FUNCTIONAL LIMIT THEOREMS FOR MULTIPARAMETER FRACTIONAL BROWNIAN MOTION

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Abstract. We prove a general functional limit theorem for multiparameter fractional Brownian motion. The functional law of the iterated logarithm, functional Lévy's modulus of continuity and many other results are its particular cases. Applications to approximation theory are discussed.

1. Introduction

Let $B(t)=B(t,\omega)$, $t\geq 0$, $\omega\in\Omega$ be the Brownian motion on the probability space $(\Omega,\mathfrak{F},\mathsf{P})$. The *law of the iterated logarithm* (Khintchin, 1924) states that

$$\mathsf{P}\left\{\omega\colon \limsup_{t\to\infty}\frac{B(t,\omega)}{\sqrt{2t\log\log t}}=1\right\}=1.$$

We abbreviate this as

$$\limsup_{t\to\infty}\frac{B(t)}{\sqrt{2t\log\log t}}=1\quad \mathsf{P}-\text{a. s.},\tag{1}$$

where a. s. stands for almost surely.

The functional counterpart to the law of the iterated logarithm was discovered by (Strassen, 1964). Let $C[0,1]$ be the Banach space of all continuous functions $f\colon[0,1]\mapsto\mathbb{R}$ with the uniform topology generated by the maximum norm

$$\|f\|_\infty=\max_{t\in[0,1]}|f(t)|.$$

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Let $\mathcal{H}_B$ be the Hilbert space of all absolutely continuous functions $f\colon[0,1]\mapsto\mathbb{R}$ with $f(0)=0$ and finite *Strassen's norm*

$$\|f\|_S=\left(\int_0^1(f'(t))^2\,dt\right)^{1/2}.$$

The centred unit ball $\mathcal{K}_B$ of the space $\mathcal{H}_B$

$$\mathcal{K}_B=\{\,f\in\mathcal{H}_B\colon\|f\|_S\leq 1\,\}$$

is called *Strassen's ball.* Define

$$\mathcal{S}=\left\{\eta_u(t)=\frac{B(ut)}{\sqrt{2u\log\log u}}\colon u>e\right\}\subset C[0,1].$$

The *functional law of the iterated logarithm* states that, in the uniform topology, the set of P-a. s. limit points of $\mathcal{S}$ as $u\to\infty$ is Strassen's ball $\mathcal{K}_B$. It follows that for any continuous functional $F\colon C[0,1]\mapsto\mathbb{R}$

$$\limsup_{u\to\infty}F(\eta_u(t))=\sup_{f\in\mathcal{K}_B}F(f)\quad\mathsf{P}-\text{a. s.}\tag{2}$$

In particular, for $F(f)=f(1)$ the supremum $\sup_{f\in\mathcal{K}_B}f(1)$ is equal to 1 and attained on the function $f(t)=t$. Therefore (2) transforms into (1).

Another interesting ordinary limit theorem is *Lévy's modulus of continuity* (Lévy, 1937). It states that

$$\limsup_{u\downarrow 0}\sup_{t\in[0,1]}\frac{|B(t+u)-B(t)|}{\sqrt{2u\log u^{-1}}}=1\quad\mathsf{P}-\text{a. s.}\tag{3}$$

The corresponding functional counterpart was discovered by (Mueller, 1981). Define

$$\mathcal{S}(u)=\left\{\eta_s(t)=\frac{B(s+ut)-B(s)}{\sqrt{2u\log u^{-1}}}\colon 0\leq s\leq 1-u\right\}\subset C[0,1].$$

Then, in the uniform topology, the set of P-a. s. limit points of $\mathcal{S}(u)$ as $u\downarrow 0$ is Strassen's ball $\mathcal{K}_B$. Lévy's modulus of continuity (3) follows from its functional counterpart in the same way as the law of the iterated logarithm (1) follows from Strassen's law.

In fact, (Mueller, 1981) contains a general functional limit theorem that includes the functional law of the iterated logaritm, the functional Lévy's modulus of continuity and many other results as particular cases. Our aim is to prove the analogue of the results of (Mueller, 1981) for the *multiparameter fractional Brownian motion.* This is the separable centred Gaussian random field $\xi(\mathbf{x})$ on the space $\mathbb{R}^N$ with the covariance function

$$R(\mathbf{x},\mathbf{y}) = \mathsf{E}\xi(\mathbf{x})\xi(\mathbf{y}) = \frac{1}{2}(\|\mathbf{x}\|^{2H} + \|\mathbf{y}\|^{2H} - \|\mathbf{x}-\mathbf{y}\|^{2H}), \tag{4}$$

where $\|\cdot\|$ denotes the usual Euclidean norm on the space $\mathbb{R}^N$. The parameter $H \in (0,1)$ is called the *Hurst parameter*. In particular, for $N = 1$ and $H = 1/2$ the multiparameter fractional Brownian motion becomes

$$\xi(t) = \begin{cases} B_1(t), & t \geq 0, \\ B_2(t), & t < 0, \end{cases}$$

where $B_1(t)$ and $B_2(t)$ are two independent copies of the Brownian motion.

In Section 2 we formulate our results. They are proved in Section 3. Examples and applications are discussed in Section 4.

2. Formulation of results

In what follows, we write $\xi_1(x) \stackrel{d}{=} \xi_2(x)$, if two random functions $\xi_1(x)$ and $\xi_2(x)$ are defined on the same space X and have the same finite-dimensional distributions. We denote by $O(N)$ the group of all orthogonal matrices on the space $\mathbb{R}^N$.

Lemma 1. *The multiparameter fractional Brownian motion has the next properties.*

(1) *It has homogeneous increments, i.e., for any* $\mathbf{y} \in \mathbb{R}^N$

$$\xi(\mathbf{x}+\mathbf{y}) - \xi(\mathbf{y}) \stackrel{d}{=} \xi(\mathbf{x}), \tag{5}$$

(2) *It is self-similar, i.e., for any* $u \in \mathbb{R}$

$$\xi(u\mathbf{x}) \stackrel{d}{=} u^H \xi(\mathbf{x}). \tag{6}$$

(3) *It is isotropic, i.e., for any* $g \in O(N)$

$$\xi(g\mathbf{x}) \stackrel{d}{=} \xi(\mathbf{x}). \tag{7}$$

Property 3 prompts us to use the $O(N)$-invariant closed unit ball of the space $\mathbb{R}^N$:

$$\mathcal{B} = \{\, \mathbf{x} \in \mathbb{R}^N \colon \|\mathbf{x}\| \leq 1 \,\}$$

(but *not* the cube $[0,1]^N$, which is not $O(N)$-invariant!) in the case of the multiparameter fractional Brownian motion instead of the interval $[0,1]$ in the case of the ordinary Brownian motion.

First of all, we need to describe Hilbert space $\mathcal{H}_\xi$ and its closed unit ball $\mathcal{K}_\xi$. In the case of the Brownian motion, the space $\mathcal{H}_B$ can be characterised as the reproducing kernel Hilbert space for the Brownian motion, or as the set of all admissible shifts of the Gaussian measure μ_B on the space $C[0,1]$ that corresponds to the Brownian motion, or as the kernel of the measure μ_B (Lifshits, 1995). In order to describe $\mathcal{H}_\xi$ we need to introduce some notations.

