q \in K^{\times}$ is fixed. We define $k := k_1$, exclude the zero map by requiring $k \in K^{\times}$, and rename α to α_k. Hence $\alpha_k(y) = ky$. Moreover, this defines α_k uniquely as a K-algebra endomorphism, and for any $m \in \mathbb{N}$, we see that $\alpha(\sigma(y^m)) = \sigma(\alpha(y^m))$ since both σ and α are multiplicative. Hence α_k and σ commute, and so we have a one-parameter family of weakly unital, hom-associative skew polynomial rings $K[y]^{\alpha_k}[x;\sigma,0]$ as the Yau twists of the quantum plane by α_k, $K[y][x;\sigma,0]^{\alpha_k}$. We shall denote $K[y][x;\sigma,0]^{\alpha_k}$ by $\mathcal{O}_q^k(K^2)$, the *hom-associative quantum planes*. Note that the quantum plane is present in the member corresponding to $k = 1_K$. If $k \neq 1_K$, then $\mathcal{O}_q^k(K^2)$ is not associative as e.g. $x * (y * y) = k^4 q^2 y^2 x$, while $(x * y) * y = k^3 q^2 y^2 x$. The defining relation $x \cdot y = qy \cdot x$ of the quantum plane reads $x * y = qy * x$, where $qy * x = kqy \cdot x$, in the hom-associative version.

Example 25 (A [9])**.** Let $U(\mathfrak{r}_2)$ be the universal enveloping algebra of the two-dimensional non-abelian Lie algebra $\mathfrak{r}_2$ in Example 18 in Chapter 1, over a field K of characteristic zero. $U(\mathfrak{r}_2)$ may be exhibited as the differential polynomial ring $K[y][x;\mathrm{id}_{K[y]}, y \cdot \mathrm{d}/\mathrm{d}y]$. We would like to find all non-zero K-algebra endomorphisms α on $K[y]$ that commute with $y \cdot \mathrm{d}/\mathrm{d}y$. First, any non-zero K-algebra endomorphism on $K[y]$ is necessarily unital (see e.g. Example 24). Now, let $\alpha(y) = k_0 + k_1 y + \ldots k_m y^m$ for some $m \in \mathbb{N}$ and $k_0, \ldots, k_m \in K$. Then $\alpha(y \cdot \mathrm{d}y/\mathrm{d}y) = \alpha(y) = k_0 + k_1 y + \ldots k_m y^m$ while $(y \cdot \mathrm{d}/\mathrm{d}y)(\alpha(y)) = k_1 y + 2k_2 y + \ldots + m k_m y^m$. Hence we must have $k_0 = k_2 = \ldots = k_m = 0$. We define $k := k_1$, exclude the zero map by requiring $k \in K^\times$, and rename α to α_k. Hence $\alpha_k(y) = ky$. Moreover, this defines α_k uniquely as a K-algebra endomorphism, and for any $m \in \mathbb{N}$, we see that $\alpha_k(y \cdot \mathrm{d}y^m/\mathrm{d}y) = \alpha_k(my^m) = mk^m y^m = (y \cdot \mathrm{d}/\mathrm{d}y)\alpha_k(y^m)$. Hence α_k and $y \cdot \mathrm{d}/\mathrm{d}y$ commute, and so we have a one-parameter family of weakly unital, hom-associative differential polynomial rings $K[y]^{\alpha_k}[x; \mathrm{id}_{K[y]}, y \cdot \mathrm{d}/\mathrm{d}y]$ as the Yau twists of $U(\mathfrak{r}_2)$ by α_k, $K[y][x; \mathrm{id}_{K[y]}, y \cdot \mathrm{d}/\mathrm{d}y]^{\alpha_k}$. We shall denote $K[y][x; \mathrm{id}_{K[y]}, y \cdot \mathrm{d}/\mathrm{d}y]^{\alpha_k}$ by $U^k(\mathfrak{r}_2)$, the *hom-associative universal enveloping algebra of* $\mathfrak{r}_2$. Note that $U(\mathfrak{r}_2)$ is present in the member corresponding to $k = 1_K$. If $k \neq 1_K$, then $U^k(\mathfrak{r}_2)$ is not associative as e.g. $(x, y, y)_* := (x*y)*y - x*(y*y) = (1_K - k)k^3 y^2(x+2)$. The defining relation $[x, y] = y$ of $U(\mathfrak{r}_2)$ becomes $[x, y]_* = ky$ in the hom-associative version.

Example 26 (**A** [9]). Consider the first Weyl algebra A_1 in Example 19 in Chapter 1 over a field K of characteristic zero. A_1 is then (isomorphic to) the differential polynomial ring $K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$. First, recall that all non-zero K-algebra endomorphisms on $K[y]$ are unital. Now, we would like to find all non-zero K-algebra endomorphisms α on $K[y]$ that commute with $\mathrm{d}/\mathrm{d}y$. To this end, let $\alpha(y) = k_0 + k_1 y + \ldots + k_m y^m$ for some $m \in \mathbb{N}$ and $k_0, \ldots, k_m \in K$. Then $\alpha(\mathrm{d}y/\mathrm{d}y) = \alpha(1_{K[y]}) = 1_{K[y]}$ while $(\mathrm{d}/\mathrm{d}y)\alpha(y) = k_1 + 2k_2 y + \ldots + m k_m y^{m-1}$. Hence $k_2 = k_3 = \ldots = k_m = 0$ while $k_1 = 1_K$. As in the previous two examples, we define $k := k_0$ and rename α to α_k. Hence $\alpha_k(y) = k + y$. Moreover, this defines α_k uniquely as a K-algebra endomorphism, and for any $m \in \mathbb{N}$, $\alpha_k(\mathrm{d}y^m/\mathrm{d}y) = \alpha_k(my^{m-1}) = m(y+k)^{m-1} = (\mathrm{d}/\mathrm{d}y)\alpha_k(y^m)$ (here, $0y^{-1}$ is defined as zero). We can thus conclude that α_k and $\mathrm{d}/\mathrm{d}y$ commute, and so we have a one-parameter family of weakly unital, hom-associative differential polynomial rings $K[y]^{\alpha_k}[x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$ as the Yau twists of A_1 by α_k, $K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]^{\alpha_k}$. We shall denote $K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]^{\alpha_k}$ by A_1^k, the *hom-associative Weyl algebras*. Note that A_1 is present in the member corresponding to $k = 0$. If $k \neq 0$, then A_1^k is not associative as for example $(1_{A_1}, 1_{A_1}, y)_* := 1_{A_1} * (1_{A_1} * y) - (1_{A_1} * 1_{A_1}) * y = k$. The defining relation $[x, y] = 1_{A_1}$ of A_1 becomes $[x, y]_* = 1_{A_1}$, where 1_{A_1} is a weak identity element in the hom-associative version.

Proposition 8 (**C** [7]). *For any* A_1^k, $\alpha_k = e^{k\frac{\partial}{\partial y}}$.

Proof. Put $p = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} p_{ij}y^i x^j$ for some $p_{ij} \in K$. Then, by defining the exponential of the partial differential operator as its formal power series and putting $0y^i$ to be zero whenever $i < 0$,

$$\begin{aligned}
\alpha_k(p) &= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} p_{ij}(y+k)^i x^j = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{l=0}^{i} p_{ij}\binom{i}{l}k^l y^{i-l}x^j \\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{l=0}^{i} p_{ij}\left(\left(k\frac{\partial}{\partial y}\right)^l \Big/ l!\right) y^i x^j \\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{N}} p_{ij}\left(\left(k\frac{\partial}{\partial y}\right)^l \Big/ l!\right) y^i x^j \\
&= \sum_{l\in\mathbb{N}}\left(\left(k\frac{\partial}{\partial y}\right)^l \Big/ l!\right)\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} p_{ij}y^i x^j =: e^{k\frac{\partial}{\partial y}}p. \qquad \square
\end{aligned}$$

Recall from Chapter 1 that A_1 cannot be deformed in the classical sense; it is formally rigid. Moreover, from the same chapter, the Lie algebra $\mathfrak{r}_2$ is strongly rigid, meaning it is formally rigid as a Lie algebra, and its universal enveloping algebra $U(\mathfrak{r}_2)$ is formally rigid as an associative algebra. Here below, we show that A_1 and $U(\mathfrak{r}_2)$ can however be formally deformed as hom-associative algebras. In more detail, we show that A_1^k, $U(\mathfrak{r}_2)$, and $\mathcal{O}_q^k(K^2)$ are one-parameter formal hom-associative deformations of A_1, $U(\mathfrak{r}_2)$, and $\mathcal{O}_q(K^2)$, respectively. Moreover, we will see that they induce formal deformations of the corresponding Lie algebras into hom-Lie algebras, when using the commutator as bracket.

Proposition 9 (**B** [4], **C** [7]). A_1^k*,* $U^k(\mathfrak{r}_2)$*, and* $\mathcal{O}_q^k(K^2)$ *are one-parameter formal hom-associative deformations of* A_1*,* $U(\mathfrak{r}_2)$*, and* $\mathcal{O}_q(K^2)$*, respectively.*

Proof. We show this for A_1^k; the other proofs are similar. Put $t := k$, and regard t as an indeterminate of the formal power series $K[\![t]\!]$ and $A_1[\![t]\!]$; this gives a deformation $(A_1[\![t]\!], \cdot_t, \alpha_t)$ of $(A_1, \cdot_0, \mathrm{id}_{A_1})$, where the latter is A_1 in the language of hom-associative algebras, $\cdot_0$ denoting the multiplication in A_1. Explicitly, with $\alpha_t = e^{t\frac{\partial}{\partial y}}$ from Proposition 8, it is clear that α_t is a formal power series in t by definition, and moreover, $\alpha_0 = \mathrm{id}_{A_1}$. Next, we extend α_t linearly over $K[\![t]\!]$ and homogeneously to all of $A_1[\![t]\!]$. To define the multiplication $\cdot_t$ in $A_1[\![t]\!]$, we first extend $\cdot_0 \colon A_1 \times A_1 \to A_1$ homogeneously to a binary operation $\cdot_0 \colon A_1[\![t]\!] \times A_1[\![t]\!] \to A_1[\![t]\!]$ linear over $K[\![t]\!]$ in both arguments. Then we simply compose α_t with $\cdot_0$, so that $\cdot_t := \alpha_t \circ \cdot_0 = e^{t\frac{\partial}{\partial y}} \circ \cdot_0$. This is again a formal power series in t by definition, and hom-associativity now follows from Proposition 2 in Chapter 1. $\square$

Proposition 10 (**B** [4], **C** [7]). *The deformations of A_1, $U(\mathfrak{r}_2)$, and $\mathcal{O}_q(K^2)$ into A_1^k, $U^k(\mathfrak{r}_2)$, and $\mathcal{O}_q^k(K^2)$, respectively, induce one-parameter formal hom-Lie deformations of the Lie algebras of A_1, $U(\mathfrak{r}_2)$, and $\mathcal{O}_q(K^2)$ into the hom-Lie algebras of A_1^k, $U^k(\mathfrak{r}_2)$, and $\mathcal{O}_q^k(K^2)$, respectively, when using the commutator as bracket.*

Proof. We show this for A_1^k; the other proofs are similar. By using the deformation $(A_1[\![t]\!], \cdot_t, \alpha_t)$ of $(A_1, \cdot_0, \alpha_0)$ in Proposition 9, we construct a hom-Lie algebra $(A_1[\![t]\!], [\cdot,\cdot]_t, \alpha_t)$ by using the commutator $[\cdot,\cdot]_t$ of the hom-associative algebra $(A_1[\![t]\!], \cdot_t, \alpha_t)$ as bracket. Indeed, by Proposition 3 in Chapter 1, this gives a hom-Lie algebra. We claim that this is also a formal hom-Lie deformation of the Lie algebra $(A_1, [\cdot,\cdot]_0, \alpha_0)$ where $[\cdot,\cdot]_0$ is the commutator in A_1, and $\alpha_0 = \mathrm{id}_{A_1}$. Since α_t is the same map as in Proposition 9, we only need to verify that $[\cdot,\cdot]_t$ is a formal power series in t, which when evaluated at $t = 0$ gives the commutator in A_1. But this is immediate since $[\cdot,\cdot]_t = \alpha_t \circ [\cdot,\cdot]_0$. $\square$

2.6 Weak unitalizations of hom-associative algebras

Let R be a unital, associative, commutative ring. A non-unital, associative R-algebra A can always be embedded into a unital, associative R-algebra called the *unitalization* of A, or the *Dorroh extension* of A after Dorroh [22] who seems to be the first who discovered it. This works as follows. Consider A as an R-module, and take the direct sum $A \oplus R$. Now endow $A \oplus R$ with the following multiplication:

$$(a_1, r_1) \bullet (a_2, r_2) := (a_1 \cdot a_2 + r_1 \cdot a_2 + r_2 \cdot a_1, r_1 \cdot r_2) \tag{2.16}$$

for any $a_1, a_2 \in A$ and $r_1, r_2 \in R$. A can then be embedded by the injection $A \to A \oplus 0$ given by $a \mapsto (a, 0)$ for any $a \in A$. This makes A an ideal in $A \oplus R$. Moreover, $A \oplus R$ is, with the above product, a unital, associative R-algebra with identity element $(0, 1_R)$. In [29], Frégier and Gohr showed that the hom-associative algebra in Example 1 in Chapter 1 cannot be embedded into a weakly unital – and hence not into a unital – hom-associative algebra. This can be shown as follows. Assume that the hom-associative algebra in Example 1 may be embedded into a weakly unital hom-associative algebra with weak identity element e. Then, on the one hand, $\alpha(v_1)\cdot\alpha(v_2) = v_1\cdot(v_1+v_2) = v_2$. On the other hand, $\alpha(v_1)\cdot\alpha(v_2) = (e\cdot v_1)\cdot\alpha(v_2) = \alpha(e)\cdot(v_1\cdot v_2) = \alpha(e)\cdot 0 = 0$, which is a contradiction. In this section, we prove that any multiplicative hom-associative algebra A can be embedded into a multiplicative, weakly unital hom-associative algebra. We prove this by twisting the unitalization (2.16) with α, and call the resulting weakly unital, hom-associative algebra a *weak unitalization* of A.

Proposition 11 (A [9]). *Let A be a non-unital, non-associative R-algebra and α an R-linear map on A. Endow $A\oplus R$ with the following multiplication:*

$$(a_1,r_1)\bullet(a_2,r_2) :=(a_1\cdot a_2 + r_1\cdot\alpha(a_2) + r_2\cdot\alpha(a_1), r_1\cdot r_2), \tag{2.17}$$

for any $a_1,a_2\in A$ and $r_1,r_2\in R$. Then $A\oplus R$ is a non-unital, non-associative R-algebra.

Proof. R can be seen as a module over itself, and since any direct sum of modules over R is again a module over R, $A\oplus R$ is a module over R. For any $a_1,a_2\in A$ and $r_1,r_2,r_3\in R$,

$$\begin{aligned}
&r_3\cdot((a_1,r_1)\bullet(a_2,r_2)) = r_3\cdot(a_1\cdot a_2 + r_1\cdot\alpha(a_2)+r_2\cdot\alpha(a_1), r_1\cdot r_2)\\
&= (r_3\cdot(a_1\cdot a_2) + r_3\cdot(r_1\cdot\alpha(a_2)) + r_3\cdot(r_2\cdot\alpha(a_1)), r_3\cdot r_1\cdot r_2)\\
&= ((r_3\cdot a_1)\cdot a_2 + (r_3\cdot r_1)\cdot\alpha(a_2) + r_2\cdot\alpha(r_3\cdot a_1), r_3\cdot r_1\cdot r_2)\\
&= (r_3\cdot a_1, r_3\cdot r_1)\bullet(a_2,r_2) = (r_3\cdot(a_1,r_1))\bullet(a_2,r_2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&((a_1,r_1)+(a_2,r_2))\bullet(a_3,r_3) = (a_1+a_2, r_1+r_2)\bullet(a_3,r_3)\\
&= ((a_1+a_2)\cdot a_3 + (r_1+r_2)\cdot\alpha(a_3) + r_3\cdot\alpha(a_1+a_2), (r_1+r_2)\cdot r_3))\\
&= (a_1\cdot a_3 + r_1\cdot\alpha(a_3) + r_3\cdot\alpha(a_1), r_1\cdot r_3)\\
&\quad + (a_2\cdot a_3 + r_2\cdot\alpha(a_3) + r_3\cdot\alpha(a_2), r_2\cdot r_3)\\
&= (a_1,r_1)\bullet(a_3,r_3) + (a_2,r_2)\bullet(a_3,r_3),
\end{aligned}$$

so the binary operation $\bullet$ is R-linear in the first argument, and by symmetry, also R-linear in the second argument. $\square$

That any weakly unital hom-associative algebra is necessarily multiplicative if also $\alpha(e) = e$, where e is a weak identity element, should be known. However, we have not been able to find this statement elsewhere, so we provide a short proof of it here.

Lemma 10 (**A** [9])**.** *If e is a weak identity element in a weakly unital hom-associative algebra A with twisting map α, and $\alpha(e) = e$, then A is multiplicative.*

Proof. For any $a, b \in A$, $\alpha(e) \cdot (a \cdot b) = e \cdot (a \cdot b) = \alpha(a \cdot b)$, and by using hom-associativity, $\alpha(e) \cdot (a \cdot b) = (e \cdot a) \cdot \alpha(b) = \alpha(a) \cdot \alpha(b)$. □

Proposition 12 (**A** [9])**.** *Let A be a multiplicative hom-associative R-algebra with twisting map α. Then $(A \oplus R, \bullet, \beta_\alpha)$ where $\bullet$ is given by* (2.17) *and β_α by $\beta_\alpha((a, r)) := (\alpha(a), r)$ for any $a \in A$ and $r \in R$, is a multiplicative, weakly unital hom-associative R-algebra with weak identity element $(0, 1_R)$ called a* weak unitalization *of A.*

Proof. We proved in Proposition 11 that the multiplication $\bullet$ made $A \oplus R$ a non-unital, non-associative algebra. Hom-associativity can be proved by the following calculation:

$$
\begin{aligned}
&\beta_\alpha((a_1, r_1)) \bullet ((a_2, r_2) \bullet (a_3, r_3)) \\
&= (\alpha(a_1), r_1) \bullet (a_2 \cdot a_3 + r_2 \cdot \alpha(a_3) + r_3 \cdot \alpha(a_2), r_2 \cdot r_3) \\
&= (\alpha(a_1) \cdot (a_2 \cdot a_3) + r_2 \cdot \alpha(a_1) \cdot \alpha(a_3) + r_3 \cdot \alpha(a_1) \cdot \alpha(a_2) \\
&\quad + r_1 \cdot \alpha(a_2 \cdot a_3 + r_2 \cdot \alpha(a_3) + r_3 \cdot \alpha(a_2)) + r_2 \cdot r_3 \cdot \alpha(\alpha(a_1)), \\
&\qquad r_1 \cdot r_2 \cdot r_3) \\
&= ((a_1 \cdot a_2) \cdot \alpha(a_3) + r_2 \cdot \alpha(a_1) \cdot \alpha(a_3) + r_3 \cdot \alpha(a_1) \cdot \alpha(a_2) \\
&\quad + r_1 \cdot \alpha(a_2 \cdot a_3 + r_2 \cdot \alpha(a_3) + r_3 \cdot \alpha(a_2)) + r_2 \cdot r_3 \cdot \alpha(\alpha(a_1)), \\
&\qquad r_1 \cdot r_2 \cdot r_3) \\
&= ((a_1 \cdot a_2 + r_1 \cdot \alpha(a_2) + r_2 \cdot \alpha(a_1)) \cdot \alpha(a_3) \\
&\quad + r_1 \cdot r_2 \cdot \alpha(\alpha(a_3)) + r_3 \cdot \alpha(a_1 \cdot a_2) + r_3 \cdot \alpha(r_1 \cdot \alpha(a_2) + r_2 \cdot \alpha(a_1)), \\
&\qquad r_1 \cdot r_2 \cdot r_3) \\
&= ((a_1, r_1) \bullet (a_2, r_2)) \bullet \beta_\alpha((a_3, r_3)).
\end{aligned}
$$

Due to the fact that α is R-linear, it follows that β_α is also R-linear. Moreover, $(a_1, r_1) \bullet (0, 1_R) = (0, 1_R) \bullet (a_1, r_1) = (1_R \cdot \alpha(a_1), 1_R \cdot r_1) = \beta_\alpha((a_1, r_1))$, so $(0, 1_R)$ is a weak identity element. We also have that $\beta_\alpha((0, 1_R)) = (0, 1_R)$, so by Lemma 10, $(A \oplus R, \bullet, \beta_\alpha)$ is multiplicative. $\square$

Remark 11. If α is the identity map, so that the algebra is associative, then the weak unitalization is the unitalization described in the beginning of this section.

Corollary 3 (A [9]). $(A, \cdot, \alpha) \cong (A \oplus 0, \bullet, \beta_\alpha)$.

Proof. The projection map $\pi\colon A \oplus 0 \to A$ is a bijective algebra homomorphism. For any $a \in A$, $\pi(\beta_\alpha(a, 0)) = \pi(\alpha(a), 0) = \alpha(a)$ and $\alpha(\pi(a, 0)) = \alpha(a)$. Therefore $\alpha \circ \pi = \pi \circ \beta_\alpha$, so by Definition 4 in Chapter 1, $(A, \cdot, \alpha) \cong (A \oplus 0, \bullet, \beta_\alpha)$. $\square$

Using Corollary 3, we may thus identify $(A, \cdot, \alpha)$ with its image in $(A \oplus R, \bullet, \beta_\alpha)$, and hence we may speak about an embedding. The next example, which is new, uses this fact.

Example 27. Let A be the multiplicative hom-associative algebra in Example 2 in Chapter 1. In Example 4, we concluded that A is not weakly unital. Moreover, we claim that A cannot be embedded into a unital, hom-associative algebra. Seeking a contradiction, assume A is embedded in the unital, hom-associative algebra B with twisting map β where $\beta|_A = \alpha$. Now, from Remark 3 in Chapter 1, $\beta(1_B)$ is a weak identity element in B. Hence, on the one hand, $\alpha(v_2 \cdot v_2) = (v_2 \cdot v_2) \cdot \beta(1_B) = \alpha(v_2) \cdot (v_2 \cdot 1_B) = (v_1 + v_2) \cdot v_2 = v_1 + v_2$. On the other hand, $\alpha(v_2 \cdot v_2) = \alpha(v_1 + v_2) = 2v_1 + v_2$. The two expressions are equal if and only if $v_1 = 0$, which is a contradiction. By Proposition 12 and Corollary 3, we may however embed A into a weakly unital hom-associative algebra.

It is clear that weak identity elements are preserved by isomorphisms of hom-associative algebras. The next lemma shows that this is true for any surjective homomorphism.

Lemma 11 (C [7]). *Surjective homomorphisms of hom-associative algebras preserve weak left (right) identity elements.*

Proof. Let $f\colon A \to B$ be a surjective homomorphism between two hom-associative algebras with twisting maps α_A and α_B, respectively, and e_A a weak left identity element of A. We show the left case; the right case is analogous. For any element $b \in B$, there is an $a \in A$ such that $b = f(a)$, so $f(e_A) \cdot b = f(e_A) \cdot f(a) = f(e_A \cdot a) = f(\alpha_A(a)) = \alpha_B(f(a)) = \alpha_B(b)$. $\square$

Lemma 12 (**A** [9]). *All ideals in a weakly unital hom-associative algebra are hom-ideals.*

Proof. Let I be an ideal, $i \in I$ and e a weak identity element in a hom-associative algebra with twisting map α. Then $\alpha(i) = e \cdot i \in I$, so $\alpha(I) \subseteq I$. □

A simple hom-associative algebra is always hom-simple. By Lemma 12, the converse is also true if the algebra has a weak identity element.

Lemma 13 (**A** [9]). *$(A, \cdot, \alpha)$ is a hom-ideal in $(A \oplus R, \bullet, \beta_\alpha)$.*

Proof. For any $a_1, a_2 \in A$ and $r_1 \in R$, $(a_1, r_1) \bullet (a_2, 0) = (a_1 \cdot a_2 + r_1 \cdot \alpha(a_2), 0) \in A$, and $(a_2, 0) \bullet (a_1, r_1) = (a_2 \cdot a_1 + r_1 \cdot \alpha(a_2), 0) \in A$, so $(A, \cdot, \alpha)$ is an ideal in a weakly unital hom-associative algebra, and by Lemma 12 therefore also a hom-ideal. □

Recall that for a non-unital, non-associative ring R, if there is a positive integer n such that $n \cdot r = 0$ for all $r \in R$, then the smallest such n is the *characteristic of* R, written $\operatorname{char}(R)$. If no such positive integer exists, then one defines $\operatorname{char}(R) = 0$.

Proposition 13 (**A** [9]). *Let R be a weakly unital hom-associative ring with weak identity element e and injective or surjective twisting map. If $n \cdot e \neq 0$ for all $n \in \mathbb{Z}_{>0}$, then $\operatorname{char}(R) = 0$. If $n \cdot e = 0$ for some $n \in \mathbb{Z}_{>0}$, then the smallest such n is the characteristic of R.*

Proof. If $n \cdot e \neq 0$ for all $n \in \mathbb{Z}_{>0}$, then clearly we cannot have $n \cdot r = 0$ for all $r \in R$, and hence $\operatorname{char}(R) = 0$. Now assume n is a positive integer such that $n \cdot e = 0$. If the twisting map α is injective, then for all $r \in R$, $\alpha(n \cdot r) = n \cdot \alpha(r) = n \cdot (e \cdot r) = (n \cdot e) \cdot r = 0 \cdot r = 0 \iff n \cdot r = 0$. On the other hand, if α is surjective, then for all $r \in R$, $r = \alpha(s)$ for some $s \in R$, and hence $n \cdot r = n \cdot \alpha(s) = n \cdot (e \cdot s) = (n \cdot e) \cdot s = 0 \cdot s = 0$. □

Proposition 14 (**A** [9]). *Let R be a hom-associative ring with twisting map α, and define*

$$S := \begin{cases} (R \oplus \mathbb{Z}, \bullet, \beta_\alpha), & \text{if } \operatorname{char}(R) = 0, \\ (R \oplus \mathbb{Z}_n, \bullet, \beta_\alpha), & \text{if } \operatorname{char}(R) = n. \end{cases}$$

Then the weak unitalization S of R has the same characteristic as R.

Proof. This follows immediately by using the definition of the characteristic. □

Chapter 3

Chapter 3

The hom-associative Weyl algebras

"A brilliant work of spies, X concluded..."

