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Perturbed Markov Chains with Damping Component and Information Networks

Benard Abola

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Abstract

This thesis brings together three thematic topics, PageRank of evolving tree graphs, stopping criteria for ranks and perturbed Markov chains with damping component. The commonality in these topics is their focus on ranking problems in information networks. In the fields of science and engineering, information networks are interesting from both practical and theoretical perspectives. The fascinating property of networks is their applicability in analysing broad spectrum of problems and well established mathematical objects. One of the most common algorithms in networks' analysis is PageRank. It was developed for web pages' ranking and now serves as a tool for identifying important vertices as well as studying characteristics of real-world systems in several areas of applications. Despite numerous successes of the algorithm in real life, the analysis of information networks is still challenging. Specifically, when the system experiences changes in vertices /edges or it is not strongly connected or when a damping stochastic matrix and a damping factor are added to an information matrix. For these reasons, extending existing or developing methods to understand such complex networks is necessary.

Chapter 2 of this thesis focuses on information networks with no bidirectional interaction. They are commonly encountered in ecological systems, number theory and security systems. We consider certain specific changes in a network and describe how the corresponding information matrix can be updated as well as PageRank scores. Specifically, we consider the graph partitioned into levels of vertices and describe how PageRank is updated as the network evolves.

In Chapter 3, we review different stopping criteria used in solving a linear system of equations and investigate each stopping criterion against some classical iterative methods. Also, we explore whether clustering algorithms may be used as stopping criteria.

Chapter 4 focuses on perturbed Markov chains commonly used for the description of information networks. In such models, the transition matrix of an information Markov chain is usually regularised and approximated by a stochastic (Google type) matrix. Stationary distribution of the stochastic matrix is equivalent to PageRank, which is very important for ranking of vertices in information networks. Determining stationary probabilities and related characteristics of singularly perturbed Markov chains is complicated; leave alone the choice of regularisation parameter. We use the procedure of artificial regeneration for the perturbed Markov chain with the matrix of transition probabilities and coupling methods. We obtain ergodic theorems, in the form of asymptotic relations. We also derive explicit upper bounds for the rate of convergence in ergodic relations. Finally, we illustrate these results with numerical examples.

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This thesis is dedicated to the memory of my beloved grandfather `Semei Abolla`.

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Västerås, November 2020

Benard Abola

Populärvetenskaplig sammanfattning

Ett nätverk är ett system av sammankopplade enheter eller objekt. Objekten representeras av hörn och sammankopplade av kanter (länkar) som representerar relationer mellan dem. En matematisk modell härledd från ett nätverk är ett av de främsta sätten att modellera komplexa system.

I sådana modeller kräver användare normalt en handfull information från ett stort dataset. Att identifiera inflytelserika enheter i ett nätverk har många användningsområden:-folkhälsofrämjande och medvetenhet, cancer-genidentifiering, utvärdera forskningsavtryck och ekologisk förvaltning, för att bara nämna några. PageRank-algoritmen är ett av de mest kända verktygen för rangordning av webbsidor i ett informationsnätverk. Den utvecklades för att hitta relativ betydelse för webbsidor; emellertid har den anpassats till många applikationer.

Avhandlingen inleds med beskrivning av metoder för uppdatering av PageRank för ett riktat träddiagram under utveckling. Sådana informationsnätverk förekommer ofta i citat och ekologiska nätverk. Att hitta viktiga noder i ekologiska nätverk är viktigt när ekologer vill undvika utrotning av kritiska arter eller fragmentering i systemet. Dessutom presenterar avhandlingen ett rankningsproblem som är associerat med konvergensen mellan PageRank och rangordningar, en uppgift som är nödvändig när rangordningen av objekt väger mer än att hitta korrekta PageRank-poäng. Till exempel i sport är fotbollsfans intresserade av en lista över bästa spelare eller lag i en viss liga, inte numeriska poäng kopplade till varje spelare eller lag.

Diskreta slumpmässiga processer är vanliga i verkliga livet och det återspeglar vårt naturliga sätt att hänvisa till tidigare, nuvarande och framtida ögonblick. Om framtida händelser av en slumpmässig process beror på nutiden men inte det förflutna, uppvisar processen vad som kallas Markov-egenskapen. En process med en sådan egenskap har många tillämpningar inom teknik och modellering av komplexa system. Vid rankning av noder i ett informationsnätverk är det vanligt att introducera en dämpande term i sådana Markov modeller (kort sagt, störda Markov modeller). Avhandlingen beskriver proceduren för inbäddning av en sådan Markov-kedja i modellen för regenerativ process (det vill säga en process som startar om sig själv), i sin tur, leder till förnyelsestypsekvationer som är enklare att hantera. För detta ändamål erhålls vissa egenskaper som är användbara för att fatta beslut. Några egenskaper hos störda Markov-kedjor såsom konvergenshastighet (hastighet) och asymptotiska förhållanden för övergångssannolikheter i tid och dämpningsparameter för Markov-kedjor med dämpningskomponent diskuteras.

Popular science summary

A network is a system of interconnected entities or objects. The objects are represented by vertices and interconnected by edges (links) which signify relations between them. A mathematical model derived from a network is one of the foremost means of modeling complex systems. In such models, users normally require a handful of information from a large dataset. Identifying influential entities in a network has many applications:- public health promotion and awareness, cancer-gene identification, evaluating research footprint and ecological management, to name but a few. One of the most well known tools for ranking webpages of an information network is the PageRank algorithm. It was developed for finding relative importance of web pages; however, it has been adapted to many applications.

The thesis begins with description of methods of updating PageRank of an evolving directed tree graph. Such information networks are commonly encountered in citation and ecological networks. Finding important nodes in ecological networks is vital when ecologists want to avoid extinction of critical species or fragmentation in the system. Furthermore, the thesis discusses convergence of PageRank and ranks, a task that is essential when ranks of objects weigh more than finding accurate PageRank scores. For example, in sport, soccer fans are interested in a list of best players or teams in a particular league, not numerical scores attached to each player or team.

Discrete time random processes are common in real life and it reflects our natural way of referring to past, present and future moments. If the future occurrences of a random process depend on the present but not the past, then the process exhibits what is called the Markov property. A process with such a property has many applications in engineering as well as modeling of complex systems. When ranking nodes in an information network, it is common to introduce a damping term in such Markov models (in short, a perturbed Markov models). The thesis describes the procedure of embedding such a Markov chain in the model of discrete time regenerative process (that is, a process that restarts itself), in turn, lead to renewal type equations that are simpler to handle. To this end, some quantities that are useful in making decision are obtained. For instance, some characteristics of the perturbed Markov chains such as the rate (speed) of convergence and asymptotic relations for transition probabilities in time and damping parameter for Markov chains with damping component are discussed.

Contents

List of Papers

This thesis is based on the following papers:

Paper A. Benard Abola, Pitos Seleka Biganda, Christopher Engström, John Magero Mango, Godwin Kakuba, Sergei Silvestrov, *PageRank in evolving tree graphs*, Stochastic Processes and Applications (S. Silvestrov, M. Rančić, A. Malyarenko, eds.), Springer Proceedings in Mathematics & Statistics, **271**, Springer, Cham, 2018, Chapter 16, 375–390.