Let $r, \varphi, \vartheta_1, \dots, \vartheta_{N-2}$ be the spherical coordinates in $\mathcal{B}$. The set of *spherical harmonics* $S^l_m(\varphi, \vartheta_1, \dots, \vartheta_{N-2})$ forms the orthonormal basis in the Hilbert space $L^2(S^{N-1}, dS)$ of all square integrable functions on the unit sphere S^{N-1} with respect to the Lebesgue measure

$$dS = \sin\vartheta_1 \sin^2\vartheta_2 \dots \sin^{N-2}\vartheta_{N-2}\, d\varphi\, d\vartheta_1\, d\vartheta_2 \dots\, d\vartheta_{N-2}.$$

Here $m \geq 0$ and $1 \leq l \leq h(m, N)$, where

$$h(m,N) = \frac{(2m+N-2)(m+N-3)!}{(N-2)!m!}.$$

Let δ^k_j denotes the Kronecker's symbol. Let ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ denotes the hypergeometric function. Let

$$\lambda_{m1} \geq \lambda_{m2} \geq \dots \geq \lambda_{mn} \geq \dots > 0$$

be the sequence of all eigenvalues (with multiplicities) of the positive definite kernel

(8)
$$\begin{aligned} b_m(r,s) = \frac{\pi^{N/2}}{\Gamma(N/2+m)} \Bigg[&(r^{2H} + s^{2H})\delta^0_m - \frac{\Gamma(m-H)}{\Gamma(-H)}(rs)^m (r+s)^{2(H-m)} \\ &\times {}_2F_1\left(m + (N-2)/2, m-H; 2m+N-1; \frac{4rs}{(r+s)^2}\right)\Bigg] \end{aligned}$$

in the Hilbert space $L^2([0,1], dr)$. Let $\psi_{mn}(r)$ be the eigenbasis of the kernel $b_m(r,s)$. The set

(9) $$\{\, \psi_{mn}(r) S^l_m(\varphi, \vartheta_1, \dots, \vartheta_{N-2}) \colon m \geq 0, n \geq 1, 1 \leq l \leq h(m,N) \,\}$$

forms the orthonormal basis in the Hilbert space $L^2(\mathcal{B}, dr\, dS)$. For any $f \in C(\mathcal{B})$, let f^l_{mn} be the Fourier coefficients of f with respect to the basis (9):

$$f^l_{mn} = \int_{S^{N-1}} \int_0^1 f(r, \varphi, \vartheta_1, \dots, \vartheta_{N-2}) \psi_{mn}(r) S^l_m(\varphi, \vartheta_1, \dots, \vartheta_{N-2})\, dr\, dS.$$

Lemma 2. *The reproducing kernel Hilbert space $\mathcal{H}_\xi$ of the multiparameter fractional Brownian motion $\xi(\mathbf{x})$, $\mathbf{x} \in \mathcal{B}$ consists of all functions $f \in C(\mathcal{B})$ with $f(\mathbf{0}) = 0$ that satisfy the condition*

$$\|f\|^2_S = \sum_{m=0}^\infty \sum_{n=1}^\infty \sum_{l=1}^{h(m,N)} \frac{(f^l_{mn})^2}{\lambda_{mn}} < \infty.$$

The scalar product in the space $\mathcal{H}_\xi$ is defined as

$$(f,g)_S=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{h(m,N)}\frac{f^l_{mn}g^l_{mn}}{\lambda_{mn}}.$$

Therefore Strassen's ball is described as

$$\mathcal{K}_\xi=\{\,f\in\mathcal{H}_\xi\colon\|f\|^2_S\le 1\,\}.$$

In what follows we write $\mathcal{K}$ instead of $\mathcal{K}_\xi$.

Let t_0 be a real number. Let for every $t\ge t_0$ there exists a non-empty set of indices $\mathcal{J}(t)$. Let every element $j\in\mathcal{J}(t)$ defines the vector $\mathbf{y}_j\in\mathbb{R}^N$ and the positive real number u_j. Let $R_r(\mathbf{y}_j,u_j)$ be the cylinder

$$R_r(\mathbf{y}_j,u_j)=\{\,(\mathbf{y},u)\colon\|\mathbf{y}-\mathbf{y}_j\|\le ru_j,e^{-r}u_j\le u\le e^ru_j\,\},\qquad r>0.$$

Now we define the function $F_r(t)$. In words: this is the volume of the union of all cylinders $R_r(\mathbf{y}_j,u_j)$ that are defined before the moment t, with respect to the measure $u^{-N-1}\,d\mathbf{y}\,du$. Formally,

$$F_r(t)=\int_{\cup_{t_0\le v\le t}\cup_{j\in\mathcal{J}(v)}R_r(\mathbf{y}_j,u_j)}u^{-N-1}\,d\mathbf{y}\,du.$$

Finally, we define

$$\mathcal{P}(t)=\{\,(\mathbf{y}_j,u_j)\colon j\in\mathcal{J}(t)\,\}$$

and the *cloud of normed increments*

$$\mathcal{S}(t)=\left\{\eta(\mathbf{x})=\frac{\xi(\mathbf{y}+u\mathbf{x})-\xi(\mathbf{y})}{\sqrt{2h(t)}u^H}\colon(\mathbf{y},u)\in\mathcal{P}(t)\right\}\subset C(\mathcal{B}).$$

Theorem 1. *Let the function $h(t)\colon[t_0,\infty)\mapsto\mathbb{R}$ satisfies the next conditions:*

(1) *$h(t)$ is increasing and $\lim_{t\to\infty}h(t)=\infty$.*
(2) *The integral $\int_{t_0}^{\infty}e^{-ah(t)}\,dF_1(t)$ converges for $a>1$ and diverges for $a<1$.*

Then, in the uniform topology, the set of P-a. s. limit points of the cloud of increments $\mathcal{S}(t)$ as $t\to\infty$ is Strassen's ball $\mathcal{K}$.

3. Proofs

3.1. **Proof of Lemmas 1 and 2.**

Proof of Lemma 1. It is enough to calculate the covariance functions of both hand sides in equations (5)–(7). Calculations are straightforward. □

Proof of Lemma 2. The covariance function (4) can be written as function of three variables:

$$R(r,s,t)=\frac{1}{2}(r^{2H}+s^{2H}-(r^2+s^2-2rst)^H),\qquad r\geq 0, s\geq 0, -1\leq t\leq 1, \tag{10}$$

where $r=\|\mathbf{x}\|$, $s=\|\mathbf{y}\|$, and t is the cosine of the angle between vectors $\mathbf{x}$ and $\mathbf{y}$. According to the general theory of isotropic random fields (Yadrenko, 1983), the multiparameter fractional Brownian motion can be written as

$$\xi(r,\varphi,\vartheta_1,\dots,\vartheta_{N-2})=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,N)}\xi^l_m(r)S^l_m(\varphi,\vartheta_1,\dots,\vartheta_{N-2}), \tag{11}$$

where $\xi^l_m(r)$ is the sequence of independent centred Gaussian processes on $[0,\infty)$ with the covariance functions

$$\mathsf{E}\xi^l_m(r)\xi^l_m(s)=\frac{2\pi^{(N-1)/2}}{\Gamma((N-1)/2)C^{(N-2)/2}_m(1)}\int_{-1}^{1}R(r,s,t)C^{(N-2)/2}_m(t)(1-t^2)^{(N-3)/2}\,dt, \tag{12}$$

and $C^{(N-2)/2}_m(t)$ are Gegenbauer's polynomials. Denote by $b_m(r,s)$ the covariance function (12). We will prove that $b_m(r,s)$ is expressed as (8).

It follows from (10) and (12) that the covariance function $b_m(r,s)$ can be written as the difference of two integrals:

$$\begin{aligned}b_m(r,s)=&\frac{\pi^{(N-1)/2}}{\Gamma((N-1)/2)C^{(N-2)/2}_m(1)}(r^{2H}+s^{2H})\int_{-1}^{1}C^{(N-2)/2}_m(t)(1-t^2)^{(N-3)/2}\,dt\\&-\frac{\pi^{(N-1)/2}}{\Gamma((N-1)/2)C^{(N-2)/2}_m(1)}\int_{-1}^{1}(r^2+s^2-2rst)^HC^{(N-2)/2}_m(t)(1-t^2)^{(N-3)/2}\,dt.\end{aligned} \tag{13}$$

The first integral is non-zero if and only if $m=0$ (Vilenkin, 1968). It follows that the first term in (13) is equal to

$$\begin{aligned}&\delta^m_0\frac{\pi^{(N-1)/2}}{\Gamma((N-1)/2)}(r^{2H}+s^{2H})\int_{-1}^{1}\frac{C^{(N-2)/2}_0(t)}{C^{(N-2)/2}_0(1)}(1-t^2)^{(N-3)/2}\,dt\\&=\frac{\delta^m_0\pi^{(N-1)/2}}{\Gamma((N-1)/2)}(r^{2H}+s^{2H})\int_{-1}^{1}(1-t^2)^{(N-3)/2}\,dt\\&=\frac{\delta^m_0\pi^{N/2}}{\Gamma(N/2)}(r^{2H}+s^{2H}).\end{aligned}$$

Here we used formula 2.2.3.1 from (Prudnikov et al., 1986).