In *The spies of Oreborg*,
by Jakob Wegelius

This chapter is based on Papers **A**, **C**, and **D**.

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3.1 The zero characteristic case

In this section, we study the hom-associative Weyl algebras A_1^k defined in Example 26 in Chapter 2 over a field K of characteristic zero.

Proposition 15 (C [7]). *K embeds as a subfield into A_1^k.*

Proof. K is embedded into the associative Weyl algebra A_1 by the isomorphism $f\colon K \to K' := \{cy^0x^0 \colon c \in K\} \subseteq A_1$ defined by $f(c) = cy^0x^0$ for any $c \in K$. One readily verifies that the same map embeds K into A_1^k, i.e. it is also an isomorphism of the hom-associative algebra K, the twisting map being id_K, and the hom-associative subalgebra $K' \subseteq A_1^k$. □

Just as in the associative case, the above proposition makes it possible to identify cy^0x^0 with c for any $c \in K$, something we will do from now on.

Lemma 14 (C [7]). *1_{A_1} is a unique weak left and a unique weak right identity element in A_1^k.*

Proof. First note that α_k is injective on A_1^k; it is injective on A_1, and since the underlying vector space is the same for the two algebras, also injective on A_1^k. Assume $e_l \in A_1^k$ is a weak left identity element. Then $e_l * 1_{A_1} = \alpha_k(1_{A_1})$. Since 1_{A_1} is a weak right identity element, $e_l * 1_{A_1} = \alpha_k(e_l)$, so $\alpha_k(e_l) = \alpha_k(1_{A_1})$. Hence $e_l = 1_{A_1}$, and analogously for the right case. □

We may define partial differential operators $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ on A_1 by $\frac{\partial}{\partial y}(y^m x^n) := my^{m-1}x^n$, $\frac{\partial}{\partial x}(y^m x^n) := ny^m x^{n-1}$ for any $m, n \in \mathbb{N}$, $0y^{-1}x^n$ and $0y^m x^{-1}$ defined to be zero, and extending linearly. If L is some linear operator on A_1^k such that for each $p \in A_1^k$, only finitely many elements $L^i p$ for $i \in \mathbb{N}$ are non-zero, then we may define e^L using ordinary formal power series. The next proposition gives an example of this.

Lemma 15 (C [7]). *For any $p, q \in A_1^k$,*

$$p * q = e^{k\frac{\partial}{\partial y}}(p \cdot q). \tag{3.1}$$

Proof. From Proposition 8 in Chapter 2, $\alpha_k = e^{k\frac{\partial}{\partial y}}$, so the result follows. □

Remark 12. The inverse of $e^{k\frac{\partial}{\partial y}}$ is simply $e^{-k\frac{\partial}{\partial y}}$, so by (3.1), $p \cdot q = e^{-k\frac{\partial}{\partial y}}(p * q)$.

Corollary 4 (C [7]). *There are no zero divisors in* A_1^k.

Proof. Using Remark 12, A_1^k cannot contain any zero divisors since A_1 does not. $\square$

Corollary 5 (C [7]). *For any polynomial* $p(x,y) \in A_1^k$,

$$[x, p(x,y)]_* = e^{k\frac{\partial}{\partial y}}[x, p(x,y)] = \frac{\partial}{\partial y} p(x, y+k), \tag{3.2}$$

$$[p(x,y), y]_* = e^{k\frac{\partial}{\partial y}}[p(x,y), y] = \frac{\partial}{\partial x} p(x, y+k). \tag{3.3}$$

Proof. We have $[x, y^m x^n] = x \cdot y^m x^n - y^m x^n \cdot x = \sum_{i\in\mathbb{N}} \binom{1}{i} \frac{\partial^{1-i} y^m}{\partial y^{1-i}} x^{n+i} - y^m x^{n+1} = m y^{m-1} x^n$ for any $m, n \in \mathbb{N}$, defining $0y^{-1}$ to be zero. By linearity in the second argument, it follows that $[x, p(x,y)] = \frac{\partial}{\partial y} p(x,y)$. By using Proposition 8,

$$\begin{aligned}
[x, p(x,y)]_* &= x * p(x,y) - p(x,y) * x = \alpha_k(x \cdot p(x,y)) - \alpha_k(p(x,y) \cdot x) \\
&= e^{k\frac{\partial}{\partial y}}(x \cdot p(x,y) - p(x,y) \cdot x) = e^{k\frac{\partial}{\partial y}}[x, p(x,y)] \\
&= e^{k\frac{\partial}{\partial y}} \frac{\partial}{\partial y} p(x,y) = \frac{\partial}{\partial y} e^{k\frac{\partial}{\partial y}} p(x,y) = \frac{\partial}{\partial y} p(y+k, x).
\end{aligned}$$

We also have $[y^m x^n, y] = y^m x^n \cdot y - y \cdot y^m x^n = y^m \sum_{i\in\mathbb{N}} \binom{n}{i} \frac{\partial^{n-i} y}{\partial y^{n-i}} x^i - y^{m+1} x^n = n y^m x^{n-1}$ for any $m, n \in \mathbb{N}$, defining $0x^{-1}$ to be zero. By linearity in the first argument, it follows that $[p(x,y), y] = \frac{\partial}{\partial x} p(x,y)$. Hence,

$$\begin{aligned}
[p(x,y), y]_* &= e^{k\frac{\partial}{\partial y}}[p(x,y), y] = e^{k\frac{\partial}{\partial y}} \frac{\partial}{\partial x} p(x,y) = \frac{\partial}{\partial x} e^{k\frac{\partial}{\partial y}} p(x,y) \\
&= \frac{\partial}{\partial x} p(y+k, x).
\end{aligned}$$

$\square$

Proposition 16 (C [7]). $C(A_1^k) = K$.

Proof. Let $c \in K$ and $q \in A_1^k$ be arbitrary. Then $[c,q]_* = \alpha_k([c,q]) = \alpha_k(0) = 0$, so $K \subseteq C(A_1^k)$. For any $p \in C(A_1^k)$, $[x,p]_* \overset{(3.2)}{=} e^{k\frac{\partial}{\partial y}}[x,p] \overset{!}{=} 0$, which implies $[x,p] = 0$. From Corollary 5, $[x,p] = \frac{\partial}{\partial y} p$, so $p \in K[x]$. Continuing, $[p,y]_* \overset{(3.3)}{=} e^{k\frac{\partial}{\partial y}}[p,y] \overset{!}{=} 0$, which implies $[p,y] = 0$. Again, from Corollary 5, $[p,y] = \frac{\mathrm{d}}{\mathrm{d}x} p$, so $p \in K$. $\square$

Corollary 6 (C [7]).

$$Z(A_1^k) = \begin{cases} K & \text{if } k = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. Recall from Chapter 1 that $Z(A_1^k) = C(A_1^k) \cap N(A_1^k)$. When $k = 0$, $N(A_1^k) = A_1^k$, and hence $Z(A_1^k) = C(A_1^k) = K$. Assume instead that $k \neq 0$, and let $c \in K$ be arbitrary. Then a straightforward calculation yields $(c, y, y)_* = -2ck^2 - cky \overset{!}{=} 0 \iff c = 0$. On the other hand, $0 \in N(A_1^k)$, so $Z(A_1^k) = \{0\}$. ☐

Proposition 17 (C [7]). *A_1^k is power associative if and only if $k = 0$.*

Proof. If $k = 0$, then A_1^k is associative and hence also power associative. On the other hand, one readily verifies that $(yx, yx, yx)_* = kx + 2k^2x^2$, so if A_1^k is power associative, then $k = 0$. ☐

Remark 13 (C [7]). Note that due to the proposition above, unless $k = 0$, A_1^k is not left alternative, right alternative, or flexible, let alone associative.

Recall from Chapter 1 that the associative Weyl algebra A_1 is simple. This fact is also true in the case of the non-associative Weyl algebras introduced in [66], and it turns out that all the hom-associative Weyl algebras have this property as well.

Proposition 18 (A [9]). *A_1^k is simple.*

Proof. The original version of this proof first appeared in [9]. Here, with the help of a couple of results from [7], we have slightly simplified that version. The main part of the proof still follows the same line of reasoning, however, and this is in turn essentially the same as in the unital, associative case.

Let I be a non-zero ideal of A_1^k and $p := \sum_{i\in\mathbb{N}} p_i(y)x^i \in I$ an arbitrary non-zero polynomial where $p_i(y) \in K[y]$. Put $m := \max_i(\deg(p_i(y)))$. Then, by (3.2) in Corollary 5, $[x, p]_* = \sum_{i\in\mathbb{N}} p_i'(y + k)x^i$ where $'$ denotes differentiation with respect to y. Since $\max_i(\deg(p_i'(y + k)) = m - 1$, by applying the commutator to the resulting polynomial with x m times, we get a polynomial $q := \sum_{j=0}^n q_j x^j$ for some $n \in \mathbb{N}$ where $q_j \in K$ and $q_n \neq 0$. By (3.3) in Corollary 5, $\deg([q, y]_*) = n-1$, where $\deg(\cdot)$ now denotes the degree of a polynomial in x. By applying the commutator to the resulting polynomial with y n times, we get $q_n = \alpha_k(q_n) = q_n * 1_{A_1} \in I \implies q_n^{-1} * q_n = \alpha_k(1_{A_1}) = 1_{A_1} \in I$.

Take any polynomial $r = \sum_{i\in\mathbb{N}} r_i(y)x^i \in A_1^k$. Then $1_{A_1} * \sum_{i\in\mathbb{N}} r_i(y-k)x^i = \sum_{i\in\mathbb{N}} r_i(y)x^i = r \in I \implies I = A_1^k$. $\square$

In Chapter 1 we said that maps of the form $[a,\cdot]\colon A \to A$ for any a in an associative algebra A are derivations called inner derivations. We also claimed that such a map need not be a derivation if A is not associative. For a concrete example of this latter fact, one can consider the map $[y^2,\cdot]_*$ in A_1^k, which is a derivation if and only if $k = 0$. The reason for this failure when $k = 0$ is due to the next lemma.

Lemma 16 **(C [7]).** *δ is a derivation on A_1^k if and only if δ is a derivation on A_1 that commutes with $e^{k\frac{\partial}{\partial y}}$.*

Proof. First, note that δ is a linear map on A_1^k if and only if it is a linear map on A_1, as the underlying vector spaces of A_1^k and A_1 are the same. Now, let δ be a derivation on A_1^k. We claim that $\delta(1_{A_1}) = 0$. First, $\delta(1_{A_1} * 1_{A_1}) = \delta(1_{A_1}) * 1_{A_1} + 1_{A_1} * \delta(1_{A_1}) = 2\alpha_k(\delta(1_{A_1})) = 2e^{k\frac{\partial}{\partial y}}\delta(1_{A_1})$, using that $\alpha_k = e^{k\frac{\partial}{\partial y}}$ from. On the other hand, $\delta(1_{A_1} * 1_{A_1}) = \delta(\alpha_k(1_{A_1})) = \delta(1_{A_1})$. The equality of the two expressions is then equivalent to the eigenvector problem $e^{k\frac{\partial}{\partial y}}p = \frac{1}{2}p$, where $p = \delta(1_{A_1})$. It turns out it has no solution, which may be seen from solving the equivalent PDE $p + 2\left(k\frac{\partial}{\partial y} + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} + \cdots + \frac{k^m}{m!}\frac{\partial^m}{\partial y^m}\right)p = 0$. To see this, let us put $p = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}y^i x^j$ for some $p_{ij} \in K$ and $m, n \in \mathbb{N}$. Then, by comparing coefficients, starting with p_{mj} for some arbitrary j and working our way down to p_{0j}, we have that $p_{ij} = 0$ for all $i, j \in \mathbb{N}$. Therefore, $\delta(1_{A_1}) = 0$ as claimed. For arbitrary $q \in A_1^k$, $\delta\left(e^{k\frac{\partial}{\partial y}}q\right) = \delta(\alpha_k(q)) = \delta(q * 1_{A_1}) = \delta(q) * 1_{A_1} + q * \delta(1_{A_1}) = \delta(q) * 1_{A_1} = \alpha_k(\delta(q)) = e^{k\frac{\partial}{\partial y}}\delta(q)$, so δ commutes with $e^{k\frac{\partial}{\partial y}}$. Now, $\alpha_k(\delta(r\cdot s)) = e^{k\frac{\partial}{\partial y}}\delta(r\cdot s) = \delta\left(e^{k\frac{\partial}{\partial y}}(r\cdot s)\right) = \delta(\alpha_k(r\cdot s)) = \delta(r*s) = \delta(r)*s + r*\delta(s) = \alpha_k(\delta(r)\cdot s) + \alpha_k(r\cdot\delta(s)) = \alpha_k(\delta(r)\cdot s + r\cdot\delta(s))$ where $r, s \in A_1$ are arbitrary. By the injectivity of α_k, $\delta(r\cdot s) = \delta(r)\cdot s + r\cdot\delta(s)$. Assume now instead that δ is a derivation on A_1 that commutes with $e^{k\frac{\partial}{\partial y}}$, and that $r, s \in A_1^k$. Then, $\delta(r * s) = \delta(\alpha_k(r\cdot s)) = \delta\left(e^{k\frac{\partial}{\partial y}}(r\cdot s)\right) = e^{k\frac{\partial}{\partial y}}\delta(r\cdot s) = \alpha_k(\delta(r\cdot s)) = \alpha_k(\delta(r)\cdot s + r\cdot\delta(s)) = \alpha_k(\delta(r)\cdot s) + \alpha_k(r\cdot\delta(s)) = \delta(r) * s + r * \delta(s)$. $\square$

Corollary 7 (C [7]). *δ is a derivation on A_1^k for k non-zero if and only if $\delta = [cy + p(x), \cdot] = e^{-k\frac{\partial}{\partial y}}[cy + p(x), \cdot]_*$ for some $c \in K$ and $p(x) \in K[x]$.*

Proof. Recall from Chapter 1 that all derivations on A_1 are inner, i.e. of the form $[q, \cdot]$ for some $q \in A_1$. From Lemma 16, there is a one-to-one correspondence between the derivations on A_1^k and the derivations on A_1 that commute with $e^{k\frac{\partial}{\partial y}}$. Hence, we are looking for $q \in A_1$ such that $e^{k\frac{\partial}{\partial y}}[q, x] = \left[q, e^{k\frac{\partial}{\partial y}}x\right] = [q, x]$ and $e^{k\frac{\partial}{\partial y}}[q, y] = \left[q, e^{k\frac{\partial}{\partial y}}y\right] = [q, y + k] = [q, y]$. We thus have two eigenvector problems of the form $e^{k\frac{\partial}{\partial y}}s = s$ with $s \in \{[q, x], [q, y]\}$. This is equivalent to the PDE $\left(k\frac{\partial}{\partial y} + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} + \cdots + \frac{k^m}{m!}\frac{\partial^m}{\partial y^m}\right)s = 0$, and by putting $s = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} s_{ij}y^ix^j$ for some $s_{ij} \in K$, we see by comparing coefficients that $s = \sum_{j\in\mathbb{N}} s_{0j}x^j$. Now, using that $s \in K[x]$, $[q, x] = -\frac{\partial}{\partial y}q$ and $[q, y] = \frac{\partial}{\partial x}q$ from Corollary 5, we get $\frac{\partial}{\partial y}q \in K[x]$ and $\frac{\partial}{\partial x}q \in K[x]$. If we put $q = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} q_{ij}y^ix^j$, then the former implies that $q = \sum_{j\in\mathbb{N}}(q_{0j} + q_{1j}y)x^j$, which upon plugging into the second yields $q = q_{10}y + \sum_{j\in\mathbb{N}} q_{0j}x^j$. We also claim that a q of this form is sufficient for fulfilling $e^{k\frac{\partial}{\partial y}}[q, u] = \left[q, e^{k\frac{\partial}{\partial y}}u\right]$ for any $u \in A_1$. First, $e^{k\frac{\partial}{\partial y}}q = kq_{10} + q$. Recalling that $e^{k\frac{\partial}{\partial y}}$ is an endomorphism on A_1, $e^{k\frac{\partial}{\partial y}}[q, u] = \left[e^{k\frac{\partial}{\partial y}}q, e^{k\frac{\partial}{\partial y}}u\right] = \left[kq_{10} + q, e^{k\frac{\partial}{\partial y}}u\right] = \left[q, e^{k\frac{\partial}{\partial y}}u\right]$. If $q_{10} := c$ and $p(x) := \sum_{j\in\mathbb{N}} q_{0j}x^j$, then $q = cy + p(x)$. By Remark 12, $[cy + p(x), \cdot] = e^{-k\frac{\partial}{\partial y}}[cy + p(x), \cdot]_*$. □

Lemma 17 (C [7]). *A non-zero map $f\colon A_1^k \to A_1^l$ is a homomorphism if and only if f is an endomorphism on A_1 such that $e^{l\frac{\partial}{\partial y}}f(x) = f(x)$ and $e^{l\frac{\partial}{\partial y}}f(y) = f(y) + k$.*

Proof. Let $f\colon A_1^k \to A_1^l$ be a non-zero homomorphism, i.e. a non-zero K-linear map such that $f \circ \alpha_k = \alpha_l \circ f$ and $f(a *_k b) = f(a) *_l f(b)$ hold for all $a, b \in A_1^k$. Since we may view the underlying vector spaces of A_1^k, A_1^l, and A_1 as the same, we only need to show that $e^{l\frac{\partial}{\partial y}}f(x) = f(x)$, $e^{l\frac{\partial}{\partial y}}f(y) = f(y) + k$, and $f(a \cdot b) = f(a) \cdot f(b)$. The former follows from $f \circ \alpha_k = \alpha_l \circ f$ with $\alpha_k = e^{k\frac{\partial}{\partial y}}$ from Proposition 8, together with the fact that $f(1_{A_1}) = 1_{A_1}$ as mentioned in Chapter 1. The latter follows from the fact that $f(a *_k b) = f(\alpha_k(a \cdot b)) = \alpha_l(f(a \cdot b))$, whereas $f(a) *_l f(b) = \alpha_l(f(a) \cdot f(b))$, and since α_l is injective,

$f(a \cdot b) = f(a) \cdot f(b)$. Assume instead that f is a non-zero endomorphism on A_1 such that $e^{l\frac{\partial}{\partial y}} f(x) = f(x)$ and $e^{l\frac{\partial}{\partial y}} f(y) = f(y) + k$. Then, with $\alpha_l = e^{l\frac{\partial}{\partial y}}$, for any $m, n \in \mathbb{N}$,

$$\begin{aligned}
\alpha_l(f(y^m x^n)) &= \alpha_l(f^m(y))\alpha_l(f^n(x)) = (\alpha_l(f(y)))^m(\alpha_l(f(x)))^n \\
&= (f(\alpha_k(y)))^m(f(\alpha_k(x)))^n = f(\alpha_k^m(y)\alpha_k^n(x)) = f(\alpha_k(y^m x^n)),
\end{aligned}$$

so $\alpha_l \circ f = f \circ \alpha_k$. Moreover, for all $a, b \in A_1^k$, we have $f(a *_k b) = f(\alpha_k(a \cdot b)) = \alpha_l(f(a \cdot b)) = \alpha_l(f(a) \cdot f(b)) = f(a) *_l f(b)$. □

Proposition 19 (C [7]). *Any non-zero homomorphism $f\colon A_1^k \to A_1^l$ where $k, l \neq 0$, is an isomorphism of the form $f(x) = \frac{l}{k}x + c$, $f(y) = \frac{k}{l}y + p(x)$ for some $c \in K$ and $p(x) \in K[x]$.*

Proof. Let us try to find all non-zero homomorphisms $f\colon A_1^k \to A_1^l$ when k and l are non-zero. By Lemma 17, this is equivalent to finding a non-zero endomorphism f on A_1 such that $e^{l\frac{\partial}{\partial y}} f(x) = f(x)$ and $e^{l\frac{\partial}{\partial y}} f(y) = f(y) + k$. The former of the two conditions was considered in the proof of Corollary 7, and it turned out to be equivalent to $f(x) \in K[x]$. The latter is equivalent to the following PDE, $\left(l\frac{\partial}{\partial y} + \frac{l^2}{2!}\frac{\partial^2}{\partial y^2} + \cdots + \frac{l^m}{m!}\frac{\partial^m}{\partial y^m}\right) f(y) = k$. If $f(y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j$ for some $c_{ij} \in K$ and $m, n \in \mathbb{N}$, then, by comparing coefficients, $f(y) = \frac{k}{l}y + p(x)$ where $p(x) := \sum_{j=0}^{n} c_{0j} x^j$. Now, note that f is an endomorphism on A_1 only if $[f(x), f(y)] = f([x, y]) = f(1_{A_1}) = 1_{A_1}$. Calculating the left-hand side, $\left[f(x), \frac{k}{l}y + p(x)\right] = \frac{k}{l}[f(x), y] \overset{(3.3)}{=} \frac{k}{l}\frac{\mathrm{d}}{\mathrm{d}x} f(x)$, which is equal to 1_{A_1} if and only if $f(x) = \frac{l}{k}x + c$ for some $c \in K$. Let us introduce the following functions:

$$\begin{aligned}
g_1(x) &= \frac{l}{k}x + y, & g_1(y) &= \frac{k}{l}y, \\
g_2(x) &= x, & g_2(y) &= y + c, \\
g_3(x) &= x - \frac{k}{l}y, & g_3(y) &= y, \\
g_4(x) &= x, & g_4(y) &= y - c + \frac{l}{k}p(x).
\end{aligned}$$

According to Theorem 2 in Chapter 1, these are all automorphisms on A_1, and moreover, $f = g_4 \circ g_3 \circ g_2 \circ g_1$ since $g_4 \circ g_3 \circ g_2 \circ g_1(x) = \frac{l}{k}x + c = f(x)$

and $g_4 \circ g_3 \circ g_2 \circ g_1(y) = \frac{k}{l}y + p(x) = f(y)$. Hence, f is an automorphism on A_1 such that $e^{l\frac{\partial}{\partial y}} f(x) = f(x)$ and $e^{l\frac{\partial}{\partial y}} f(y) = f(y) + k$, and therefore an isomorphism from A_1^k to A_1^l. $\square$

Remark 14. By the above proposition, we also get a classification of the hom-associative Weyl algebras up to isomorphism. We see that there are precisely two isomorphism classes: the first consists of the associative Weyl algebra A_1, and the other consists of all the hom-associative Weyl algebras that are not associative, i.e. A_1^k where $k \neq 0$.

Corollary 8 (C [7]). *Any non-zero endomorphism f on A_1^k where $k \neq 0$ is an automorphism of the form $f(x) = x + c$ and $f(y) = y + p(x)$ for some $c \in K$ and $p(x) \in K[x]$.*

Proof. This follows from Proposition 19 with $k = l$. $\square$

By the above corollary, a hom-associative analogue of the Dixmier conjecture (Conjecture 1 in Chapter 1) is true.

3.2 The prime characteristic case

In this section, we introduce and study hom-associative Weyl algebras over a field K of prime characteristic p.

3.2.1 Modular arithmetic

Recall that if p is a prime, then any number $n \in \mathbb{N}$ can be written as $n_0 + n_1p + n_2p^2 + \cdots + n_jp^j$ for some non-negative integers $n_0, n_1, n_2, \ldots, n_j < p$. This is often referred to as the *base p expansion* of n.

Theorem 4 (Lucas's theorem [53]). *Assume p is a prime, $m, n \in \mathbb{N}$, and that $m_0 + m_1p + m_2p^2 + \cdots + m_jp^j$ and $n_0 + n_1p + n_2p^2 + \cdots + n_jp^j$ are the base p expansions of m and n, respectively. Then $\binom{m}{n} \equiv \prod_{i=0}^{j} \binom{m_i}{n_i} \pmod{p}$ where $\binom{m_i}{n_i} := 0$ if $m_i < n_i$.*

The next result is also known. However, since we have not found the original source in which it was first proved, we provide a proof of it here.

Corollary 9. *If p is a prime and $m \in \mathbb{N}_{>0}$, then $\binom{m}{n} \equiv 0 \pmod{p}$ for all $n \in \mathbb{N}_{>0}$ with $n < m$ if and only if $m = p^q$ for some $q \in \mathbb{N}_{>0}$.*

Proof. Assume $m = p^q$ for some $q \in \mathbb{N}_{>0}$, and let the base p expansion of m be $m_0 + m_1p + m_2p^2 + \cdots + m_qp^q$. That is, $m_0 = m_1 = m_2 = \ldots = m_{q-1} = 0$ and $m_q = 1$. Any $n \in \mathbb{N}_{>0}$ such that $n < m$ can be written as $n = n_0 + n_1p + n_2p^2 + \cdots + n_{q-1}p^{q-1}$ where at least one of $n_0, n_1, n_2, \ldots, n_{q-1}$ is non-zero, say n_i. Then $\binom{m_i}{n_i} = \binom{0}{n_i} = 0$. By Theorem 4, $\binom{m}{n} \equiv 0 \pmod{p}$.

Assume m is not a power of p. We wish to show that there is some $n \in \mathbb{N}_{>0}$ with $n < m$ such that $\binom{m}{n}$ is not divisible by p. To this end, let $m_0 + m_1p + m_2p^2 + \cdots + m_rp^r$ be the base p expansion of m where $m_r \neq 0$. If $m_r = 1$, then there is some $j < r$ such that $m_j \neq 0$, since otherwise $m = p^r$. If $n := m_0 + m_1p + m_2p^2 + \cdots + m_jp^j < m$, then by Theorem 4, $\binom{m}{n} \equiv \binom{1}{0}\binom{m_{r-1}}{0}\cdots\binom{m_j}{m_j}\cdots\binom{m_0}{m_0} \pmod{p} = 1$. If $m_r > 1$, then, with $n := m_0 + m_1p + m_2p^2 + \cdots + (m_r - 1)p^r < m$, Theorem 4 gives $\binom{m}{n} \equiv \binom{m_r}{m_r-1}\binom{m_{r-1}}{m_{r-1}}\cdots\binom{m_0}{m_0} \pmod{p} = m_r < p$. □

3.2.2 Seven little lemmas on Yau twisted algebras

In this subsection, we introduce some lemmas describing properties of Yau twisted algebras in terms of properties of the underlying associative algebras. These results will then be used in proving properties of the hom-associative Weyl algebras in the succeeding subsection. Throughout this subsection, R is a unital, associative, commutative ring.

