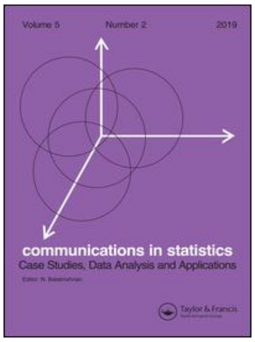

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Properties of American options under a Markovian Regime Switching Model

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ABSTRACT

In this article, a model under which the underlying asset follows a Markov regime-switching process is considered. The underlying economy is partially observable in a form of a signal stochastically related to the actual state of the economy. The American option pricing problem is formulated using a partially observable Markov decision process (POMDP). Through the article, a three-state economy is assumed with a focus on the threshold for the early exercise, hold regions and its monotonicity. An extensive numerical experimental study is conducted in order to clarify the relationship between the monotonicity of the exercising strategy and the sufficient conditions which are obtained in Jin, Dimitrov, and Ni. In this article, the effect of sufficient conditions is confirmed. It was shown that sufficient conditions are not necessary for the monotonicity of the exercising strategy, and a discussion including milder conditions is presented based on the numerical studies.

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Hidden Markov Chain; optimal strategy; partially observable; Markov decision process; totally positive of order 2

1. Introduction

An American call (put) option is a financial contract that gives its holder the right but not obligation to buy (sell) the so-called underlying asset for a predetermined strike price K on or before a maturity time T. Note that American options can be exercised any time before or at maturity, which makes the pricing problem of an American option difficult. Knowing some analytical properties of the American option price and the structural properties of the optimal exercising strategies is very useful for the pricing problem. We refer to Jönsson (2005) for a good survey on optimal exercising regions for American options and books by Silvestrov (2014a,b) for more comprehensive bibliographic remarks on American options.

Since Black and Scholes proposed their celebrated option pricing model where the underlying asset follows a geometric Brownian motion with constant

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volatility, numerous extensions have appeared to relax the unrealistic assumption of constant volatility. Option pricing under Markov-modulated regime-switching models has been actively studied in the last decades. Under such a model, model parameters, for instance, the volatility, can be modulated by a hidden Markov chain whose states refer to the hidden states of an economy.

The regime-switching model was first introduced to economic analysis by Hamilton (1989) with the purpose of explaining business cycles. There are many sources on option pricing of which we name a few. Naik (1993), Guo (2001) and Buffington and Elliott (2002) have priced European options, Guo and Shepp (2001) have discussed pricing problems for European, perpetual American and look-back options. Under the assumption of a two-state economy, an analytical approximation for the price of an American option has also been obtained in Buffington and Elliott (2002) and an explicit formula, in terms of infinite series, in Chan and Zhu (2021) by means of the homotopy analysis. As regime-switching models often result in an incomplete market, Liew and Siu (2010) have investigated two approaches in selecting the equivalent martingale measure to option valuation. We refer to the collective books by Mamon and Elliott (2007, 2014) and references therein for an overview of hidden Markov models in Finance.

In using regime-switching models, parameter estimation and calibration is an important issue. To obtain an estimate of the state of the economy, filtering methods can be applied to the historical time series of the underlying asset. For example, Elliott, Malcolm, and Tsoi (2003) have proposed such an algorithm and tested it successfully on simulated data. More recent work is given by He and Zhu (2017, 2021) in which the authors proposed a new algorithm for calibrating a local regime-switching model using both simulated and real market data.

As pointed out in Elliott and Siu (2013), most of the earlier work on option pricing seems to assume an observed underlying Markov chain and does not address the estimation of the supposedly hidden states. However, in reality, the economic states are often not directly observable. Some studies consider partially observable hidden economy states. Elliott and Siu (2013) have investigated option pricing in a continuous-time, hidden Markov-modulated, pure-jump asset price model without the observable assumption. In the previously mentioned work by Liew and Siu (2010), the option price is obtained by conditioning on observable information for the hidden states. In addition, Liew and Siu (2010) have estimated the unknown transitional probability matrix of the hidden Markov chain and asset price model parameters under a two-state economy, using real market time series of IBM from the year 2008 to 2010. This further confirms the practical use of hidden Markov models.

Our work, in the present article and an earlier one Jin, Dimitrov, and Ni (2019), is closely related to Sato and Sawaki (2014, 2018). In these articles, the so-called callable American option has been studied. The economy states of an N-state regime-switching model are assumed to be observable in Sato and Sawaki

(2014) but partially observable through some signal in Sato and Sawaki (2018). Moreover, the probabilistic relationship between the observable signal and the hidden states of the economy is assumed to be known. For a callable American option, both the buyer and seller can terminate the option before the maturity, Sato and Sawaki (2018) have investigated the early optimal strategies for both the buyer and seller and the analytical properties of early exercising decisions.

As the hidden Markov model is partially observable and is given in a discrete-time setting, moreover as the American option pricing problem involves decisions of holding and exercising, a partially observable Markov decision problem can be formulated. We note that American option pricing usually is considered as an optimal stopping problem, which in a discrete-time setting is a special case of the more general Markov decision problem.

In our earlier article Jin, Dimitrov, and Ni (2019), we have followed the framework in Sato and Sawaki (2018) and considered valuation and optimal strategies using a partially observable Markov decision process. However, we note that Assumption 3.1 in Sato and Sawaki (2018) is too stringent for most of the applications in option pricing. This assumption requires that the random variable of the underlying asset return can be ordered in terms of stochastic increasing across different economic states, which is very difficult to satisfy with state-inhomogenous volatilities.