Rewrite the second term as

$$-\frac{\pi^{(N-1)/2}m!(N-3)!}{\Gamma((N-1)/2)(m+N-3)!}(2rs)^H\lim_{\alpha\to(N-1)/2}\int_{-1}^{1}\left(\frac{r^2+s^2}{2rs}-t\right)^H\times$$
$$C_m^{(N-2)/2}(t)(1+t)^{\alpha-1}(1-t)^{(N-3)/2}\,dt.$$

Using formula 2.21.4.15 from (Prudnikov et al., 1988), we can express this limit as

$$-\frac{(-1)^m 2^{N-2}\pi^{(N-1)/2}\Gamma((N-1)/2)(m-1)!(r+s)^{2H}}{(m+N-2)!}\lim_{\alpha\to(N-1)/2}\frac{\Gamma(\alpha-(N-3)/2-m)}{\Gamma((N-1)/2-\alpha)}$$
$$\times\lim_{\alpha\to(N-1)/2}\frac{{}_3F_2\left(\frac{N-1}{2},-H,1;\alpha-\frac{N-3}{2}-m,m+N-1;\frac{4rs}{(r+s)^2}\right)}{\Gamma(\alpha-\frac{N-3}{2}-m)}.$$

The first limit is calculated as

$$\begin{aligned}\lim_{\alpha\to(N-1)/2}\frac{\Gamma(\alpha-(N-3)/2-m)}{\Gamma((N-1)/2-\alpha)}&=\lim_{\beta\to0}\frac{\Gamma(-\beta-m+1)}{\Gamma(\beta)}\\&=\lim_{\beta\to0}\frac{(-1)^{m-1}\Gamma(-\beta)}{(1+\beta)(2+\beta)\dots(m-1+\beta)\Gamma(\beta)}\\&=\frac{(-1)^{m-1}}{(m-1)!}\lim_{\beta\to0}\frac{\Gamma(-\beta)}{\Gamma(\beta)}\\&=\frac{(-1)^{m-1}}{(m-1)!}.\end{aligned}$$

For the second limit we use formula 7.2.3.6 from (Prudnikov et al., 1990):

$$\lim_{\alpha\to(N-1)/2}\frac{{}_3F_2\left(\frac{N-1}{2},-H,1;\alpha-\frac{N-3}{2}-m,m+N-1;\frac{4rs}{(r+s)^2}\right)}{\Gamma(\alpha-\frac{N-3}{2}-m)}$$
$$=\frac{(4rs)^m\Gamma((N-1)/2+m)\Gamma(m-H)(m+N-2)!}{(r+s)^{2m}\Gamma((N-1)/2)\Gamma(-H)(2m+N-2)!}$$
$$\times{}_2F_1\left(m+(N-2)/2,m-H;2m+N-1;\frac{4rs}{(r+s)^2}\right).$$

Collecting all terms together, we obtain (8).

By Mercer's theorem, function $b_m(r,s)$ may be written as uniformly and absolutely convergent series

$$b_m(r,s)=\sum_{n=1}^{\infty}\lambda_{mn}\psi_n(r)\psi_n(s),\qquad r,s\in[0,1].$$

It follows that the random process $\xi^l_m(r)$ has the form

$$\xi^l_m(r)=\sum_{n=1}^\infty \sqrt{\lambda_{mn}}\xi^l_{mn}\psi_n(r),\qquad r\in[0,1],$$

where ξ^l_{mn} are independent standard normal random variables. Substituting this representation to (11), we obtain:

(14)

$$\xi(r,\varphi,\vartheta_1,\dots,\vartheta_{N-2})=\sum_{m=0}^\infty\sum_{l=1}^{h(m,N)}\sum_{n=1}^\infty\sqrt{\lambda_{mn}}\xi^l_{mn}\psi_n(r)S^l_m(\varphi,\vartheta_1,\dots,\vartheta_{N-2}).$$

We call (14) the *local spectral representation* of the multiparameter fractional Brownian motion, because it is valid only for $r\in[0,1]$, i.e., in $\mathcal{B}$.

Now Lemma 2 follows from (14) and the general theory of Gaussian measures (Lifshits, 1995). □

3.2. **Asymptotic relations that are equivalent to Theorem 1.** In this subsection, we formulate two asymptotic relations and prove that they are equivalent to Theorem 1.

Lemma 3. *The cloud of increments $\mathcal{S}(t)$ is P-a. s. almost inside $\mathcal{K}$, i.e.,*

$$\lim_{t\to\infty}\sup_{\eta\in\mathcal{S}(t)}\inf_{f\in\mathcal{K}}\|\eta-f\|_\infty=0\quad \mathsf{P}-a.\ s. \tag{15}$$

Lemma 4. *Any neighbourhood of any element $f\in\mathcal{K}$ is caught by the cloud of increments $\mathcal{S}(t)$ infinitely often, i.e.,*

$$\sup_{f\in\mathcal{K}}\liminf_{t\to\infty}\inf_{\eta\in\mathcal{S}(t)}\|\eta-f\|_\infty=0. \tag{16}$$

It is obvious that (15) and (16) follow from Theorem 1.

Conversely, on the one hand, it follows from (15) that the set of P-a.s. limit points of $\mathcal{S}(t)$ contains in the closure of $\mathcal{K}$. On the other hand, it follows from (16) that $\mathcal{K}$ contains in the set of P-a.s. limit points of $\mathcal{S}(t)$. According to general theory (Lifshits, 1995), $\mathcal{K}$ is compact. Therefore it is closed, and we are done.

3.3. **Construction of the auxiliary sequences.** We divide the set $\mathbb{R}^N\times(0,\infty)$ onto parallelepipeds

$$R_{\mathbf{k}p}=\{\,(\mathbf{y},u)\colon k_jre^{pr}\le y_j\le(k_j{+}1)re^{pr}\quad\text{for}\quad 1\le j\le N, e^{pr}\le u\le e^{(p+1)r}\,\},$$

where $\mathbf{k}\in\mathbb{Z}^N$ and $p\in\mathbb{Z}$. The next Lemma describes the most important property of the parallelepipeds $R_{\mathbf{k}p}$.

Lemma 5. *For any $t \in [t_0, \infty)$ the union of all cylinders $R_r(\mathbf{y}_j, u_j)$ that are defined before the moment t contains in the union of finitely many parallelepipeds $R_{\mathbf{k}p}$.*

Proof. It follows from condition 2 of Theorem 1 that for any $t \in [t_0, \infty)$ the volume of all cylinders $R_r(\mathbf{y}_j, u_j)$ that are defined before the moment t with respect to the measure $u^{-N-1}\,d\mathbf{y}\,du$ is finite. So it is enough to prove that the volume of any parallelepiped $R_{\mathbf{k}p}$ with respect to the above mentioned measure is also finite. We have

$$\begin{aligned}\int_{R_{\mathbf{k}p}} u^{-N-1}\,d\mathbf{y}\,du &\sim \frac{r^N e^{Npr}[e^{(p+1)r} - e^{pr}]}{e^{(N+1)pr}} \\ &\sim r^N(e^r - 1) \\ &\sim r^{N+1} \qquad (r \downarrow 0).\end{aligned}$$

Here and in what follows we write $f(r) \sim g(r)$ $(r \downarrow 0)$ if

$$\lim_{r\downarrow 0} \frac{f(r)}{g(r)} = 1.$$

□

Lemma 6. *There exist the sequence of real numbers t_q and the sequence of parallelepipeds $R_{\mathbf{k}_q p_q}$, $q \geq 0$, that satisfy the next conditions.*

(1) *For any $q \geq 0$ and for any $\varepsilon > 0$ there exists a real number $t \in (t_q, t_q + \varepsilon)$ such that*

$$\mathcal{P}(t) \cap R_{\mathbf{k}_q p_q} \neq \varnothing.$$

(2) *If $r < 2/\sqrt{N}$ and $a > 1$, then*

$$\sum_{q=0}^{\infty} \exp(-ah(t_q)) < \infty.$$

Proof. We use mathematical induction.

The real number t_0 is already constructed (it is involved in the formulation of Theorem 1). According to Lemma 5, the union of all cylinders $R_r(\mathbf{y}_j, u_j)$ that are defined before the moment $t_0 + 1$, contains in the union of finitely many parallelepipeds $R_{\mathbf{k}p}$. Therefore there exists a parallelepiped $R_{\mathbf{k}_0 p_0}$ which intersects with infinitely many sets from the sequence $\mathcal{P}(t_0 + 1)$, $\mathcal{P}(t_0 + 1/2)$, …, $\mathcal{P}(t_0 + 1/n)$, ….