Lemma 18 (**D** [8]). *Let A be a unital, associative R-algebra with identity element 1_A, and let $\alpha \in \operatorname{End}_R(A)$ be injective. Then 1_A is a unique weak left and a unique weak right identity element in A^α.*

Proof. Let A be a unital, associative R-algebra with identity element 1_A, and assume that $\alpha \in \operatorname{End}_R(A)$ is injective. By Corollary 1 in Chapter 1, 1_A is a weak identity element in A^α. For uniqueness, assume e_l is a weak left identity element in A^α. Then $\alpha(e_l) = e_l * 1_A = \alpha(1_A)$, so $e_l = 1_A$ by the injectivity of α. Similarly, if e_r is a weak right identity element, then $\alpha(e_r) = 1_A * e_r = \alpha(1_A)$, so $e_r = 1_A$. □

Lemma 19 (**D** [8]). *Let A be a non-unital, associative R-algebra, and let $\alpha \in \mathrm{End}_R(A)$. Then $D_l(A) \subseteq D_l(A^\alpha)$ and $D_r(A) \subseteq D_l(A^\alpha)$, with equality if α is injective.*

Proof. We show the left case; the right case is similar. Let A be a non-unital, associative R-algebra, and let $\alpha \in \mathrm{End}_R(A)$, $a \in D_l(A)$, and $b \in A^\alpha$. Then $a * b = \alpha(a \cdot b) = \alpha(0) = 0$, so $a \in D_l(A^\alpha)$, and hence $D_l(A) \subseteq D_l(A^\alpha)$.

Now, assume that α is injective, $c \in D_l(A^\alpha)$ and $d \in A$. Then $0 = c * d = \alpha(c \cdot d) \iff c \cdot d = 0$, so $c \in D_l(A)$, and hence $D_l(A^\alpha) \subseteq D_l(A)$. □

Lemma 20 (**D** [8]). *Let A be a non-unital, associative R-algebra, and let $\alpha \in \mathrm{End}_R(A)$. Then $C(A) \subseteq C(A^\alpha)$, with equality if α is injective.*

Proof. Let A be a non-unital, associative R-algebra, and let $\alpha \in \mathrm{End}_R(A)$, $a \in C(A)$ and $b \in A^\alpha$. Then $[a, b]_* = \alpha([a, b]) = \alpha(0) = 0$, so $a \in C(A^\alpha)$, and hence $C(A) \subseteq C(A^\alpha)$.

Now, assume that α is injective, $c \in C(A^\alpha)$ and $d \in A$. Then $\alpha([c, d]) = [c, d]_* = 0 \iff [c, d] = 0$, so $c \in C(A)$, and hence $C(A^\alpha) \subseteq C(A)$. □

Lemma 21 (**D** [8]). *Let A be a non-unital, associative R-algebra, and let $\alpha, \beta \in \mathrm{End}_R(A)$. Then $C_{\mathrm{End}_R(A)}(\alpha, \beta) \subseteq \mathrm{Hom}_R(A^\alpha, A^\beta)$, with equality if β is injective.*

Proof. Let A be a non-unital, associative R-algebra, and let $\alpha, \beta \in \mathrm{End}_R(A)$ and $f \in C_{\mathrm{End}_R(A)}(\alpha, \beta)$. Denote by $*_\alpha$ the multiplication in A^α, and by $*_\beta$ the multiplication in A^β. Then, for any $a, b \in A$, $f(a *_\alpha b) = f(\alpha(a \cdot b)) = \beta(f(a \cdot b)) = \beta(f(a) \cdot f(b)) = f(a) *_\beta f(b)$. Since f is R-linear, multiplicative, and satisfies $f \circ \alpha = \beta \circ f$ by assumption, it follows that $f \in \mathrm{Hom}_R(A^\alpha, A^\beta)$, and hence $C_{\mathrm{End}_R(A)}(\alpha, \beta) \subseteq \mathrm{Hom}_R(A^\alpha, A^\beta)$.

Now, assume that β is injective and $g \in \mathrm{Hom}_R(A^\alpha, A^\beta)$. Then, for any $c, d \in A$, $g(c *_\alpha d) = g(\alpha(c \cdot d)) = \beta(g(c \cdot d))$. On the other hand, $g(c *_\alpha d) = g(c) *_\beta g(d) = \beta(g(c) \cdot g(d))$, so by the injectivity of β, $g(c \cdot d) = g(c) \cdot g(d)$. By assumption, g is R-linear and $g \circ \alpha = \beta \circ g$, so $g \in C_{\mathrm{End}_R(A)}(\alpha, \beta)$, and hence $\mathrm{Hom}_R(A^\alpha, A^\beta) \subseteq C_{\mathrm{End}_R(A)}(\alpha, \beta)$. □

By setting $\alpha = \beta$ in Lemma 21, we have the following lemma:

Lemma 22 (**D** [8]). *Let A be a non-unital, associative R-algebra, and let $\alpha \in \mathrm{End}_R(A)$. Then $C_{\mathrm{End}_R(A)}(\alpha) \subseteq \mathrm{End}_R(A^\alpha)$, with equality if α is injective.*

Lemma 23 (**D** [8])**.** *Let A be a non-unital, associative R-algebra, and let $\alpha \in \operatorname{End}_R(A)$. Then $C_{\operatorname{Der}_R(A)}(\alpha) \subseteq \operatorname{Der}_R(A^\alpha)$.*

Proof. Let A be a non-unital, associative R-algebra, and let $\alpha \in \operatorname{End}_R(A)$, $\delta \in C_{\operatorname{Der}_R(A)}(\alpha)$, and $a, b \in A^\alpha$. Then $\delta(a * b) = \delta(\alpha(a \cdot b)) = \alpha(\delta(a \cdot b)) = \alpha(\delta(a) \cdot b + a \cdot \delta(b)) = \alpha(\delta(a) \cdot b) + \alpha(a \cdot \delta(b)) = \delta(a) * b + a * \delta(b)$, so $\delta \in \operatorname{Der}_R(A^\alpha)$. □

Lemma 24 (**D** [8])**.** *Let A be a unital, associative R-algebra with identity element 1_A, and let $\alpha \in \operatorname{End}_R(A)$ be injective. Then $\operatorname{Der}_R(A^\alpha) = C_{\operatorname{Der}_R(A)}(\alpha)$ if and only if $\delta(1_A) = 0$ for any $\delta \in \operatorname{Der}_R(A^\alpha)$.*

Proof. Let A be a unital, associative R-algebra with identity element 1_A, and let $\alpha \in \operatorname{End}_R(A)$ be injective and $\delta \in \operatorname{Der}_R(A^\alpha)$. From Lemma 23, $C_{\operatorname{Der}_R(A)}(\alpha) \subseteq \operatorname{Der}_R(A^\alpha)$. Assume $\operatorname{Der}_R(A^\alpha) \subseteq C_{\operatorname{Der}_R(A)}(\alpha)$. Then, since $\delta \in \operatorname{Der}_R(A^\alpha) \subseteq C_{\operatorname{Der}_R(A)}(\alpha) \subseteq \operatorname{Der}_R(A)$, $\delta(1_A) = \delta(1_A \cdot 1_A) = \delta(1_A) \cdot 1_A + \delta(1_A) \cdot 1_A = 2\delta(1_A) \iff \delta(1_A) = 0$.

Now, assume instead that $\delta(1_A) = 0$. Then, for any $a \in A$, $\delta(\alpha(a)) = \delta(1_A * a) = \delta(1_A) * a + 1_A * \delta(a) = 1_A * \delta(a) = \alpha(\delta(a))$. Hence, for any $b, c \in A$, $\alpha(\delta(b \cdot c)) = \delta(\alpha(b \cdot c)) = \delta(b * c) = \delta(b) * c + b * \delta(c) = \alpha(\delta(b) \cdot c + b \cdot \delta(c)$, and since α is injective, $\delta(b \cdot c) = \delta(b) \cdot c + b \cdot \delta(c)$. We can thus conclude that $\delta \in C_{\operatorname{Der}_R(A)}(\alpha)$, so that $\operatorname{Der}_R(A^\alpha) \subseteq C_{\operatorname{Der}_R(A)}(\alpha)$. □

3.2.3 Morphisms, derivations, commutation and association relations

In this subsection, we first single out the possible endomorphisms that may twist the associativity condition of A_1 over a field K of prime characteristic p, in view of Proposition 7 in Chapter 2. We then use these endomorphisms to define the hom-associative Weyl algebras as Yau twists of A_1. With the help of the results from the previous subsection, we then determine basic properties of these Yau twisted Weyl algebras.

Lemma 25 (**D** [8])**.** *Let α be a non-zero endomorphism on $K[y]$. Then α commutes with $\mathrm{d}/\mathrm{d}y$ if and only if $\alpha(y) = k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots$, where only finitely many $k_0, k_p, k_{2p}, \ldots \in K$ are non-zero.*

Proof. Let α be a non-zero endomorphism on $K[y]$. As $K[y]$ contains no zero divisors, α is unital (see Chapter 1). Now, assume $\alpha(y) = k_0 + k_1 y + k_2 y^2 + \cdots$, where only finitely many $k_0, k_1, k_2, \ldots \in K$ are non-zero, and put $\delta := \mathrm{d}/\mathrm{d}y$. Then $\alpha(\delta(y)) = \alpha(1_{K[y]}) = 1_{K[y]}$, and $\delta(\alpha(y)) = k_1 + 2k_2 y + 3k_3 y^2 + \cdots$. Hence $\alpha(\delta(y)) = \delta(\alpha(y))$ if and only if $\alpha(y) = k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots$. We claim that this is also a sufficient condition for α to commute with δ on $K[y]$. Now, as α and δ are both linear, we only need to verify that $\alpha(\delta(y^m)) = \delta(\alpha(y^m))$ for all $m \in \mathbb{N}$. We prove this by induction over m. The case $m = 0$ holds since $\delta\left(\alpha\left(1_{K[y]}\right)\right) = \delta\left(1_{K[y]}\right) = 0$ and $\alpha\left(\delta\left(1_{K[y]}\right)\right) = \alpha(0) = 0$, and the case $m = 1$ was proven above. Continuing, assume that $\alpha(\delta(y^m)) = \delta(\alpha(y^m))$ for $m \in \mathbb{N}$. Then, $\delta\left(\alpha(y^{m+1})\right) = \delta\left(\alpha(y^m) \cdot \alpha(y)\right) = \delta(\alpha(y^m)) \cdot \alpha(y) + \alpha(y^m) \cdot \delta(\alpha(y)) = \alpha(\delta(y^m)) \cdot \alpha(y) + \alpha(y^m) \cdot \alpha(\delta(y)) = \alpha(\delta(y^m) \cdot y + y^m \cdot \delta(y)) = \alpha(\delta(y^{m+1}))$. □

By applying Lemma 25 together with Proposition 7 (and Remark 9) in Chapter 2 to A_1, we may now define a family of (weakly unital) hom-associative Ore extensions.

Definition 17 (The hom-associative Weyl algebras, **D** [8]). Let $k_0, k_p, k_{2p}, \ldots \in K$ where only finitely many of the elements are non-zero, and then define $k := (k_0, k_p, k_{2p}, \ldots)$. Name the map in Lemma 25 α_k, emphasizing its dependence on k. The *hom-associative Weyl algebras* are the weakly unital, hom-associative Ore extensions $A_1^{\alpha_k} = K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]^{\alpha_k} = K[y]^{\alpha_k}[x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$.

For convenience, instead of $A_1^{\alpha_k}$, we shall write A_1^k.

Remark 15 (**D** [8]). The associative Weyl algebra A_1 is the hom-associative Weyl algebra A_1^0. Note that as vector spaces over K, all the A_1^k are the same.

We denote by $\deg(q)$ the total degree of a polynomial $q \in A_1$, and define $\deg(0) := -\infty$.

Lemma 26 (**D** [8]). *α_k is injective.*

Proof. Let $q \in A_1$ be arbitrary. Then $q = \sum_{i \in \mathbb{N}} q_i(y) x^i$ for some $q_i(y) \in K[y]$, so $\alpha_k(q) = \sum_{i \in \mathbb{N}} q_i(\alpha_k(y)) x^i$. Assume that $\alpha_k(q) = 0$. Then $-\infty = \deg(\alpha_k(q)) = \deg(\alpha_k(y)) \deg(q)$, and since $\deg(\alpha_k(y)) > 0$, we must have $\deg(q) = -\infty$. Hence $q = 0$, so α_k is injective. □

Lemma 27 (**D** [8]). *α_k is surjective if and only if $k = (k_0, 0, 0, 0, \ldots)$.*

Proof. Assume that α_k is surjective. Then $y = \alpha_k(q)$ for some $q = \sum_{i\in\mathbb{N}} q_i(y)x^i$ where $q_i(y) \in K[y]$. As $1 = \deg(y) = \deg(\alpha_k(q)) = \deg(\alpha_k(y))\deg(q)$, we must have $\deg(\alpha_k(y)) = 1$. Hence $\alpha_k(y) = k_0 + y$, so that $k = (k_0, 0, 0, 0, \ldots)$.

Now, assume that $k = (k_0, 0, 0, 0, \ldots)$. Then α_k is the endomorphism on A_1 defined by $\alpha_k(x) = x$ and $\alpha_k(y) = k_0 + y$. By Theorem 2 in Chapter 1, α_k is then a triangular automorphism, and hence surjective. □

Proposition 20 (**D** [8]). *K embeds as a subfield into A_1^k.*

Proof. The proof is identical to the proof in the zero characteristic case (see Proposition 15 in Section 3.1). □

We define the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in the obvious way on $K[x]$ and $K[y]$, respectively, and then extend to elements $q = \sum_{i\in\mathbb{N}} q_i x^i$, $q_i \in K[y]$ by $\frac{\partial}{\partial x} q = \sum_{i\in\mathbb{N}} i q_i(y) x^{i-1}$ and $\frac{\partial}{\partial y} q = \sum_{i\in\mathbb{N}} \left(\frac{\partial}{\partial y} q_i\right) x^i$. Here, we set $0x^{-1} := 0$ and $0y^{-1} := 0$.

Lemma 28 (**D** [8]). *For any polynomial $q(x, y) \in A_1^k$,*

$$[x, q(x, y)]_* = \frac{\partial}{\partial y} q\left(x, k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots\right),$$
$$[q(x, y), y]_* = \frac{\partial}{\partial x} q\left(x, k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots\right).$$

Proof. The proof is similar to the case when K has characteristic zero. In A_1, $[x, y^m x^n] = x \cdot y^m x^n - y^m x^n \cdot x = \sum_{i\in\mathbb{N}} \binom{1}{i} \frac{\partial^{1-i} y^m}{\partial y^{1-i}} x^{n+i} - y^m x^{n+1} = m y^{m-1} x^n$ for any $m, n \in \mathbb{N}$, where we define $0y^{-1}$ to be zero. We have that $[x, y^m x^n]_* = \alpha_k([x, y^m x^n]) = \alpha_k\left(m y^{m-1} x^n\right) = m(k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots)^{m-1} x^n$, and so by using that $[\cdot, \cdot]_*$ is linear in the second argument, $[x, q(x, y)]_* = \frac{\partial}{\partial y} q\left(x, k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots\right)$. Similarly, $[y^m x^n, y] = y^m x^n \cdot y - y \cdot y^m x^n = \sum_{i\in\mathbb{N}} \binom{n}{i} \frac{\partial^{n-i} y}{\partial y^{n-i}} x^i - y^{m+1} x^n = n y^m x^{n-1}$ for any $m, n \in \mathbb{N}$ with $0x^{-1}$ defined to be zero. Hence $[y^m x^n, y]_* = \alpha_k([y^m x^n, y]) = \alpha_k\left(n y^m x^{n-1}\right) = n(k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots)^m x^{n-1}$. Using the linearity of $[\cdot, \cdot]_*$, we have $[q(x, y), y]_* = \frac{\partial}{\partial x} q\left(x, k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots\right)$. □

Lemma 29 (**D** [8]). *1_{A_1} is a unique weak left and a unique weak right identity element in A_1^k.*

Proof. Since α_k is injective, this follows from Lemma 18. $\square$

Corollary 10 (**D** [8])**.** *There are no zero divisors in A_1^k.*

Proof. Since α_k is injective and since there are no zero divisors in A_1, by Lemma 19 there are no zero divisors in A_1^k. $\square$

Corollary 11 (**D** [8])**.** $C(A_1^k) = K[x^p, y^p]$.

Proof. We know that $C(A_1) = K[x^p, y^p]$. Since α_k is injective, $C(A_1^k) = C(A_1)$ by Lemma 20. $\square$

In Section 3.1, we saw that A_1^k is simple whenever $\mathrm{char}(K) = 0$ (cf. Proposition 18). The next proposition demonstrates that in prime characteristic, this is not the case.

Proposition 21 (**D** [8])**.** *A_1^k is not simple.*

Proof. By Corollary 11, $x^p \in C(A_1^k)$, so the left ideal I of A_1^k generated by x^p is also a right ideal. If $i \in I$ is non-zero, then $i = q * x^p = \alpha_k(q \cdot x^p) = \alpha_k(q) \cdot x^p$ for some non-zero q, so $\deg(i) = \deg(\alpha_k(q)) + p \geq p$. Hence I does not contain all elements of A_1^k, and since I is non-zero, A_1^k is not simple. $\square$

Lemma 30 (**D** [8])**.** *For any $q, r, s \in A_1^k$, $(q, r, s)_* = 0 \iff q \cdot \alpha_k(r \cdot s) = \alpha_k(q \cdot r) \cdot s$.*

Proof. $(q, r, s)_* = q*(r*s) - (q*r)*s = \alpha_k(q \cdot \alpha_k(r \cdot s)) - \alpha_k(\alpha_k(q \cdot r) \cdot s) = \alpha_k(q \cdot \alpha_k(r \cdot s) - \alpha_k(q \cdot r) \cdot s)$. Since α_k is injective, the lemma follows. $\square$

Proposition 22 (**D** [8])**.** *A_1^k is power associative if and only if $k = 0$.*

Proof. If $k = 0$, then A_1^k is associative, and hence also power associative. To show the converse, we note that it follows from Lemma 30 that $(yx, yx, yx)_* = 0$ if and only if $yx \cdot \alpha_k((yx) \cdot (yx)) = \alpha_k((yx) \cdot (yx)) \cdot yx$. However,

$$
\begin{aligned}
& yx \cdot \alpha_k((yx) \cdot (yx)) = \alpha_k((yx) \cdot (yx)) \cdot yx \\
\iff & yx \cdot \alpha_k(y^2x^2 + yx) = \alpha_k(y^2x^2 + yx) \cdot yx \\
\iff & yx \cdot (\alpha_k(y)^2 \cdot x^2 + \alpha_k(y) \cdot x) = (\alpha_k(y)^2 \cdot x^2 + \alpha_k(y) \cdot x) \cdot yx \\
\iff & y \cdot \alpha_k(y)^2 \cdot x^3 + 2y \cdot \alpha_k(y) \cdot x^2 + y \cdot \alpha_k(y) \cdot x^2 + yx \\
& \quad = y \cdot \alpha_k(y)^2 \cdot x^3 + 2\alpha_k(y)^2 \cdot x^2 + y \cdot \alpha_k(y) \cdot x^2 + \alpha_k(y) \cdot x \\
\iff & 2y \cdot \alpha_k(y) \cdot x^2 + yx = 2\alpha_k(y)^2 \cdot x^2 + \alpha_k(y) \cdot x.
\end{aligned}
$$

The last equality clearly holds only if $k = 0$. □

From the above result, we can also conclude that A_1^k is left alternative, right alternative, flexible, and associative if and only if $k = 0$.

Proposition 23 (**D** [8]). *$N_l(A_1^k) = N_m(A_1^k) = N_r(A_1^k) = \{0\}$ if and only if $k \neq 0$.*

Proof. If $k = 0$, then A_1^k is associative, so $N_l(A_1^k) = N_m(A_1^k) = N_r(A_1^k) = A_1^k$. Now, assume $k \neq 0$. Let $r = \sum_{i\in\mathbb{N}} r_i x^i$ be some element in $N_r(A_1^k)$ where $r_i \in K[y]$. Then, $(1_{A_1}, 1_{A_1}, r)_* = 0$. By Lemma 30, this is equivalent to $\alpha_k(r) = r \iff \alpha_k(r_i) = r_i$ for all $i \in \mathbb{N}$, so $r_i \in K$. We also have $(y, 1_{A_1}, r) = 0$, which is equivalent to $y \cdot \alpha_k(r) = \alpha_k(y) \cdot r$. We first rewrite the right-hand side of the preceding equation: $\alpha_k(y) \cdot r = \sum_{i\in\mathbb{N}} r_i \alpha_k(y) x^i$. Then we rewrite the left-hand side: $y \cdot \alpha_k(r) = y \cdot r = \sum_{i\in\mathbb{N}} r_i y x^i$. Hence, for all $i \in \mathbb{N}$, it must be true that $r_i \cdot \alpha_k(y) = r_i y$. Since $k \neq 0$ implies $\alpha_k(y) \neq y$, we have $r_i = 0$ for all $i \in \mathbb{N}$. In other words, $r = 0$, so $N_r(A_1^k) \subseteq \{0\}$. Since $\{0\} \subseteq N_r(A_1^k)$, we have $N_r(A_1^k) = \{0\}$.

Assume $k \neq 0$ and let $s = \sum_{i\in\mathbb{N}} s_i x^i$ be some element in $N_m(A_1^k)$ where $s_i \in K[y]$. Then, $(y, s, 1_{A_1})_* = 0$. By Lemma 30, this is equivalent to $y \cdot \alpha_k(s) = \alpha_k(y) \cdot \alpha_k(s)$. This can be rewritten as $y \cdot \alpha_k(s_i) = \alpha_k(y) \cdot \alpha_k(s_i) \iff (y - \alpha_k(y)) \cdot \alpha_k(s_i) = 0$ for all $i \in \mathbb{N}$. Again, $k \neq 0$ implies $\alpha_k(y) \neq y$, and by Corollary 10, there are no zero divisors in A_1^k. Hence $\alpha_k(s_i) = 0$ for all $i \in \mathbb{N}$. By Lemma 26, α_k is injective, and so $s_i = 0$ for all $i \in \mathbb{N}$, and $N_m(A_1^k) = \{0\}$.

By similar calculations, $N_l(A_1^k) = \{0\}$. □

Corollary 12 (**D** [8]). *$N(A_1^k) = \{0\}$ if and only if $k \neq 0$.*

Corollary 13 (**D** [8]). *$Z(A_1^k) = \{0\}$ if and only if $k \neq 0$.*

Lemma 31 (**D** [8]). *The following statements are equivalent:*

(i) $\delta \in \mathrm{Der}_K(A_1^k)$.

(ii) $\delta \in C_{\mathrm{Der}_K(A_1)}(\alpha_k)$.

(iii) $\delta \in \mathrm{Der}_K(A_1)$ *satisfying* $\alpha_k(\delta(x)) = \delta(x)$ *and* $\alpha_k(\delta(y)) = \delta(y) + k_p\delta(y^p) + k_{2p}\delta(y^{2p}) + \cdots$.

(iv) $\delta = uE_x + vE_y + \mathrm{ad}_q$, *where* $E_x, E_y \in \mathrm{Der}_K(A_1)$ *are defined by* $E_x(x) = y^{p-1}$, $E_x(y) = E_y(x) = 0$, $E_y(y) = x^{p-1}$, *and* $u, v \in K[x^p, y^p]$ *and* $q \in A_1$ *satisfy* $\alpha_k(u) = u$, $\alpha_k(v) = v$, $\frac{\partial}{\partial x}(\alpha_k(q) - q) = v \cdot \frac{\mathrm{d}}{\mathrm{d}(y^p)}(y - \alpha_k(y))$, *and* $\frac{\partial}{\partial y}(\alpha_k(q) - q) = u \cdot (\alpha_k(y)^{p-1} - y^{p-1})$.

Proof. (i) $\iff$ (ii): We wish to show that $\mathrm{Der}_K(A_1^k) = C_{\mathrm{Der}_K(A_1)}(\alpha_k)$. Since α_k is injective, this holds if and only if $\delta(1_{A_1}) = 0$ for all $\delta \in \mathrm{Der}_K(A_1^k)$ by Lemma 24. Now, we have that $\delta(1_{A_1}) = \delta(\alpha_k(1_{A_1})) = \delta(1_{A_1} * 1_{A_1}) = \delta(1_{A_1}) * 1_{A_1} + 1_{A_1} * \delta(1_{A_1}) = 2\alpha_k(\delta(1_{A_1})$. Hence, if $p = 2$, then $\delta(1_{A_1}) = 0$. Therefore, assume $p \neq 2$ and put $\delta(1_{A_1}) = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} d_{ij} y^i x^j$ for $d_{ij} \in K$. The previous equation then reads $\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} d_{ij} y^i x^j = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} 2 d_{ij} \alpha_k(y)^i x^j$, so $\sum_{i\in\mathbb{N}} d_{ij} y^i = \sum_{i\in\mathbb{N}} 2 d_{ij} \alpha_k(y)^i$ for all $j \in \mathbb{N}$. Now, $\alpha_k(y) = k_0 + y + k_p y^p + k_{2p} y^{2p} + \cdots$, so by comparing the degrees of the two sums, $k_p = k_{2p} = k_{3p} = \ldots = 0$ unless all d_{ij} are zero. If $k_p = k_{2p} = k_{3p} = \ldots = 0$ we are left with the equation $\sum_{i\in\mathbb{N}} d_{ij} y^i = \sum_{i\in\mathbb{N}} 2 d_{ij} (k_0 + y)^i$, which has no solution unless $d_{ij} = 0$ for all $i, j \in \mathbb{N}$. So $\delta(1_{A_1}) = 0$, and therefore $\mathrm{Der}_K(A_1^k) = C_{\mathrm{Der}_K(A_1)}(\alpha_k)$.