Paper B. Benard Abola, Pitos Seleka Biganda, Christopher Engström, John Magero Mango, Godwin Kakuba, Sergei Silvestrov, *Updating of PageRank in Evolving Tree Graphs*, Data Analysis and Applications 3 (A. Makrides, A. Karagrigoriou, C.H. Skiadas, eds.), Computational, Classification, Financial, Statistical and Stochastic Methods, **5**, ISTE, Wiley, 2020, Chapter 2, 35–51.

Paper C. Benard Abola, Pitos Seleka Biganda, Christopher Engström, John Magero Mango, Godwin Kakuba, Sergei Silvestrov, *A Variant of updating PageRank in evolving tree graphs*, Proceedings of the 18th Conference of the Applied Stochastic Models and Data Analysis International Society and Demographics 2019 Workshop. Florence, Italy, 2019 (C.H. Skiadas, ed.), ISAST: International Society for the Advancement of Science and Technology, 2019, 31–49.

Paper D. Benard Abola, Pitos Seleka Biganda, Christopher Engström, Sergei Silvestrov, *Evaluation of stopping criteria for ranks in solving linear systems*, Data Analysis and Applications 1 (C. H. Skiadas, J. R. Bozeman, eds.), Clustering and Regression, Modeling-estimating, Forecasting and Data Mining, **2**, ISTE, Wiley, 2019, Chapter 10, 137–152.

Paper E. Benard Abola, Pitos Seleka Biganda, Dmitrii Silvestrov, Sergei Silvestrov, Christopher Engström, John Magero Mango, Godwin Kakuba,

Perturbed Markov chains and information networks. arXiv:1901.11483v 3 [math.PR] 2 May 2019, 60 pp.

Paper F. Dmitrii Silvestrov, Sergei Silvestrov, Benard Abola, Pitos Seleka Biganda, Christopher Engström, John Magero Mango, Godwin Kakuba, *Coupling and ergodic theorems for Markov chains with dampling component.* Theory of Probability and Mathematical Statistics, **101** (2019), no. 2, 212-231.

Paper G. Benard Abola, Pitos Seleka Biganda, Dmitrii Silvestrov, Sergei Silvestrov, Christopher Engström, John Magero Mango, Godwin Kakuba, *Nonlinearly Perturbed Mar-kov Chains and Information Networks*, Proceedings of the 18th Conference of the Applied Stochastic Models and Data Analysis International Society and Demographics 2019 Workshop. Florence, Italy, 2019 (C.H. Skiadas, ed.), ISAST: International Society for the Advancement of Science and Technology, 2019, 51–79.

Paper H. Dmitrii Silvestrov, Sergei Silvestrov, Benard Abola, Pitos Seleka Biganda, Christopher Engström, John Magero Mango, Godwin Kakuba, *Perturbed Markov chains with damping component*, Methodology and Computing in Applied Probability, 31 pp. (2020). https://doi.org/10.1007/s11009-020-09815-9

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Parts of this thesis have been presented in communications given at the following international conferences:

1: ASMDA2017–17th Applied Stochastic Models and Data Analysis International Conference with the 6th Demographics Workshop, 6–9 June 2017, London, UK.

2: SPAS2017–International Conference on Stochastic Processes and Algebraic Structures–From Theory Towards Applications, 4–6 October 2017, Västerås and Stockholm, Sweden.

3: SMTDA2018–5th Stochastic Modeling Techniques and Data Analysis International Conference with Demographics Workshop, 12–15 June 2018, Chania, Crete, Greece.

4: ASMDA2019–18th Conference of the Applied Stochastic Models and Data Analysis International Society and Demographics 2019 Workshop, 11–14 June 2019, Florence, Italy.

5: SPAS2019–International Conference on Stochastic Processes and Algebraic Structures–From Theory Towards Applications, 30th September–2nd October 2019. Västerås, Sweden.

The following papers are not included in the thesis:

- Pitos Seleka Biganda, Benard Abola, Christopher Engström, Sergei Silvestrov, *PageRank, connecting a line of nodes with multiple complete graphs*, Proceedings of the 17th Applied Stochastic Models and Data Analysis International Conference with the 6th Demographics Workshop. London, UK, 2017. ISAST: International Society for the Advancement of Science and Technology, 2017, 113–126.

- Pitos Seleka Biganda, Benard Abola, Christopher Engström, John Magero Mango, Godwin Kakuba, Sergei Silvestrov, *Traditional and lazy PageRanks for a line of nodes connected with complete graphs*, Stochastic Processes and Applications (S. Silvestrov, M. Rančić, A. Malyarenko, eds.), Springer Proceedings in Mathematics & Statistics, **271**, Springer, Cham, 2018, Chapter 17, 391–412.

- Pitos Seleka Biganda, Benard Abola, Christopher Engström, John Magero Mango, Godwin Kakuba, Sergei Silvestrov, *Exploring the relationship between ordinary PageRank, lazy PageRank and random walk with backstep PageRank for different graph structures*, Data Analysis and Applications 3 (A. Makrides, A. Karagrigoriou, C. H. Skiadas, eds.), Computational, Classification, Financial, Statistical and Stochastic Methods, **5**, ISTE, Wiley, 2020, Chapter 3, 53–73.

- Pitos Seleka Biganda, Benard Abola, Christopher Engström, John Magero Mango, Godwin Kakuba, Sergei Silvestrov, *PageRank and Perturbed Markov chains*. Proceedings of the 18th Conference of the Applied Stochastic Models and Data Analysis International Society and Demographics 2019 Workshop. Florence, Italy, 2019 (C.H. Skiadas, ed.), ISAST: International Society for the Advancement of Science and Technology, 2019, 233–247.

• Dmitrii Silvestrov, Sergei Silvestrov, Benard Abola, Pitos Seleka Biganda, Christopher Engström, John Magero Mango, Godwin Kakuba, *Perturbation analysis for stationary distribution of Markov chains with damping component*, Algebraic Structures and Applications (S. Silvestrov, A. Malyarenko, M. Rančić, eds.), Springer Proceedings in Mathematics and Statistics, **317**, Springer, Cham, 2020, Chapter 38, 903–933.

Chapter 1

Introduction

Information retrieval refers to a process of representing, storing, organising and accessing information from documents, web pages and many others. The process of retrieving relevant information that satisfies users' need can be traced back about 5 millennia years ago [12, 13, 119]. In that era, pieces of information were documented in either papyrus rolls or clay tablets, and accessing relevant information was a great challenge [13, 88, 98]. This has persisted up to the present time. In view of [5], filtering information from a web-graph is complicated because the environment can not be controlled. In some instances, one needs to pay attention to redundant attributes, composition of the structure(s), users' behaviour and partial or overall changes in the network [1, 2, 8, 35, 36, 38].