Assume that the real numbers t_0, t_1, …, t_q, and the parallelepipeds $R_{\mathbf{k}_0 p_0}$, $R_{\mathbf{k}_1 p_1}$, …, $R_{\mathbf{k}_q p_q}$ are already constructed. Define

$$t_{q+1} = \inf\{\, t > t_q \colon \mathcal{P}(t) \nsubseteq R_{\mathbf{k}_0 p_0} \cup R_{\mathbf{k}_1 p_1} \cup \cdots \cup R_{\mathbf{k}_q p_q} \,\}.$$

Parallelepiped $R_{\mathbf{k}_{q+1}p_{q+1}}$ is defined as a parallelepiped that intersects with infinitely many sets from the sequence $\mathcal{P}(t_q+1)$, $\mathcal{P}(t_q+1/2)$, …, $\mathcal{P}(t_q+1/n)$, …. It means that condition 1 is satisfied.

In order to prove condition 2, define the function $F'_r(t)$ as the volume of the union of all the parallelepipeds $R_{\mathbf{k}p}$, that intersect with at least one set $\mathcal{P}(v)$ for $v\in[t_0,t]$, with respect to the measure $u^{-N-1}\,d\mathbf{y}\,du$. Formally,

$$F'_r(t)=\int_{\cup_{(\mathbf{k},p)\in\mathbb{Z}^{N+1}\colon R_{\mathbf{k}p}\cap(\cup_{t_0\le v\le t}\mathcal{P}(v))\neq\varnothing}R_{\mathbf{k}p}}u^{-N-1}\,d\mathbf{y}\,du.$$

The length of a side of a cube, which is inscribed in the ball of radius 1 in the space $\mathbb{R}^N$, is equal to $2/\sqrt{N}$. It follows that if $r<2/\sqrt{N}$ and $(\mathbf{y},u)\in R_{\mathbf{k}p}$, then $R_{\mathbf{k}p}\subset R_1(\mathbf{y},u)$. Therefore we have $F'_r(t)\le F_1(t)$ and

$$\begin{aligned}\sum_{q=0}^{\infty}\exp(-ah(t_q))&\sim\frac{1}{r^{N+1}}\int_{t_0}^{\infty}\exp(-ah(t))\,dF'_r(t)\\&\le\frac{1}{r^{N+1}}\int_{t_0}^{\infty}\exp(-ah(t))\,dF_1(t)\\&<\infty.\end{aligned}$$

□

3.4. **Proof of Lemma 3.** Denote

$$\eta_{\mathbf{y},u}(\mathbf{x})=\frac{\xi(\mathbf{y}+u\mathbf{x})-\xi(\mathbf{y})}{\sqrt{2}u^H},\qquad\mathbf{x}\in\mathcal{B}.$$

Using properties 1 and 2 (Lemma 1), we obtain

$$\eta_{\mathbf{y},u}(\mathbf{x})\stackrel{d}{=}\frac{1}{\sqrt{2}}\xi(\mathbf{x}).$$

Let $(\mathbf{y}_q,u_q)$ be the centre of the parallelepiped $R_{\mathbf{k}_qp_q}$. Let $(r,\varphi,\vartheta_1,\dots,\vartheta_{N-2})$ be the spherical coordinates of a point $\mathbf{x}\in\mathcal{B}$. Let $(r_2,\varphi_2,\vartheta_{1,2},\dots,\vartheta_{N-2,2})$ be the spherical coordinates of a point $\mathbf{z}\in\mathcal{B}$. Let $b^{l'l''}_{m'n'm''n''qs}$ be the Fourier coefficients of the function

$$b_{qs}(\mathbf{x},\mathbf{z})=\mathsf{E}\eta_{\mathbf{y}_q,u_q}(\mathbf{x})\eta_{\mathbf{y}_s,u_s}(\mathbf{z})$$

with respect to the orthonormal basis

$$\psi_{m'n'}(r)S^{l'}_{m'}(\varphi,\vartheta_1,\dots,\vartheta_{N-2})\psi_{m''n''}(r_2)S^{l''}_{m''}(\varphi_2,\vartheta_{1,2},\dots,\vartheta_{N-2,2}).$$

Let $\{\xi^{lq}_{mn}\}$, $q\ge0$ be the sequence of series of standard normal random variables, that are independent in every series, with the next correlation between series:

$$\mathsf{E}\xi^{l'q}_{m'n'}\xi^{l''s}_{m''n''} = \lambda^{-1/2}_{m'n'}\lambda^{-1/2}_{m''n''}b^{l'l''}_{m'n'm''n''}, \qquad q \neq s.$$

Then we have:

$$\eta_{\mathbf{y}_q,u_q}(\mathbf{x}) = \frac{1}{\sqrt{2}}\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,N)}\sum_{n=1}^{\infty}\sqrt{\lambda_{mn}}\xi^{lq}_{mn}\psi_n(r)S^l_m(\varphi,\vartheta_1,\dots,\vartheta_{N-2}).$$

Let m_0 and n_0 be two natural numbers. Denote

$$\eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}(\mathbf{x}) = \frac{1}{\sqrt{2}}\sum_{m=0}^{m_0}\sum_{l=1}^{h(m,N)}\sum_{n=1}^{n_0}\sqrt{\lambda_{mn}}\xi^{lq}_{mn}\psi_n(r)S^l_m(\varphi,\vartheta_1,\dots,\vartheta_{N-2}).$$

For any $\varepsilon > 0$ consider the next three events:

$$\begin{aligned}
A_{1q}(\varepsilon) &= \left\{\frac{\eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}}{\sqrt{h(t_q)}} \notin \mathcal{K}_{\varepsilon/3}\right\},\\
A_{2q}(\varepsilon) &= \left\{\frac{\|\eta_{\mathbf{y}_q,u_q} - \eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}\|_\infty}{\sqrt{h(t_q)}} > \frac{\varepsilon}{3}\right\},\\
A_{3q}(\varepsilon) &= \left\{\sup_{(\mathbf{y},u)\in R_{\mathbf{k}_q p_q}}\frac{\|\eta_{\mathbf{y}_q,u_q} - \eta_{\mathbf{y},u}\|_\infty}{\sqrt{h(t_q)}} > \frac{\varepsilon}{3}\right\},
\end{aligned}$$

where $\mathcal{K}_{\varepsilon/3}$ denotes the $\varepsilon/3$-neighbourhood of Strassen's ball $\mathcal{K}$ in the space $C(\mathcal{B})$. To prove Lemma 3, it is enough to prove, that for any $\varepsilon > 0$ there exist natural numbers $m_0 = m_0(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ such that the events $A_{1q}(\varepsilon)$, $A_{2q}(\varepsilon)$, and $A_{3q}(\varepsilon)$ occur only finitely many times P-a. s. In other words,

$$\mathsf{P}\left\{\limsup_{q\to\infty} A_{1q}(\varepsilon)\right\} = \mathsf{P}\left\{\limsup_{q\to\infty} A_{2q}(\varepsilon)\right\} = \mathsf{P}\left\{\limsup_{q\to\infty} A_{3q}(\varepsilon)\right\} = 0.$$

By Borel–Cantelli lemma, it is enough to prove that

$$\sum_{q=1}^{\infty}\mathsf{P}\{A_{1q}(\varepsilon)\} < \infty, \tag{17a}$$

$$\sum_{q=1}^{\infty}\mathsf{P}\{A_{2q}(\varepsilon)\} < \infty, \tag{17b}$$

$$\sum_{q=1}^{\infty}\mathsf{P}\{A_{3q}(\varepsilon)\} < \infty. \tag{17c}$$

We prove (17b) first. Denote

$$\sigma^2_{m_0n_0}(\mathbf{x})=\mathsf{E}\left[\eta_{\mathbf{y}_q,u_q}(\mathbf{x})-\eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}(\mathbf{x})\right]^2,\qquad \sigma^2_{m_0n_0}=\max_{\mathbf{x}\in\mathcal{B}}\sigma^2_{m_0n_0}(\mathbf{x}).$$

Using the large deviations estimate (Lifshits, 1995, Section 12, (11)), we obtain

$$\mathsf{P}\left\{\|\eta_{\mathbf{y}_q,u_q}-\eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}\|_\infty>\frac{\varepsilon\sqrt{h(t_q)}}{3}\right\}\leq\exp\left[-\frac{\varepsilon^2h(t_q)}{18\sigma^2_{m_0n_0}}+o\left(\frac{\varepsilon\sqrt{h(t_q)}}{3}\right)\right].$$