In Jin, Dimitrov, and Ni (2019), we have proved the analytical properties of American options under a set of sufficient conditions without this key assumption as in Sato and Sawaki (2018). However, relaxing this key assumption has posed a challenge, so at this stage of research, our model had to be more explicit. The asset return random variable is also assumed to be discrete with two possible values, as in a binomial tree. We consider the standard plain-vanilla American options instead of callable American options with general payoff functions. Nevertheless, our N-state partially observable Markov decision model is still novel and the corresponding results are new to the literature. A follow-up work, for a general asset return distribution with an arbitrary distribution function $F(\cdot)$, is under preparation.

It should be pointed out that most of the earlier studies on American option pricing under the regime-switching model deal with approximating the price by an explicit valuation formula, under the assumption of a two-state economy, as in Buffington and Elliott (2002) and Chan and Zhu (2021). There is so far very little work on the analytical structural properties for the American option prices and optimal exercising strategies under a partially observable hidden Markov model with an N-state economy.

This present article is a continuation of Jin, Dimitrov, and Ni (2019) with the following purpose. In this previous work, we have proved that the optimal exercising strategy is monotone in asset price, time-to-maturity, and economic information if both the transition probability matrix (TPM) for the hidden

Markov chain and the observable conditional probabilities (CPM) has a property of totally positive of order two (TP_2). However, the TP_2 properties of TPM and CPM are proved to be sufficient conditions. This means TP_2 may be relaxed to a milder condition.

As analytical proofs have limitations, in this work, we conduct extensive numerical experimental studies on randomly simulated matrices for TPM and CPM, in order to further clarify the relationship between the monotonicity of the exercising strategy and the properties of TPM and CPM. We assume a three-state economy and focus on the monotonicity of the threshold for the early exercise and hold regions. We confirm the effect of the sufficient conditions based on the results of our numerical experiments. Moreover, our results show that TPM affects the monotonicity of the threshold more than CPM. In addition, we add a numerical discussion on TPM with the stochastic increasing property introduced in Marshall, Olkin, and Arnold (1979), which is a milder condition than TP_2, and found that it also establishes the monotonicity of threshold with a large range.

This article proceeds as follows. The Markovian regime-switching model is presented in Sec. 2 and the American option pricing problem is formulated in Sec. 3. In Sec. 4, a summary without proofs of some analytical results from our previous research Jin, Dimitrov, and Ni (2019) is given. In the end, in Sec. 5, results of numerical experimental studies are presented.

2. Partially Observable Markovian Regime-Switching Model

Consider an American option with a strike price K and a maturity time T. The payoff function for call (put) is given by $v^e(s) = \max\{s-K, 0\}$ ($v^e(s) = \max\{K-s, 0\}$) with domain $s \in [0, \infty)$.

Make a grid on $[0, T]$ using $M+1$ grid points $\{0, 1h, \dots, th, \dots, Mh = T\}$. Here, M is an integer and h is the length of period. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{M}}, Q)$, $\mathbb{M} = \{0, 1, \dots, M\}$ be a complete filtered probability space where the probability measure Q is the market-chosen risk-neutral probability measure which we take as given. All expectations are taken with respect to Q hereafter, all stochastic processes and probability distributions are defined in the above probability space.

In the formulated model, the asset price process with a deterministic initial value S_0, defined on times epochs $1, 2, \dots, t, \dots, M$, is given by

$$S_t = S_{t-1}X_t, \quad t = 1, \dots, M,$$

where the *price relative* X_t depends on a variable economic situation Z at time t.

Next, suppose that economy Z_t takes a value from a finite state space $\mathbb{Z} = \{1, 2, \dots, n\}$. The economy states are ordered in an ascending order with 1 being the worst economic situation and n being the best. Let the changing of economic situation Z be defined by known a TPM $\mathbf{P} = [p_{ij}]_{i,j\in\mathbb{Z}}$ where p_{ij} refers to the probability that the economic situation transits from level i to level j.

At each period, consider a random variable Y that provides incomplete information related to the real economic situation Z. An observation of Y comes from a finite set $\mathbb{Y} = \{1, 2, \ldots, m\}$. Let $\mathbf{\Gamma} = [\gamma_{j\theta}]_{j\in\mathbb{Z},\theta\in\mathbb{Y}}$ be a conditional probability matrix (CPM) that describes the relationships between the economic situation and the observations. Here, $\gamma_{j\theta} = P(Y = \theta|Z = j)$ is the element of $\mathbf{\Gamma}$ in jth row and θth column.

Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ be a probability vector, called the *economy information vector*, that expresses the information about the economic situation. Here,

$$\pi_j = P\left(Z = j\right), \quad j = 1, 2, \ldots, n, \quad \sum_{j=1}^{n} \pi_j = 1.$$

At any period, the pair $(s, \boldsymbol{\pi})$ is called a process state, meaning that the current asset price is s and the economy information vector is $\boldsymbol{\pi}$.

3. American option pricing

It is well known that it is never optimal to early exercise an American call option on a non-dividend-paying underlying asset. Therefore, we consider a dividend-paying underlying asset with a continuously compounded dividend yield, denoted by δ, throughout the paper.

At each time epoch t, the option holder can choose to early exercise or hold the option. If the holder decides to early exercise, then a payoff of $v^e(s)$ is received where s is the underlying asset price at time t.