(ii) $\iff$ (iii): Both α_k and δ are linear, and therefore (ii) is equivalent to $\alpha_k(\delta(y^m x^n)) = \delta(\alpha_k(y^m x^n))$ for $m, n \in \mathbb{N}$. Now, we claim that this in turn is equivalent to $\alpha_k(\delta(x)) = \delta(\alpha_k(x))$ and $\alpha_k(\delta(y)) = \delta(\alpha_k(y))$. Clearly the former condition implies the latter condition. We wish to show, by induction over m and n, that also the latter condition implies the former condition. To this end, assume that $\alpha_k(\delta(x)) = \delta(\alpha_k(x))$ and $\alpha_k(\delta(y)) = \delta(\alpha_k(y))$. First, $\alpha_k(\delta(1_{A_1})) = \alpha_k(0) = 0$ and $\delta(\alpha_k(1_{A_1})) = \delta(1_{A_1}) = 0$, so the base case $m = n = 0$ holds. Now, let n be fixed, and assume that the induction hypothesis holds. Then,

$$\begin{aligned}
\alpha_k(\delta(y^{m+1} x^n)) &= \alpha_k(\delta(y) \cdot y^m x^n + y \cdot \delta(y^m x^n)) \\
&= \alpha_k(\delta(y)) \cdot \alpha_k(y^m x^n) + \alpha_k(y) \cdot \alpha_k(\delta(y^m x^n)) \\
&= \delta(\alpha_k(y)) \cdot \alpha_k(y^m x^n) + \alpha_k(y) \cdot \delta(\alpha_k(y^m x^n)) \\
&= \delta(\alpha_k(y) \cdot \alpha_k(y^m x^n)) = \delta(\alpha_k(y^{m+1} x^n)).
\end{aligned}$$

Similarly, $\alpha_k(\delta(y^m x^{n+1})) = \delta(\alpha_k(y^m x^{n+1}))$ whenever m is fixed. By using that $\delta(\alpha_k(x)) = \delta(x)$ and $\delta(\alpha_k(y)) = k_0 \delta(1_{A_1}) + \delta(y) + k_p \delta(y^p) + k_{2p} \delta(y^{2p}) + \cdots = \delta(y) + k_p \delta(y^p) + k_{2p} \delta(y^{2p}) + \cdots$, the end result follows.

(iii) $\Longrightarrow$ (iv): Let $\delta \in \mathrm{Der}_K(A_1)$, and assume $\alpha_k(\delta(x)) = \delta(x)$ and $\alpha_k(\delta(y)) = \delta(y) + k_p\delta(y^p) + k_{2p}\delta(y^{2p}) + \cdots$. By the proof of the preceding equivalence, we then have $\delta \in C_{\mathrm{Der}_K(A_1)}(\alpha_k)$, so $\alpha_k(\delta(x^p)) = \delta(\alpha_k(x^p))$ and $\alpha_k(\delta(y^p)) = \delta(\alpha_k(y^p))$. By Theorem 1 in Chapter 1, $\delta \in \mathrm{Der}_K(A_1) \Longrightarrow \delta = uE_x + vE_y + \mathrm{ad}_q$, where $E_x, E_y \in \mathrm{Der}_K(A_1)$ are defined by $E_x(x) = y^{p-1}$, $E_x(y) = E_y(x) = 0$, $E_y(y) = x^{p-1}$, and $u, v \in K[x^p, y^p]$ and $q \in A_1$. By Lemma 3.6 in [11], $E_x = -\frac{\mathrm{d}}{\mathrm{d}(x^p)}$ on $K[x^p]$ and $E_y = -\frac{\mathrm{d}}{\mathrm{d}(y^p)}$ on $K[y^p]$, so by using that $x^p, y^p \in K[x^p, y^p] = C(A_1)$, we have $\alpha_k(\delta(x^p)) = \alpha_k(-u)$, $\delta(\alpha_k(x^p)) = -u$, $\alpha_k(\delta(y^p)) = \alpha_k(-v)$, and $\delta(\alpha_k(v)) = \delta(-v)$. Hence we must have $\alpha_k(u) = u$ and $\alpha_k(v) = v$. Now, put $q = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} q_{ij}y^ix^j$ for some $q_{ij} \in K$. Then $\alpha_k\left(\frac{\partial q}{\partial y}\right) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} q_{ij}i\alpha_k(y)^{i-1}x^j$
$= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} q_{ij}i(k_0 + y + k_py^p + k_{2p}y^{2p} + \cdots)^{i-1}x^j = \frac{\partial\alpha_k(q)}{\partial y}$ and
$\alpha_k\left(\frac{\partial q}{\partial x}\right) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} q_{ij}j\alpha_k(y)^ix^{j-1} = \frac{\partial\alpha_k(q)}{\partial x}$. Using this and Lemma 28, $\alpha_k(\delta(x)) = \alpha_k(u\cdot y^{p-1} + [q,x]) = u\cdot\alpha_k(y)^{p-1} - \alpha_k\left(\frac{\partial q}{\partial y}\right) = u\cdot\alpha_k(y)^{p-1} - \frac{\partial\alpha_k(q)}{\partial y}$, while $\delta(x) = u\cdot y^{p-1} - \frac{\partial q}{\partial y}$. We also have $\alpha_k(\delta(y)) = \alpha_k(v\cdot x^{p-1} + [q,y]) = v\cdot x^{p-1} + \alpha_k\left(\frac{\partial q}{\partial x}\right) = v\cdot x^{p-1} + \frac{\partial\alpha_k(q)}{\partial x}$, and since $y^p, y^{2p}, y^{3p}, \ldots \in K[y^p] \subset C(A_1)$, we have $\delta(y) + k_p\delta(y^p) + k_{2p}\delta(y^{2p}) + \cdots = \delta(\alpha_k(y)) = v\cdot x^{p-1} - v\cdot(k_p + 2k_{2p}y^p + 3k_{3p}y^{2p} + \cdots) + \frac{\partial q}{\partial x} = v\cdot x^{p-1} - v\cdot\frac{\mathrm{d}}{\mathrm{d}(y^p)}(\alpha_k(y) - y) + \frac{\partial q}{\partial x}$. Hence, we can conclude that $\frac{\partial}{\partial x}(\alpha_k(q) - q) = v\cdot\frac{\mathrm{d}}{\mathrm{d}(y^p)}(y - \alpha_k(y))$ and $\frac{\partial}{\partial y}(\alpha_k(q) - q) = u\cdot(\alpha_k(y)^{p-1} - y^{p-1})$.

(iv) $\Longrightarrow$ (iii): Assume that (iv) holds. Then, by Theorem 1 in Chapter 1, $\delta \in \mathrm{Der}_K(A_1)$. Moreover, by the very same calculation as in the proof of the preceding implication, $\alpha_k(\delta(x)) = \delta(x)$ and $\alpha_k(\delta(y)) = \delta(y) + k_p\delta(y^p) + k_{2p}\delta(y^{2p}) + \cdots$. $\square$

When $k = 0$, all the derivations of A_1^k are described in Theorem 1 in Chapter 1. The next two propositions deal with the case when $k \neq 0$.

Proposition 24 (**D** [8]). *If $k = (k_0, 0, 0, 0, \ldots) \neq 0$, then $\delta \in \mathrm{Der}_K(A_1^k)$ if and only if $\delta = uE_x + vE_y + \mathrm{ad}_q$. Here $E_x, E_y \in \mathrm{Der}_K(A_1)$ are defined by $E_x(x) = y^{p-1}$, $E_x(y) = E_y(x) = 0$, $E_y(y) = x^{p-1}$, $u = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} u_{ij}y^ix^j \in K[x^p, y^p]$, $v = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} v_{ij}y^ix^j \in K[x^p, y^p]$, and $q = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} q_{ij}y^ix^j$*

$\in A_1$ *for some* $q_{ij}, u_{ij}, v_{ij} \in K$*, satisfying, for all* $j, l \in \mathbb{N}$ *and* $m \in \mathbb{N}_{>0}$*,*

$$\sum_{l+1 \leq i} \binom{i}{l} v_{ij} k_0^i = \sum_{l+1 \leq i} \binom{i}{l} u_{ij} k_0^i = \sum_{l+1 \leq i} \binom{i}{l} j q_{ij} k_0^i = 0,$$

$$\sum_{m+1 \leq i} \binom{i}{m} m q_{ij} k_0^i = \sum_{i=0}^{m-1} \binom{p-1}{m-i} u_{ij} k_0^{i+p-1}.$$

Proof. Let $k = (k_0, 0, 0, 0, \ldots) \neq 0$. By Lemma 31, $\delta \in \mathrm{Der}_K(A_1^k) \iff \delta = uE_x + vE_y + \mathrm{ad}_q$, where $E_x, E_y \in \mathrm{Der}_K(A_1)$ are defined by $E_x(x) = y^{p-1}$, $E_x(y) = E_y(x) = 0$, $E_y(y) = x^{p-1}$, and $u, v \in K[x^p, y^p]$ and $q \in A_1$ satisfy $\alpha_k(u) = u$, $\alpha_k(v) = v$, $\frac{\partial}{\partial x}(\alpha_k(q) - q) = 0$, and $\frac{\partial}{\partial y}(\alpha_k(q) - q) = u \cdot ((k_0+y)^{p-1} - y^{p-1})$. We wish to rewrite these last four conditions. To this end, let $u = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} u_{ij} y^i x^j$ for some $u_{ij} \in K$. Then, by the binomial theorem, $\alpha_k(u) = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} u_{ij}(k_0 + y)^i x^j = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{l=0}^{i} \binom{i}{l} u_{ij} k_0^{i-l} y^l x^j$. Hence, we have that $\alpha_k(u) = u \iff \sum_{i \in \mathbb{N}_{>0}} \sum_{j \in \mathbb{N}} \sum_{l=0}^{i-1} \binom{i}{l} u_{ij} k_0^{i-l} y^l x^j = 0 \iff \sum_{i \in \mathbb{N}_{>0}} \sum_{l=0}^{i-1} \binom{i}{l} u_{ij} k_0^{i-l} y^l = 0$ for all $j \in \mathbb{N}$ $\iff \sum_{l \in \mathbb{N}} \sum_{l+1 \leq i} \binom{i}{l} u_{ij} k_0^{i-l} y^l = 0$ for all $j \in \mathbb{N}$ $\iff \sum_{l+1 \leq i} \binom{i}{l} u_{ij} k_0^i = 0$ for all $j, l \in \mathbb{N}$. Similarly, if $v = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} v_{ij} y^i x^j$ for some $v_{ij} \in K$, then $\alpha_k(v) = v \iff \sum_{l+1 \leq i} \binom{i}{l} v_{ij} k_0^i = 0$ for all $j, l \in \mathbb{N}$. Now, put $q = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} q_{ij} y^i x^j$ for some $q_{ij} \in K$. Then, $\alpha_k(q) - q = \sum_{i \in \mathbb{N}_{>0}} \sum_{j \in \mathbb{N}} \sum_{l=0}^{i-1} \binom{i}{l} q_{ij} k_0^{i-l} y^l x^j$, so by defining $0x^{-1} := 0$, $\frac{\partial}{\partial x}(\alpha_k(q) - q) = 0 \iff \sum_{i \in \mathbb{N}_{>0}} \sum_{j \in \mathbb{N}} \sum_{l=0}^{i-1} \binom{i}{l} j q_{ij} k_0^{i-l} y^l x^{j-1} = 0$ $\iff \sum_{l+1 \leq i} \binom{i}{l} j q_{ij} k_0^i = 0$ for all $j, l \in \mathbb{N}$. We have that $\frac{\partial}{\partial y}(\alpha_k(q) - q) = \sum_{i \in \mathbb{N}_{>0}} \sum_{j \in \mathbb{N}} \sum_{l=1}^{i-1} \binom{i}{l} l q_{ij} k_0^{i-l} y^{l-1} x^j$, and since $(k_0 + y)^{p-1} = \sum_{n=0}^{p-1} \binom{p-1}{n} k_0^{p-1-n} y^n$, we get $u \cdot ((k_0 + y)^{p-1} - y^{p-1}) = u \cdot \sum_{n=0}^{p-2} \binom{p-1}{n} k_0^{p-1-n} y^n$. Using that $u \in K[x^p, y^p] = C(A_1)$, we can rewrite the last condition as $\sum_{i \in \mathbb{N}_{>0}} \sum_{j \in \mathbb{N}} \sum_{l=1}^{i-1} \binom{i}{l} l q_{ij} k_0^{i-l} y^{l-1} x^j = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{n=0}^{p-2} \binom{p-1}{n} u_{ij} k_0^{p-1-n} y^{i+n} x^j$. Continuing, this is in turn equivalent to $\sum_{i \in \mathbb{N}_{>0}} \sum_{l=1}^{i-1} \binom{i}{l} l q_{ij} k_0^{i-l} y^{l-1} = \sum_{i \in \mathbb{N}} \sum_{n=0}^{p-2} \binom{p-1}{n} u_{ij} k_0^{p-1-n} y^{i+n}$ for all $j \in \mathbb{N}$. Now, put $m := i + n + 1$ in the latter double sum, so that the above equality reads $\sum_{i \in \mathbb{N}_{>0}} \sum_{l=1}^{i-1} \binom{i}{l} l q_{ij} k_0^{i-l} y^{l-1} = \sum_{i \in \mathbb{N}} \sum_{m=i+1}^{i+p-1} \binom{p-1}{m-i} u_{ij} k_0^{i+p-m-1} y^{m-1}$ for all $j \in \mathbb{N}$. Last, we define $\binom{i}{l} :=$

0 whenever $i < l$, and then change the order of summation in both double sums, resulting in $\sum_{l\in\mathbb{N}_{>0}} \sum_{l+1\le i} \binom{i}{l} l q_{ij} k_0^{i-l} y^{l-1}$ $= \sum_{m\in\mathbb{N}_{>0}} \sum_{i=0}^{m-1} \binom{p-1}{m-i} u_{ij} k_0^{i+p-m-1} y^{m-1}$ for all $j \in \mathbb{N}$. By comparing coefficients, $\sum_{m+1\le i} \binom{i}{m} m q_{ij} k_0^{i} = \sum_{i=0}^{m-1} \binom{p-1}{m-i} u_{ij} k_0^{i+p-1}$ for all $j \in \mathbb{N}$ and $m \in \mathbb{N}_{>0}$. $\square$

Proposition 25 (**D** [8]). *If* $k = (k_0, k_p, k_{2p}, \ldots, k_{Mp}, 0, 0, 0, \ldots)$ *for some* $M \in \mathbb{N}_{>0}$ *where* $k_{Mp} \neq 0$*, then* $\delta \in \mathrm{Der}_K(A_1^k)$ *if and only if*

$$\delta = \begin{cases} vE_y + \mathrm{ad}_q & \text{if } k = (k_0, 0, \ldots, 0, k_{p^2}, 0, \ldots, 0, k_{2p^2}, 0, \ldots), \\ \mathrm{ad}_q & \text{otherwise.} \end{cases}$$

Here, $E_y \in \mathrm{Der}_K(A_1)$ *is defined by* $E_y(x) = 0$*,* $E_y(y) = x^{p-1}$*,* $v \in K[x^p]$ *and* $q = q_{11}yx + \sum_{i\equiv 0 \pmod p} q_{i1}y^i x + q_{1i}yx^i$ *for some* $q_{ij} \in K$*.*

Proof. Let $k = (k_0, k_p, k_{2p}, \ldots, k_{Mp}, 0, 0, 0, \ldots)$ for some $M \in \mathbb{N}_{>0}$ where $k_{Mp} \neq 0$. By Lemma 31, $\delta \in \mathrm{Der}_K(A_1^k) \iff \delta = uE_x + vE_y + \mathrm{ad}_q$, where $E_x, E_y \in \mathrm{Der}_K(A_1)$ are defined by $E_x(x) = y^{p-1}$, $E_x(y) = E_y(x) = 0$, $E_y(y) = x^{p-1}$, and $u, v \in K[x^p, y^p]$ and $q \in A_1$ satisfy $\alpha_k(u) = u$, $\alpha_k(v) = v$, $\frac{\partial}{\partial x}(\alpha_k(q) - q) = -v \cdot (k_p + 2k_{2p}y^p + 3k_{3p}y^{3p} + \cdots + Mk_{Mp}y^{(M-1)p})$, and $\frac{\partial}{\partial y}(\alpha_k(q) - q) = u \cdot ((k_0 + y + k_p y^p + k_{2p}y^{2p} + \cdots + k_{Mp}y^{Mp})^{p-1} - y^{p-1})$. We wish to rewrite these last four conditions. Since $k_{Mp} \neq 0$ where $M \in \mathbb{N}_{>0}$, $\alpha_k(u) = u \iff u \in K[x^p]$ and $\alpha_k(v) = v \iff v \in K[x^p]$. Continuing, we put $q = \sum_{i,j\in\mathbb{N}} q_{ij}y^i x^j$ for some $q_{ij} \in K$ and examine the last condition. Since $u \in K[x^p] \subset C(A_1)$, by comparing the coefficients for y^{p-1}, we see that $u = 0$, which in turn gives $q = \sum_{i\equiv 0 \pmod p} \sum_{j\in\mathbb{N}} (q_{1j}y + q_{ij}y^i)x^j$. Now, α_k and $\frac{\partial}{\partial x}$ commute (see the second last part of the proof of Lemma 31), so the third condition is $\alpha_k\left(\frac{\partial q}{\partial x}\right) = \frac{\partial q}{\partial x} - v \cdot \left(k_p + 2k_{2p}y^p + 3k_{3p}y^{2p} + \cdots + Mk_{Mp}y^{(M-1)p}\right)$. Assume $k_p = 2k_{2p} = 3k_{3p} = \ldots = Mk_{Mp} = 0$, which is equivalent to $k = (k_0, 0, \ldots, 0, k_{p^2}, 0, \ldots, 0, k_{2p^2}, 0, \ldots)$, or that $v = 0$. The previous equality then reads $\alpha_k\left(\frac{\partial q}{\partial x}\right) = \frac{\partial q}{\partial x}$, which is equivalent to $\frac{\partial q}{\partial x} \in K[x]$, so $q = \sum_{i,j\equiv 0 \pmod p} (q_{11}y + q_{i1}y^i)x + (q_{1j}y + q_{ij}y^i)x^j$. Now, assume instead that $v \neq 0$ and that not all of $k_p, 2k_{2p}, 3k_{3p}, \ldots, Mk_{Mp}$ are zero. Let L be such that $Lk_{Lp} = 0$ and $\ell k_{\ell p} = 0$ for $\ell > L$. Then, with q_{ij} possibly non-zero only if $i = 1$

or $i \equiv 0 \pmod p$,

$$
\begin{aligned}
&\alpha_k\left(\frac{\partial q}{\partial x}\right) = \frac{\partial q}{\partial x} - v \cdot \left(k_p + 2k_{2p}y^p + 3k_{3p}y^{2p} + \cdots + Mk_{Mp}y^{(M-1)p}\right) \\
&\iff \sum_{i,j\in\mathbb{N}} jq_{ij}\alpha_k(y)^i x^{j-1} \\
&= -v \cdot \left(k_p + 2k_{2p}y^p + 3k_{3p}y^{2p} + \cdots + Lk_{Lp}y^{(L-1)p}\right) + \sum_{i,j\in\mathbb{N}} jq_{ij}y^i x^{j-1} \\
&\iff \sum_{i,j\in\mathbb{N}} jq_{ij}\left(k_0 + y + k_p y^p + k_{2p}y^{2p} + \cdots + k_{Mp}y^{Mp}\right)^i x^{j-1} \\
&= -v \cdot \left(k_p + 2k_{2p}y^p + 3k_{3p}y^{2p} + \cdots + Lk_{Lp}y^{(L-1)p}\right) + \sum_{i,j\in\mathbb{N}} jq_{ij}y^i x^{j-1}
\end{aligned}
$$

By comparing the degrees of the left- and right-hand side, we realize that the above equation has no solution. Last, $q_{ij}y^i x^j \in C(A_1)$ for all $i, j \equiv 0 \pmod p$. $\square$

Lemma 32 (**D** [8]). *The following statements are equivalent:*

(i) $f \in \mathrm{Hom}_K(A_1^k, A_1^l)$ *and* $f \neq 0$.

(ii) $f \in C_{\mathrm{End}_K(A_1)}(\alpha_k, \alpha_l)$ *and* $f \neq 0$.

(iii) $f \in \mathrm{End}_K(A_1)$ *satisfying* $f(x) = \alpha_l(f(x))$ *and* $k_0 + f(y) + k_p f(y)^p + k_{2p}f(y)^{2p} + \cdots = \alpha_l(f(y))$.

Proof. (i) $\iff$ (ii): By Lemma 26, α_l is injective, and so the statement follows from Lemma 21.

(ii) $\iff$ (iii): Using that $\alpha_k(x) = x$, $\alpha_k(y) = k_0 + y + k_p y^p + k_{2p}y^{2p} + \cdots$, and $f(1_{A_1}) = 1_{A_1}$, it is immediate that (ii) implies (iii). We now wish to show that (iii) implies (ii). To this end, assume that (iii) holds, i.e. that $f(\alpha_k(x)) = \alpha_l(f(x))$ and $f(\alpha_k(y)) = \alpha_l(f(y))$. Now, since α_k, α_l, and f are linear, it is sufficient to show that $f(\alpha_k(y^m x^n)) = \alpha_l(f(y^m x^n))$ for any $m, n \in \mathbb{N}$. Since $f(\alpha_k(1_{A_1})) = f(1_{A_1}) = 1_{A_1}$ and $\alpha_l(f(1_{A_1})) = \alpha_l(1_{A_1}) = 1_{A_1}$, we have that

$$
\begin{aligned}
&f(\alpha_k(y^m x^n)) = f(\alpha_k(y)^m \cdot \alpha_k(x)^n) = f(\alpha_k(y))^m \cdot f(\alpha_k(x))^n \\
&= \alpha_l(f(y))^m \cdot \alpha_l(f(x))^n = \alpha_l(f(y)^m) \cdot \alpha_l(f(x)^n) \\
&= \alpha_l(f(y^m)) \cdot \alpha_l(f(x^n)) = \alpha_l(f(y^m) \cdot f(x^n)) = \alpha_l(f(y^m x^n)). \qquad \square
\end{aligned}
$$

Proposition 26 (**D** [8]). *Every non-zero endomorphism on A_1^k is injective.*

Proof. With $k = l$ in Lemma 32, every endomorphism on A_1^k is an endomorphism on A_1, and by Lemma 17 in [76], every non-zero endomorphism on A_1 is injective. □

In characteristic zero, if $k \neq 0$, then every non-zero endomorphism on A_1^k is an automorphism (cf. Corollary 8 in Section 3.1). If $k = 0$, then any non-zero endomorphism on A_1^k is injective since $A_1^0 = A_1$ is simple. The Dixmier conjecture (Conjecture 1 in Chapter 1) then states that every non-zero endomorphism on A_1 is also an automorphism. In prime characteristic, it is known that not every non-zero endomorphism on A_1 is an automorphism, however. The next proposition demonstrates that this fact generalizes to A_1^k for arbitrary k.

Proposition 27 (**D** [8]). *Not every non-zero endomorphism on A_1^k is an automorphism.*

Proof. Since A_1 is a free algebra on the letters x and y modulo the commutation relation $x \cdot y - y \cdot x = 1_{A_1}$, we may define an endomorphism on A_1 by defining it arbitrary on x and y and then extending it linearly and multiplicatively, as long as it respects the above commutation relation. Since $x^p \in C(A_1)$, f defined by $f(x) = x + x^p$ and $f(y) = y$ satisfies $f(x \cdot y - y \cdot x) = f(1_{A_1})$ and hence defines an endomorphism on A_1. However, f is not surjective. Assume the contrary, and let $x = f(q)$ for some $q = \sum_{i \in \mathbb{N}} q_i(y)x^i$, where $q_i(y) \in K[y]$. Then $1 = \deg(x) = \deg(f(q)) = \deg(q)\deg(f(x)) = \deg(q)p$, which is a contradiction. We claim that $f \in \mathrm{End}_K(A_1^k)$. We have that $f(x) = \alpha_k(f(x))$ and $k_0 + f(y) + k_p f(y)^p + k_{2p} f(y)^{2p} + \cdots = \alpha_k(f(y))$, so with $k = l$ in Lemma 32, $f \in \mathrm{End}_K(A_1^k)$. □

It is clear, for instance from Proposition 22, that A_1^k is associative if and only if $k = 0$, and hence $A_1^0 \cong A_1^k$ if and only if $k = 0$. The next two propositions deal with the case when $k \neq 0$.

Proposition 28 (**D** [8]). *If $k = (k_0, 0, 0, 0, \ldots) \neq 0$, then $A_1^k \cong A_1^l$ if and only if $l = (l_0, 0, 0, 0, \ldots) \neq 0$.*

Proof. Let $k = (k_0, 0, 0, 0, \ldots) \neq 0$ and assume $A_1^k \cong A_1^l$. Since A_1^k is not associative, A_1^l cannot be associative, so $l \neq 0$. Put $l = (l_0, l_p, l_{2p}, \ldots, l_{Np}, 0, 0, 0)$

for some $N \in \mathbb{N}$, where $l_{Np} \neq 0$. We now wish to show that there is no non-zero homomorphism, let alone isomorphism, from A_1^k to A_1^l unless $N = 0$. To this end, assume $f\colon A_1^k \to A_1^l$ is a non-zero homomorphism. By Lemma 32, we must have $k_0 + f(y) = \alpha_l(f(y))$. Now, put $f(y) = \sum_{i=0}^m \sum_{j=0}^n c_{ij} y^i x^j$ for some $m, n \in \mathbb{N}$ and $c_{ij} \in K$. Then, $\alpha_l(f(y)) = \sum_{i=0}^m \sum_{j=0}^n c_{ij}(l_0 + y + l_p y^p + l_{2p} y^{2p} + \cdots + l_{Np} y^{Np})^i x^j$. Hence $k_0 + f(y) = \alpha_l(f(y))$ if and only if $k_0 + \sum_{i=0}^m \sum_{j=0}^n c_{ij} y^i x^j = \sum_{i=0}^m \sum_{j=0}^n c_{ij}(l_0 + y + l_p y^p + l_{2p} y^{2p} + \cdots + l_{Np} y^{Np})^i x^j$. We note that if $m = 0$, then $k_0 + \sum_{j=0}^n c_{0j} x^j = \sum_{j=0}^n c_{0j} x^j \iff k_0 = 0$, which is a contradiction. Hence $m \neq 0$, so by comparing degrees in y, we must have $N = 0$.