By Lemma 6, condition 2, it is sufficient to prove that for any $\varepsilon>0$ there exist natural numbers $m_0=m_0(\varepsilon)$ and $n_0=n_0(\varepsilon)$ such that, say,

$$\frac{\varepsilon^2}{9\sigma^2_{m_0n_0}}<1.$$

Denote

$$\eta^{(m_0)}_{\mathbf{y}_q,u_q}(\mathbf{x})=\frac{1}{\sqrt{2}}\sum_{m=0}^{m_0}\sum_{l=1}^{h(m,N)}\sum_{n=1}^{\infty}\sqrt{\lambda_{mn}}\xi^{lq}_{mn}\psi_n(r)S^l_m(\varphi,\vartheta_1,\dots,\vartheta_{N-2})$$

and

$$\sigma^2_{m_0}(\mathbf{x})=\mathsf{E}\left[\eta_{\mathbf{y}_q,u_q}(\mathbf{x})-\eta^{(m_0)}_{\mathbf{y}_q,u_q}(\mathbf{x})\right]^2,\qquad \sigma^2_{m_0}=\max_{\mathbf{x}\in\mathcal{B}}\sigma^2_{m_0}(\mathbf{x}).$$

The sequence $\sigma^2_{m_0}(\mathbf{x})$ converges to zero as $m_0\to\infty$ for all $\mathbf{x}\in\mathcal{B}$. Moreover, $\sigma^2_{m_0}(\mathbf{x})\geq\sigma^2_{m_0+1}(\mathbf{x})$ for all $\mathbf{x}\in\mathcal{B}$ and all natural m_0. Functions $\sigma^2_{m_0}(\mathbf{x})$ are non-negative and continuous. By Dini's theorem, the sequence $\sigma^2_{m_0}(\mathbf{x})$ converges to zero uniformly on $\mathcal{B}$, i.e.,

$$\lim_{m_0\to\infty}\sigma^2_{m_0}=0,$$

and we choose such an m_0, that for any $m>m_0$, $\sigma^2_m<\varepsilon^2/18$.

In the same way, we can apply Dini's theorem to the sequence of functions

$$\mathsf{E}\left[\eta^{(m_0)}_{\mathbf{y}_q,u_q}(\mathbf{x})-\eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}(\mathbf{x})\right]^2,\quad n_0\geq 1$$

and find such n_0 that

$$\sup_{\mathbf{x}\in\mathcal{B}}\mathsf{E}\left[\eta^{(m_0)}_{\mathbf{y}_q,u_q}(\mathbf{x})-\eta^{(m_0,n)}_{\mathbf{y}_q,u_q}(\mathbf{x})\right]^2\leq\varepsilon^2/18$$

for all $n>n_0$. (17b) is proved.

Now we prove (17a). In what follows we denote by C a constant depending only on N, H and that may vary at each occurrence. Specific constants will be denote by C_1, C_2, ...

Consider the finite-dimensional subspace E of the space $C(\mathcal{B})$ spanned by the functions

$$\psi_n(r)S^l_m(\varphi,\vartheta_1,\dots,\vartheta_{N-2}),$$

for $1\le n\le n_0$, $0\le m\le m_0$, and $1\le l\le h(m,N)$. All norms on E are equivalent. In particular, there exists a constant $C_1=C_1(m_0,n_0)$ such that the $\varepsilon/3$-neighbourhood of Strassen's ball $\mathcal{K}$ in the space E equipped by the uniform norm contains in the ball of radius $1+C_1\varepsilon$ with respect to Strassen's norm. Then we have

$$\begin{aligned}\mathsf{P}\{A_{1q}(\varepsilon)\}&\le\mathsf{P}\left\{\left\|\frac{\eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}}{\sqrt{h(t_q)}}\right\|^2_S>(1+C_1\varepsilon)^2\right\}\\&=\mathsf{P}\left\{\left\|\eta^{(m_0,n_0)}_{\mathbf{y}_q,u_q}\right\|^2_S>(1+C_1\varepsilon)^2h(t_q)\right\}\\&=\mathsf{P}\left\{\sum_{m=0}^{m_0}\sum_{l=1}^{h(m,N)}\sum_{n=1}^{n_0}\lambda_{mn}(\xi^{lq}_{mn})^2>2(1+C_1\varepsilon)^2h(t_q)\right\}.\end{aligned}$$

Denote

$$\lambda=\min_{\substack{0\le m\le m_0\\1\le n\le n_0}}\lambda_{mn},\qquad M=n_0\sum_{m=0}^{m_0}h(m,N)$$

and let χ_M denotes a random variable that has χ^2 distribution with M degrees of freedom. Using standard probability estimates for χ_M, we can write

$$\begin{aligned}\mathsf{P}\{A_{1q}(\varepsilon)\}&\le\mathsf{P}\{\chi_M>2(1+C_1\varepsilon)^2\lambda^{-1}h(t_q)\}\\&\le\exp\{-(1+C_1\varepsilon)^2\lambda^{-1}h(t_q)\}\end{aligned}$$

for large enough $h(t_q)$. Applying Lemma 6, condition 2 concludes proof.

Now we prove (17a). Denote

$$\zeta_1=\sup_{\substack{(\mathbf{y}_1,u_1)\in R_r(\mathbf{y},u)\\(\mathbf{y}_2,u_2)\in R_r(\mathbf{y},u)}}\frac{|\xi(\mathbf{y}_1)-\xi(\mathbf{y}_2)|}{\sqrt{2}u_1^H},$$

$$\zeta_2=\sup_{\substack{(\mathbf{y}_1,u_1)\in R_r(\mathbf{y},u)\\(\mathbf{y}_2,u_2)\in R_r(\mathbf{y},u)}}\frac{\sup_{\mathbf{x}\in\mathcal{B}}|\xi(\mathbf{y}_1+u_1\mathbf{x})-\xi(\mathbf{y}_2+u_2\mathbf{x})|}{\sqrt{2}u_1^H},$$

$$\zeta_3=\sup_{\substack{(\mathbf{y}_1,u_1)\in R_r(\mathbf{y},u)\\(\mathbf{y}_2,u_2)\in R_r(\mathbf{y},u)}}\left|\frac{1}{\sqrt{2}u_1^H}-\frac{1}{\sqrt{2}u_2^H}\right|\sup_{\mathbf{x}\in\mathcal{B}}|\xi(\mathbf{y}_2+u_2\mathbf{x})-\xi(\mathbf{y}_2)|.$$

It is easy to see that $\zeta_1 \leq \zeta_2$ and $\|\eta_{\mathbf{y}_q,u_q} - \eta_{\mathbf{y},u}\|_\infty \leq \zeta_1 + \zeta_2 + \zeta_3$. It follows that

$$\mathsf{P}\{A_{3q}(\varepsilon)\} \leq \mathsf{P}\left\{\zeta_2 > \varepsilon\sqrt{h(t_q)}/12\right\} + \mathsf{P}\left\{\zeta_3 > \varepsilon\sqrt{h(t_q)}/6\right\}. \tag{18}$$

The second term in the right hand side may be estimated as

$$\begin{aligned}\mathsf{P}\left\{\zeta_3 > \varepsilon\sqrt{h(t_q)}/6\right\} &= \mathsf{P}\left\{\sup_{\substack{\mathbf{x}\in\mathcal{B}\\ (\mathbf{y}_1,u_1)\in R_r(\mathbf{y},u)\\ (\mathbf{y}_2,u_2)\in R_r(\mathbf{y},u)}} |\xi(x)||[(u_2/u_1)^H - 1] > \frac{\varepsilon\sqrt{2h(t_q)}}{6}\right\}\\ &\leq \mathsf{P}\left\{\sup_{\mathbf{x}\in\mathcal{B}} \xi(\mathbf{x}) > \frac{\varepsilon\sqrt{2h(t_q)}}{6\delta}\right\},\end{aligned} \tag{19}$$

where $\delta = \delta(r) = \max\{r, e^{2Hr} - 1, e^r - 1\}$.