Currently, the need for easy access to relevant information within optimal time is great. The demand seems to overweigh the existing classical methods of information retrieval such as Boolean, vector space models and text frequency-inverse document frequency and the probabilistic method [5, 77].

To deal with the challenge of ranking or retrieving relevant pieces of information in graph models, PageRank algorithm, developed by Brin and Page in late 1990s for Google search engine [23], is one of the most preferred. The novelty of Brin and Page's search algorithm forms one of the greatest applications of Markov chains, pioneered by Andrei A. Markov [130]. Importantly, research areas of Markov chain and semi Markov chain processes, perturbed stochastic processes and their applications have been extensively investigated for decades [103]. However there is still much more to be done.

Essentially, perturbed Markov chains have applications in engineering

such as communication, reliability and queuing. The stationary distribution and related characteristics of the aforementioned systems vary with time as well as the perturbation parameter, resulting in triangular array mode which is challenging to understand. The problem gets more complicated for perturbed Markov chains with finite phase space, where interaction within a subsystem is stronger than between subsystems [103, 139].

In addition to the above, updating the stationary distribution of Markov chains when edges or vertices of such a system change comes with numerical computational bottlenecks. However, such Markov chain models have various applications in internet search engines, biology, finance, transport, queuing networks and telecommunications [3, 4, 23, 33, 36, 47, 76, 125].

To contribute to information network analysis, this thesis focuses on:

1. PageRank of evolving tree graphs,
2. Evaluation of stopping criteria for ranks in solving a linear system of equations,
3. Perturbed Markov chains with damping component.

The thesis is organised as follows: Chapter 1 introduces the general historic background, applications and recent advances in PageRank and networks. This will be equipped with numerous concepts of graph partition, properties of non-negative matrices, discrete time Markov process, coupling methods for Markov chains and PageRank computation.

Chapter 2, which is composed of Papers **A**, **B** and **C**, gives the explicit formulae to update transition matrices of evolving directed tree graphs. In addition, the chapter presents algorithms of updating a matrix of transition matrices and PageRank of an evolving tree graph.

Chapter 3, which is based on Paper **D**, explores the numerical experiments on stopping criteria in solving linear systems with link-based ranking problem. Also, the chapter illustrates how the clustering algorithm can be used as a stopping criterion.

Chapter 4, which is based on Papers **E**, **F**, **G** and **H**, focuses on perturbed Markov chain models applied in description of information networks and PageRank. We describe coupling explicit estimates for the rates of convergence in ergodic theorems for Markov chains with damping component. Furthermore, discusses ergodic theorems for Markov chains with damping component in triangular array mode. We illustrate with numerical examples some of the results.

1.1 PageRank and its Applications

This subsection presents a brief historical background, applications, some advances in the computation of PageRank as well as the common variants of PageRank.

PageRank is an algorithm that uses link structure of the web to assign relative importance to each web page. According to [23, 93], a web page is assumed to be important if some other important pages point to it.

Since the related work on the World Wide Web Worm by McBryan Oliver A. in the early 1990s [89], research on web search engines has attracted considerable attention among scientific communities. One noticeable research work in this direction was the development of the Google search engine algorithm. The ranking algorithm is referred to as PageRank and the main idea of the algorithm is to determine the stationary distribution (Perron vector) of the Google matrix. The rank algorithm outperforms other centrality measures because it is based on a natural perception that an object is more important (relevant) if it is referenced by other important objects, for example in opinion leader detection [94]. Importantly, in biological systems, the stability of the algorithm has attracted attention because it gives a modest number of false positives and false negatives [64]. Additionally, the algorithm can be used to compare the ranks of two disconnected networks, where centrality measures such as an eigenvector cannot be applied. We recommend the following papers and books where details of these mathematical objects are discussed [5, 38, 76, 92, 102, 120–122].

1.1.1 Methods of computing PageRank

In many instances, PageRank problem is approached as an eigenvector problem and can be computed using the power method. Because of other technical issues, for instance, when damping factor c is very close to 1, convergence of the power method slows down [137]. Besides, such a value of c can not be applied in fast–changing Markov chains or processor-shared queues [69, 70]. Most commonly, the damping factor $c = 0.85$ is recommended for web graphs. With this choice of c, power method can effectively be used.

Several computational methods have been developed, namely Power series [5], Monte Carlo methods [10], aggregation/disaggregation method [19], the inner-outer methods [48], adaptive method [66], extrapolation method

[67] and Arnoldi-type algorithm [137]. We would like to mention the following recent research papers [37, 55], where PageRank algorithm has been improved. Specifically, in [55] an algorithm, known as Arnoldi-Inout can efficiently compute the PageRank vector in a case where damping factor is close to 1, while [37] developed a new method that handles changing networks.

1.1.2 The applications and variants of PageRank

Recently, several applications of PageRank have been identified, for example, in natural language processing and biology [33], green supply chain management [40], number theory [43], video object tracking [49], ecology and dynamical systems [47] community detection [61], complex products and systems in mechanical engineering design [80], gene prioritization in biological networks [132] and combating web spam [54,141]. These signify that PageRank algorithm has plentiful of applications in real life.

Not withstanding, there are several variants of PageRank, namely personalized PageRank [60], topical PageRank [58], Ranking with backstep [123, 124], Query-Dependent PageRank [46], lazy PageRank [26] and many others. It should be pointed out that algorithms that are similar to PageRank exist. For instance, EigenTrust algorithm [68], DeptRank algorithm [15] and PageRank-based vulnerable transmission lines screening algorithm [84]. As indicated in [17], bibliography concerning different aspects of PageRank exist in literature.

1.1.3 Forms of information networks

It is well known that information networks take several forms. Networks such as citation, social networks, protein-protein interaction and communication include either strongly connected components or connected acyclic components or both. On the other hand, there are systems that are modeled as tree graphs. For example networks of integers, ecological systems and triangular network coding [38, 126].

Motivated by applications of directed tree graphs and drawbacks usually encountered during updating PageRank when changes occur in the information network, developing an algorithm that focuses on such a structure should be of interest. We propose an algorithm for recalculating PageRank of evolving tree graphs. Specifically, an algorithm that relies on the levels

of vertices to update PageRank. This way one can update PageRank of the affected vertices only. We also look at how to maintain cyclic components in an evolving tree graph. The discussion on these subjects is presented in Chapter 2.

1.2 PageRank Scores and Ranks

It is well known that PageRank is a solution of a large sparse linear system. While many iterative schemes to handle such systems exist, such numerical computations are becoming demanding [22]. Moreover, only a small number of relevant pages are provided by search engines, which is termed as ranking of web pages [71]. Several researchers have questioned the need to determine accurate PageRank scores [19, 22, 49, 57, 134–136].

In the view of Haveliwala [57], if the PageRank vector is to be used for establishing the importance of pages, the convergence should be measured based on how the ordering changes as the number of iterations increases. While Berkhin [19] questioned the sense to iterate beyond the accuracy that establishes the order of the pages in a search engine analysis.