Assume that if a holder decides to hold the option, then the information vector at the beginning of the next period is updated to $\mathbf{T}(\boldsymbol{\pi}, \theta)$, given the observation θ with probability $\psi(\theta|\boldsymbol{\pi})$. The probability $\psi(\theta|\boldsymbol{\pi})$ is given by

$$\psi(\theta|\boldsymbol{\pi}) = \sum_{j=1}^{n}\sum_{i=1}^{n} \pi_i p_{ij} \gamma_{j\theta},$$

and the jth element of the updated information vector $\mathbf{T}(\boldsymbol{\pi}, \theta)$ is

$$T_j(\boldsymbol{\pi}, \theta) = \frac{\sum_{i=1}^{n} \pi_i p_{ij} \gamma_{j\theta}}{\psi(\theta|\boldsymbol{\pi})}.$$

In the next step, we formulate the optimal stopping problem using a partially observable Markov decision process.

Denote N be the remaining periods to maturity, for example $N = M = T/h$ at the beginning of option transaction and $N = 0$ at maturity.

Let $r > 0$ be the continuous compounded risk-free interest rate. Assume that X_j is distributed as the well-known risk-neutral distribution in a binomial tree,

$$P(X_j = x_j) = \begin{cases} q_j, & x_j = u_j \\ 1 - q_j, & x_j = d_j \end{cases}$$

where

$$q_j = \frac{e^{(r-\delta)h} - d_j}{u_j - d_j}, \quad u_j = e^{\sigma_j \sqrt{h}}, \quad d_j = \frac{1}{u_j},$$

and $\sigma_j > \sigma_{j'}$ for $j < j'$. The arbitrage opportunities are excluded when

$$d_j < e^{(r-\delta)\sqrt{h}} < u_j, \quad j \in \mathbb{Z}.$$

Note that the economy is ordered according to the volatility σ_i, where higher volatility is linked to a worse economy and lower volatility to a better economy. Indeed, low volatility indicates usually a stable market, on the other hand, most of the assets show very high volatility under a bad economy.

Remark 1. In our model, it can be calculated that $E[X^j], j = 1, 2, \ldots, N$ is the same for all states. This is because, under the risk-neutral probability, the underlying asset drift at the state-homogeneous risk-free interest rate subtracting the state-homogeneous dividend yield rate. At the same time $Var[X^j] > Var[X^{j'}]$ for states $j < j'$. Hence, X^j and $X^{j'}$ cannot be ordered in terms of stochastic ordering. Therefore, as mention in the introduction, Assumption 3.1 (i) in Sato and Sawaki (2014) does not apply, implying that our model is not a particular case of the generic model in their work.

Consider an American option with the current process state $(s, \boldsymbol{\pi})$ and remaining periods to maturity N. The option price $v_N(s, \boldsymbol{\pi})$, is given by

$$v_N(s, \boldsymbol{\pi}) = \max \begin{cases} \max\{K - s, 0\} = v_N^e(s) \\ \beta \sum_{\theta=1}^{m} \psi(\theta|\boldsymbol{\pi}) \sum_{k=1}^{2} v_{N-1} \left[s x_j^k, \mathbf{T}(\boldsymbol{\pi}, \theta) \right] P(x_j^k) = v_N^h(s, \boldsymbol{\pi}) \end{cases}$$

where $x_j^1 = u_j, x_j^2 = d_j$, and $\beta = e^{-rh}$ $(0 < \beta < 1)$ is the discount factor.

The quantity denoted by $v_N^e(s, \boldsymbol{\pi})$ is the payoff when the holder exercises the option at the beginning of the current period, and denoted by $v_N^h(s, \boldsymbol{\pi})$ is the value when the holder decides to hold and follow the optimal strategy in the remaining periods. As the payoff of early exercise does not depend on the remaining periods, notation $v^e(s)$ is used instead of $v_N^e(s)$.

The hold value $v_0^h(s, \boldsymbol{\pi})$ is zero at the maturity. So the option value at the maturity is simply the payoff function, i.e.,

$$v_0(s, \boldsymbol{\pi}) = \max\{v^e(s), v_0^h(s, \boldsymbol{\pi})\} = v^e(s).$$

4. Analytical structural properties of American options

First, we review the following definitions of totally positive property of order 2 (Karlin 1968) and stochastic increasing property (Marshall, Olkin, and Arnold 1979).

Definition 4.1. If for two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \geq 0, \quad 1 \leq i < j \leq n,$$

holds, then it is said that $\mathbf{y}$ dominates $\mathbf{x}$ in the sense of totally positive ordering of order 2, denoted by $\mathbf{x} \overset{\mathrm{TP}_2}{\prec} \mathbf{y}$.

Definition 4.2. Let $\mathbf{X} = [x_{ij}]_{ij}$ be an $n \times m$ matrix for which

$$\det(\mathbf{B}) \geq 0$$

for every submatrix $\mathbf{B} = [x_{i_k j_l}]_{kl}$ of dimensions 2×2 where $1 \leq i_1 < i_2 \leq n, 1 \leq j_1 < j_2 \leq m$. Matrix $\mathbf{X}$ is said to have a property of totally positive of order two, denoted by $\mathbf{X} \in \mathrm{TP}_2$.

Definition 4.3. If for two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$

$$\sum_{i=k}^{n} x_i \leq \sum_{i=k}^{n} y_i \quad \text{for} \quad k = 1, \ldots, n,$$

holds, then it is said that $\mathbf{y}$ dominates $\mathbf{x}$ in the sense of stochastic increasing order, denoted by $\mathbf{x} \overset{\mathrm{SI}}{\prec} \mathbf{y}$.