Now, assume that $k = (k_0, 0, 0, 0, \ldots) \neq 0$ and $l = (l_0, 0, 0, 0, \ldots) \neq 0$. Define a homomorphism g on A_1 by $g(x) := \frac{l_0}{k_0} x$ and $g(y) := \frac{k_0}{l_0} y$. By Theorem 2 in Chapter 1, g is then not only a homomorphism, but also a linear automorphism. Moreover, $g(x) = \alpha_l(g(x))$ and $k_0 + g(y) = \alpha_l(g(y))$, so by Lemma 32, $g\colon A_1^k \to A_1^l$ is an isomorphism. $\square$

Proposition 29 (D [8]). *Let $k = (k_0, k_p, k_{2p}, \ldots, k_{Mp}, 0, 0, 0, \ldots)$ for some $M \in \mathbb{N}_{>0}$ where $k_{Mp} \neq 0$ and let $l = (l_0, l_p, l_{2p}, \ldots, l_{Np}, 0, 0, 0, \ldots)$ for some $N \in \mathbb{N}_{>0}$ where $l_{Np} \neq 0$. Then a map $f\colon A_1^k \to A_1^l$ is an isomorphism if and only if $M = N$, $f \in \mathrm{Aut}_K(A_1)$ of the form $f(x) = d_0 + c_1^{-1} x$ and $f(y) = c_0 + c_1 y$ for $c_0, d_0 \in K$, $c_1 \in K^\times$, satisfying*

$$\sum_{i=j}^{M} \binom{i}{j} k_{ip} c_0^{(i-j)p} c_1^{jp-1} = l_{jp}, \quad 0 \leq j \leq M. \tag{3.4}$$

Proof. First, let us define $k := (k_0, k_p, k_{2p}, \ldots, k_{Mp}, 0, 0, 0, \ldots)$, and then $l := (l_0, l_p, l_{2p}, \ldots, l_{Np}, 0, 0, 0, \ldots)$ for some $M, N \in \mathbb{N}_{>0}$ where $k_{Mp}, l_{Np} \neq 0$, and assume that $f\colon A_1^k \to A_1^l$ is a map. By Lemma 32, f is a non-zero homomorphism if and only if f is an endomorphism on A_1 satisfying $f(x) = \alpha_l(f(x))$ and $k_0 + f(y) + k_p f(y)^p + k_{2p} f(y)^{2p} + \cdots + k_{Mp} f(y)^{Mp} = \alpha_l(f(y))$. Since $l_{Np} \neq 0$ where $N \in \mathbb{N}_{>0}$, from the definition of α_l, $f(x) = \alpha_l(f(x)) \iff f(x) \in K[x]$. Let us put $f(y) = \sum_{i=0}^m \sum_{j=0}^n c_{ij} y^i x^j$ for some $m, n \in \mathbb{N}$ and $c_{ij} \in K$

where $c_{mn} \neq 0$. Then,

$$
\begin{aligned}
&k_0 + f(y) + k_p f(y)^p + k_{2p} f(y)^{2p} + \cdots + k_{Mp} f(y)^{Mp} \\
&= k_0 + \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j + k_p \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j\right)^p \\
&+ k_{2p} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j\right)^{2p} + \cdots + k_{Mp} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j\right)^{Mp}, \\
&\alpha_l(f(y)) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} \alpha_l(y)^i x^j \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} (l_0 + y + l_p y^p + l_{2p} y^{2p} + \cdots + l_{Np} y^{Np})^i x^j \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{\ell=0}^{i} \binom{i}{\ell} c_{ij} (l_0 + l_p y^p + l_{2p} y^{2p} + \cdots + l_{Np} y^{Np})^{i-\ell} y^\ell x^j.
\end{aligned}
$$

Hence $k_0 + f(y) + k_p f(y)^p + k_{2p} f(y)^{2p} + \cdots + k_{Mp} f(y)^{Mp} = \alpha_l(f(y))$ if and only if

$$
\begin{aligned}
&k_0 + k_p \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j\right)^p + k_{2p} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j\right)^{2p} \\
&+ \cdots + k_{Mp} \left(\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij} y^i x^j\right)^{Mp} \\
&= \sum_{i=1}^{m}\sum_{j=0}^{n}\sum_{\ell=0}^{i-1} \binom{i}{\ell} c_{ij} (l_0 + l_p y^p + l_{2p} y^{2p} + \cdots + l_{Np} y^{Np})^{i-\ell} y^\ell x^j.
\end{aligned}
$$

By comparing degrees in y, we realize that $M = N$, which in turn gives $n = 0$, so

$$k_0 + k_p\left(\sum_{i=0}^{m} c_{i0}y^i\right)^p + k_{2p}\left(\sum_{i=0}^{m} c_{i0}y^i\right)^{2p} + \cdots + k_{Mp}\left(\sum_{i=0}^{m} c_{i0}y^i\right)^{Mp}$$
$$= \sum_{i=1}^{m}\sum_{\ell=0}^{i-1}\binom{i}{\ell}c_{i0}(l_0 + l_p y^p + l_{2p}y^{2p} + \cdots + l_{Mp}y^{Mp})^{i-\ell}y^{\ell}. \tag{3.5}$$

Hence $f\colon A_1^k \to A_1^l$ is a non-zero homomorphism if and only if $M = N$, $f \in \mathrm{End}_K(A_1)$ satisfying $f(x) \in K[x]$, $f(y) \in K[y]$, and (3.5). Now, we wish to know when f is an isomorphism. To this end, first assume that f is a surjective homomorphism. Then there is some $r := \sum_{i=0}^{m'}\sum_{j=0}^{n'} r_{ij}y^i x^j$ with $m', n' \in \mathbb{N}$ and $r_{ij} \in K$, such that $y = f(r)$. Since $y = f(r) = \sum_{i=0}^{m'}\sum_{j=0}^{n'} r_{ij}f(y)^i \cdot f(x)^j$ where $f(x) \in K[x]$ and $f(y) \in K[y]$, by comparing degrees, $n' = 0$ and $m' = 1$. This in turn gives $m = 1$, so that $f(y) = c_{00} + c_{10}y$ where $c_{10} \neq 0$. After some reindexing, (3.5) now reads

$$\sum_{i=0}^{M} k_{ip}(c_{00} + c_{10}y)^{ip} = c_{10}\sum_{i=0}^{M} l_{ip}y^{ip}$$
$$\iff \sum_{i=0}^{M}\sum_{j=0}^{i}\binom{i}{j}k_{ip}c_{00}^{(i-j)p}c_{10}^{jp}y^{jp} = c_{10}\sum_{i=0}^{M} l_{ip}y^{ip}$$
$$\iff \sum_{i=j}^{M}\binom{i}{j}k_{ip}c_{00}^{(i-j)p}c_{10}^{jp-1} = l_{jp}, \quad 0 \le j \le M. \tag{3.6}$$

By a similar argument, $f(x) = d_0 + d_1x$ for some $d_0, d_1 \in K$ where $d_1 \neq 0$. From $f(x)\cdot f(y) - f(y)\cdot f(x) = f(1_{A_1}) = 1_{A_1}$, it follows that $d_1 = c_{10}^{-1}$. Now, does this define an isomorphism? Yes, under the assumption that (3.6) is satisfied and that f is an automorphism on A_1. The latter can for instance be shown by first introducing the following functions, all being automorphisms on A_1 by Theorem 2,

$$\begin{aligned} g_1(x) &:= c_{10}^{-1}x, & g_2(x) &:= x, & g_3(x) &:= y, & g_4(x) &:= x,\\ g_1(y) &:= c_{10}y, & g_2(y) &:= y + c_{00}c_{10}^{-1}, & g_3(y) &:= -x, & g_4(y) &:= y + d_0c_{10}, \end{aligned}$$

$g_5 := -g_3$, and then noting that $f = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1$. The result now follows with $c_0 := c_{00}$ and $c_1 := c_{10}$. □

The two preceding propositions implicitly classify all hom-associative Weyl algebras up to isomorphism. However, it is not obvious under what circumstances there exist (do not exist) constants c_0, c_1 that solve (3.4), hence giving rise to isomorphic (non-isomorphic) hom-associative Weyl algebras. In the next corollary, we study a particular family of hom-associative Weyl algebras, over a finite field. Even in this particular case, we see that there do indeed exist many non-isomorphic hom-associative Weyl algebras, as opposed to the characteristic zero case.

Corollary 14 (**D** [8]). *If* $k = (k_0, 0, \ldots, 0, k_{p^n}, 0, \ldots, 0, k_{p^{2n}}, 0, \ldots, 0, k_{p^{mn}}, 0, 0, \ldots)$ *where* $k_{p^{mn}} \neq 0$ *for some* $m, n \in \mathbb{N}_{>0}$ *and* $K = \mathbb{F}_{p^n}$*, then* $A_1^k \cong A_1^l$ *if and only if* $k_{jp} = l_{jp}$ *for all* $j \in \mathbb{N}_{>0}$.

Proof. Let k be as above, and assume $K = \mathbb{F}_{p^n}$ for some $n \in \mathbb{N}_{>0}$ and $A_1^k \cong A_1^l$. By Proposition 29, $k_{jp} = l_{jp} = 0$ whenever $p^{mn-1} < j$. Now, by Corollary 9, (3.4) in Proposition 29 reads

$$k_{jp} c_1^{jp-1} = l_{jp}, \quad 1 \leq j \leq p^{mn-1}, \tag{3.7}$$

$$k_0 + k_{p^n} c_0^{p^n} + k_{p^{2n}} c_0^{p^{2n}} + \cdots + k_{p^{mn}} c_0^{p^{mn}} = l_0 c_1. \tag{3.8}$$

By Fermat's little theorem for finite fields, $c_0^{p^n} = c_0$ and $c_1^{p^n-1} = 1$, and so by induction $c_0^{p^n} = c_0^{p^{2n}} = c_0^{p^{3n}} = \ldots = c_0^{p^{mn}} = c_0$ and $c_1^{p^n-1} = c_1^{p^{2n}-1} = c_1^{p^{3n}-1} = \ldots = c_1^{p^{mn}-1} = 1$. Hence (3.7) is equivalent to $k_{jp} = l_{jp}$, $1 \leq j \leq p^{mn-1}$, and (3.8) to $k_0 + k_{p^n} c_0 + k_{p^{2n}} c_0 + \cdots + k_{p^{mn}} c_0 = l_0 c_1$. The two equations thus have a solution $c_0 = \frac{l_0 c_1 - k_0}{k_{p^n} + k_{p^{2n}} + k_{p^{3n}} + \cdots + k_{p^{mn}}}$ and $c_1 \in \mathbb{F}_{p^n}^{\times}$. □

3.2.4 Multi-parameter formal deformations

In Chapter 1, we described one-parameter formal hom-associative deformations and one-parameter formal-hom-Lie deformations, which originally were introduced by Makhlouf and Silvestrov [55]. In Proposition 9 in Chapter 2, we then saw that the hom-associative Weyl algebras in characteristic zero are a one-parameter formal deformation of the first associative Weyl algebra, and in Proposition 10 that this deformation induces a one-parameter formal hom-Lie deformation of the corresponding Lie algebra, when using the commutator as bracket. In this subsection, we generalize the two notions above and introduce *multi-parameter formal hom-associative deformations* and *multi-parameter formal hom-Lie deformations* (in [3], a similar notion for ternary hom-Nambu-Lie algebras was introduced, together with examples thereof). We then show that the hom-associative Weyl algebras over a field of prime characteristic are a multi-parameter formal hom-associative deformation of the first Weyl algebra in prime characteristic, inducing a multi-parameter formal hom-Lie deformation of the corresponding Lie algebra, when using the commutator as bracket. To this end, let R be a unital, associative, commutative ring, and M an R-module. We denote by $R[\![t_1,\ldots,t_n]\!]$ the formal power series ring in the indeterminates $t_1,\ldots,t_n$ for some $n \in \mathbb{N}_{>0}$, and by $M[\![t_1,\ldots,t_n]\!]$ the $R[\![t_1,\ldots,t_n]\!]$-module of formal power series in the same indeterminates, but with coefficients in M. This allows us to define a hom-associative algebra $(M[\![t_1,\ldots,t_n]\!],\cdot_t,\alpha_t)$ over $R[\![t_1,\ldots,t_n]\!]$ where $t := (t_1,\ldots,t_n)$. With some abuse of notation, we say that we extend a map $f\colon M \to M$ *homogeneously* to a map $f\colon M[\![t_1,\ldots,t_n]\!] \to M[\![t_1,\ldots,t_n]\!]$ by putting $f(at_1^{i_1}\cdots t_n^{i_n}) := f(a)t_1^{i_1}\cdots t_n^{i_n}$ for all $a \in M$ and $i_1,\ldots,i_n \in \mathbb{N}$. The case for binary maps is analogous.

Definition 18 (Multi-parameter formal hom-associative deformation, **D** [8])**.** A *multi-parameter*, or an *n-parameter formal hom-associative deformation* of a hom-associative algebra $(M,\cdot_0,\alpha_0)$ over R, is a hom-associative algebra $(M[\![t_1,\ldots,t_n]\!],\cdot_t,\alpha_t)$ over $R[\![t_1,\ldots,t_n]\!]$ where $n \in \mathbb{N}_{>0}$, $t := (t_1,\ldots,t_n)$, and

$$\cdot_t = \sum_{i\in\mathbb{N}^n} \cdot_i t^i, \quad \alpha_t = \sum_{i\in\mathbb{N}^n} \alpha_i t^i.$$

Here, $i := (i_1,\ldots,i_n) \in \mathbb{N}^n$ and $t^i := t_1^{i_1}\cdots t_n^{i_n}$. Moreover, $\cdot_i\colon M\times M \to M$ is an R-bilinear map, extended homogeneously to an $R[\![t_1,\ldots,t_n]\!]$-bilinear map $\cdot_i\colon M[\![t_1,\ldots,t_n]\!]\times M[\![t_1,\ldots,t_n]\!] \to M[\![t_1,\ldots,t_n]\!]$. Similarly, $\alpha_i\colon M \to M$ is an R-linear map, extended homogeneously to an $R[\![t_1,\ldots,t_n]\!]$-linear map denoted by $\alpha_i\colon M[\![t_1,\ldots,t_n]\!] \to M[\![t_1,\ldots,t_n]\!]$.

Proposition 30 (**D** [8])**.** *A_1^k is a multi-parameter formal deformation of A_1.*

Proof. Let $k = (k_0,k_p,k_{2p},\ldots,k_{Mp},0,0,0,\ldots)$ for some $M \in \mathbb{N}$ and put, for any $i \in \mathbb{N}$, $k_i = 0$ unless $i = 0,p,2p,\ldots,Mp$, or $i = 1$, in which case

$k_1 = 1_K$. By using the multinomial theorem, for an arbitrary monomial $y^m x^n$ where $m, n \in \mathbb{N}$,

$$\begin{aligned}
&\alpha_k(y^m x^n) = (k_0 + k_1 y + k_2 y^2 + \cdots + k_{Mp} y^{Mp})^m x^n \\
&= \sum_{i_0+i_1+i_2+\cdots i_{Mp}=m} \frac{m!}{i_0! i_1! i_2! \cdots i_{Mp}!} \left(\prod_{j=0}^{Mp} (k_j y^j)^{i_j} \right) x^n \\
&= \sum_{i_0+i_1+i_2+\cdots i_{Mp}=m} \frac{m!}{i_0! i_1! i_2! \cdots i_{Mp}!} \left(\prod_{j=0}^{Mp} (y^j)^{i_j} \right) x^n \prod_{j=0}^{Mp} k_j^{i_j} \\
&= \sum_{i_0+i_1+i_2+\cdots i_{Mp}=m} \frac{m!}{i_0! i_1! i_2! \cdots i_{Mp}!} \left(\prod_{j=0}^{Mp} y^{j i_j} \right) x^n \left(k_0^{i_0} k_1^{i_1} k_2^{i_2} \cdots k_{Mp}^{i_{Mp}} \right) \\
&= \sum_{i_0+i_p+i_{2p}+\cdots+i_{Mp}\leq m} \frac{m!}{i_0! i_p! i_{2p}! \cdots i_{Mp}! \cdot (m - i_0 - i_p - i_{2p} - \ldots - i_{Mp})!} \\
&\quad \cdot \left(\prod_{j=1}^{M} y^{pj i_{jp}} \right) x^n \left(k_0^{i_0} k_p^{i_p} k_{2p}^{i_{2p}} \cdots k_{Mp}^{i_{Mp}} \right)
\end{aligned}$$

Now define $t_1 := k_0$, $t_2 := k_p$, $t_3 := k_{2p}, \ldots, t_{M+1} := k_{Mp}$ and regard $t_1, \ldots, t_{M+1}$ as indeterminates of the formal power series $K[\![t_1, \ldots, t_{M+1}]\!]$ and $A_1[\![t_1, \ldots, t_{M+1}]\!]$. Then, by the above calculation, α_t is a formal power series in $t_1, \ldots, t_{M+1}$ where $t := (t_1, \ldots, t_{M+1})$. For any specific element $q \in A_1$, $\alpha_t(q)$ will be a polynomial. Moreover, $\alpha_0 = \mathrm{id}_{A_1}$. Next, we extend α_t linearly over $K[\![t_1, \ldots, t_{M+1}]\!]$ and homogeneously to all of $A_1[\![t_1, \ldots, t_{M+1}]\!]$. To define the multiplication $\cdot_t$ in $A_1[\![t_1, \ldots, t_{M+1}]\!]$, we first extend the multiplication $\cdot_0$ in A_1 homogeneously to a binary operation $\cdot_0 \colon A_1[\![t_1, \ldots, t_n]\!] \times A_1[\![t_1, \ldots, t_n]\!] \to A_1[\![t_1, \ldots, t_n]\!]$ linear over $K[\![t_1, \ldots, t_n]\!]$ in both arguments. Then we compose α_t with $\cdot_0$, so that $\cdot_t := \alpha_t \circ \cdot_0$. This is again a formal power series in $t_1, \ldots, t_{M+1}$, and hom-associativity follows from Proposition 2 in Chapter 1. Hence we have a formal deformation $(A_1[\![t_1, \ldots, t_{M+1}]\!], \cdot_t, \alpha_t)$ of $(A_1, \cdot_0, \alpha_0)$, where the latter is A_1 in the language of hom-associative algebras. $\square$

Definition 19 (Multi-parameter formal hom-Lie deformation, **D** [8]). A *multi-parameter*, or an *n-parameter formal hom-Lie deformation* of a hom-Lie algebra $(M, [\cdot,\cdot]_0, \alpha_0)$ over R, is a hom-Lie algebra $(M[\![t_1,\dots,t_n]\!], [\cdot,\cdot]_t, \alpha_t)$ over $R[\![t_1,\dots,t_n]\!]$ where $n \in \mathbb{N}_{>0}$, $t := (t_1,\dots,t_n)$, and

$$[\cdot,\cdot]_t = \sum_{i\in\mathbb{N}^n} [\cdot,\cdot]_i t^i, \quad \alpha_t = \sum_{i\in\mathbb{N}^n} \alpha_i t^i.$$

Here, $i := (i_1,\dots,i_n) \in \mathbb{N}^n$ and $t^i := t_1^{i_1}\cdots t_n^{i_n}$. Moreover, $[\cdot,\cdot]_i \colon M \times M \to M$ is an R-bilinear map, extended homogeneously to an $R[\![t_1,\dots,t_n]\!]$-bilinear map $[\cdot,\cdot]_i \colon M[\![t_1,\dots,t_n]\!] \times M[\![t_1,\dots,t_n]\!] \to M[\![t_1,\dots,t_n]\!]$. Also, $\alpha_i \colon M \to M$ is an R-linear map, extended homogeneously to an $R[\![t_1,\dots,t_n]\!]$-linear map denoted by $\alpha_i \colon M[\![t_1,\dots,t_n]\!] \to M[\![t_1,\dots,t_n]\!]$.

Proposition 31 (**D** [8]). *The deformation of A_1 into A_1^k induces a multi-parameter formal hom-Lie deformation of the Lie algebra of A_1 into the hom-Lie algebra of A_1^k, when using the commutator as bracket.*

Proof. Let $k = (k_0, k_p, k_{2p}, \dots, k_{Mp}, 0, 0, 0, \dots)$. Then, by using the deformation $(A_1[\![t_1,\dots,t_{M+1}]\!], \cdot_t, \alpha_t)$ of $(A_1, \cdot_0, \alpha_0)$ in Proposition 30, we construct a hom-Lie algebra $(A_1[\![t_1,\dots,t_{M+1}]\!], [\cdot,\cdot]_t, \alpha_t)$ by using the commutator $[\cdot,\cdot]_t$ of the hom-associative algebra $(A_1[\![t_1,\dots,t_{M+1}]\!], \cdot_t, \alpha_t)$ as bracket. Indeed, by Proposition 3 in Chapter 1 this gives a hom-Lie algebra. We claim that this is also a formal hom-Lie deformation of the Lie algebra $(A_1, [\cdot,\cdot]_0, \alpha_0)$ where $[\cdot,\cdot]_0$ is the commutator in A_1, and $\alpha_0 = \mathrm{id}_{A_1}$. Since α_t is the same map as in Proposition 30, we only need to verify that $[\cdot,\cdot]_t$ is a formal power series in $t_1,\dots,t_n$, which when evaluated at $t = 0$ gives the commutator in A_1. But this is immediate since $[\cdot,\cdot]_t = \alpha_t \circ [\cdot,\cdot]_0$. $\square$

Chapter 4

Chapter 4

Hilbert's basis theorem for non-associative and hom-associative Ore extensions

"Then, I can already draw a conclusion, X continued."

In *The spies of Oreborg*,
by Jakob Wegelius

This chapter is based on Paper **E**.

E P. Bäck and J. Richter,
Hilbert's basis theorem for non-associative and hom-associative Ore extensions,
Algebr. Represent. Theory (2022), `arXiv:1804.11304`.

4.1 Hom-module theory

In this section, we develop theory of hom-modules over non-unital, hom-associative rings. Many of the results and proofs are nearly identical to the classical ditto of the associative case, and for this reason, some of them have been left out in the journal version of Paper **E**. They are, however, all present in the `arXiv` version of the paper, and we have also provided them here for the convenience of the reader.

4.1.1 Basic definitions and theorems

Definition 20 (Hom-module, **E** [6])**.** Let R be a non-unital, hom-associative ring with twisting map α_R and multiplication written with juxtaposition. Let M be an additive group with a group homomorphism $\alpha_M\colon M \to M$, also called a twisting map. A *right R-hom-module* M_R consists of M and an operation $\cdot\colon M \times R \to M$, called *scalar multiplication*, such that for all $r_1, r_2 \in R$ and $a_1, a_2 \in M$, the following hold:

$$(a_1 + a_2) \cdot r_1 = a_1 \cdot r_1 + a_2 \cdot r_1 \qquad \text{(right-distributivity)}, \tag{M1}$$
$$a_1 \cdot (r_1 + r_2) = a_1 \cdot r_1 + a_1 \cdot r_2 \qquad \text{(left-distributivity)}, \tag{M2}$$
$$\alpha_M(a_1) \cdot (r_1 r_2) = (a_1 \cdot r_1) \cdot \alpha_R(r_2) \qquad \text{(hom-associativity)}. \tag{M3}$$

A *left R-hom-module* is defined analogously and written ${}_RM$.

For the sake of brevity, we also allow ourselves to write M in case it does not really matter whether it is a right or a left R-hom-module we are dealing with, and simply call it an R-hom-module. Furthermore, any two right (left) R-hom-modules are assumed to be equipped with the same twisting map α_R on R.

Remark 16 (**E** [6]). A hom-associative ring R is both a right R-hom-module R_R and a left R-hom-module ${}_RR$.

Definition 21 (Homomorphism of hom-modules, **E** [6])**.** A *homomorphism* from a right (left) R-hom-module M to a right (left) R-hom-module M' is an additive map $f\colon M \to M'$ such that $f \circ \alpha_M = \alpha_{M'} \circ f$ and $f(a \cdot r) = f(a) \cdot r$ ($f(r \cdot a) = r \cdot f(a)$) hold for all $a \in M$ and $r \in R$. If f is also bijective, the two are *isomorphic*, written $M \cong M'$.

Definition 22 (Hom-submodule, **E** [6])**.** Let M be a right (left) R-hom-module. An *R-hom-submodule*, or just *hom-submodule*, is an additive subgroup N of M that is closed under scalar multiplication and invariant under α_M.

In the above definition, N is then a right (left) R-hom-module with twisting maps α_R and α_N, the latter being given by the restriction of α_M to N. We denote that N is a hom-submodule of M by $N \leq M$ or $M \geq N$, and in case N is a proper subgroup of M, by $N < M$ or $M > N$.

Proposition 32 (**E** [6])**.** *Let $f: M \to M'$ be a homomorphism of right (left) R-hom-modules, $N \leq M$ and $N' \leq M'$. Then $f(N)$ and $f^{-1}(N')$ are hom-submodules of M' and M, respectively.*

Proof. We see that $f(N)$ and $f^{-1}(N')$ are additive subgroups when considering f as a group homomorphism. Let $r \in R$ and $a' \in f(N)$ be arbitrary. Then there is some $a \in N$ such that $a' = f(a)$, so $a' \cdot r = f(a) \cdot r = f(a \cdot r) \in f(N)$ since $a \cdot r \in N$. Moreover, $\alpha_{M'}(a') = \alpha_{M'}(f(a)) = f(\alpha_M(a)) = f(\alpha_N(a)) \in f(N)$. Now, take any $b \in f^{-1}(N')$. Then there is some $b' \in N'$ such that $f(b) = b'$, so $f(b \cdot r) = f(b) \cdot r = b' \cdot r \in N'$ since $b' \in N'$, and hence $b \cdot r \in f^{-1}(N')$. Last, $f(\alpha_M(b)) = \alpha_{M'}(f(b)) = \alpha_{M'}(b') = \alpha_{N'}(b') \in N'$, so $\alpha_M(b) \in f^{-1}(N')$. The left case is analogous. $\square$

Proposition 33 (**E** [6])**.** *The intersection of any set of hom-submodules of a right (left) R-hom-module is a hom-submodule.*

Proof. We show the case of right R-hom-modules; the case of left R-hom-modules is analogous. Let $N = \cap_{i \in I} N_i$ be an intersection of hom-submodules N_i of a right R-hom-module M, where I is some index set. Take any $a, b \in N$ and $j \in I$. Since $a, b \in N_j$ and N_j is an additive subgroup, $(a - b) \in N_j$, and therefore $(a - b) \in N$. For any $r \in R$, $a \cdot r \in N_j$ since N_j is a hom-submodule, and therefore $a \cdot r \in N$. Last, $\alpha_M(a) = \alpha_{N_j}(a) \in N_j$ for the same reason, so $\alpha_M(N)$ is a subset of N. $\square$

Definition 23 (Generating set of hom-submodule, **E** [6])**.** Let S be a non-empty subset of a right (left) R-hom-module M. The intersection N of all hom-submodules of M that contain S is called the *hom-submodule generated by* S, and S is called a *generating set* of N. If there is a finite generating set of N, then N is called *finitely generated*.

Remark 17 (**E** [6]). The hom-submodule N of a right (left) R-hom-module M generated by a non-empty subset S is the smallest hom-submodule of M that contains S in the sense that any other hom-submodule of M that contains S also contains N.