Boldi, Santini and Vigna [22] pointed out that it is the relative order of pages for PageRank that is interesting for search engines not the accuracy of the PageRank vector itself. This seems to suggest that rank-ordering as an algorithm should provide a few number of iterations that are sufficient for identifying important web pages. Consequently, the time complexity of the algorithm can be drastically reduced. As a matter of fact, paying much attention to the accurate solution of a linear system underscores the practical significance for ranking in areas where it is applied. This motivates the need to evaluate stopping criteria concerning ranks. In Chapter 3, we propose clustering algorithms as stopping criteria.

1.3 Perturbed Markov Chains with Damping Component

Perturbed Markov chains are important objects of study in the theory of Markov processes and their application to stochastic networks, queuing and reliability models, bio-stochastic systems and many other stochastic models.

We refer here to some recent books and papers devoted to perturbation problems for Markov type processes, [3,4,6,7,9,11,21,31,35,45,52,53,56,72–74,85,90,91,95,106,109,112,114–118,120–122,138,140].

In particular, we would like to mention works [9,53,114,115], where extended bibliographies of works in the area and the corresponding methodological and historical remarks can be found. We are especially interested in models of Markov chains commonly used for description of information networks. In such models, an information network is represented by the Markov chain associated with the corresponding node-link graph. Stationary distributions and other related characteristics of information Markov chains usually serve as basic tools for ranking of nodes in information networks.

The ranking problem may be complicated by singularity of the corresponding information Markov chain, where its phase space is split into several weakly or completely non-communicating groups of states. In such models, the matrix of transition probabilities $\mathbf{P}_0$ of the information Markov chain is usually regularised and approximated by a matrix $\mathbf{P}_\varepsilon = (1-\varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D}$, where $\mathbf{D}$ is a so-called damping stochastic matrix with identical rows and all elements positive, while $\varepsilon \in [0,1]$ is a damping (perturbation) parameter.

Power method is one of the most convenient and robust methods to approximate the corresponding stationary distribution $\vec{\pi}_\varepsilon$ by rows of the matrix $\mathbf{P}_\varepsilon^n$, for some $n \geq 1$. The damping parameter ε should be chosen neither too small nor too large. In the first case, where ε takes too small values, the damping effect will not work against absorbing and pseudo-absorbing effects, since the second eigenvalue for such matrices (determining the rate of convergence in the above mentioned ergodic approximation) takes values approaching 1. In the second case, the ranking information (accumulated by the matrix $\mathbf{P}_0$ via the corresponding stationary distribution) may be partly lost, due to the deviation of the matrix $\mathbf{P}_\varepsilon$ from matrix $\mathbf{P}_0$.

This actualises the problem of estimating rate of convergence for perturbed stationary distribution $\vec{\pi}_\varepsilon$ with respect to damping parameter ε, as well as studies of asymptotic behaviour of matrices $\mathbf{P}_\varepsilon^n$ in triangular array mode, where $\varepsilon \to 0$ and $n \to \infty$, simultaneously. We would like to mention some works [95,105–111], which have results related to ergodic theorems in triangular array mode and the so–called quasi-stationary ergodic theorems for perturbed regenerative processes, Markov chains and semi-Markov processes.

1.3.1 Google matrix and damping parameter

It is well known that associated to a set of web pages of an information network is the Google matrix denoted by $\mathbf{G}$ [9, 23, 33, 34, 76]. The matrix of transition probabilities is usually defined as $\mathbf{G} = c\mathbf{P} + (1-c)\mathbf{E}$, where $\mathbf{P}$ is an $m \times m$ row-stochastic matrix (also called hyperlink matrix), $\mathbf{E}$ (the damping matrix) is the $m \times m$ rank-one stochastic matrix and $c \in (0, 1)$ is the damping parameter. We refer you to the following papers [5, 59, 77] and a classical textbook by Langville and Meyer [76], where properties of the matrices $\mathbf{G}$ and $\mathbf{P}$ are presented. Note that if $c = 1$, then $\mathbf{P}$ corresponds to $\mathbf{G}$. The analysis of Google matrix of information networks can become challenging when the links of networks are very large and damping parameter is close to 1 [20, 38, 44].

The damping factor c may be denoted and interpreted differently depending on the model being studied. For instance, a model of Markov chain with restart is considered in [11], where parameter p is the probability to restart the move and $1 - p$ is the probability to follow the link according to the corresponding transition probability of the above Markov chain. Hence, one can argue that the parameter p has the same interpretation as the damping factor in Google's PageRank problem.

Our representation of perturbed Markov chains is traditional for perturbed stochastic processes. In Chapter 4, the damping factor commonly denoted as c is rewritten as perturbation parameter $\varepsilon = 1 - c$. Also, we denote the damping matrix $\mathbf{E}$ previously mentioned by $\mathbf{D}$. Hence, It is essential to point out here that, representation of information network model by a Markov chain with a matrix of transition probabilities $\mathbf{P}_\varepsilon = (1-\varepsilon)\mathbf{P}_0 + \varepsilon\mathbf{D}$ should not create any confusion to readers.

1.4 Preliminaries

In this section, we will highlight some key definitions, notations and concepts necessary for readers to follow-up the thesis. Material presented here is partly based on [24, 27–29, 62, 78, 131, 133]. We begin with useful concepts in graph theory.

1.4.1 Some basic definitions in graph theory

The terms graph and information network will be used interchangeably. Modeling information networks using graphs is common and ranges from simple to complex graphs, directed or undirected graphs depending on the assignments on the edges. However, we are interested in directed graphs, unless otherwise specified.

An example of a directed graph is a transition diagram of a Markov chain where the vertices and edges represent the states and transition from one state to another, respectively. In this case, each edge is assigned a transition probability (weight) if there is a direct edge from one state to the neighbouring state. Such directed information networks have various applications in modern information systems.

Let $\mathcal{G} := (V, E)$ be a directed graph, a set of vertices and edges are denoted by V and E, respectively. Hereafter, any graph with loops or cyclic components will be denoted by $\mathcal{G}$ and a graph without cycles will be denoted by $\mathcal{T}$. We represent the number of vertices by $|V|$.

A path consists of a sequence of distinct vertices $v_1, v_2, \ldots, v_l$ such that for any two vertices v_i and v_{i+1} there is an edge starting from v_i and ending at v_{i+1}. Suppose $u = v_1$ and $v = v_l$ such that in the sequence $v_1, v_2, \ldots, v_l$, we have $u = v$, then the subgraph consisting of these vertices and edges is a cycle.

Definition 1.4.1. A directed graph $\mathcal{G}$ is called strongly connected if for every pair of vertices (v_i, v_j), there is an edge from v_i to v_j and an edge from v_j to v_i.

On the other hand, a directed graph is said to be a directed acyclic graph (DAG) if it does not contain cycles. In other words, DAG is a finite directed graph with no directed cycles. In a case where two vertices do not have an edge from either direction then it is disconnected.