Definition 4.4. Let $\mathbf{X} = [x_{ij}]_{ij}$ be an $n \times m$ for which

$$\sum_{j=k}^{m} x_{ij} \leq \sum_{j=k}^{m} x_{i'j} \quad \text{for} \quad 1 \leq i < i' \leq n \quad \text{and} \quad k = 1, \ldots, m.$$

Matrix $\mathbf{X}$ is said to have a stochastic increasing (SI) property, denoted by $\mathbf{X} \in SI$.

To obtain structural properties of American options, let us impose the following assumptions.

(A-1) TPM $\mathbf{P}$, corresponding to economic situation, has a TP_2 property.
(A-2) Conditional probability matrix $\boldsymbol{\Gamma}$, corresponding to signal, has a TP_2 property.

The TP_2 property of TPM implies that a better economy in a period moves to a more progressive situation in the next period. The TP_2 property of CPM implies that a better economic situation gives rise to higher output levels for the observations probabilistically.

In the authors' previous research Jin, Dimitrov, and Ni (2019), analytical structural properties of American options were derived using the sufficient conditions given by assumptions (A-1) and (A-2). Some results that are important for the experimental studies in this article are given below without proof. Note that functions are considered as increasing or decreasing in the weak sense throughout the article.

Proposition 4.5. For a put (call) American option, if assumptions (A-1) and (A-2) hold, then hold value of the option $v_N^h(s,\boldsymbol{\pi})$ is

1. decreasing (increasing) in asset price s for every N and $\boldsymbol{\pi}$;
2. increasing in remaining period N for every s and $\boldsymbol{\pi}$;
3. decreasing in information vector $\boldsymbol{\pi}$ in the sense of TP_2 for every s and N.

Proposition 4.5 establishes the monotonicity of $v_N^h(s,\boldsymbol{\pi})$ in N, s, and $\boldsymbol{\pi}$, that is, to be monotone in remaining time to maturity, asset price and the economic situation.

Proposition 4.6. For an American put (call) option, (i) $v_N^h(s,\boldsymbol{\pi})$ is a convex function of s, (ii) the decreasing (increasing) rate of $v_N^h(s,\boldsymbol{\pi})$ in s is less than 1 for any $\boldsymbol{\pi}$ under the assumptions (A-1) and (A-2).

Proposition 4.7. For an American put or call option, the difference between $v_N^h(s,\boldsymbol{\pi})$ and $v^e(s)$ is increasing in N for any s and $\boldsymbol{\pi}$ under the assumptions (A-1) and (A-2).

Proposition 4.8. For an American put or call option, the difference between $v_N^h(s,\boldsymbol{\pi})$ and $v^e(s)$ is decreasing in $\boldsymbol{\pi}$ in the sense of TP_2 ordering for any N and s under the assumptions (A-1) and (A-2).

Propositions 4.6–4.8 provide some properties of the relationship between $v_N^h(s,\boldsymbol{\pi})$ and $v^e(s)$, and these properties are important for the following discussion on the thresholds for the following two regions.

The early exercising region and holding region for every remaining periods to maturity N are defined by

- Exercise region

$$
\begin{aligned}
D_N^e &= \left\{(s,\boldsymbol{\pi}) \mid v_N^h(s,\boldsymbol{\pi}) < v^e(s)\right\} \\
&= \left\{(s,\boldsymbol{\pi}) \mid v_N(s,\boldsymbol{\pi}) = v^e(s)\right\}
\end{aligned}
$$

- Hold region

$$
\begin{aligned}
D_N^h &= \left\{(s,\boldsymbol{\pi}) \mid v_N^h(s,\boldsymbol{\pi}) > v^e(s)\right\} \\
&= \left\{(s,\boldsymbol{\pi}) \mid v_N(s,\boldsymbol{\pi}) = v_N^h(s,\boldsymbol{\pi})\right\}
\end{aligned}
$$

Next, we investigate the thresholds in both $\boldsymbol{\pi}$ and s for the above two regions. To discuss the threshold in $\boldsymbol{\pi}$, one needs to define the set of all TP_2-ordered sets of economy information vectors, denoted by Π, as follows

$$\Pi = \bigcup \{\boldsymbol{\pi}^i, i \in \mathfrak{I} : \boldsymbol{\pi}^k \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}^l \text{ for } k \leq l, k, l \in \mathfrak{I}\},$$

where $\mathfrak{I}$ refers to the enumeration of information vectors in each TP_2-ordered set. There are infinitely many elements, i.e., TP_2-ordered sets, in Π. The set Π, which is the union of all such ordered-sets, matches the space of all economy information vectors $\{(\pi_1, \ldots, \pi_n) : \sum_{i=1}^n \pi_i = 1\}$.

Notation TP_2^* is used to denote an arbitrary TP_2-ordered set in Π.

Proposition 4.9. For an American put or call option, there exists at most one threshold $\boldsymbol{\pi}_N(s)$ for any s and N which separates an ordered set TP_2^* into two regions: an (early) exercise for any $\boldsymbol{\pi}$ $(\in \mathrm{TP}_2^*)$ less than $\boldsymbol{\pi}_N(s)$, and a hold region otherwise. Moreover, $\boldsymbol{\pi}_N(s') \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}_N(s)$ for $s < s'$, and $\boldsymbol{\pi}_{N^1}(s) \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}_{N^2}(s)$ for $N^1 < N^2$.