Another large deviation estimate (Lifshits, 1995, Section 14, (12)) states that there exists a constant $C = C(H)$ such that for all $K > 0$

$$\mathsf{P}\left\{\sup_{\mathbf{x}\in\mathcal{B}} \xi(x) > K\right\} \leq CK^{N/H-1}\exp(-K^2/2). \tag{20}$$

Using this fact, we can continue estimate (19) as follows

$$\mathsf{P}\left\{\zeta_3 > \varepsilon\sqrt{h(t_q)}/6\right\} \leq C\frac{\varepsilon^{N/H-1}[h(t_q)]^{(N-H)/(2H)}}{\delta^{N/H-1}}\exp\left(-\frac{2\varepsilon^2 h(t_q)}{72\delta^2}\right).$$

If we choose such a small r that $\delta < \varepsilon/6$, then by Lemma 6, condition 2 the series

$$\sum_{q=1}^\infty \mathsf{P}\left\{\zeta_3 > \varepsilon\sqrt{h(t_q)}/6\right\}$$

converges.

Using (6), we write random variable ζ_2 as follows

$$\zeta_2 = \sup_{\substack{\mathbf{x}\in\mathcal{B}\\ (\mathbf{y}_1,u_1)\in R_r(\mathbf{y},u)\\ (\mathbf{y}_2,u_2)\in R_r(\mathbf{y},u)}} \left|\xi\left(\frac{\mathbf{y}_1 + u_1\mathbf{x}}{2^{1/(2H)}u_1}\right) - \xi\left(\frac{\mathbf{y}_2 + u_2\mathbf{x}}{2^{1/(2H)}u_1}\right)\right|.$$

The right hand side can be estimated as follows.

$$\begin{aligned}\frac{\|(\mathbf{y}_1+u_1\mathbf{x})-(\mathbf{y}_2+u_2\mathbf{x})\|}{2^{1/(2H)}u_1} &\leq \frac{\|\mathbf{y}_1-\mathbf{y}_2\|}{2^{1/(2H)}u_1}+\frac{\|\mathbf{x}\|\cdot|u_1-u_2|}{2^{1/(2H)}u_1}\\ &\leq \frac{2ru}{2^{1/(2H)}u_1}+2^{-1/(2H)}\left(1-\frac{u_2}{u_1}\right)\\ &\leq 2^{1-1/(2H)}re^r+2^{-1/(2H)}(e^{2r}-1)\\ &\leq 2^{1-1/(2H)}\delta e^r+2^{-1/(2H)}(\delta^2+2\delta)\\ &\leq 2^{1-1/(2H)}(\delta^2+\delta)+2^{-1/H}(\delta^2+\delta)\\ &\leq C_2\delta.\end{aligned}$$

Let $\mathbf{z}_j$, $1\leq j\leq C(C_2\delta)^{-N}$ be the $C_1\delta$-net in $\mathcal{B}$. Standard entropy estimate for the first term in the right hand side of (18) gives

$$\begin{aligned}\mathsf{P}\left\{\zeta_2>\varepsilon\sqrt{h(t_q)}/12\right\} &\leq \mathsf{P}\left\{\sup_{\substack{\mathbf{x}\in\mathcal{B}\\ \|\mathbf{y}\|\leq C_2\delta}}|\xi(\mathbf{x}+\mathbf{y})-\xi(\mathbf{x})|>\frac{\varepsilon\sqrt{h(t_q)}}{12}\right\}\\ &\leq \mathsf{P}\left\{\sup_{\substack{1\leq j\leq C(C_2\delta)^{-N}\\ \|\mathbf{y}\|\leq C_2\delta}}|\xi(\mathbf{z}_j+\mathbf{y})-\xi(\mathbf{z}_j)|>\frac{\varepsilon\sqrt{h(t_q)}}{24}\right\}\\ &\leq C\delta^{-N}\mathsf{P}\left\{\sup_{\|\mathbf{y}\|\leq 1}|\xi(\mathbf{y}|>\frac{\varepsilon\sqrt{h(t_q)}}{24(C_2\delta)^H}\right\}\\ &\leq C\delta^{-N}\cdot\frac{\varepsilon^{N/H-1}[h(t_q)]^{(N-H)/(2H)}}{\delta^{N-H}}\exp\left(-\frac{\varepsilon^2h(t_q)}{1152(C_2\delta)^{2H}}\right).\end{aligned}$$

Here we used Lemma 1 and (20). If we choose such a small r that

$$\frac{\varepsilon^2}{1152(C_2\delta)^{2H}}>1,$$

then by Lemma 6, condition 2 the series

$$\sum_{q=1}^{\infty}\mathsf{P}\left\{\zeta_2>\varepsilon\sqrt{h(t_q)}/12\right\}$$

converges. This concludes proof of Lemma 3.

3.5. **Proof of Lemma 4.** By Lemma 6, condition 1, for any $q\geq 0$ there exists a number $t'_q\in[t_q,t_{q+1})$ such that $\mathcal{P}(t)\cap R_{\mathbf{k}_qp_q}\neq\varnothing$. Choose arbitrary points $(\mathbf{y}'_q,u'_q)\in\mathcal{P}(t)\cap R_{\mathbf{k}_qp_q}$. It is easy to see that the sequence t'_q satisfies Lemma 6, condition 2 as well.

The set of all $f \in C(\mathcal{B})$ with $0 < \|f\|_S < 1$ is dense in $\mathcal{K}$. It is enough to prove that for any such f we have

$$\liminf_{q\to\infty} \left\| \frac{\xi(\mathbf{y}'_q + u'_q\mathbf{x}) - \xi(\mathbf{y}'_q)}{(u'_q)^H\sqrt{2h(t'_q)}} - f \right\|_\infty = 0 \quad \mathsf{P}-\text{a.s.}$$

According to (Li and Shao, 2001, Theorem 5.1), there exists a constant $C_3 = C_3(N, H)$ such that for all $\varepsilon \in (0, 1]$

$$\mathsf{P}\left\{\sup_{\mathbf{x}\in\mathcal{B}} |\xi(\mathbf{x})| \le \varepsilon\right\} \ge \exp\left(-C_3\varepsilon^{-N/H}\right).$$

Denote $\beta = (C_3)^{H/N}(1 - \|f\|_S^2)^{-H/N}$. We will prove that

$$\liminf_{q\to\infty}[h(t'_q)]^{(N+2H)/(2N)} \left\| \frac{\xi(\mathbf{y}'_q + u'_q\mathbf{x}) - \xi(\mathbf{y}'_q)}{(u'_q)^H\sqrt{2h(t'_q)}} - f \right\|_\infty \le \frac{1}{\sqrt{2}}\beta \quad \mathsf{P}-\text{a.s.}$$

Consider the event

$$\tilde{A}_{1q}(\varepsilon) = \left\{ \left\| \frac{\xi(\mathbf{y}'_q + u'_q\mathbf{x}) - \xi(\mathbf{y}'_q)}{(u'_q)^H} - \sqrt{2h(t'_q)}f \right\|_\infty \le \beta(1+\varepsilon)[h(t'_q)]^{-H/N} \right\}.$$

Using (Monrad and Rootzén, 1995, Proposition 4.2), we obtain

$$\begin{aligned}\log \mathsf{P}\{\tilde{A}_{1q}(\varepsilon)\} &\ge 2h(t'_q)(-1/2)\|f\|_S^2 - C_3\beta^{-N/H}(1+\varepsilon)^{-N/H}h(t'_q)\\ &= -h(t'_q)\left[\|f\|_S^2 + (1 - \|f\|_S^2)(1+\varepsilon)^{-N/H}\right].\end{aligned}$$

The multiplier in square brackets is less than 1. It follows that

$$\sum_{q=0}^{\infty} \mathsf{P}\{\tilde{A}_{1q}(\varepsilon)\} = \infty. \tag{21}$$

If the events $\tilde{A}_{1q}(\varepsilon)$ were independent, the usage of the second Borel–Cantelli lemma would conclude the proof. However, they are dependent.

In order to create independence, we use another spectral representation of the multiparameter fractional Brownian motion, as (Monrad and Rootzén, 1995; Li and Shao, 2001) did. Let $\overline{W}$ denotes a complex-valued scattered Gaussian random measure on $\mathbb{R}^N$ with Lebesgue measure as its control measure.