1.4.2 Graph partition

A subgraph $\mathcal{G}_1 = (V_1, E_1)$ of $\mathcal{G}$ is said to be a maximal strongly connected component if after removal of one of its edges, the subgraph ceases to be a strongly connected component. That is to say, every subgraph $\mathcal{G}_1 \subset \mathcal{G}$ is a strongly connected component if it is a maximal strongly connected subgraph. Obviously, by maximal property each vertex belongs to a single

strongly connected component (SCC). Take a case where a vertex v_s belongs to two SCCs $\mathcal{G}_1$ and $\mathcal{G}_2$, then the union is a SCC (by definition of strongly connected component of a graph).

On the other hand, if we assume that a graph $\mathcal{G}$ consists of strongly connected components $\mathcal{G}_1$ and $\mathcal{G}_2$, where there is an edge from $\mathcal{G}_1$ to $\mathcal{G}_2$, this could only happen if there exist at least one vertex $v_s \in \mathcal{G}_1,\ v_t \in \mathcal{G}_2$ and $v_s \neq v_t$. It turns out that the corresponding graph $\mathcal{G}$ is a directed acyclic graph. For more discussion, one is referred to [33, 51].

A strongly connected graph $\mathcal{G} = (V, E)$ can be partitioned into a finite number of subgraphs $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_r$, for $r \geq 1$ such that the corresponding sets of vertices are ordered as $V_1, V_2 \ldots, V_r$, where $V_i \bigcap V_j = \emptyset$, for $i \neq j$.

In the case where $\mathcal{G}$ has several SCCs, the components can be ordered as a directed acyclic graph, such that each component is represented by a single vertex, that is, we merge a subset of vertices of each component into one vertex.

The overall adjacency matrix associated to $\mathcal{G}$ takes a block triangular form $\mathbf{A}$ as in relation (1.1) [127].

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \cdots & \mathbf{A}_{1r} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} & \cdots & \mathbf{A}_{2r} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{A}_{r-1r} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_{rr} \end{pmatrix}, \tag{1.1}$$

where $\mathbf{A}_{ii}$, $i = 1, \ldots, r$ are block square matrices corresponding to strongly connected components, $\mathbf{A}_{ij}$, for $i < j,\ i, j = 1, \ldots, r$ are not square matrices and $\mathbf{0}$ is a matrix with all entries zero.

The case where $\mathbf{A}_{ij} = \mathbf{0}$, for $i < j$, $i, j = 1, \ldots, r$, in relation (1.1), implies that the graph $\mathcal{G}$ consists of disjoint strongly connected components, while $\mathbf{A} = \mathbf{A}_{11}$ indicates that the graph consists of only one strongly connected component. Clearly, $\mathbf{A}_{11}$ is linked to sub-matrices $\mathbf{A}_{ii}$, $i = 2, \ldots, r-1$, however the sets of vertices $V_2, V_3 \ldots, V_r$ are disjointed. Such a matrix representation is common in Markov diagrams.

Essentially, any matrix can be partitioned as matrix $\mathbf{A}$ provided it is decomposable. This reordering of vertices into block triangular form is useful

in computing sparse matrices, finding inverses as well as determining eigenvalues [18, 51, 101]. We use this reordering technique to study evolving tree graphs and perturbed Markov chains with damping component in Chapter 2 and Chapter 4, respectively.

In a directed tree graph, the associated information matrix is nilpotent. For such a graph, a vertex with no incoming edge is called a root and a vertex with no outgoing edge is referred to as a leaf (dangling vertex, as usually called in PageRank problem).

Note that reordering vertices of a graph leads to grouping vertices into levels. If a graph consists of connected components such as $SCCs$ and connected acyclic components ($CACs$), then we talk of level of a component because all the vertices that form the component have the same level. On the other hand, if a graph consists of connected 1-vertex $SCCs$, it is appropriate to say level of a vertex. To shed light on what a level of a vertex is, let us define the term connected acyclic component [36].

Definition 1.4.2. Let $\mathcal{G}$ be a simple directed connected graph. A connected acyclic component (CAC) of $\mathcal{G}$ is a subgraph $\mathcal{G}_1$ of $\mathcal{G}$ such that its vertices are not part of a non-loop cycle in $\mathcal{G}$ and the underlying graph is connected. Furthermore, any edge in $\mathcal{G}$ that exist between any two vertices in $\mathcal{G}_1$ belong to $\mathcal{G}_1$. A vertex in the CAC with no edge to any other edge in the CAC, we refer to it as a leaf of the CAC.

Partitioning a graph into $SCCs/CACs$ have the following features:- each vertex $v \in \mathcal{G}$ belongs to one and only one component. If each component is merged into a single vertex, the resultant directed graph forms a directed acyclic graph. This is a connected directed graph with no loop. Furthermore, since the vertices of DAG correspond to components of $\mathcal{G}$, any edge between any two of such vertices corresponds to edges between associated components. In this case, the edges between the components should have the same direction as in DAG. Based on Definition 1.4.2, a CAC can be viewed as a linked graph of 1-vertex $SCCs$ which is essentially a tree.

Definition 1.4.3. A level of a component L_C of a graph $\mathcal{G}$ is the length of the longest path in the underlying DAG starting in component C to any other components. On the other hand, a level of a vertex $L(v_i)$ of a graph $\mathcal{G}$ is the length of the longest path in the underlying DAG starting at v_i to any other vertices.

To determine levels of vertices of a graph from another vertex with a known level can be achieved as follows:

Example 1.4.1. Assume that vertices v_s and v_t belong to a tree graph $\mathcal{T}(V, E)$. Suppose the level of the vertex v_t is known, $L(v_t)$, say. Let us denote the length of the longest path from v_t to v_s by $k(\geq 0)$. In this case, the level of v_s is expressed as

$$L(v_s) = \min(L(v_t), L(v_t) - k), \text{ for } L(v_t) \geq k.$$

We would like to remark that negative levels are also possible.

1.4.3 Some properties of nonegative matrices

We present some properties of nonnegative matrices and explore connections between such matrices and corresponding graphs.

A matrix $\mathbf{A}$ of order $m \times m$ is referred to as nonnegative if each entry $a_{ij} \geq 0$, $i, j = 1, 2, \ldots m$. This class of matrices has many applications in real life, for instance economics [18], internet search engines [23, 77, 93] and biological fields [33].

Significant progress in the field of nonnegative matrices has been accredited to Perron-Frobenius theory [75, 76] and combinatorial properties, such as irreducibility and primitivity [24]. An example of a nonnegative matrix in practice is the transition matrix of a Markov chain. A nonnegative matrix derived from a directed network can be a stochastic or sub-stochastic matrix. A sub-stochastic matrix arises if some vertices do not have outgoing edges. The following definition gives a scaled adjacency matrix.

Definition 1.4.4. Let $\mathcal{G} = (V, E)$ be a directed graph with the total number of vertices $m = |V|$ and $|E|$ the number of vertices and edges, respectively. A weighted adjacency matrix $\mathbf{A} = [a_{ij}]$, for $i, j = 1, \ldots, m$, of $\mathcal{G}$ is the $m \times m$ matrix with elements

$$a_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } (v_i, v_j) \in E, \\ 0, & \text{otherwise,} \end{cases}$$

where d_i is the number of outgoing edges from vertex v_i and (v_i, v_j) denotes an edge from vertex v_i to v_j in E.