Proposition 4.9 focuses on economy situation and presents a property of the threshold in $\boldsymbol{\pi}$ $(\in \mathrm{TP}_2^*)$. Figures 1 and 2 are used to illustrate the thresholds for different s and N.

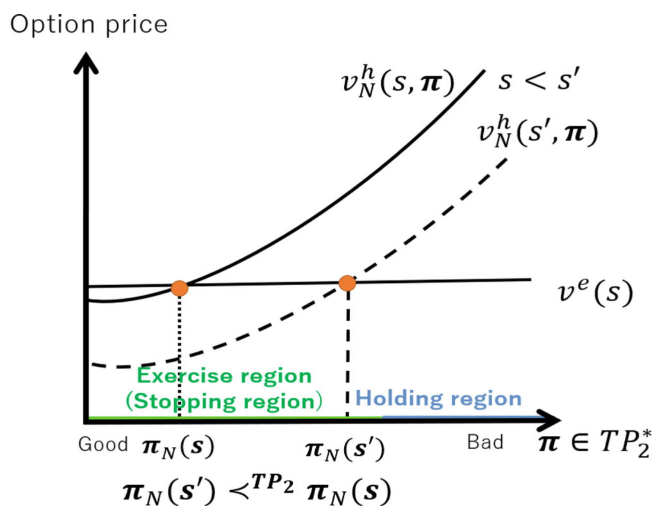


Figure 1. Threshold $\boldsymbol{\pi}_N(s)$ with different asset prices s for the case of an American put option.

Next, focus on the asset price s and obtain a similar property of the threshold in s as given in Proposition 4.10.

Proposition 4.10. For an American put or call option, there exists at most one threshold $s_N(\boldsymbol{\pi})$ for any $\boldsymbol{\pi}$ and N which separates the space of s into two regions: (early) exercise region $(s < s_N(\boldsymbol{\pi}))$ and holding region $(s > s_N(\boldsymbol{\pi}))$. Moreover, $s_N(\boldsymbol{\pi}^1) \leq s_N(\boldsymbol{\pi}^2)$ for $\boldsymbol{\pi}^1 \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}^2$, and $s_{N^1}(\boldsymbol{\pi}) \geq s_{N^2}(\boldsymbol{\pi})$ for $N^1 < N^2$.

From Propositions 4.9 and 4.10, we know that the information space $(s, \boldsymbol{\pi})$ for any $\boldsymbol{\pi}$ $(\in \mathrm{TP}_2^*)$ is divided into at most two regions, D_N^e and D_N^h, and the area of

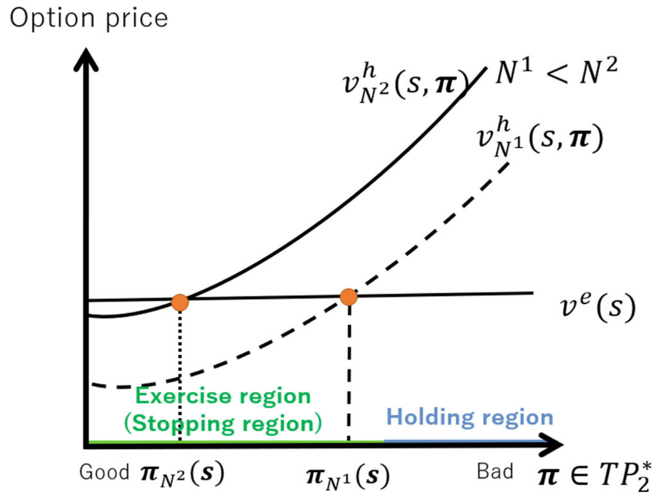


Figure 2. Threshold $\boldsymbol{\pi}_N(s)$ with different remaining periods N for the case of an American put option.

holding region D^h_N increases with the remaining periods N as shown in Figure 3. This means it is preferable to hold the option if more time periods remain.

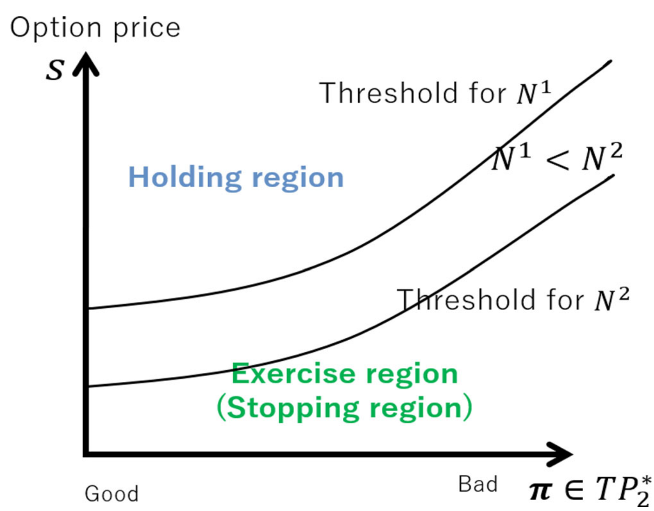


Figure 3. Threshold on space $\boldsymbol{\pi}_N(s)$ with different remaining periods N for the case of an American put option.

Remark 2. It should be noted that the monotonicity of the threshold with respect to the information vector $\boldsymbol{\pi}$ is not naturally established. It is necessary to confirm, under which conditions, the optimal strategy is increasingly or decreasingly monotone with respect to the information vector. This was addressed in the present section. As shown in the next section, in Case 3 with a violation of the TP_2 property of TPM, the monotonicity is not guaranteed.