Lemma 7 (Global spectral representation)**.** *There exists a constant $C_4 = C_4(N, H)$ such that*

$$\xi(\mathbf{x}) = C_4 \int_{\mathbb{R}^N} \left(e^{i(\mathbf{p},\mathbf{x})} - 1\right) \|\mathbf{p}\|^{-(N/2)-H}\, d\overline{W}(\mathbf{p}).$$

This result is well-known. Using formulas 2.2.3.1, 2.5.6.1, and 2.5.3.13 from (Prudnikov et al., 1986), one can prove that

$$C_4 = 2^H \sqrt{\frac{H\Gamma((N+H)/2)}{\Gamma(N/2)\Gamma(1-H)}}.$$

Let $0 < a < b$ be two real numbers. Denote

$$\xi^{(a,b)}(\mathbf{x}) = C_4 \int_{\|\mathbf{p}\|\in(a,b]} \left(e^{i(\mathbf{p},\mathbf{x})}-1\right)\|\mathbf{p}\|^{-(N/2)-H}\, d\overline{W}(\mathbf{p}), \qquad \tilde{\xi}^{(a,b)}(\mathbf{x}) = \xi(\mathbf{x}) - \xi^{(a,b)}(\mathbf{x}).$$

Lemma 8. *The random field $\tilde{\xi}^{(a,b)}(\mathbf{x})$ has the next properties.*

(1) *It has homogeneous increments.*
(2) *For any $u \in \mathbb{R}$*

$$\tilde{\xi}^{(a,b)}(u\mathbf{x}) \stackrel{d}{=} u^H \tilde{\xi}^{(ua,ub)}(\mathbf{x}). \tag{22}$$

(3) *It is isotropic.*

This Lemma can be proved exactly in the same way, as Lemma 1.

Put

$$d_q = (u'_q)^{-1}\exp\{h(t'_q)[\exp(h(t'_q)+1-H]\}$$

and consider the events

$$\tilde{A}_{2q}(\varepsilon) = \left\{\left\|\frac{\xi^{(d_{q-1},d_q)}(\mathbf{y}'_q + u'_q\mathbf{x}) - \xi^{(d_{q-1},d_q)}(\mathbf{y}'_q)}{(u'_q)^H} - \sqrt{2h(t'_q)}f\right\|_\infty \le \beta(1+\varepsilon)[h(t'_q)]^{-H/N}\right\},$$

$$\tilde{A}_{3q}(\varepsilon) = \left\{\left\|\frac{\tilde{\xi}^{(d_{q-1},d_q)}(\mathbf{y}'_q + u'_q\mathbf{x}) - \tilde{\xi}^{(d_{q-1},d_q)}(\mathbf{y}'_q)}{(u'_q)^H}\right\|_\infty \ge \varepsilon\beta[h(t'_q)]^{-H/N}\right\}.$$

Lemma 9. *We have*

$$\sum_{q=0}^{\infty} \mathsf{P}\{\tilde{A}_{3q}(\varepsilon)\} < \infty.$$

Proof. Using Lemma 8, one can write

$$\mathsf{P}\{\tilde{A}_{3q}(\varepsilon)\} = \mathsf{P}\left\{\|\tilde{\xi}^{(u'_q d_{q-1}, u'_q d_q)}(\mathbf{x})\|_\infty \ge \varepsilon\beta[h(t'_q)]^{-H/N}\right\}.$$

Put

$$x_q = \exp\{-\exp[(1-\|f\|_S^2)h(t'_q)]\}.$$

Using Lemma 8 once more, we have

$$\mathsf{P}\{\tilde{A}_{3q}(\varepsilon)\} = \mathsf{P}\left\{\sup_{\|\mathbf{x}\|\le x_q} |\tilde{\xi}^{(x_q^{-1}u'_q d_{q-1}, x_q^{-1}u'_q d_q)}(\mathbf{x})| \ge \varepsilon x_q^H \beta[h(t'_q)]^{-H/N}\right\}.$$

Denote

$$\zeta_q(\mathbf{x}) = \tilde{\xi}^{(x_q^{-1}u'_q d_{q-1}, x_q^{-1}u'_q d_q)}(\mathbf{x}).$$

We estimate the variance of the random field $\zeta_q(\mathbf{x})$ for $\|\mathbf{x}\|\leq x_q$. We have

$$\begin{aligned}\mathsf{E}[\zeta_q(\mathbf{x})]^2=2C_4^2&\int_{\|\mathbf{p}\|\leq x_q^{-1}u'_qd_{q-1}}(1-\cos(\mathbf{p},\mathbf{x}))\|\mathbf{p}\|^{-N-2H}\,d\mathbf{p}\\&+2C_4^2\int_{\|\mathbf{p}\|>x_q^{-1}u'_qd_q}(1-\cos(\mathbf{p},\mathbf{x}))\|\mathbf{p}\|^{-N-2H}\,d\mathbf{p}.\end{aligned}$$

In the first integral, we bound $1-\cos(\mathbf{p},\mathbf{x})$ by $\|\mathbf{p}\|^2\cdot\|\mathbf{x}\|^2/2$. In the second integral, we bound it by 2. Then we have

$$\mathsf{E}[\zeta_q(\mathbf{x})]^2\leq C_4^2x_q^2\int_{\|\mathbf{p}\|\leq x_q^{-1}u'_qd_{q-1}}\|\mathbf{p}\|^{2-N-2H}\,d\mathbf{p}+4C_4^2\int_{\|\mathbf{p}\|>x_q^{-1}u'_qd_q}\|\mathbf{p}\|^{-N-2H}\,d\mathbf{p}.$$

Now we pass to spherical coordinates and obtain

$$\begin{aligned}\mathsf{E}[\tilde{\xi}^{(x_q^{-1}u'_qd_{q-1},x_q^{-1}u'_qd_q)}(\mathbf{x})]^2&\leq Cx_q^2\int_0^{x_q^{-1}u'_qd_{q-1}}p^{1-2H}\,dp+C\int_{x_q^{-1}u'_qd_q}^\infty p^{-1-2H}\,dp\\&=Cx_q^{2H}[(u'_qd_{q-1})^{2-2H}+(u'_qd_q)^{-2H}].\end{aligned}$$

Substituting definitions of d_q and x_q to the last inequality, we obtain

$$\mathsf{E}[\zeta_q(\mathbf{x})]^2\leq C\exp\left\{-2H\{\exp[(1-\|f\|_S^2)h(t'_q)]+(1-H)h(t'_q)\}\right\}$$

or,

$$\mathsf{E}[\tilde{\xi}^{(x_q^{-1}u'_qd_{q-1},x_q^{-1}u'_qd_q)}(\mathbf{x})-\tilde{\xi}^{(x_q^{-1}u'_qd_{q-1},x_q^{-1}u'_qd_q)}(\mathbf{y})]^2\leq\varphi_q^2(\|\mathbf{x}-\mathbf{y}\|),$$

for $\|\mathbf{x}-\mathbf{y}\|=\delta\leq x_q$, where

$$\varphi_q^2(\delta)=C\min\left\{\delta^{2H},\exp\left\{-2H\{\exp[(1-\|f\|_S^2)h(t'_q)]+(1-H)h(t'_q)\}\right\}\right\}.$$

We need the next lemma (Fernique, 1975)

Lemma 10. *Let $\zeta(\mathbf{x})$, $\mathbf{x}\in[0,x]^N$ be a separable centred Gaussian random field. Assume that*

$$\sup_{\substack{\mathbf{x},\mathbf{y}\in[0,x]^N\\\|\mathbf{x}-\mathbf{y}\|\leq\delta}}\mathsf{E}(\zeta(\mathbf{x})-\zeta(\mathbf{y}))^2\leq\varphi^2(\delta).$$

Then, for any sequence of positive real numbers y_0, y_1, …, y_p, …, and for any sequence of integer numbers m_1, m_2, …, m_p, …, every of which can be divided by previous,

$$\mathsf{P}\left\{\sup_{\mathbf{x}\in[0,x]^N}|\zeta(\mathbf{x})|\geq y_0\varphi(x)+\sum_{p=1}^\infty y_p\varphi(x/2m_p)\right\}\leq\sqrt{2/\pi}\sum_{p=0}^\infty(m_{p+1})^N\int_{y_p}^\infty e^{-u^2/2}\,du.$$

Put $m_{p,q} = q^{2^p}$, $y_{0q} = 2\sqrt{(N+1)h(t'_q)}$, and

$$y_{p,q} = \varepsilon(p+1)^{-2} x_q^H \beta [h(t'_q)]^{-H/N} / \varphi_q(2x_q \cdot q^{-2^p}), \qquad q \geq 1.$$

For large enough q,

$$y_{p,q} > 2\sqrt{(N+3)h(t'_q)} 2^{p/2}$$

for all $p \geq 1$. Moreover,

$$y_{0,q}\varphi_q(x_q) + \sum_{p=1}^{\infty} y_{p,q}\varphi_q(x_q/2m_{p,q}) < \varepsilon x_q^H \beta [h(t'_q)]^{-H/N}.$$