Thus, the quantity a_{ij} denotes a chance that a random walker starting from a vertex v_i follows an edge to v_j and $a_{ij} = 0$, otherwise. If the matrix $\mathbf{A}$ takes the form of one of the block matrices $\mathbf{A}_{ii}$, for $i = 1, \ldots, r$ in relation (1.1), then it is an irreducible matrix. In such a matrix, the corresponding graph consists of strongly connected components. Notably, it has been proven that a square matrix is irreducible if and only if its directed graph is strongly connected [51]. Specifically, a matrix $\mathbf{A}$ in (1.1) is reducible. Such a partition indicates that the associated graph is not strongly connected. In other words, some vertices can not be reached when a random walk is performed [39].

The property of irreducibility of nonnegative matrices has been intensively studied [5, 23, 24, 51, 75, 97, 122]. One very important mathematical result in the study of nonnegative matrices is the Perron-Frobenius theorem.

Theorem 1.4.1 (The Perron-Frobenius theorem [86]). *Suppose* $\mathbf{A}_{m\times m} \geq \mathbf{0}$ *and irreducible. Then each of the following is true.*

(i) *Eigenvalue* $\lambda = \rho(\mathbf{A})$ *and* $\lambda > 0$, *where* $\rho(\mathbf{A})$ *is the spectral radius of* $\mathbf{A}$.

(ii) *Algebraic multiplicity of* λ, $alg\, mult_A(\lambda) = 1$.

(iii) *There is an eigenvector* $\vec{\pi} > 0$ *such that* $\mathbf{A}\vec{\pi} = \lambda\vec{\pi}$.

(iv) *The unique vector defined by*

$$\mathbf{A}\vec{\nu} = \lambda\vec{\nu},\ \vec{\nu} > \vec{0} \text{ and } \|\vec{\nu}\|_1 = 1. \tag{1.2}$$

is called Perron vector. There are no nonnegative eigenvectors for $\mathbf{A}$ *except for positive multiples of* $\vec{\nu}$, *regardless of the eigenvalues.*

(v) *The Collatz-Wielandt formula,* $\lambda = \max_{x\in\mathcal{N}} f(\vec{x})$,
where $f(\vec{x}) = \min_{\substack{1\leq i\leq n \\ x_i\neq 0}} \frac{[\mathbf{A}\vec{x}]_i}{x_i}$ *and* $\mathcal{N} = \{\vec{x} \in \mathbb{R}^n \,|\, \vec{x} \geq 0, \text{ and } \vec{x} \neq 0\}$.

Theorem 1.4.1 reveals important spectra properties of irreducible matrices. We discuss the properties in the context of nonnegative stochastic matrices.

Assume that the matrix $\mathbf{A}$ in Definition 1.4.4 is an irreducible row stochastic. Then by appropriate norm, one can note that it is bounded above by a spectral radius $\rho(\mathbf{A}) = 1$, such that for any eigenvalue $\bar{\lambda}$ of $\mathbf{A}$, inequality

$|\bar{\lambda}| \leq 1$ holds. This is the implication of Collatz-Wielandt formula in Theorem 1.4.1. Hence, the eigenvalue (Perron root) 1 is the dominant eigenvalue of $\mathbf{A}$ with a corresponding dominant positive eigenvector. On the other hand, if the matrix $\mathbf{A}$ is stochastic, the positive eigenvector corresponding to the dominant eigenvalue is a stationary distribution.

Essentially, it is not enough to assume irreducibility of the matrix but also to ensure that the algebraic multiplicity of $\rho(\mathbf{A})$ for a nonnegative matrix $\mathbf{A}$ to be equal to 1. In other words, one can not guarantee one eigenvalue only on the spectral circle of radius 1 unless algebraic multiplicity of $\rho(\mathbf{A})$ is 1. For this reason, in addition to nonnegative irreducible matrices, one needs to further classify such matrices as primitive and imprimitive. Nonnegative irreducible matrix $\mathbf{A}$, which is classified as a primitive matrix, has only one dominant eigenvalue, $\lambda = \rho(\mathbf{A})$, on its spectral circle. Otherwise, it is imprimitive.

Consequently, if a stochastic matrix $\mathbf{A}$ is primitive, then by Perron-Frobenius theorem, one can show that $\lim_{n\to\infty} \mathbf{A}^n$ exists. It is well known that long term behaviour defined by large enough n of primitive matrix $\mathbf{A}$ converge to the product of right eigenvector $\vec{e}$ and left eigenvector $\vec{\pi}^\top > \vec{0}$ of $\mathbf{A}$, that is, $\lim_{n\to\infty} \mathbf{A}^n = \vec{e}\vec{\pi}^\top$ [24, 75, 92]. Let $\mathbb{N}$ and $\mathbb{S}$ denote the set of natural numbers and a countable set, respectively. Suppose $n \in \mathbb{N}$ and $i_1, \cdots, i_{n-1} \in \mathbb{S}$, then the entries of $\mathbf{A}^n$ take the form

$$a_{ij}^n = \sum_{i_1,\cdots,i_{n-1}\in\mathbb{S}} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-1}j} > 0. \tag{1.3}$$

The above relation (1.3) holds only if the directed graph of $\mathbf{A}$ is strongly connected.

1.5 Discrete Time Markov Processes

Here, we present informally discrete time Markov chains and construction of Markov chains. The purpose is to keep the presentation simple for different categories of readers. For more discussion, we refer you to the following textbooks [28, 78, 79, 97, 102, 131].

Definition 1.5.1. [28] Denote a set of phase space of a Markov chain by $\mathbb{Y}$. A process $\{Y_n\}_{n\in\mathbb{N}}$ is a discrete time Markov chain if

$$P\{Y_n = j \,|Y_{n-1} = i, Y_{n-2} = i_{n-2}, \ldots, Y_0 = i_0\} = P\{Y_n = j \,|Y_{n-1} = i\}, \tag{1.4}$$

for states $i_0, i_1, \ldots, i_{n-1}, i, j, \in \mathbb{Y}$, $n \geq 1$.

Let us denote relation (1.4) by p_{ij}, then the corresponding matrix of transition probabilities is $\mathbf{P} = [p_{ij}]_{i,j\in\mathbb{Y}}$, $i, j = 1, \cdots, m$. The matrix $\mathbf{P}$ is row stochastic if

$$\sum_{j\in\mathbb{Y}} p_{ij} = 1,$$

while $\mathbf{P}$ is substochastic if $\sum_{j\in\mathbb{Y}} p_{ij} < 1$, for at least one j. Following relation (1.4), the states i and j are thought of as a current and future states at times $n-1$ and n, respectively. The stochastic processes that possess property (1.4) are said to have Markov property. In the case where equality

$$P\{Y_n = j \,|Y_{n-1} = i\} = P\{Y_1 = j \,|Y_0 = i\}$$

holds, the Markov chain is referred to as homogeneous Markov chain. Throughout this thesis, whenever we mention Markov chains we mean homogeneous Markov chains.