5. Numerical experimental studies

In this section, results of experimental studies are presented. The object of interest is the behavior of the analytical properties of optimal exercising strategies under some deviations from the strict assumptions (A-1) and (A-2).

5.1. Three-state model

Assume that the information vector consists of three pieces of information. Hence, consider a three-state model. For the details of the model implementation refer to the previous research by Jin, Dimitrov, and Ni (2019).

The thresholds in the numerical examples are obtained as follows. Let S be a set of initial asset prices S_0. First, take a finite subset $S^* \subseteq S$, and an order-set $\mathrm{TP}_2^* \in \Pi$ for $n = 3$. Then, for a fixed economy information vector $\boldsymbol{\pi} \in \mathrm{TP}_2^*$ compute the option price for every $S_0 \in S^*$. Initial asset price $s^* \in S^*$ is the one-threshold that splits the set S^* into a (early) exercise region D_N^e and a hold region D_N^h.

5.2. TP_2^*-ordered set of economy information vectors

To determine a TP_2 ordered set of economy information vectors, that is a set $\mathrm{TP}_2^* \in \Pi$ for $n = 3$, the following proposition was used.

Proposition 5.1. Let $\boldsymbol{\pi}_1 = (p_1, p_2, 1 - p_1 - p_2)$ and $\boldsymbol{\pi}_2 = (q_1, q_2, 1 - q_1 - q_2)$ such that $\boldsymbol{\pi}_1 \neq \boldsymbol{\pi}_2$. If $p_1 \neq 0$ and $p_1 = q_1$, then $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are not TP_2 comparable. If $p_1 \geq q_1$ and $p_2 = q_2$, then $\boldsymbol{\pi}_1 \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}_2$.

Proof. Let $p_1 = q_1$,

$$\boldsymbol{\pi}_1 = (p_1, p_2, 1 - p_1 - p_2),$$
$$\boldsymbol{\pi}_2 = (p_1, q_2, 1 - p_1 - q_2).$$

Assume opposite that $\boldsymbol{\pi}_1 \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}_2$. Then $q_2 \geq p_2$ from

$$\begin{vmatrix} p_1 & p_2 \\ p_1 & q_2 \end{vmatrix} \geq 0.$$

However, $q_2 \geq p_2$ leads to

$$\begin{vmatrix} p_1 & 1 - p_1 - p_2 \\ p_1 & 1 - p_1 - q_2 \end{vmatrix} \leq 0,$$

Therefore, $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are not TP_2 comparable if $p_1 = q_1$. Assume that $p_2 = q_2$, by the definition of totally positive of order two, it follows that $\boldsymbol{\pi}_1 \overset{\mathrm{TP}_2}{\prec} \boldsymbol{\pi}_2$ because $p_1 \geq q_1$. $\square$

Table 1. General model test parameters.

Name	Notation	Parameters
Maturity time	T	$8/252$
Number of steps	M	4
Time duration of a step	h	$2/252$
Volatility vector	$\boldsymbol{\sigma}$	$(0.5, 0.3, 0.1)$
Strike price	K	100
Interest rate	r	0.02
Dividend yield (American call)	δ	0.1
TPM	$\mathbf{P}$	$[p_{ij}]_{i,j=1,2,3}$
CPM	$\mathbf{\Gamma}$	$[\gamma_{ij}]_{i,j=1,2,3}$

5.3. Numerical results

In this subsection, unless it is said otherwise, the parameters used for computation are given in Table 1.

The choice of parameters has to satisfy assumptions (A-1) and (A-2). Hence, both the TPM and CPM matrices should have the property of TP_2.

The experimental study in this paper will be concentrated on the threshold, that is the early exercise and hold regions. Other results may be found in Jin, Dimitrov, and Ni (2019). To show the early exercise and hold regions, as well as the monotonicity of threshold in the information vectors $\boldsymbol{\pi}$, a set $\mathrm{TP}_2^* \in \Pi$ and a set S^* of initial asset prices are required. Denote the following set with TP_2^*:

$$\begin{aligned}
\pi_2 &= 0.05, \\
\pi_1 &= \pi_2 + 0.03 \times i, \quad i = 0, 1, \ldots, 30, \\
\pi_3 &= 1 - \pi_1 - \pi_2.
\end{aligned}$$

As explained in Sec. 5.2, this is a set of TP_2 ordered information vectors. Thus, $\mathrm{TP}_2^* \in \Pi$. Next, the set of initial asset prices used for the experimental studies is

$$S_p^* = \left\{ \left(0.7 + \frac{0.3}{5000} \times i \right) \times K : i \in \{0, 1, \ldots, 5000\} \right\}$$

for an American put, and

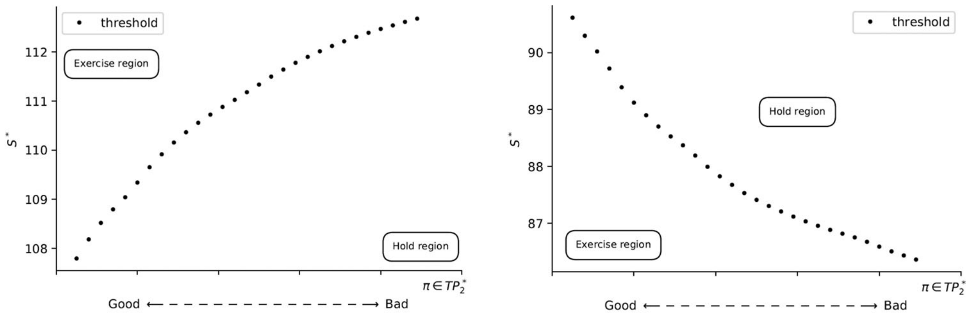
$$S_c^* = \left\{ \left(1 + \frac{0.3}{5000} \times i \right) \times K : i \in \{0, 1, \ldots, 5000\} \right\}$$

for an American call option with dividend yield $\delta = 0.1$ were used. Let TPM and CPM be