Proposition 34 (**E** [6]). *Let M be a right (left) R-hom-module, and consider an ascending chain $N_1 \leq N_2 \leq \ldots$ of hom-submodules of M. Then $\cup_{i=1}^{\infty} N_i$ is a hom-submodule of M.*

Proof. Denote $\cup_{i=1}^{\infty} N_i$ by N, and let $a, b \in N$. Then $a \in N_j$ and $b \in N_k$ for some $j, k \in \mathbb{N}_{>0}$, and since $N_j \leq N_{\max(j,k)}$ and $N_k \leq N_{\max(j,k)}$, we have $a, b \in N_{\max(j,k)}$. Hence $(a-b) \in N_{\max(j,k)} \subseteq N$, so $(a-b) \in N$. Take any $r \in R$. Then, since $a \in N_j$, $a \cdot r \in N_j \subseteq N$, so $a \cdot r \in N$ for the right case, and analogously for the left case. Finally, $\alpha_M(a) = \alpha_{N_j}(a) \in N_j \subseteq N$, so N is invariant under α_M. $\square$

Proposition 35 (**E** [6]). *Let M be a right (left) R-hom-module and $N_1, N_2, \ldots, N_n$ any finite number of hom-submodules of M. Then $\sum_{i=1}^{n} N_i = N_1 + N_2 + \cdots + N_n$ is a hom-submodule of M.*

Proof. We prove the right case; the left case is analogous. Let $N := \sum_{i=1}^{n} N_i$ and take any $r \in R$, $a_i, b_i \in N_i$. Then $\left(\sum_{i=1}^{n} a_i\right) \cdot r = \sum_{i=1}^{n} a_i \cdot r \in N$, and $\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i - b_i) \in N$. Last, N is invariant under $\alpha_N := \alpha_M|_N$ since $\alpha_M\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} \alpha_M(a_i) = \sum_{i=1}^{n} \alpha_N(a_i) \in N$. $\square$

Corollary 15 (**E** [6]). *Let M be a right (left) R-hom-module and M_1, M_2, and M_3 hom-submodules of M with $M_3 \leq M_1$. Then the* modular law $(M_1 \cap M_2) + M_3 = M_1 \cap (M_2 + M_3)$ *holds.*

Proof. The modular law holds for M_1, M_2 and M_3 when considered as additive groups. By Proposition 33 and Proposition 35, the intersection and sum of any two hom-submodules of M are also hom-submodules of M, and hence the modular law holds for M_1, M_2 and M_3 as hom-modules as well. $\square$

Proposition 36 (**E** [6]). *Let $M_1, M_2, \ldots, M_m$ be any finite number of right R-hom-modules. Endowing the (external) direct sum $M := \bigoplus_{i=1}^{m} M_i = M_1 \oplus M_2 \oplus \cdots \oplus M_m$ with the scalar multiplication $\bullet \colon M \times R \to M$ and twisting map $\alpha_M \colon M \to M$ defined here below, makes it a right R-hom-module:*

$$(a_1, a_2, \ldots, a_m) \bullet r := (a_1 \cdot r, a_2 \cdot r, \ldots, a_m \cdot r),$$
$$\alpha_M((a_1, a_2, \ldots, a_m)) := (\alpha_{M_1}(a_1), \alpha_{M_2}(a_2), \ldots, \alpha_{M_m}(a_m)).$$

Here, $(a_1, a_2, \ldots, a_m) \in M$, $r \in R$, and α_{M_i} is the twisting map on M_i for $1 \leq i \leq m$.

Proof. Since M is an additive group, what is left to check is that α_M is a group homomorphism, i.e. an additive map, and that (M1), (M2) and (M3) in Definition 20 holds. Let us start with the former. For any $a_i, b_i \in M_i$,

$$\begin{aligned}
&\alpha_M((a_1, a_2, \ldots, a_m) + (b_1, b_2, \ldots, b_m)) \\
&= (\alpha_{M_1}(a_1 + b_1), \alpha_{M_2}(a_2 + b_2), \ldots, \alpha_{M_m}(a_m + b_m)) \\
&= (\alpha_{M_1}(a_1), \alpha_{M_2}(a_2), \ldots, \alpha_{M_m}(a_m)) + (\alpha_{M_1}(b_1), \alpha_{M_2}(b_2), \ldots, \alpha_{M_m}(b_m)) \\
&= \alpha_M((a_1, a_2, \ldots, a_m)) + \alpha_M((b_1, b_2, \ldots, b_m)).
\end{aligned}$$

Let us now continue with (M1), (M2), and (M3). For any $r_1, r_2 \in R$,

$$\begin{aligned}
&((a_1, a_2, \ldots, a_m) + (b_1, b_2, \ldots, b_m)) \bullet r_1 \\
&= (a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m) \bullet r_1 \\
&= (a_1 \cdot r_1 + b_1 \cdot r_1, a_2 \cdot r_1 + b_2 \cdot r_1, \ldots, a_m \cdot r_1 + b_m \cdot r_1) \\
&= (a_1 \cdot r_1, a_2 \cdot r_1, \ldots, a_m \cdot r) + (b_1 \cdot r_1, b_2 \cdot r_1, \ldots, b_m \cdot r_1) \\
&= (a_1, a_2, \ldots, a_m) \bullet r_1 + (b_1, b_2, \ldots, b_m) \bullet r_1, \\
&(a_1, a_2, \ldots, a_m) \bullet (r_1 + r_2) \\
&= (a_1 \cdot (r_1 + r_2), a_2 \cdot (r_1 + r_2), \ldots, a_m \cdot (r_1 + r_2)) \\
&= (a_1 \cdot r_1 + a_1 \cdot r_2, a_2 \cdot r_1 + a_2 \cdot r_2, \ldots, a_m \cdot r_1 + a_m \cdot r_2) \\
&= (a_1, a_2, \ldots, a_m) \bullet r_1 + (a_1, a_2, \ldots, a_m) \bullet r_2, \\
&\alpha_M((a_1, a_2, \ldots, a_m)) \bullet (r_1 r_2) \\
&= (\alpha_{M_1}(a_1) \cdot (r_1 r_2), \alpha_{M_2}(a_2) \cdot (r_1 r_2), \ldots, \alpha_{M_m}(a_m) \cdot (r_1 r_2)) \\
&= ((a_1 \cdot r_1) \cdot \alpha_R(r_2), (a_2 \cdot r_1) \cdot \alpha_R(r_2), \ldots, (a_m \cdot r_1) \cdot \alpha_R(r_2)) \\
&= ((a_1, a_2, \ldots, a_m) \bullet r_1) \bullet \alpha_R(r_2).
\end{aligned}$$ □

An analogous result holds for left R-hom-modules.

Corollary 16 (E [6])**.** *For any right (left) R-hom-modules M_1, M_2, and M_3, $(M_1 \oplus M_2) \oplus M_3 \cong M_1 \oplus M_2 \oplus M_3 \cong M_1 \oplus (M_2 \oplus M_3)$.*

Proof. We prove the right case of the first isomorphism. The proof of the second isomorphism is similar, as are all the left cases. Considered as additive groups, $M := (M_1 \oplus M_2) \oplus M_3 \cong M_1 \oplus M_2 \oplus M_3 =: M'$ by the natural isomorphism

$f(((a_1, a_2), a_3)) = (a_1, a_2, a_3)$ for any $((a_1, a_2), a_3) \in M$. Let $r \in R$ be arbitrary. Then,

$$\begin{aligned}
&f(((a_1, a_2), a_3) \bullet r) = f(((a_1, a_2) \bullet r, a_3 \cdot r)) \\
&= f(((a_1 \cdot r, a_2 \cdot r), a_3 \cdot r)) = (a_1 \cdot r, a_2 \cdot r, a_3 \cdot r) \\
&= f(((a_1, a_2), a_3)) \bullet r, \\
&f(\alpha_M((a_1, a_2), a_3)) = f(((\alpha_{M_1}(a_1), \alpha_{M_2}(a_2)), \alpha_{M_3}(a_3))) \\
&= (\alpha_{M_1}(a_1), \alpha_{M_2}(a_2), \alpha_{M_3}(a_3)) = \alpha_{M'}(f(((a_1, a_2), a_3))).
\end{aligned}$$

□

Proposition 37 (**E** [6]). *Let M_R be a right R-hom-module with twisting map α_M. Let $N_R \leq M_R$ and consider the additive groups M and N of M_R and N_R, respectively. Form the quotient group M/N with elements of the form $a + N$ for $a \in M$. Then M/N becomes a right R-hom-module when endowed with the following twisting map and scalar multiplication for $a \in M$ and $r \in R$:*

$$\begin{aligned}
&\bullet\colon M/N \times R \to M/N, && (a + N) \bullet r := a \cdot r + N, \\
&\alpha_{M/N}\colon M/N \to M/N, && \alpha_{M/N}(a + N) := \alpha_M(a) + N.
\end{aligned}$$

Proof. First, let us make sure that the scalar multiplication and twisting map are both well-defined. To this end, take two arbitrary elements of M/N. They are of the form $a_1 + N$ and $a_2 + N$ for some $a_1, a_2 \in M$. If $a_1 + N = a_2 + N$, then $(a_1 - a_2) \in N$, and since N_R is a right R-hom-module, $(a_1 - a_2) \cdot r_1 \in N$ for any $r_1 \in R$. Then $(a_1 \cdot r_1 - a_2 \cdot r_1) \in N$, so $a_1 \cdot r_1 + N = a_2 \cdot r_1 + N$, and hence $(a_1 + N) \bullet r_1 = (a_2 + N) \bullet r_1$, so the scalar multiplication is well-defined. Now, since $(a_1 - a_2) \in N$, $\alpha_M(a_1 - a_2) \in N$ due to the fact that $N_R \leq M_R$. On the other hand, $\alpha_M(a_1 - a_2) = \alpha_M(a_1) - \alpha_M(a_2)$, so $(\alpha_M(a_1) - \alpha_M(a_2)) \in N$. Then $\alpha_M(a_1) + N = \alpha_M(a_2) + N$, and therefore $\alpha_{M/N}(a_1 + N) = \alpha_{M/N}(a_2 + N)$, which proves that $\alpha_{M/N}$ is well-defined. Furthermore, $\alpha_{M/N}$ is a group homomorphism since for any $(a_3 + N), (a_4 + N) \in M/N$ where $a_3, a_4 \in M$,

$$\begin{aligned}
&\alpha_{M/N}\left((a_3 + N) + (a_4 + N)\right) = \alpha_{M/N}\left((a_3 + a_4) + N\right) \\
&= \alpha_M(a_3 + a_4) + N = (\alpha_M(a_3) + \alpha_M(a_4)) + N \\
&= (\alpha_M(a_3) + N) + (\alpha_M(a_4) + N) \\
&= \alpha_{M/N}\left(a_3 + N\right) + \alpha_{M/N}\left(a_4 + N\right).
\end{aligned}$$

For any r_2 and r_3 in R,

$$
\begin{aligned}
&((a_3 + N) + (a_4 + N)) \bullet r_2 = ((a_3 + a_4) + N) \bullet r_2 \\
&= (a_3 + a_4) \cdot r_2 + N = (a_3 \cdot r_2 + a_4 \cdot r_2) + N \\
&= (a_3 \cdot r_2 + N) + (a_4 \cdot r_2 + N) = (a_3 + N) \bullet r_2 + (a_4 + N) \bullet r_2, \\
&(a_3 + N) \bullet (r_2 + r_3) = a_3 \cdot (r_2 + r_3) + N = (a_3 \cdot r_2 + a_3 \cdot r_3) + N \\
&= (a_3 \cdot r_2 + N) + (a_3 \cdot r_3 + N) = (a_3 + N) \bullet r_2 + (a_3 + N) \bullet r_3, \\
&\alpha_{M/N}(a_3 + N) \bullet (r_2 r_3) = (\alpha_M(a_3) + N) \cdot (r_2 r_3) \\
&= \alpha_M(a_3) \cdot (r_2 r_3) + N = (a_3 \cdot r_2) \cdot \alpha_R(r_3) + N \\
&= (a_3 \cdot r_2 + N) \bullet \alpha_R(r_3) = ((a_3 + N) \bullet r_2) \bullet \alpha_R(r_3).
\end{aligned}
$$ □

Again, an analogous result holds for left R-hom-modules as well.

Corollary 17 (**E** [6])**.** *Let M be a right (left) R-hom-module with $N \leq M$. Then the* natural projection $\pi\colon M \to M/N$ *defined by $\pi(a) = a + N$ for any $a \in M$ is a surjective homomorphism of hom-modules.*

Proof. We know that π is a surjective group homomorphism, and for any $a \in M$ and $r \in R$, $\pi(a \cdot r) = a \cdot r + N = (a + N) \bullet r = \pi(a) \bullet r$ for the right case, and analogously for the left case. We also have that $\pi(\alpha_M(a)) = \alpha(a) + N = \alpha_{M/N}(a + N) = \alpha_{M/N}(\pi(a))$, which completes the proof. □

Corollary 18 (**E** [6])**.** *Let M be a right (left) R-hom-module with $N \leq M$. If L is a hom-submodule of M/N, then $L = K/N$ for some hom-submodule K of M that contains N.*

Proof. Let L be a hom-submodule of M/N. If we use the natural projection $\pi\colon M \to M/N$ from Corollary 17, we know that $K = \pi^{-1}(L)$ is a hom-submodule of M since it is the preimage of a homomorphism of hom-submodules (cf. Proposition 32). By the surjectivity of π, $\pi(K) = \pi(\pi^{-1}(L)) = L$, so $L = \pi(K) = K/N$. □

Theorem 5 (The first isomorphism theorem for hom-modules, **E** [6])**.** *Let $f\colon M \to M'$ be a homomorphism of right (left) R-hom-modules. Then* $\ker f$ *is a hom-submodule of M,* $\operatorname{im} f$ *is a hom-submodule of M', and $M/\ker f \cong \operatorname{im} f$.*

Proof. We prove the case of right R-hom-modules; the case of left R-hom-modules is analogous. By definition, $\ker f$ is the preimage of the hom-submodule 0 of M', and hence it is a hom-submodule of M by Proposition 32. Now, $\operatorname{im} f = f(M)$, so by the same proposition, $\operatorname{im} f$ is a hom-submodule of M'. The map $g\colon M/\ker f \to \operatorname{im} f$ defined by $g(a + \ker f) = f(a)$ for any $(a + \ker f) \in M/\ker f$ is a well-defined group isomorphism. Furthermore, $g((a+\ker f)\bullet r) = g(a \cdot r + \ker f) = f(a \cdot r) = f(a) \cdot r = g(a + \ker f) \cdot r$ for any $r \in R$. Last, $g(\alpha_{M/\ker f}(a + \ker f)) = g(\alpha_M(a) + \ker f) = f(\alpha_M(a)) = \alpha_{M'}(f(a)) = \alpha_{\operatorname{im} f}(f(a)) = \alpha_{\operatorname{im} f}(g(a + \ker f))$, which completes the proof. □

Theorem 6 (The second isomorphism theorem for hom-modules, **E** [6]). *Let M be a right (left) R-hom-module with $N \leq M$ and $L \leq M$. Then $N/(N \cap L) \cong (N + L)/L$.*

Proof. By Proposition 33, $N \cap L$ is a hom-submodule of N and by Proposition 35, $N + L$ is a hom-module with $L = (0 + L) \leq (N + L)$, so the expression makes sense. The map $f\colon N \to (N+L)/L$ defined by $f(a) = a+L$ for any $a \in N$ is a group homomorphism. Furthermore, it is surjective since for any $((a+b)+L) \in (N+L)/L$. We have that $(a+b)+L = (a+L)+(b+L) = a+L+(0+L) = a+L = f(a)$. For any $r \in R$, $f(a\cdot r) = a\cdot r+L = (a+L)\bullet r = f(a)\bullet r$ (and similarly for the left case), and moreover, $f(\alpha_N(a)) = \alpha_N(a) + L = (\alpha_N(a) + \alpha_L(0)) + L = \alpha_{N+L}(a+0) + L = \alpha_{(N+L)/L}(a+L) = \alpha_{(N+L)/L}(f(a))$. We also see that $\ker f = N \cap L$, so by Theorem 5, $N/(N \cap L) \cong (N + L)/L$. □

Theorem 7 (The third isomorphism theorem for hom-modules, **E** [6]). *Let M be a right (left) R-hom-module with $L \leq N \leq M$. Then N/L is a hom-submodule of M/L and $(M/L)/(N/L) \cong M/N$.*

Proof. According to Corollary 17, the natural projection $\pi\colon M \to M/L$ is a homomorphism of right (left) hom-modules, so hom-submodules of M are mapped to hom-submodules of M/L. Since $N \leq M$, $N/L = \pi(N) \leq \pi(M) = M/L$, using that π is surjective. The map $f\colon M/L \to M/N$ defined by $f(a + L) = a + N$ for any $(a + L) \in M/L$ is a well-defined surjective group homomorphism. Moreover, for any $r \in R$, $f((a + L) \bullet r) = f(a \cdot r + L) = a \cdot r + N = (a + N) \bullet r = f(a + L) \bullet r$ (and analogously for the left case), and $f(\alpha_{M/L}(a + L)) = f(\alpha_M(a) + L) = \alpha_M(a) + N = \alpha_{M/N}(a + N) = \alpha_{M/N}(f(a + L))$. We also see that $\ker f = N/L$, so by using Theorem 5, $(M/L)/\ker f = (M/L)/(N/L) \cong \operatorname{im} f = M/N$. □

4.1.2 The hom-Noetherian conditions

Recall from Chapter 1 that a family $\mathcal{F}$ of subsets of a set S satisfies the *ascending chain condition* if there is no properly ascending infinite chain $S_1 \subset S_2 \subset \ldots$ of subsets from $\mathcal{F}$. Also recall that an element in $\mathcal{F}$ is called a *maximal element* of $\mathcal{F}$ provided there is no element in $\mathcal{F}$ that properly contains that element.

Proposition 38 (E [6]). *Let M be a right (left) R-hom-module. Then the following conditions are equivalent:*

(HNM1) M satisfies the ascending chain condition on its hom-submodules.

(HNM2) Any non-empty family of hom-submodules of M has a maximal element.

(HNM3) Any hom-submodule of M is finitely generated.

Proof. The following proof is an adaptation of a proof that can be found in [32] to the hom-associative setting.

(HNM1) $\Longrightarrow$ (HNM2): Let $\mathcal{F}$ be a non-empty family of hom-submodules of M that does not have a maximal element and pick an arbitrary hom-submodule N_1 in $\mathcal{F}$. Since N_1 is not a maximal element, there exists some $N_2 \in \mathcal{F}$ such that $N_1 < N_2$. Now, N_2 is not a maximal element either, so there exists some $N_3 \in \mathcal{F}$ such that $N_2 < N_3$. Continuing in this manner we get an infinite chain of hom-submodules $N_1 < N_2 < \ldots$, which proves the contrapositive statement.

(HNM2) $\Longrightarrow$ (HNM3): Assume (HNM2) holds, let N be an arbitrary hom-submodule of M, and $\mathcal{G}$ the family of all finitely generated hom-submodules of N. Since the zero module is a hom-submodule of N that is finitely generated, $\mathcal{G}$ is clearly non-empty and thus contains a maximal element L by assumption. If $N = L$, we are done, so assume the opposite and take some $a \in N \backslash L$. Now, let K be the hom-submodule of N generated by the set $L \cup \{a\}$. Then K is finitely generated as well, so $K \in \mathcal{G}$. Moreover, $L < K$, which is a contradiction since L is a maximal element in $\mathcal{G}$. Therefore, $N = L$, and N is finitely generated.

(HNM3) $\Longrightarrow$ (HNM1): Assume (HNM3) holds, let $N_1 \leq N_2 \leq \ldots$ be an ascending chain of hom-submodules of M, and $N = \cup_{i=1}^{\infty} N_i$. By Proposition 34, N is a hom-submodule of M, and hence it is finitely generated by some set S which by Definition 23 is contained in N. Moreover, since S is finite, it needs to be contained in N_j for some $j \in \mathbb{N}_{>0}$. However, $N_j = N$ by Remark 17, so $N_k = N_j$ for all $k \geq j$, and hence the ascending chain condition holds. $\square$

Definition 24 (Hom-Noetherian module, **E** [6])**.** A right (left) R-hom-module is called *hom-Noetherian* if it satisfies the three equivalent conditions of Proposition 38 on its hom-submodules.

Appealing to Remark 16, all properties that hold for right (left) hom-modules necessarily also hold for hom-associative rings, replacing "hom-submodule" by "right (left) hom-ideal". Hence we have the following:

Corollary 19 (**E** [6])**.** *Let R be a non-unital, hom-associative ring. Then the following conditions are equivalent:*

(HNR1) R satisfies the ascending chain condition on its right (left) hom-ideals.

(HNR2) Any non-empty family of right (left) hom-ideals of R has a maximal element.

(HNR3) Any right (left) hom-ideal of R is finitely generated.

Definition 25 (Hom-Noetherian ring, **E** [6])**.** A non-unital, hom-associative ring R is called *right (left) hom-Noetherian* if it satisfies the three equivalent conditions of Corollary 19 on its right (left) hom-ideals. If R satisfies the conditions on both its right and its left hom-ideals, it is called *hom-Noetherian*.

Remark 18 (**E** [6]). If the twisting map is either the identity map or the zero map, a right (left) hom-Noetherian ring is simply a right (left) Noetherian ring (cf. Remark 2 in Chapter 1). If R is a unital, hom-associative ring, then by Lemma 12 in Chapter 2, all right (left) ideals of R are also right (left) hom-ideals. In particular, R is right (left) hom-Noetherian if and only if R is right (left) Noetherian.

Proposition 39 (**E** [6])**.** *The hom-Noetherian conditions are preserved by surjective homomorphisms of right (left) R-hom-modules.*

Proof. It is sufficient to prove this for any of the equivalent conditions (HNM1), (HNM2), or (HNM3) in Proposition 38. Let us choose (HNM2). To this end, let $f\colon M \to M'$ be a surjective homomorphism of right (left) R-hom-modules where M is hom-Noetherian. Let $\mathcal{F}'$ be a non-empty family of right (left) hom-submodules of M'. Now, consider the corresponding family in M, namely $\mathcal{F} = \{f^{-1}(N')\colon N' \in \mathcal{F}'\}$. By the surjectivity of f, this family is non-empty, and since M is Noetherian, it has a maximal element $f^{-1}(N_0')$ for some $N_0' \in \mathcal{F}'$. We would like to show that N_0' is a maximal element of $\mathcal{F}'$. Assume there exists an element

$N' \in \mathcal{F}'$ such that $N'_0 < N'$. We know that the operation of taking preimages under any function preserves inclusions sets. We also know that the preimage of any hom-submodule is again a hom-submodule by Proposition 32, so taking preimages under a hom-homomorphism preserves the inclusions on the hom-submodules, and therefore $N'_0 < N'$ implies that $f^{-1}(N'_0) < f^{-1}(N')$, which contradicts the maximality of $f^{-1}(N'_0)$ in $\mathcal{F}$. Hence N'_0 is a maximal element of $\mathcal{F}'$, and M' is hom-Noetherian. □

Proposition 40 (E [6]). *Let M be a right (left) R-hom-module, and $N \leq M$. Then M is hom-Noetherian if and only if M/N and N are hom-Noetherian.*

Proof. This is again an adaptation of a proof that can be found in [32] to the hom-associative setting.

$(\Longrightarrow)$: Assume M is hom-Noetherian and $N \leq M$. Then any hom-submodule of N is also a hom-submodule of M, and hence it is finitely generated, and N therefore also hom-Noetherian. If $L_1 \leq L_2 \leq \ldots$ is an ascending chain of hom-submodules of M/N, then from Corollary 18, each $L_i = M_i/N$ for some M_i with $N \leq M_i \leq M$. Furthermore, $M_1 \leq M_2 \leq \ldots$, but since M is hom-Noetherian, there is some n such that $M_i = M_n$ for all $i \geq n$. Then $L_i = M_n/N = L_n$ for all $i \geq n$, so M/N is hom-Noetherian.

$(\Longleftarrow)$: Assume M/N and N are hom-Noetherian. Let $M_1 \leq M_2 \leq \ldots$ be an ascending chain of hom-submodules of M. By Proposition 33, $M_i \cap N$ is a hom-submodule of N for every $i \in \mathbb{N}_{>0}$, and furthermore $M_i \cap N \leq M_{i+1} \cap N$. We thus have an ascending chain $M_1 \cap N \leq M_2 \cap N \leq \ldots$ of hom-submodules of N. By Proposition 35, $M_i + N$ is a hom-submodule of M, and moreover, $N = 0 + N$ is a hom-submodule of $M_i + N$, so we can consider $(M_i + N)/N$. Now, $(M_i + N)/N \leq (M_{i+1} + N)/N$ by Corollary 18, so we have an ascending chain $(M_1 + N)/N \leq (M_2 + N)/N \leq \ldots$ of hom-submodules of M/N. Since both N and M/N are hom-Noetherian, there is some k such that $M_j \cap N = M_k \cap N$ and $(M_j + N)/N = (M_k + N)/N$ for all $j \geq k$. The latter equation implies that for any $a_j \in M_j$ and $b \in N$, there are $a_k \in M_k$ and $b' \in N$ such that $(a_j + b) + N = (a_k + b') + N$. Hence $x := ((a_j + b) - (a_k + b')) \in N$, and therefore $a_j + b = (a_k + (x + b')) \in (M_k + N)$, so that $(M_j + N) \leq (M_k + N)$, and by a similar argument, $(M_k + N) \leq (M_j + N)$, so $M_j + N = M_k + N$ for all $j \geq k$. Using this and the modular law for hom-modules (Corollary 15), $M_k = (M_k \cap N) + M_k = (M_j \cap N) + M_k = M_j \cap (N + M_k) = M_j \cap (M_k + N) = M_j \cap (M_j + N) = M_j$ for all $j \geq k$, and hence M is hom-Noetherian. □

Corollary 20 (E [6]). *Any finite direct sum of hom-Noetherian modules is hom-Noetherian.*

Proof. We prove this by induction.

Base case ($\mathsf{P}(2)$): Let M_1 and M_2 be two hom-Noetherian modules and consider the direct sum $M = M_1 \oplus M_2$, which is a right (left) R-hom-module by Proposition 36. Moreover, $M_1 \cong M_1 \oplus 0$ as additive groups, and for any $r \in R$, $f((a_1, 0) \bullet r) = f((a_1 \cdot r, 0 \cdot r)) = f((a_1 \cdot r, 0)) = a_1 \cdot r = f((a_1, 0)) \cdot r$. Now, $f(\alpha_{M_1 \oplus 0}((a_1, 0))) = f((\alpha_{M_1}(a_1), 0)) = \alpha_{M_1}(a_1) = \alpha_{M_1}(f(a_1, 0))$, so as right (left) R-hom-modules, $M_1 \cong M_1 \oplus 0 \leq M$. Similarly, the projection $g \colon M \to M_2$ is a surjective homomorphism of right (left) R-hom-modules with $\ker g = M_1 \oplus 0$, so by Theorem 5, $M/(M_1 \oplus 0) \cong M_2$. Due to Proposition 39, $M_1 \oplus 0$ and $M/(M_1 \oplus 0)$ are both hom-Noetherian, and so by Proposition 40, M is hom-Noetherian.