It is a common practice to derive a matrix of transition probabilities of a Markov chain from a directed graph by arguing from the traditional random walk perspective [33]. By traditional random walk, we refer to a user starting at a particular node and randomly jumps to any vertex with uniform probability if the vertex has no outgoing edges, otherwise chooses with uniform probability one of the outgoing edges and goes to the target vertex. However, this does not occasionally address practical or technical issues as in the case of PageRank. For example, it may not be feasible to guarantee the ergodic theorem for Markov chains. To this end, a matrix of transition probabilities is regularised by a damping matrix that has a positive transition probability between any pair of vertices. Hence, it is natural to ask: Is it possible to get a Markov chain associated with such a particular transition matrix? If it is possible; How does one construct such a Markov chain?

1.5.1 Construction of a Markov chain

It is well known that Markov chains can be constructed by some recursive relations. A simple way of constructing a Markov chain is as follows. Suppose $\{Y_n\}_{n\in\mathbb{N}}$ is a sequence of stochastic processes on a finite phase space $\mathbb{Y}$ such

that

$$Y_n = g(Y_{n-1}, W_n),\ n \geq 1, \tag{1.5}$$

where real valued function $g : \mathbb{Y} \times \mathbb{W} \to \mathbb{Y}$, $W_n, n \geq 1$ are independent, uniformly distributed random variables, in interval $\mathbb{W} = [0, 1]$. In this formulation, one can obtain Y_n if the previous value of Y_{n-1} is known.

Following relation (1.5), we note clearly that Y_{n-1} is a function of $(Y_0, W_1, \ldots, Y_{n-2}, W_{n-1})$, which is independent of W_n, and the conditional probabilities $p_{ij} = P\{Y_n = j/Y_{n-1} = i\} = P\{g(i, W_n) = j\}$, for $i, j \in \mathbb{Y}$ possess the Markov property. Useful discussion on the content can be found in [97].

1.5.2 Decomposition of finite state space

Let us have a look at how the phase space $\mathbb{Y}$ of an information network can be decomposed. If a matrix of transition probabilities corresponding to a network is irreducible, then it cannot be decomposed any more. This is the same as saying that the phase space of the Markov chain consists of a single communicative class. Generally, two classifications of states of Markov chain can be achieved.

Decomposition of a finite phase space $\mathbb{Y}$ is usually as follows:

$$\mathbb{Y} = \mathbb{Y}^{(0)} \bigcup \left(\bigcup_{g=1}^{h} \mathbb{Y}^{(g)} \right), \tag{1.6}$$

where $\mathbb{Y}^{(0)}$ is set of transient states and $\mathbb{Y}^{(g)}, g = 1, \ldots, h$ are disjoint closed communicative classes of states.

In Chapter 4, we will consider the following cases:

(1) There is no class of transient states $\left(\mathbb{Y}^{(0)} = \emptyset\right)$ and $h \geq 1$

(2) There is a class of transient states $\left(\mathbb{Y}^{(0)} \neq \emptyset\right)$ and $h \geq 1$.

Equivalently, case (1) corresponds to one or several disjoint strongly connected components, while case (2) represents a weakly connected subgraph plus several disjoint strongly connected components. More precisely, the phase space consists of one (or several) strongly connected components together with a set of phase states corresponding to transient states. The existence of limiting probability distribution in case of (2) can be sophisticated.

However, if certain assumptions are made, then one can use probabilistic ideas to obtain the asymptotic distribution and other characteristics of perturbed Markov chains with damping component as it will be described in Chapter 4.

1.5.3 Discrete time renewal processes and regenerative processes

We would like to mention that renewal equations have several applications in science and engineering [42]. The concept was first introduced by Doob in the late 1930s. Since then, a considerable amount of work has been done. For instance, in [42] discrete time and continuous time renewal processes are discussed in detail; perturbed renewal equations by Silvestrov, D. [104, 110, 111], and more works on the subject can be found, for example, in the following books and papers [14, 32, 52, 90, 91, 95].

A renewal process is the process where some so-called recurrent events occur at random times and inter-event times are independently and identically distributed, while a regenerative process is that process where there exist a sequence of random times such that after one of the random times, the system restarts itself and the future of the process is independent of the past.

More formally, a regenerative process is understood as follows [97]:

Definition 1.5.2. A discrete time stochastic process $Y_\tau, \tau = 0, 1, \ldots$ with a measurable phase space $\mathbb{X}$ is referred to as regenerative if there exists T_n such that

1. $\{T_n\}$ is a renewal process, that is, $T_k = \sum_{i=1}^{k} S_i$, $k \geq 0$, where $S_i, i \geq 1$ are positive integer valued independent and identically distributed random variables,

2. the process $\{Y_{T_k+\tau} : \tau \geq 0\}$ is independent of $\{Y_\tau : \tau < T_k\}$, for every $k \geq 1$,

3. the process $\{Y_{T_k+\tau} : \tau \geq 0\}$ is stochastically equivalent to process $\{Y_\tau, \tau \geq 0\}$, for every $k \geq 1$.

By stochastically equivalence, we mean that the corresponding processes have the same finite–dimensional distributions.

Let us denote the probability that a recurrent event $\{Y_n \in A\}$ (A is a measurable subset of $\mathbb{X}$) occurs at moment n by $p(n)$. Suppose $f(n)$ is the probability that the first regenerative time $T_1 = n$, where $\sum_{n=1}^{\infty} f(n) = 1$. That is, $\{f(n)\}$ is the probability distribution of the first regenerative time T_1. Then the sequences $\{p(n)\}$ and $\{f(n)\}$ are connected by following recurrent relation

$$p(n) = q(n) + \sum_{l=1}^{n} p(n-l)f(l),\ n \geq 0, \tag{1.7}$$

where $q(n) = P\{Y_n \in A, T_1 > n\}$ is the probability that the recurrent event $\{Y_n \in A\}$ occurs at moment n and the first regenerative time T_1 takes a value larger than n.

Relation (1.7) is referred to as renewal equation. In our case, the stochastic process Y is a Markov chain with finite state space and recurrence times are inter-regenerative moments. The asymptotic behavior and other properties of (1.7) are discussed in the textbooks by Feller [41, 42].

1.5.4 Coupling methods for Markov chains

Coupling methods first appeared in the probability domain in the late 1930s. The idea was introduced by Wolfgang Doeblin [30]. A paper by Lindvall [83] presents a collection of Doeblin's works. The method was advanced by several researchers [50,65,79,81,82,96,113,129], and it is one of the important mathematical objects in analysing Markov chains and other stochastic processes. We also want to make reference to the work by Baxendale [16], Roberts [99], Rosenthal [100] and Silvestrov, D. [104], where coupling Markov renewal processes and related algorithms have been investigated.