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.5 \\ 0.05 & 0.25 & 0.7 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.4 & 0.5 \\ 0.05 & 0.4 & 0.55 \end{bmatrix}.$$

respectively. This choice of matrices satisfies TP_2 property, $\mathbf{P}, \mathbf{\Gamma} \in \mathrm{TP}_2$.

Figure 4(a) and (b) show that the threshold is decreasing and increasing in $\boldsymbol{\pi}$ for $N = 4$, as well as the exercise and hold regions for both American call option with dividend yield and American put, respectively.



(a) American call option with dividend yield. (b) American put option.

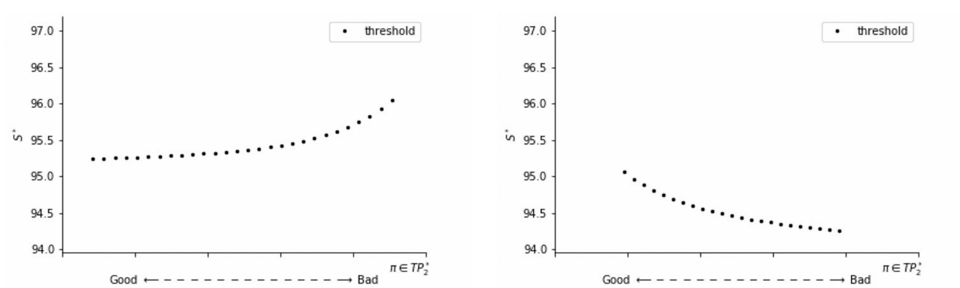
Figure 4. An example of the optimal stopping regions for an American option and the monotonicity of the threshold in $\boldsymbol{\pi}$ with parameters given in Table 1, TP_2^*, S_c^*, and S_p^*.

Consider an American put option in the following until the end. To see the model behavior when $\mathbf{P}$, $\boldsymbol{\Gamma}$ or both do not have TP_2 property, a various sets $\mathrm{TP}_2^i \in \Pi, i = 1, 2, \ldots, 10,000$ were used. To do such Monte Carlo simulation, sets $\mathrm{TP}_2^i \in \Pi, i = 1, 2, \ldots, 10,000$ were randomly generated, as well as $\mathbf{P}$, $\boldsymbol{\Gamma}$ or both. The following three cases were considered

1. both TPM and CPM are randomly generated;
2. CPM is randomly generated;
3. TPM is randomly generated.

In all three cases sets of TP_2 ordered vectors were randomly generated according to the property explained in Sec. 5.2.

Case 1: TPM and CPM are randomly generated matrices that satisfy conditions of the TPM of a Markov chain. As expected, varying both matrices, TPM and CPM, the claim that the threshold is monotonically increasing does not stand. Figure 5(a) shows that the threshold is decreasing in information vector $\boldsymbol{\pi}$ under assumptions of Case 1, on the other hand, Figure 5(b) shows that the threshold is increasing in information vector $\boldsymbol{\pi}$ under assumptions of Case 1.



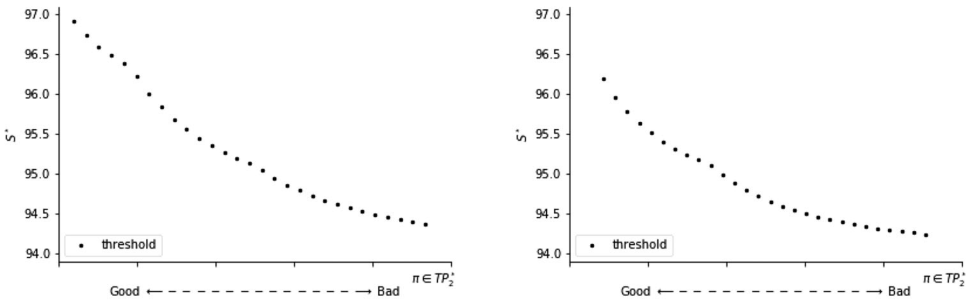
(a) The threshold is monotonically decreasing in information vector $\boldsymbol{\pi}$. (b) The threshold is monotonically increasing in information vector $\boldsymbol{\pi}$.

Figure 5. An example of Case 1, where matrices TPM and CPM were randomly generated and the threshold is monotone in information vector $\boldsymbol{\pi}$.