We have

$$\sum_{q=1}^{\infty} q^2 e^{-y_{0q}^2/2} + \sum_{q=1}^{\infty}\sum_{p=1}^{\infty} q^{N\cdot 2^{p+1}} e^{-y_{pq}^2/2} < \infty,$$

and application of Lemma 10 finishes the proof. □

It follows from definition of the events $\tilde{A}_{1q}(\varepsilon)$, $\tilde{A}_{2q}(\varepsilon)$, and $\tilde{A}_{2q}(\varepsilon)$, that

$$\tilde{A}_{1q}(\varepsilon) \subset \tilde{A}_{2q}(2\varepsilon) \cup \tilde{A}_{3q}(\varepsilon) \subset \tilde{A}_{1q}(2\varepsilon) \cup \tilde{A}_{2q}(\varepsilon). \tag{23}$$

Combining (21), (23), and Lemma 9, we get

$$\sum_{q=0}^{\infty} \mathsf{P}\{\tilde{A}_{2q}(\varepsilon)\} = \infty. \tag{24}$$

Now we prove that the events $\tilde{A}_{2q}(\varepsilon)$ are independent. It is enough to prove that $d_{q-1} < d_q$. Using the definition of d_q, this inequality becomes

$$u'_q < e^{1-H} u'_{q-1}.$$

By our choice of u'_q, we have $e^{p_q r} \leq u'_q \leq e^{p_q+1} r$. Since by construction of the parallelepipeds $R_{\mathbf{k}_q p_q}$ two adjacent parallelepipeds can lie in the same u-layer or in adjacent u-layers, we have

$$u'_{q-1} \leq e^{2r} u'_q.$$

We choose $r < (1-H)/2$, and we are done.

It follows from the second Borel–Cantelli lemma that

$$\mathsf{P}\left\{\limsup_{q\to\infty} \tilde{A}_{2q}(\varepsilon)\right\} = 1. \tag{25}$$

Combining (23), (25), and Lemma 9, we get

$$\mathsf{P}\left\{\limsup_{q\to\infty} \tilde{A}_{1q}(3\varepsilon)\right\} = 1.$$

Since ε can be chosen arbitrarily close to 0, Lemma 4 is proved.

4. Examples

4.1. **Local functional law of the iterated logarithm.** Let $t_0 = 3$. Let $\mathcal{J}(t)$ contains only one element 0. Let $\mathbf{y}_0 = \mathbf{0}$ and $u_0 = t^{-1}$. Then we have

$$R_1(\mathbf{0}, u) = \{\, (\mathbf{y}, v) \colon \|\mathbf{y}\| \leq u, e^{-1}u \leq v \leq eu \,\}.$$

It is easy to see that $dA_1(u)$ is comparable to

$$du \int_{\|\mathbf{y}\| \leq u} \frac{d\mathbf{y}}{u^{N+1}} = C\frac{du}{u}.$$

The function $h(u) = \log\log u$ satisfies the conditions of Theorem 1. We obtain, that, in the uniform topology, the set of P-a. s. limit points of the cloud of increments

$$\frac{\xi(t\mathbf{x})}{\sqrt{2\log\log t^{-1}}t^H}$$

as $t \downarrow 0$ is Strassen's ball $\mathcal{K}$. For the case of $N = 1$ and $H = 1/2$, this result is due to (Gantert, 1993).

Let $F(f) = \|f\|_\infty$, $f \in C(\mathcal{B})$. On the one hand, we have

$$\limsup_{t\downarrow 0} \frac{\xi(t)}{\sqrt{2\log\log t^{-1}}t^H} = \sup_{f\in\mathcal{K}} \|f\| \qquad \mathsf{P}-\text{a.s.}$$

On the other hand, according to (Benassi et al., 1997), we have

$$\limsup_{t\downarrow 0} \frac{\xi(t)}{\sqrt{2\log\log t^{-1}}t^H} = 1 \qquad \mathsf{P}-\text{a.s.}$$

It follows that

$$\sup_{f\in\mathcal{K}} \|f\| = 1. \tag{26}$$

4.2. **Global functional law of the iterated logarithm.** Let $t_0 = 3$. Let $\mathcal{J}(t)$ contains only one element 0. Let $\mathbf{y}_0 = \mathbf{0}$ and $u_0 = t$. It is easy to check, that $dA_1(t)$ is comparable to $t^{-1}\,dt$. It follows that, in the uniform topology, the set of P-a. s. limit points of the cloud of increments

$$\frac{\xi(t\mathbf{x})}{\sqrt{2\log\log t}t^H}$$

as $t \to \infty$ is Strassen's ball $\mathcal{K}$. For the case of $N = 1$ and $H = 1/2$, this result is due to (Strassen, 1964). Using the continuous functional $F(f) = \|f\|_\infty$, we obtain

$$\lim_{t\to\infty} \sup_{\|\mathbf{x}\|\le t} \frac{\xi(\mathbf{x})}{\sqrt{2\log\log t}\,t^H} = \sup_{f\in\mathcal{K}} \|f\| \qquad \mathsf{P}-\text{a.s.}$$

or, by (26),

$$\lim_{t\to\infty} \sup_{\|\mathbf{x}\|\le t} \frac{\xi(\mathbf{x})}{\sqrt{2\log\log t}\,t^H} = 1 \qquad \mathsf{P}-\text{a.s.}$$

4.3. **Functional Lévy modulus of continuity.** Let $t_0 = 2$. Let $\mathcal{J}(t) = \{\, \mathbf{y} \in \mathbb{R}^N \colon \|\mathbf{y}\| \le 1 - t^{-1} \,\}$ and $u = t^{-1}$ for any $\mathbf{y} \in \mathcal{J}(t)$. Then we have

$$\mathcal{P}(t) = \{\, (\mathbf{y}, u) \colon \|\mathbf{y}\| \le 1 - t^{-1}, u = t^{-1} \,\}$$

and

$$\cup_{t\le u}\mathcal{P}(t) = \{\, (\mathbf{y}, v) \colon \|\mathbf{y}\| \le 1 - u^{-1}, u^{-1} \le v \le 1 \,\}.$$

It is easy to see that $dA_1(u)$ is comparable to

$$(1 - u^{-1})^N \int_1^{u^{-1}} v^{-N-1}\, dv \sim u^N.$$

The function $h(u) = N \log u$ satisfies the conditions of Theorem 1. It follows that, in the uniform topology, the set of P-a. s. limit points of the cloud of increments

$$\mathcal{S}(t) = \left\{ \eta(\mathbf{x}) = \frac{\xi(\mathbf{y} + t\mathbf{x}) - \xi(\mathbf{y})}{\sqrt{2N\log t^{-1}}\,t^H} \colon \|\mathbf{y}\| \le 1 - t \right\}$$

as $t \downarrow 0$ is Strassen's ball $\mathcal{K}$. For the case of $N = 1$ and $H = 1/2$, this result is due to (Mueller, 1981). Using the continuous functional $F(f) = \|f\|_\infty$ and (26), we obtain:

$$\limsup_{\|\mathbf{y}\|\downarrow 0} \sup_{\mathbf{x}\in\mathcal{B}} \frac{|\xi(\mathbf{x} + \mathbf{y}) - \xi(\mathbf{x})|}{\sqrt{2N\log\|\mathbf{y}\|^{-1}}\,\|\mathbf{y}\|^H} = 1 \qquad \mathsf{P}-\text{a.s.}, \tag{27}$$

which coincides with the results by (Benassi et al., 1997).

Let L be the linear space of all deterministic functions $f \in C(\mathcal{B})$ satisfying the condition

$$\limsup_{\|\mathbf{y}\|\downarrow 0} \sup_{\mathbf{x}\in\mathcal{B}} \frac{|f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})|}{\sqrt{2N\log\|\mathbf{y}\|^{-1}}\,\|\mathbf{y}\|^H} = 1. \tag{28}$$

It follows from (27) that $\mu(L) = 1$. By (Lifshits, 1995, Section 9, Proposition 1) $\mathcal{H}_\xi \subset L$. From Lemma 2 we obtain the following Bernstein-type theorem from approximation theory:

Theorem 2. *Let $f \in C(\mathcal{B})$ with $f(\mathbf{0}) = 0$ satisfies the condition*

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{h(m,N)} \frac{(f^l_{mn})^2}{\lambda_{mn}} < \infty.$$

Then f satisfies (28).

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