Induction step ($\forall k \in \mathbb{N}_{>1}$ $(\mathsf{P}(k) \to \mathsf{P}(k+1))$): Assume $M' = \bigoplus_{i=1}^{k} M_i$ is hom-Noetherian for $2 \leq k$, where each M_i is a hom-Noetherian right (left) R-hom-module. Let M_{k+1} be a hom-Noetherian right (left) R-hom-module. Then $\bigoplus_{i=1}^{k+1} M_i \cong M' \oplus M_{k+1}$ by Corollary 16. The latter of the two is hom-Noetherian by the base case, and by Proposition 39 the former as well. $\square$

4.2 Hilbert's basis theorem for hom-associative Ore extensions

In this section, we consider unital, hom-associative Ore extensions $R[x; \sigma, \delta]$ over unital, hom-associative rings R. First, recall from Proposition 6 in Chapter 2 that if R is a unital, hom-associative ring with twisting map α, then $R[x; \sigma, \delta]$ is a unital, hom-associative ring if σ is a unital endomorphism, δ is a σ-derivation, and σ and δ commute with α, the latter extended homogeneously to a twisting map on $R[x; \sigma, \delta]$. Now, any unital, non-associative ring may be regarded as a unital, hom-associative ring R with twisting map equal to the zero map. Hence, in this case, if σ is a unital endomorphism and δ is a σ-derivation on R, then σ and δ commute with the zero map, and so we may speak about the unital, non-associative Ore extension $R[x; \sigma, \delta]$.

Recall from Remark 18 that for unital, hom-associative rings, being right (left) hom-Noetherian is the same as being right (left) Noetherian. Also recall from Chapter 1 that the associator $(\cdot, \cdot, \cdot) \colon R \times R \times R \to R$ is defined by $(r, s, t) = (r \cdot s) \cdot t - r \cdot (s \cdot t)$ for any $r, s, t \in R$, and that the left, middle, and right nucleus of R are defined as $N_l(R) := \{r \in R \colon (r,s,t) = 0,\ s,t \in R\}$, $N_m(R) := \{s \in R \colon (r,s,t) = 0,\ r,t \in R\}$, and $N_r(R) := \{t \in R \colon (r,s,t) = 0,\ r,s \in R\}$, respectively. The nucleus of R, $N(R)$, is defined as $N_l(R) \cap N_m(R) \cap N_r(R)$. $N_l(R), N_m(R), N_r(R)$, and hence also $N(R)$, are all associative subrings of R.

Lemma 33 (E [6]). *Let R be a unital, non-associative ring, σ a unital endomorphism and δ a σ-derivation on R. Then, in $R[x;\sigma,\delta]$, the following hold for all $r, s \in R$ and $l, m, n \in \mathbb{N}$:*

(i) $\sum_{i\in\mathbb{N}} \pi_i^m \left(r \cdot \pi_{l-i}^n(s)\right) = \sum_{i\in\mathbb{N}} \pi_i^m(r) \cdot \pi_l^{i+n}(s)$.

(ii) $\pi_l^{m+1} = \pi_{l-1}^m \circ \sigma + \pi_l^m \circ \delta = \sigma \circ \pi_{l-1}^m + \delta \circ \pi_l^m$.

Proof. A proof of (i) in the associative setting can be found in [63]. However, the proof makes no use of associativity, so we can conclude that (i) holds in the non-associative setting as well.

Regarding (ii), we first recall that π_l^{m+1} consists of the sum of all $\binom{m+1}{l}$ possible compositions of l copies of σ and $m+1-l$ copies of δ. Therefore, we can split the sum into a part containing σ innermost (outermost) and a part containing δ innermost (outermost). When $l = 0$, we immediately see that the result holds as $\pi_{-1}^m := 0$. When $l > m$, $\pi_l^m := 0$, and in case also $l > m+1$, $\pi_l^{m+1} = \pi_{l-1}^m := 0$. In case $l = m+1$, $\pi_l^l = \pi_{l-1}^{l-1} \circ \sigma = \sigma \circ \pi_{l-1}^{l-1}$, so we can conclude that (ii) holds when $l = 0$ and when $l > m$. For the remaining case $1 \leq l \leq m$, we use the recursive formula for binomial coefficients $\binom{m+1}{l} = \binom{m}{l-1} + \binom{m}{l}$ and simply count the terms in the two parts of the sum. □

Proposition 41 (E [6]). *Let R be a unital, non-associative ring, σ a unital endomorphism and δ a σ-derivation on R. Then $x^k \in N(R[x;\sigma,\delta])$ for any $k \in \mathbb{N}$.*

Proof. By identifying x^0 with $1_R \in R$, $x^0 \in N(R[x;\sigma,\delta])$. We now wish to show that $x \in N(R[x;\sigma,\delta])$. In order to do that, we must show that x associates with all polynomials in $R[x;\sigma,\delta]$. Due to distributivity, it is however sufficient to prove that x associates with arbitrary monomials rx^m and sx^n in $R[x;\sigma,\delta]$. To this end, first note that $rx^m \cdot x = \sum_{i\in\mathbb{N}} \left(r \cdot \pi_i^m(1_R)\right) x^{i+1} = rx^{m+1}$ since σ is unital by assumption, and $\delta(1_R) = 0$ by Remark 6 in Chapter 1. Then,

$$(rx^m \cdot sx^n) \cdot x = \left(\sum_{i\in\mathbb{N}} \left(r \cdot \pi_i^m(s)\right) x^{i+n}\right) \cdot x = \sum_{i\in\mathbb{N}} \left(\left(r \cdot \pi_i^m(s)\right) x^{i+n}\right) \cdot x$$

$$= \sum_{i\in\mathbb{N}} \left(r \cdot \pi_i^m(s)\right) x^{i+n+1} = rx^m \cdot sx^{n+1} = rx^m \cdot \left(sx^n \cdot x\right),$$

so $x \in N_r(R[x;\sigma,\delta])$. Also, by using (ii) in Lemma 33,

$$\begin{aligned}
&(rx^m \cdot x) \cdot sx^n = rx^{m+1} \cdot sx^n = \sum_{i\in\mathbb{N}} \left(r \cdot \pi_i^{m+1}(s)\right) x^{i+n} \\
&= \sum_{i\in\mathbb{N}} \left(r \cdot \left(\pi_{i-1}^m \circ \sigma(s) + \pi_i^m \circ \delta(s)\right)\right) x^{i+n} \\
&= \sum_{j\in\mathbb{N}} \left(r \cdot \pi_j^m(\sigma(s))\right) x^{j+n+1} + \sum_{i\in\mathbb{N}} \left(r \cdot \pi_i^m(\delta(s))\right) x^{i+n} \\
&= rx^m \cdot \sigma(s)x^{n+1} + rx^m \cdot \delta(s)x^n = rx^m \cdot \left(\sigma(s)x^{n+1} + \delta(s)x^n\right) \\
&= rx^m \cdot \sum_{i\in\mathbb{N}} \left(1_R \cdot \pi_i^1(s)\right) x^{n+i} = rx^m \cdot \left(x \cdot sx^n\right),
\end{aligned}$$

so $x \in N_m(R[x;\sigma,\delta])$. Last,

$$\begin{aligned}
&(x \cdot rx^m) \cdot sx^n = \left(\sum_{i\in\mathbb{N}} \left(1_R \cdot \pi_i^1(r)\right) x^{i+m}\right) \cdot sx^n \\
&= \left(\delta(r)x^m + \sigma(r)x^{m+1}\right) \cdot sx^n = \delta(r)x^m \cdot sx^n + \sigma(r)x^{m+1} \cdot sx^n \\
&= \sum_{i\in\mathbb{N}} \left(\delta(r) \cdot \pi_i^m(s)\right) x^{i+n} + \sum_{j\in\mathbb{N}} \Big(\sigma(r) \cdot \pi_j^{m+1}(s)\Big) x^{j+n} \\
&= \sum_{i\in\mathbb{N}} \left(\delta(r) \cdot \pi_i^m(s)\right) x^{i+n} + \sum_{j\in\mathbb{N}} \left(\sigma(r) \cdot \left(\sigma \circ \pi_{j-1}^m(s) + \delta \circ \pi_j^m(s)\right)\right) x^{j+n} \\
&= \sum_{i\in\mathbb{N}} \left(\sigma(r) \cdot \delta\left(\pi_i^m(s)\right) + \delta(r) \cdot \pi_i^m(s)\right) x^{i+n} \\
&\quad + \sum_{k\in\mathbb{N}} \left(\sigma(r) \cdot \sigma\left(\pi_k^m(s)\right)\right) x^{k+n+1} \\
&= \sum_{i\in\mathbb{N}} \delta\left(r \cdot \pi_i^m(s)\right) x^{i+n} + \sum_{k\in\mathbb{N}} \sigma\left(r \cdot \pi_k^m(s)\right) x^{k+n+1} \\
&= \sum_{i\in\mathbb{N}} x \cdot \left(r \cdot \pi_i^m(s)\right) x^{i+n} = x \cdot \sum_{i\in\mathbb{N}} \left(r \cdot \pi_i^m(s)\right) x^{i+n} = x \cdot \left(rx^m \cdot sx^n\right).
\end{aligned}$$

Hence $x \in N_l(R[x;\sigma,\delta])$, and so $x \in N(R[x;\sigma,\delta])$. Since $N(R[x;\sigma,\delta])$ is a ring it also contains all powers of x, so $x^k \in N(R[x;\sigma,\delta])$ for any $k \in \mathbb{N}$. $\square$

Proposition 42 (E [6]). *Let R be a unital, hom-associative, Noetherian ring with twisting map α, σ a unital endomorphism and δ a σ-derivation that both commute with α. Extend α homogeneously to $R[x;\sigma,\delta]$. Then, for any $m \in \mathbb{N}$, $\sum_{i=0}^m x^i R$ ($\sum_{i=0}^m Rx^i$) is a hom-Noetherian right (left) R-hom-module.*

Proof. Let us prove the right case; the left case is similar, but slightly simpler. Put $M = \sum_{i=0}^m x^i R$. First note that M really is a subset of $R[x;\sigma,\delta]$, where the elements are of the form $\sum_{i=0}^m 1_R x^i \cdot r_i x^0$ with $r_i \in R$. When identifying $1_R x^i$ with x^i and r_i with $r_i x^0$, this gives us elements of the form $\sum_{i=0}^m x^i \cdot r_i$. Using this identification also allows us to write the multiplication in R, which in Definition 20 is done by juxtaposition, by "$\cdot$" instead. The purpose of this is do be consistent with our previous notation.

Since distributivity follows from that in $R[x;\sigma,\delta]$, it suffices to show that the multiplication in $R[x;\sigma,\delta]$ is a scalar multiplication, and that we have twisting maps α_M and α_R that give us hom-associativity. To this end, for any $r \in R$ and any element in M (which is of the form described above), by using Proposition 41,

$$\left(\sum_{i=0}^m x^i \cdot r_i\right) \cdot r = \sum_{i=0}^m \left(x^i \cdot r_i\right) \cdot r = \sum_{i=0}^m x^i \cdot (r_i \cdot r), \tag{4.1}$$

and the latter is clearly an element of M. Now, we claim that M is invariant under the homogeneously extended twisting map on $R[x;\sigma,\delta]$. To follow the notation in Definition 20, let us denote this map when restricted to M by α_M, and that of R by α_R. Then, by using the additivity of α_M and α_R, as well as the fact that the latter commutes with σ and δ, we get

$$\begin{aligned}\alpha_M\left(\sum_{i=0}^m x^i \cdot r_i\right) &= \alpha_M\left(\sum_{i=0}^m \sum_{j\in\mathbb{N}} \pi_j^i(r_i)x^j\right) = \sum_{i=0}^m \sum_{j\in\mathbb{N}} \alpha_M\left(\pi_j^i(r_i)x^j\right) \\ &= \sum_{i=0}^m \sum_{j\in\mathbb{N}} \alpha_R\left(\pi_j^i(r_i)\right)x^j = \sum_{i=0}^m \sum_{j\in\mathbb{N}} \pi_j^i(\alpha_R(r_i))x^j = \sum_{i=0}^m X^i \cdot \alpha_R(r_i),\end{aligned} \tag{4.2}$$

which again is an element of M. Last, let $r, s \in R$ be arbitrary. Then,

$$\begin{aligned}
\alpha_M\left(\sum_{i=0}^m x^i \cdot r_i\right) \cdot (r \cdot s) &\overset{(4.2)}{=} \left(\sum_{i=0}^m x^i \cdot \alpha_R(r_i)\right) \cdot (r \cdot s) \\
&\overset{(4.1)}{=} \sum_{i=0}^m x^i \cdot (\alpha_R(r_i) \cdot (r \cdot s)) = \sum_{i=0}^m x^i \cdot ((r_i \cdot r) \cdot \alpha_R(s)) \\
&\overset{(4.1)}{=} \left(\sum_{i=0}^m x^i \cdot (r_i \cdot r)\right) \cdot \alpha_R(s) \overset{(4.1)}{=} \left(\left(\sum_{i=0}^m x^i \cdot r_i\right) \cdot r\right) \cdot \alpha_R(s),
\end{aligned}$$

which proves hom-associativity. What is left to prove is that M is hom-Noetherian. Now, let us define $f\colon \bigoplus_{i=0}^m R \to M$ by $(r_0, r_1, \ldots, r_m) \mapsto \sum_{i=0}^m x^i \cdot r_i$ for any $(r_0, r_1, \ldots, r_m) \in \bigoplus_{i=0}^m R$. We see that f is additive, and for any $r \in R$, we have $f((r_0, r_1, \ldots, r_m) \bullet r) = f((r_0, r_1, \ldots, r_m)) \cdot r$. A similar argument gives $f(\alpha_{\bigoplus_{i=0}^m R}((r_0, r_1, \ldots, r_m))) = \alpha_M(f((r_0, r_1, \ldots, r_m)))$, which shows that f is a homomorphism of two right R-hom-modules. Moreover, f is surjective, and so by Proposition 39, M is hom-Noetherian. $\square$

Lemma 34 (**E** [6]). *Let R be a unital, hom-associative ring with twisting map α, σ an automorphism and δ a σ-derivation that both commute with α. Extend α homogeneously to $R[x; \sigma, \delta]$. Then the following hold:*

(i) σ^{-1} *is an automorphism on R^{op} that commutes with α.*

(ii) $-\delta \circ \sigma^{-1}$ *is a σ^{-1}-derivation on R^{op} that commutes with α.*

(iii) $R[x; \sigma, \delta]^{op} \cong R^{op}[x; \sigma^{-1}, -\delta \circ \sigma^{-1}]$.

Proof. That σ^{-1} is an automorphism and $-\delta \circ \sigma^{-1}$ a σ^{-1}-derivation on R^{op} is an exercise in [32] that can be solved without any use of associativity. Now, since α commutes with σ and δ, for any $r \in R^{\mathrm{op}}$, $\sigma(\alpha(\sigma^{-1}(r))) = \alpha(\sigma(\sigma^{-1}(r))) = \alpha(r)$, so by applying σ^{-1} to both sides, $\alpha(\sigma^{-1}(r)) = \sigma^{-1}(\alpha(r))$. From this, it follows that $-\delta(\sigma^{-1}(\alpha(r))) = -\delta(\alpha(\sigma^{-1}(r))) = \alpha(-\delta(\sigma^{-1}(r)))$, which proves the first and second statement.

For the third statement, let us start by putting $S := R^{\mathrm{op}}[x; \sigma^{-1}, -\delta \circ \sigma^{-1}]$ and $S' := R[x; \sigma, \delta]^{\mathrm{op}}$, and then define a map $f\colon S \to S'$ by $\sum_{i=0}^n r_i x^i \mapsto$

$\sum_{i=0}^{n} r_i \cdot_{\text{op}} x^i$ for any $n \in \mathbb{N}$. We claim that f is an isomorphism of hom-associative rings. First, note that an arbitrary element of S' by definition is of the form $p := \sum_{i=0}^{m} p_i x^i$ for some $m \in \mathbb{N}$ and $p_i \in R^{\text{op}}$. Then,

$$\begin{aligned} p &= \underbrace{x^m \cdot \sigma^{-m}(p_m) + q_{m-1}x^{m-1} + \cdots + q_0}_{=p_m x^m} + \cdots \\ &\quad + \underbrace{x \cdot \sigma^{-1}(p_1) + \delta(\sigma^{-1}(p_1))}_{=p_1 x} + p_0 \\ &= x^m \cdot \sigma^{-m}(p_m) + x^{m-1} \cdot p'_{m-1} + \cdots + x \cdot p'_1 + p'_0 \\ &= \sigma^{-m}(p_m) \cdot_{\text{op}} x^m + p'_{m-1} \cdot_{\text{op}} x^{m-1} + \cdots + p'_1 \cdot_{\text{op}} x + p'_0 \in \operatorname{im} f, \end{aligned}$$

for some $p'_{m-1}, q_{m-1}, \ldots, p'_0, q_0 \in R^{\text{op}}$, so f is surjective. The second last step also shows that $\sum_{i=0}^{m} Rx^i \subseteq \sum_{i=0}^{m} x^i R$ as sets, and a similar calculation shows that $\sum_{i=0}^{m} x^i R \subseteq \sum_{i=0}^{m} Rx^i$, so that as sets, $\sum_{i=0}^{m} Rx^i = \sum_{i=0}^{m} x^i R$. Hence, if $\sum_{i=0}^{m} r_i \cdot_{\text{op}} x^i = \sum_{j=0}^{m'} r'_j \cdot_{\text{op}} x^j$ for some $r_i, r'_j \in R^{\text{op}}$ and $m, m' \in \mathbb{N}$, then $m = m'$ and so

$$\begin{aligned} 0 &= \sum_{i=0}^{m} (r_i - r'_i) \cdot_{\text{op}} x^i = \sum_{i=0}^{m} x^i \cdot (r_i - r'_i) = \sum_{i=0}^{m} \sum_{j \in \mathbb{N}} \pi_j^i (r_i - r'_i) x^j \\ &= \sum_{j=0}^{m} \sum_{i=0}^{m} \pi_j^i (r_i - r'_i) x^j \implies 0 = \sum_{i=0}^{m} \pi_j^i (r_i - r'_i) x^j \quad \text{for } 0 \leq j \leq m, \end{aligned} \tag{4.3}$$

where the implication comes from comparing coefficients with the left-hand side, which is equal to zero. Let us prove by induction that $r_j = r'_j$ for $0 \leq j \leq m$. Put $k = m - j$, where m is fixed, and consider the statement $\mathsf{P}(k)$: $r_{m-k} = r'_{m-k}$ for $0 \leq k \leq m$.

Base case ($\mathsf{P}(0)$): $k = 0 \iff j = m$, so using that σ is an automorphism,

$$0 \overset{(4.3)}{=} \sum_{i=0}^{m} \pi_m^i (r_i - r'_i) x^m = \sigma^m (r_m - r'_m) x^m \implies 0 = r_m - r'_m.$$

Induction step (For $0 \leq k \leq m$: $(\mathsf{P}(k) \to \mathsf{P}(k+1))$): By putting $j = m - (k+1)$ and then using the induction hypothesis,

$$0 \overset{(4.3)}{=} \sum_{i=0}^{m} \pi_{m-(k+1)}^i (r_i - r'_i) x^{m-(k+1)} = \sigma^{m-(k+1)} (r_{m-(k+1)} - r'_{m-(k+1)}),$$

which implies $0 = r_{m-(k+1)} = r'_{m-(k+1)}$. Hence $r_j = r'_j$ for $0 \le j \le m$, so $\sum_{i=0}^{m} r_i \cdot_{\mathrm{op}} x^i = \sum_{j=0}^{m'} r'_j \cdot_{\mathrm{op}} x^j \implies \sum_{i=0}^{m} r_i x^i = \sum_{j=0}^{m'} r'_j x^j$, proving that f is injective. Additivity of f follows immediately from the definition by using distributivity. Using additivity also makes it sufficient to consider only two arbitrary monomials rx^m and sx^n in S when proving that f is multiplicative. To this end, let us use the following notation for multiplication in S: $rx^m \bullet sx^n := \sum_{i\in\mathbb{N}} (r \cdot_{\mathrm{op}} \bar{\pi}_i^m(s))\, x^{i+n}$, and then use induction over m and n;

Base case $(\mathsf{P}(0,0))$: $f(r \bullet s) = f(r \cdot_{\mathrm{op}} s) = r \cdot_{\mathrm{op}} s = f(r) \cdot_{\mathrm{op}} f(s)$.

Induction step over n $(\forall (m,n) \in \mathbb{N}\times\mathbb{N}\ (\mathsf{P}(m,n) \to \mathsf{P}(m,n+1)))$: By Proposition 41, we know that $x \in N(S')$, and so

$$
\begin{aligned}
f\left(rx^m \bullet sx^{n+1}\right) &= f\left(\sum_{i\in\mathbb{N}} (r \cdot_{\mathrm{op}} \bar{\pi}_i^m(s))\, x^{i+n+1}\right) \\
&= \sum_{i\in\mathbb{N}} (r \cdot_{\mathrm{op}} \bar{\pi}_i^m(s)) \cdot_{\mathrm{op}} x^{i+n+1} = \left(\sum_{i\in\mathbb{N}} (r \cdot_{\mathrm{op}} \bar{\pi}_i^m(s)) \cdot_{\mathrm{op}} x^{i+n}\right) \cdot_{\mathrm{op}} x \\
&= f(rx^m \bullet sx^n) \cdot_{\mathrm{op}} x = (f(rx^m) \cdot_{\mathrm{op}} f(sx^n)) \cdot_{\mathrm{op}} x \\
&= f(rx^m) \cdot_{\mathrm{op}} (f(sx^n) \cdot_{\mathrm{op}} x) = f(rx^m) \cdot_{\mathrm{op}} ((s \cdot_{\mathrm{op}} x^n) \cdot_{\mathrm{op}} x) \\
&= f(rx^m) \cdot_{\mathrm{op}} (s \cdot_{\mathrm{op}} (x^n \cdot_{\mathrm{op}} x)) = f(rx^m) \cdot_{\mathrm{op}} \left(s \cdot_{\mathrm{op}} x^{n+1}\right) \\
&= f(rx^m) \cdot_{\mathrm{op}} f(sx^{n+1}).
\end{aligned}
$$

Induction step over m $(\forall (m,n) \in \mathbb{N}\times\mathbb{N}\ (\mathsf{P}(m,n) \to \mathsf{P}(m+1,n)))$: By Proposition 41, we know that $x \in N(S'^{\mathrm{op}}) \cap N(S)$, and so

$$
\begin{aligned}
f\left(rx^{m+1} \bullet sx^n\right) &= f\left((rx^m \bullet x) \bullet sx^n\right) = f\left(rx^m \bullet (x \bullet sx^n)\right) \\
&= f\left(rx^m \bullet \left(\left(\sigma^{-1}(s)x - \delta\circ\sigma^{-1}(s)\right) \bullet x^n\right)\right) \\
&= f\left(rx^m \bullet \sigma^{-1}(s)x^{n+1}\right) - f\left(rx^m \bullet \delta\circ\sigma^{-1}(s)x^n\right) \\
&= f\left(rx^m \bullet \sigma^{-1}(s)x^n\right) \cdot_{\mathrm{op}} x - f\left(rx^m \bullet \delta\circ\sigma^{-1}(s)x^n\right) \\
&= \left(f\left(rx^m\right) \cdot_{\mathrm{op}} f\left(\sigma^{-1}(s)x^n\right)\right) \cdot_{\mathrm{op}} x - f\left(rx^m\right) \cdot_{\mathrm{op}} f\left(\delta\circ\sigma^{-1}(s)x^n\right) \\
&= f\left(rx^m\right) \cdot_{\mathrm{op}} \left(f\left(\sigma^{-1}(s)x^n\right) \cdot_{\mathrm{op}} x\right) - f\left(rx^m\right) \cdot_{\mathrm{op}} f\left(\delta\circ\sigma^{-1}(s)x^n\right) \\
&= f\left(rx^m\right) \cdot_{\mathrm{op}} f\left(\sigma^{-1}(s)x^{n+1}\right) - f\left(rx^m\right) \cdot_{\mathrm{op}} f\left(\delta\circ\sigma^{-1}(s)x^n\right) \\
&= f\left(rx^m\right) \cdot_{\mathrm{op}} f\left(\sigma^{-1}(s)x^{n+1} - \delta\circ\sigma^{-1}(s)x^n\right) \\
&= f\left(rx^m\right) \cdot_{\mathrm{op}} f\left(\left(\sigma^{-1}(s)x - \delta\circ\sigma^{-1}(s)\right) \bullet x^n\right)
\end{aligned}
$$

$$
\begin{aligned}
&= f(rx^m) \cdot_{\text{op}} f((x \bullet s) \bullet x^n) = f(rx^m) \cdot_{\text{op}} f(x \bullet (s \bullet x^n)) \\
&= f(rx^m) \cdot_{\text{op}} f(x \bullet sx^n) = f(rx^m) \cdot_{\text{op}} (f(x) \cdot_{\text{op}} f(sx^n)) \\
&= f(rx^m) \cdot_{\text{op}} (x \cdot_{\text{op}} f(sx^n)) = f((rx^m) \cdot_{\text{op}} x) \cdot_{\text{op}} f(sx^n) \\
&= f\left(rx^{m+1}\right) \cdot_{\text{op}} f(sx^n),
\end{aligned}
$$

Now, we are done if we can prove that $f \circ \alpha = \alpha \circ f$ for the homogeneously extended map α. Since both α and f are additive, it again suffices to prove that $f((\alpha(rx^m)) = \alpha(f(rx^m))$ for some arbitrary monomial rx^m in $R[x;\sigma,\delta]$.

$$
f(\alpha(rx^m)) = f(\alpha(r)x^m) = \alpha(r) \cdot_{\text{op}} x^m = x^m \cdot \alpha(r) = \sum_{i\in\mathbb{N}} \pi_i^m(\alpha(r))x^i
$$

$$
= \sum_{i\in\mathbb{N}} \alpha(\pi_i^m(r))\, x^i = \alpha\left(\sum_{i\in\mathbb{N}} \pi_i^m(r)x^i\right) = \alpha(x^m \cdot r) = \alpha(f(rx^m)). \quad \square
$$

Theorem 8 (Hilbert's basis theorem for hom-associative Ore extensions, **E** [6])**.** *Let R be a unital, hom-associative ring with twisting map α, σ an automorphism and δ a σ-derivation that both commute with α. Extend α homogeneously to $R[x;\sigma,\delta]$. If R is right (left) Noetherian, then so is $R[x;\sigma,\delta]$.*

Proof. This proof is an adaptation of a proof in [32] to the hom-associative setting. Let us begin with the right case, and therefore assume that R is right Noetherian. We wish to show that any right ideal of $R[x;\sigma,\delta]$ is finitely generated. Since the zero ideal is finitely generated, it is sufficient to show that any non-zero right ideal I of $R[x;\sigma,\delta]$ is finitely generated. Let $J := \{r \in R\colon rx^d + r_{d-1}x^{d-1} + \cdots + r_1x + r_0 \in I, r_{d-1},\ldots,r_0 \in R\}$, i.e. J consists of the zero element and all leading coefficients of polynomials in I. We claim that J is a right ideal of R. First, one readily verifies that J is an additive subgroup of R. Now, let $r \in J$ and $s \in R$ be arbitrary. Then there is some polynomial $p = rx^d +$ [lower order terms] in I. Moreover, $p \cdot \sigma^{-d}(s) = rx^d \cdot \sigma^{-d}(s) +$ [lower order terms] $= \left(r \cdot \sigma^d(\sigma^{-d}(s))\right) x^d +$ [lower order terms] $= (r \cdot s)x^d +$ [lower order terms], which is an element of I since p is. Therefore, $r \cdot s \in J$, so J is a right ideal of R.