Case 2: In this case, TPM $\mathbf{P}$ is fixed and given by

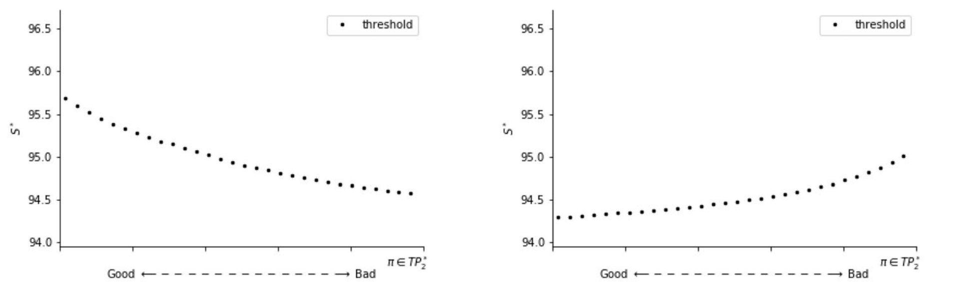
$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.5 \\ 0.05 & 0.25 & 0.7 \end{bmatrix}$$

and CPM $\mathbf{\Gamma}$ is a randomly generated matrix that satisfies conditions of the TPM of a Markov chain. Based on the simulation, deviation from the sufficient assumption $\mathbf{\Gamma} \in \mathrm{TP}_2$ does not affect the monotonicity of the threshold as much as the deviation from the sufficient conditions presented in Case 1. Figures 6(a) and 7(b) show that the threshold is increasing in the information vector $\boldsymbol{\pi}$ under assumptions of Case 2, that is, CPM is a randomly generated matrix.



(a) Example 1. (b) Example 2.

Figure 6. Two examples of Case 2, where matrix CPM was randomly generated and the threshold is increasing in information vector $\boldsymbol{\pi}$.



(a) The threshold is monotonically increasing in information vector $\boldsymbol{\pi}$. (b) The threshold is monotonically decreasing in information vector $\boldsymbol{\pi}$.

Figure 7. An example of Case 3, where matrices TPM and CPM were randomly generated and the threshold is monotone in information vector $\boldsymbol{\pi}$.

Case 3: Assume now that CPM $\mathbf{\Gamma}$ is fixed and given by

$$\mathbf{\Gamma} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.4 & 0.5 \\ 0.05 & 0.4 & 0.55 \end{bmatrix}$$

and TPM $\mathbf{P}$ is randomly generated matrix that satisfies conditions of TPM of a Markov chain. Violation of assumption $\mathbf{P} \in \mathrm{TP}_2$ influence the results obtained for the threshold after a couple of simulations. It is clear that the condition $\mathbf{P} \in \mathrm{TP}_2$ affects the monotonicity of threshold more than the condition $\boldsymbol{\Gamma} \in \mathrm{TP}_2$. Figure 5(a) shows that the threshold is increasing in information vector $\boldsymbol{\pi}$ under assumptions given in Case 3, and Figure 5(b) shows that the threshold is decreasing in information vector $\boldsymbol{\pi}$ under assumptions of Case 3.

Henceforth, a milder condition for TPM will be investigated. Particularly, assume that TPM has stochastic increasing property, but it does not have TP_2 property. The following matrix is an example of such case:

$$\mathbf{X} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.5 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}.$$

The parameters from Table 1 are used for the numerical study with a change of TPM matrix $\mathbf{P}$ which is in the following form:

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.5 \\ p^x & 1 - p^x - p^y & p^y \end{bmatrix}$$

where $0 \le p^x \le 0.1$ and $0.5 \le p^y \le 1$. It can be shown that matrix $\mathbf{P}$ has stochastic increasing property. A couple choices of p^x and p^y contemplated that yield matrix of interest, that is, a matrix which has stochastic increasing but not TP_2 property, are given in Table 2. In combination with the previously chosen set $TP_2^* \in \Pi$, as well as other randomly chosen sets from Π, the extensive numerical study suggest that the monotonicity of threshold is established under the milder condition—stochastic increasing property. The analytical proof is yet to be derived.

Table 2. Choices of probabilities in the last row of $\mathbf{P}$ that yield a matrix with stochastic increasing property and without TP_2 property.

p^x	$1 - p^x - p^y$	p^y
0.1	0.3	0.6
0.1	0.2	0.9
0.1	0.1	0.8
0.1	0	0.9
0	0.5	0.5
0.05	0.15	0.8
0.05	0.05	0.9

6. Conclusion and future research

Review of the previous research and new experimental results of the American option pricing and the corresponding optimal exercising strategies under a novel model were presented. Under this model, the asset price follows an extended binomial tree with the volatility parameter governed by a discrete-time hidden Markov chain.

The numerical results indicate that violation of sufficient conditions affect the structural properties of American put and call options with dividend yield. That is, the TP_2 property of the TPM for the economic situations is important for having the monotonically decreasing (increasing) property of the exercising threshold. One of the assumptions may be relaxed in practice as it does not affect the monotonicity to great extent. Specifically, the TP_2 property of CPM may be ignored as it has little effect on the monotonicity.

For future research, the model shall be generalized by permitting a more general probability distribution for the asset price dynamics. In particular, a follow-up work including an arbitrary distribution function $F(\cdot)$ is under preparation. As the results of this research are limited to the pricing of short-maturity options future research may contain extensions of the model for options with longer maturities.

In addition, estimation and calibration of the model shall be implemented. In the recent years, machine learning had a great impact on the development of model estimation. With that said, to estimate the parameters of the model considered in this paper the online Hidden Markov model estimation-based Q-learning algorithm for partially observable Markov decision process studied in Yoon, Lee, and Hovakimyan (2019) may be used. A modification of the algorithm for regime-switching calibration that uses the Tikhonov regularization approach provided in He and Zhu (2021) may be considered in combination with the mentioned estimation procedure. Real market data will be used for both estimation and calibration.

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