Since R is right Noetherian and J is a right ideal of R, J is finitely generated, say by $\{r_1,\ldots,r_k\} \subseteq J$. All the elements $r_1,\ldots,r_k$ are assumed to be non-zero, and moreover, each of them is a leading coefficient of some polynomial $p_i \in I$ of degree n_i. Put $n = \max(n_1,\ldots,n_k)$. Then each r_i is the leading coefficient of

$p_i \cdot x^{n-n_i} = r_i x^{n_i} \cdot x^{n-n_i} + [\text{lower order terms}] = r_i x^n + [\text{lower order terms}]$, which is an element of I of degree n.

Let $N := \sum_{i=0}^{n-1} Rx^i$. Then calculations similar to those in the proof of the third statement of Lemma 34 show that as sets, $N = \sum_{i=0}^{n-1} Rx^i = \sum_{i=0}^{n-1} x^i R$. By Proposition 42, N is then a hom-Noetherian right R-hom-module. Now, since I is a right ideal of the ring $R[x;\sigma,\delta]$ which contains R, in particular, it is also a right R-hom-module. By Proposition 33, $I \cap N$ is then a hom-submodule of N, and since N is a hom-Noetherian right R-hom-module, $I \cap N$ is finitely generated, say by the set $\{q_1, q_2, \ldots, q_l\}$.

Let I_0 be the right ideal of $R[x;\sigma,\delta]$ generated by

$$\left\{p_1 \cdot x^{n-n_1}, p_2 \cdot x^{n-n_2}, \ldots, p_k \cdot x^{n-n_k}, q_1, q_2, \ldots, q_l\right\}.$$

Since all the elements in this set belong to I, we have that $I_0 \subseteq I$. We claim that $I \subseteq I_0$. In order to prove this, pick any element $p' \in I$.

Base case ($\mathsf{P}(n)$): If $\deg p' < n$, $p' \in N = \sum_{i=0}^{n-1} Rx^i$, so $p' \in I \cap N$. On the other hand, the generating set of $I \cap N$ is a subset of the generating set of I_0, so $I \cap N \subseteq I_0$, and therefore $p' \in I_0$.

Induction step ($\forall m \geq n\ (\mathsf{P}(m) \to \mathsf{P}(m+1))$): Assume $\deg p' = m \geq n$ and that I_0 contains all elements of I with $\deg < m$. Does I_0 contain all elements of I with $\deg < m+1$ as well? Let r' be the leading coefficient of p', so that we have $p' = r'x^m + [\text{lower order terms}]$. Since $p' \in I$ by assumption, $r' \in J$. We then claim that $r' = \sum_{i=1}^{k} \sum_{j=1}^{k'} (\cdots((r_i \cdot s_{ij1}) \cdot s_{ij2}) \cdot \cdots) \cdot s_{ijk''}$ for some $k', k'' \in \mathbb{N}_{>0}$ and some $s_{ij1}, s_{ij2}, \ldots, s_{ijk''} \in R$. First, we note that since J is generated by $\{r_1, r_2, \ldots, r_k\}$, it is necessary that J contains all elements of that form. Secondly, we see that subtracting any two such elements or multiplying any such element from the right with one from R again yields such an element, and hence the set of all elements of this form is not only a right ideal containing $\{r_1, r_2, \ldots, r_k\}$, but also the smallest such to do so.

Recall that $p_i \cdot x^{n-n_i} = r_i x^n + [\text{lower order terms}]$, and hence we have $(p_i \cdot x^{n-n_i}) \cdot \sigma^{-n}(s_{ij1}) = (r_i \cdot s_{ij1})x^n + [\text{lower order terms}]$. By iterating this multiplication from the right, we set
$c_{ij} := (\cdots(((p_i \cdot x^{n-n_i}) \cdot \sigma^{-n}(s_{ij1})) \cdot \sigma^{-n}(s_{ij2})) \cdot \cdots) \cdot \sigma^{-n}(s_{ijk''})$. Since $p_i \cdot x^{n-n_i}$ is a generator of I_0, c_{ij} is an element of I_0 as well, and therefore also $q := \sum_{i=1}^{k} \sum_{j=1}^{k'} c_{ij} \cdot x^{m-n} = r'x^m + [\text{lower order terms}]$. However, as $I_0 \subseteq I$, we also have that $q \in I$, and since $p' \in I$, $(p' - q) \in I$. Now, $p' = r'x^m +$

[lower order terms], so $\deg(p' - q) < m$, and therefore $(p' - q) \in I_0$. This shows that $p' = (p' - q) + q$ is an element of I_0 as well, and thus $I = I_0$, which is finitely generated.

For the left case, first note that any hom-associative ring S is right (left) Noetherian if and only if S^{op} is left (right) Noetherian, due to the fact that any right (left) ideal of S is a left (right) ideal of S^{op}, and vice versa. Now, assume that R is left Noetherian. Then, R^{op} is right Noetherian, and using (i) and (ii) in Lemma 34, σ^{-1} is an automorphism and $-\delta \circ \sigma^{-1}$ a σ^{-1}-derivation on R^{op}, and both commute with α. Hence, by the previously proved right case, $R^{\mathrm{op}}[x; \sigma^{-1}, -\delta \circ \sigma^{-1}]$ is right Noetherian. By (iii) in Lemma 34, $R^{\mathrm{op}}[x; \sigma^{-1}, -\delta \circ \sigma^{-1}] \cong R[x; \sigma, \delta]^{\mathrm{op}}$. One readily verifies that surjective homomorphisms between hom-associative rings preserve the Noetherian conditions (HNR1), (HNR2), and (HNR3) in Corollary 19 by examining the proof of Proposition 39, changing the module homomorphism to that between rings instead, and "submodule" to "ideal". Therefore, $R[x; \sigma, \delta]^{\mathrm{op}}$ is right Noetherian, so $R[x; \sigma, \delta]$ is left Noetherian. □

Remark 19 (**E** [6]). By putting $\alpha = \mathrm{id}_R$ in Theorem 8, we recover the classical Hilbert's basis theorem for associative Ore extensions.

Corollary 21 (Hilbert's basis theorem for non-associative Ore extensions, **E** [6]). *Let R be a unital, non-associative ring, σ an automorphism and δ a σ-derivation on R. If R is right (left) Noetherian, then so is $R[x; \sigma, \delta]$.*

Proof. Put $\alpha = 0$ in Theorem 8. □

4.3 Examples

Here we provide some examples of unital, non-associative and hom-associative Ore extensions which are all Noetherian by the above theorem and its corollary. First, note that any unital division algebra D is Noetherian: let I be any non-zero right ideal of D. If $a \in D$ is an arbitrary non-zero element, then $1_D = a \cdot a^{-1} \in I$, so $I = D$, and analogously for the left case. As an ideal of itself, D is finitely generated (by 1_D, for instance), as is the zero ideal. The derivations on any normed division algebra D is a linear combination of derivations $\delta_{a,b}$ where $a, b \in D$, defined by $\delta_{a,b}(c) := [[a, b], c] - 3(a, b, c)$ for all $c \in D$ [71]. These derivations are called *inner*, and in particular, all derivations on $\mathbb{H}$ are of the form $[a, \cdot]$ for some $a \in \mathbb{H}$.

Given a unital, associative algebra A with product $\cdot$ over a field of characteristic different from two, one may define a unital, non-associative algebra A^+ by using the *Jordan product* $\{\cdot,\cdot\}\colon A^+ \to A^+$. This is given by $\{a,b\} := \frac{1}{2}(a\cdot b + b\cdot a)$ for any $a, b \in A$. A^+ is then a *Jordan algebra*, that is, a commutative algebra where any two elements a and b satisfy the *Jordan identity*, given by $\{\{a,b\}\{a,a\}\} = \{a,\{b,\{a,a\}\}\}$. Since inverses on A extend to inverses on A^+, one may infer that if $A = \mathbb{H}$, then A^+ is also Noetherian. Using the standard notation i, j, k for the *quaternion units* in $\mathbb{H}$ with defining relations $i^2 = j^2 = k^2 = ijk = -1$, one can see that $\mathbb{H}^+$ is not associative as e.g. $(i,i,j)_{\mathbb{H}^+} := \{\{i,i\},j\} - \{i,\{i,j\}\} = -j$.

Example 28 (**E** [6])**.** Let σ be the automorphism on $\mathbb{H}$ defined by $\sigma(i) = -i$, $\sigma(j) = k$, and $\sigma(k) = j$. Any automorphism on $\mathbb{H}$ is also an automorphism on $\mathbb{H}^+$, and hence $\mathbb{H}^+[x;\sigma,0]$ is a unital, non-associative skew polynomial ring. $\mathbb{H}^+[x;\sigma,0]$ is then Noetherian by Corollary 21.

Example 29 (**E** [6])**.** Let $[j,\cdot]_{\mathbb{H}}$ be the inner derivation on $\mathbb{H}$ induced by j. Any derivation on $\mathbb{H}$ is also a derivation on $\mathbb{H}^+$, and so we may form the unital, non-associative differential polynomial ring $\mathbb{H}^+[x;\mathrm{id}_{\mathbb{H}},[j,\cdot]_{\mathbb{H}}]$ which is Noetherian by Corollary 21.

Example 30 (**E** [6])**.** From the Jordan identity one may infer that a map $\delta_{a,c}\colon J \to J$ defined by $\delta_{a,c}(b) := (a,b,c)_J$ for any $a,b,c \in J$ where J is a Jordan algebra, is a derivation, called an inner derivation. On $\mathbb{H}^+$ one could for instance take $a = i$ and $c = j$, resulting in $\delta_{i,j}(b) = \{\{i,b\},j\} - \{i,\{b,j\}\}$ for any $b \in \mathbb{H}^+$. Then $\mathbb{H}^+[x;\mathrm{id}_{\mathbb{H}},\delta_{i,j}]$ is a unital, non-associative differential polynomial ring which is Noetherian by Corollary 21.

Example 31 (**E** [6])**.** Take any derivation on $\mathbb{O}$, e.g. $\delta_{i,j}$ defined by $\delta_{i,j}(c) := [[i,j],c] - 3(i,j,c)$ for any $c \in \mathbb{O}$. Then $\mathbb{O}[x;\mathrm{id}_{\mathbb{O}},\delta_{i,j}]$ is a unital, non-associative Ore extension which is Noetherian by Corollary 21.

Example 32 (**E** [6])**.** (The following example is only present in the `arXiv` version of Paper **E**.) One may define an *octonionic Weyl algebra* $A_1(\mathbb{O})$ as the tensor product of the (associative) first Weyl algebra $A_1(\mathbb{R})$ over $\mathbb{R}$, and $\mathbb{O}$. $A_1(\mathbb{O})$ is then a free module of finite rank over $A_1(\mathbb{R})$ (see e.g. Example 33 here below), and hence it is Noetherian. One may also define an octonionic Weyl algebra as an iterated Ore extension of $\mathbb{O}$. Using this latter approach, let us first mention that for any unital, non-associative ring R, the *non-associative Weyl algebra over R* was introduced in

[66] as the iterated, unital, non-associative Ore extension $R[y][x; \mathrm{id}_R, \delta]$ where $\delta\colon R[y] \to R[y]$ is an R-linear map such that $\delta(1_{R[y]}) = 0$. Now, let $\delta\colon \mathbb{O}[y] \to \mathbb{O}[y]$ be the $\mathbb{O}$-linear map defined on monomials by $\delta(aY^m) = maY^{m-1}$ for arbitrary $a \in \mathbb{O}$ and $m \in \mathbb{N}$, with the interpretation that $0aY^{-1}$ is 0. One readily verifies that δ is an $\mathbb{O}$-linear derivation on $\mathbb{O}[y]$, and by Remark 6 in Chapter 1, $\delta(1_{\mathbb{O}[y]}) = 0$. We define an octonionic Weyl algebra $\mathbb{O}[y][x; \mathrm{id}_{\mathbb{O}[y]}, \delta]$, where δ is the aforementioned derivation. By using Corollary 21 twice, $\mathbb{O}[y][x; \mathrm{id}_{\mathbb{O}[y]}, \delta]$ is Noetherian. Moreover, $\mathbb{O}[y][x; \mathrm{id}_{\mathbb{O}[y]}, \delta] \cong A_1(\mathbb{O})$ (see e.g. Example 33 here below).

Example 33 (**E** [6]). For any $q \in \mathbb{R}\backslash\{0,1\}$, one may define an *octonionic q-Weyl algebra* $A_q(\mathbb{O})$ as the tensor product of the usual q-Weyl algebra $A_q(\mathbb{R})$ over $\mathbb{R}$, and $\mathbb{O}$. In particular, $A_q(\mathbb{O})$ is a free module of finite rank over $A_q(\mathbb{R})$, and hence it is Noetherian. Alternatively, one may see that $A_q(\mathbb{O})$ is Noetherian by noting that it may be constructed as an iterated Ore extension as follows. First, by using the fact that $Z(\mathbb{O}) = \mathbb{R}$, we may define an $\mathbb{O}$-automorphism on $\mathbb{O}[y]$ by $\sigma(y) = qy$ for some $q \in \mathbb{R}\backslash\{0,1\}$. By Proposition 41, $y^k \in N(\mathbb{O}[y])$ for any $k \in \mathbb{N}$, and so $y^k \in Z(\mathbb{O}[y])$. From this we may infer that for each polynomial $p(y) \in \mathbb{O}[y]$, there is a unique polynomial $r(y) \in \mathbb{O}[y]$ such that $p(qy) - p(y) = (qy - y)r(y) = r(y)(qy - y)$. It thus makes sense to define a map δ on $\mathbb{O}[y]$ by

$$\delta(p(y)) := \frac{p(qy) - p(y)}{qy - y} = \frac{\sigma(p(y)) - p(y)}{\sigma(y) - y},$$

the *q-derivative*. By a straightforward calculation, δ is an $\mathbb{O}$-linear σ-derivation. By using Corollary 21 twice, $\mathbb{O}[y][x; \sigma, \delta]$ is Noetherian. Moreover, $\mathbb{O}[y][x; \sigma, \delta] \cong A_q(\mathbb{O})$.

We here include a proof of the statement that $A_q(\mathbb{O})$ is a free module of finite rank over $A_q(\mathbb{R})$, and that $\mathbb{O}[y][x; \sigma, \delta] \cong A_q(\mathbb{O})$. We begin with the latter statement. As a vector space over $\mathbb{R}$, $A_q(\mathbb{R})$ has a basis $\{Y^m X^n : m, n \in \mathbb{N}\}$. $\mathbb{O}$ is also an $\mathbb{R}$-vector space, with basis $B := \{1, i, j, k, l, li, lj, lk\}$. Hence the $\mathbb{R}$-algebra $\mathbb{O} \otimes_{\mathbb{R}} A_q(\mathbb{R})$ has a basis, as an $\mathbb{R}$-vector space, equal to $\{b \otimes Y^m X^n : b \in B, m, n \in \mathbb{N}\}$. We want to define an $\mathbb{R}$-algebra isomorphism $f\colon \mathbb{O} \otimes_{\mathbb{R}} A_q(\mathbb{R}) \to \mathbb{O}[y][x; \sigma, \delta]$. We define f on an arbitrary basis element as $f(b \otimes Y^m X^n) = by^m x^n$, which makes f bijective. Moreover, all $b' \in B$ commute with y^m and x^n for any $m, n \in \mathbb{N}$, and y^m and x^n in turn associate with all elements in $\mathbb{O}[y][x; \sigma, \delta]$. Therefore, we have that $f((b \otimes Y^m X^n) \cdot (b' \otimes Y^{m'} X^{n'})) =$

$f(bb' \otimes Y^m X^n Y^{m'} X^{n'}) = (bb')y^m x^n y^{m'} x^{n'} = by^m x^n b' y^{m'} x^{n'} = f(b \otimes Y^m X^n) \cdot f(b' \otimes Y^{m'} X^{n'})$ for any $m, n, m', n' \in \mathbb{N}$ and $b, b' \in B$. Hence f is an $\mathbb{R}$-automorphism. We now turn $\mathbb{O} \otimes_{\mathbb{R}} A_q(\mathbb{R})$ into an $A_q(\mathbb{R})$-module by defining $a \cdot (b \otimes_{\mathbb{R}} Y^m X^n) = b \otimes_{\mathbb{R}} aY^m X^n$ and $(b \otimes_{\mathbb{R}} Y^m X^n) \cdot a = b \otimes_{\mathbb{R}} Y^m X^n a$ on arbitrary basis elements and then extend linearly to all elements. This is now a free $A_q(\mathbb{R})$-module of rank eight with basis $\{1 \otimes_{\mathbb{R}} 1, i \otimes_{\mathbb{R}} 1, j \otimes_{\mathbb{R}} 1, \ldots, lk \otimes_{\mathbb{R}} 1\}$.

Example 34 (**E** [6])**.** This example is a slight generalization of Example 1.1 in [29]. Let R and S be unital, associative, commutative rings, and $f\colon R \to S$ a homomorphism. Further assume that R is Noetherian. Let A be a non-unital, non-associative, Noetherian S-algebra, and define a multiplication $\cdot$ on $U := A \times R$ by $(a_1, r_1) \cdot (a_2, r_2) := (f(r_1)a_2 + f(r_2)a_1 + a_1 a_2, r_1 r_2)$ for any $r_1, r_2 \in R$ and $a_1, a_2 \in A$. U is then unital with identity element $(0, 1_R)$, and by defining a twisting map α on U by $\alpha(a, r) := (0, pr)$ for any $r \in R$, $a \in A$, and $p \in \ker f$, U is hom-associative. Moreover, U is Noetherian, and if A is not associative, then U is not associative. Now, let σ_A be an automorphism on A. Then σ defined by $\sigma(a, r) := (\sigma_A(a), r)$ is an automorphism on U. Moreover, if δ_A is a σ_A-derivation on A, then δ defined by $\delta(a, r) := (\delta_A(a), 0)$ is a σ-derivation on U, and both δ and σ commute with α. Hence, by Theorem 8, $U[x; \sigma, \delta]$ is Noetherian. Here, one could e.g. take $R = \mathbb{R}[y]$, $S = \mathbb{R}$, $f\colon \mathbb{R}[y] \to \mathbb{R}$ the evaluation homomorphism at zero, $p \in \mathbb{R}[y]$ any polynomial without a constant term, and A, σ_A, and δ_A any $\mathbb{R}$-algebra, σ, and δ, respectively, from the previous examples.

We here include a proof that U is Noetherian. Suppose we have an ascending chain of right (left) ideals, $I_1 \subseteq I_2 \subseteq \ldots$, in U. Define $J_j = \{r \in R \mid \exists a \in A : (a, r) \in I_j\}$. This is an ideal in R. Also define $H_j = \{a \in A \mid (a, 0) \in I_j\}$. This is a right (left) ideal in A. We thus have two ascending chains, $J_1 \subseteq J_2 \subseteq \ldots$ and $H_1 \subseteq H_2 \subseteq \ldots$, in R and A, respectively. Since R and A are Noetherian there is some integer n such that if $k > n$ then $J_k = J_n$ and $H_k = H_n$. We claim that in fact also $I_k = I_n$. Let $(a, r) \in I_k$. Then $r \in J_k = J_n$ so there is $a' \in A$ such that $(a', r) \in I_n$. It follows that $a - a' \in H_k = H_n$, which implies $(a - a', 0) \in I_n$. Hence $(a, r) = (a', r) + (a - a', 0)$ is a sum of two elements in I_n and therefore belongs to I_n.

Example 35 (**E** [6])**.** Let R be a unital, non-associative, Noetherian ring, and denote by I the ideal of R generated by all expressions of the form $(rs)t - r(st)$ where $r, s, t \in R$. Define $S := R/I$ and let $\pi\colon R \to S$ be the natural homomorphism. Set $U := S \times R$ and define a multiplication $\cdot$ on U by $(s_1, r_1) \cdot (s_2, r_2) :=$

$(\pi(r_1)s_2 + s_1\pi(r_2) + s_1s_2, r_1r_2)$ for all $r_1, r_2 \in R$ and $s_1, s_2 \in R$. U is unital with identity element $(0, 1_R)$, and the map α defined by $\alpha(s,r) := (\pi(r)+s, 0)$ for all $r \in R$ and $s \in S$ is a well-defined twisting map that makes U hom-associative. Since R is Noetherian, so is S, and by the same argument as in Example 34, U is Noetherian. Moreover, if R is not associative, then U is not associative. Now, let σ_R be an endomorphism on R. Then $\sigma_R(I) \subseteq I$, which guarantees the naturally extended endomorphism σ_S on S to be well-defined. By defining $\sigma(s,r) := (\sigma_S(s), \sigma_R(r))$, we get an endomorphism σ on U. Similarly, any σ_R-derivation δ_R satisfies $\delta_R(I) \subseteq I$, and hence the naturally extended σ_S-derivation δ_S is well-defined, and in turn gives rise to a σ-derivation δ on U defined by $\delta(s,r) := (\delta_S(s), \delta_R(r))$. Now, assume that σ_R is an automorphism. Then it is clear that σ_S is surjective. Moreover, $\sigma_R^{-1}(I) \subseteq I$, which in turn implies that σ_S is injective. Hence σ is an automorphism. Moreover, α commutes with both δ and σ, and therefore $U[x;\sigma,\delta]$ is Noetherian. Here, one could e.g. take R to be any base ring together with σ and δ from the previous examples.

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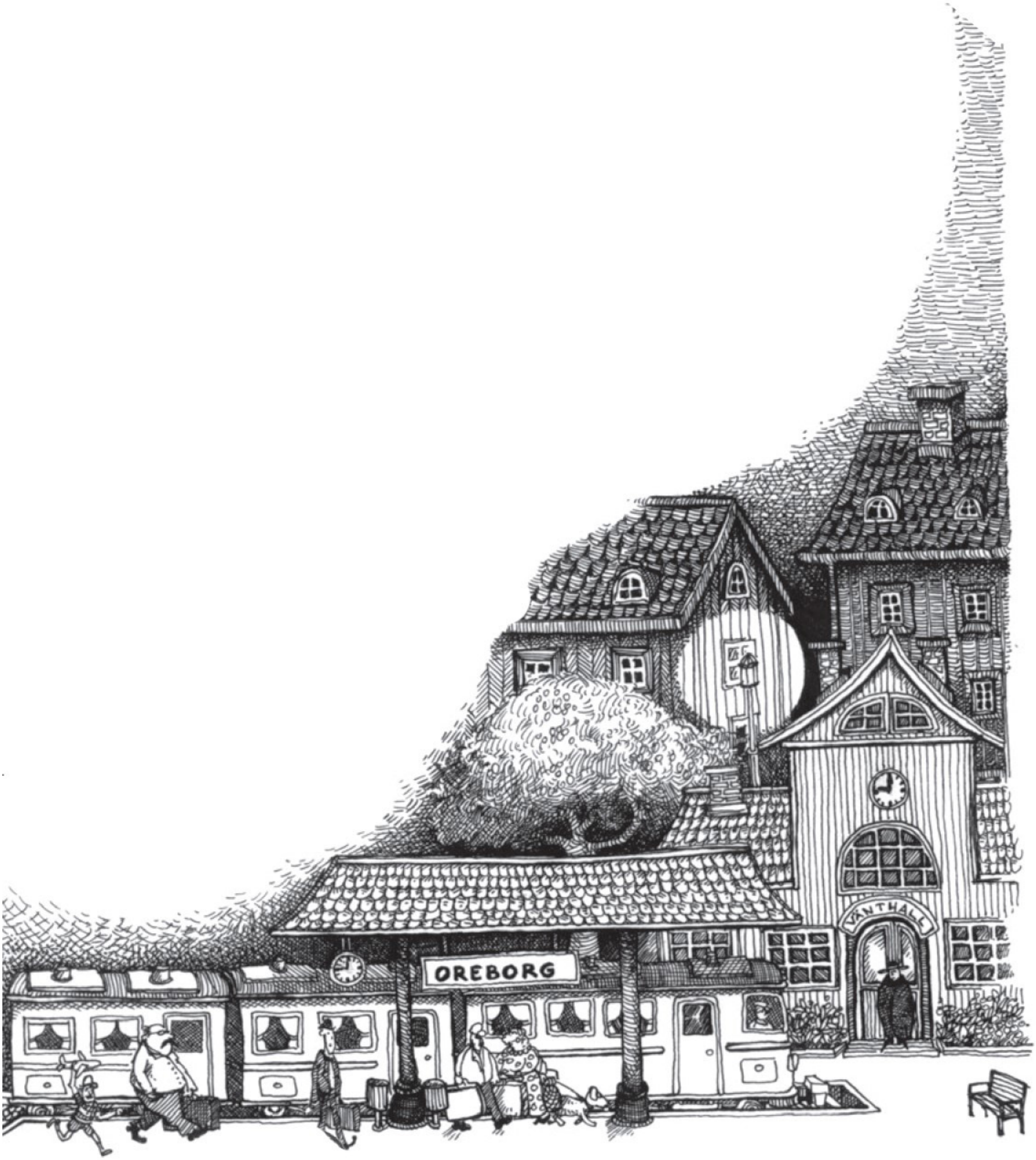

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