By coupling of two Markov chains, we mean two independent chains are constructed on the same probability space. The Markov chains are allowed to evolve simultaneously but independently until they both hit the same state, then the chains combine and move on as a single process, see Figure 1.1. The evolution of the two processes commences with different probability measures $\vec{\pi}$ and $\vec{p}$, where $\vec{\pi}$ and $\vec{p}$ are stationary and initial distributions of the Markov chain, respectively. The time instance might occur if the two states have an inclination to get in contact. It is well known that coupling method is one of the important tools in ergodic theory and determining the rate of convergence of stochastic processes.

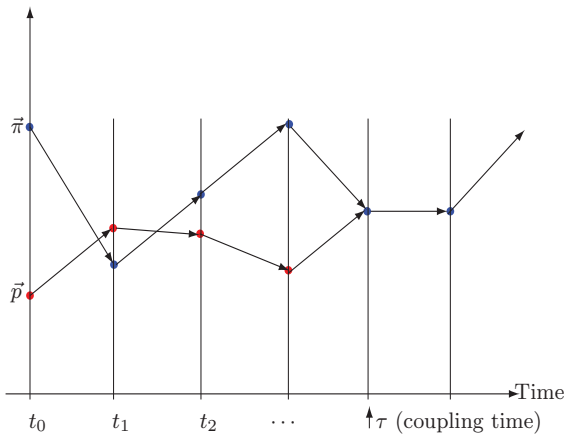

Figure 1.1: Coupling Markov chains initiated at two distinct states with initial distributions $\vec{p}$ and $\vec{\pi}$.

It is worth noting that there are several constructions of coupling in literature [25, 50, 128, 129]. For instance, distributional coupling is preferred when dealing with some continuous time stochastic processes. In Chapter 4, we use a maximal coupling type algorithm to determine effective upper bounds for the rate of convergence of perturbed Markov chains with damping component.

1.6 PageRank Computation

PageRank is an algorithm to measure the relative importance of web pages [23, 26, 77, 93]. It originated as a mathematical tool for Google search engine and has since then, gained great fame as it is the basic algorithm that reveals how the search engine weighs the information value of different websites against each other. In the last two decades, several research directions have emerged, which can be categorised into three, namely methods of computation, applications and variants of PageRank. Computing PageRank in web information retrieval is implicated by the rapid growth of the size of data and change of the web structure such as edges and vertices [5].

In some applications, PageRank is viewed as the stationary distribution of the matrix of transition probabilities of a directed graph $\mathcal{G} = (V, E)$. In [23] and [77], this version of PageRank is referred to as normalised PageRank. In [33] non-normalised PageRank has been discussed and its advantages have been pointed out. The followings are common to both variants: they require a weighted adjacency matrix $\mathbf{P}$ and random walk theory. The idea of using random walk in PageRank problem is traditional as stated by the pioneers of PageRank [23, 26].

Let us denote by $\mathbb{Y}$ a set of all vertices in a directed graph $\mathcal{G}$. To construct a matrix of transition probabilities (the so-called hyperlink matrix) $\mathbf{P}$, whose elements are probabilities denoted by p_{ij} is carried out as follows. A random walker at vertex v_i moves one step to the new vertex v_j, if the weight of an edge $v_i \to v_j$ is given by a_{ij}, then the one-step transition probability is expressed as

$$p_{ij} = \frac{a_{ij}}{\sum_{j \in \mathbb{Y}} a_{ij}}. \tag{1.8}$$

For $a_{ij} = 1$ and $\sum_{j \in \mathbb{Y}} a_{ij} = d(v_i)$, for $i = 1, \ldots, m$, relation (1.8) simplifies to Definition 1.4.4 and $\mathbf{P} = [p_{ij}],\ i, j = 1, \ldots, m$ is the matrix of transition probabilities.

Suppose $\mathbf{P}$ is sub-stochastic matrix because of dangling vertices, a remedy is to add artificial edges from dangling vertices to all vertices in the graph. This is the same as adding a sub-stochastic matrix $\vec{g}\vec{u}^{\top}$, where $\vec{g}$ is a non-negative column vector with entries defined as follows:

$$g_k = \begin{cases} 1, & \text{if } k \in \mathbb{Y}_D, \\ 0, & \text{otherwise}, \end{cases} \tag{1.9}$$

where $\mathbb{Y}_{\mathrm{D}}$ is a set of dangling vertices in $\mathbb{Y}$ and $\vec{u}^{\top}$ is a row vector with entries equal to $\frac{1}{m}$. The new matrix $\mathbf{P} + \vec{g}\vec{u}^{\top}$ is stochastic but may not have a unique stationary distribution. To ensure primitivity, a damping (teleportation) matrix $\mathbf{D} = \vec{u}\vec{e}^{\top}$, where $\vec{e}^{\top}$ is a row vector of ones, is added to $\mathbf{P} + \vec{g}\vec{u}^{\top}$. The resulting matrix takes the form

$$\mathbf{G} = c\left(\mathbf{P} + \vec{g}\vec{u}^{\top}\right)^{\top} + (1 - c)\mathbf{D}, \tag{1.10}$$

where $c \in (0, 1)$ is the probability of following an outgoing edge (in short c is called the damping parameter), $1 - c$ is the probability that a web user jumps randomly to an arbitrary web page (vertex) according to the teleportation

matrix $\mathbf{D}$. The matrix $\mathbf{G}$ is stochastic matrix, aperiodic and irreducible [10] therefore by Perron-Frobenius theorem, the dominant eigenvalue is 1 with corresponding positive eigenvector (stationary distribution) $\vec{\pi}$ [75, 76].

In the case where $\mathbb{Y}_{\mathrm{D}} = \emptyset$, relation (1.10) reduces to

$$\mathbf{G} = c\mathbf{P}^{\top} + (1-c)\mathbf{D}. \tag{1.11}$$

Relation (1.11) is, for instance, the same as a Google matrix of irreducible matrix $\mathbf{P}^{\top}$. Since $\mathbf{G}$ is a primitive matrix, then by Perron-Frobenius theorem, the dominant eigenvalue is 1. Note that $\mathbf{D} = [d_{ij}]_{i,j=1,\ldots,m}$ (with elements $d_{ij} = d_j$) is a stochastic matrix with $\vec{d} = \langle d_j, j \in \mathbb{Y} \rangle$ as stationary distribution.

Let us look at another variant of PageRank, that is, non-normalised PageRank.

Definition 1.6.1. Consider a random walk on a graph described by $\mathbf{P}$, which is the adjacency matrix weighted such that the sum over every non-zero row is equal to one. In each step with probability $c \in (0,1)$, move to a new vertex from the current vertex by traversing a random outgoing edge from the current vertex with a probability equal to the weights on the corresponding edge. With probability $1-c$ or if the current vertex has no outgoing edge we stop the random walk. Then PageRank π_j for a single vertex v_j can be written as

$$\pi_j = \left(w_j + \sum_{v_i \in V,\, v_i \neq v_j} w_i p_{ij} \right) \sum_{k=0}^{\infty} (p_{jj})^k, \tag{1.12}$$

where p_{ij} is the probability to hit vertex v_j in a random walk starting in vertex v_i and w_j is the weight of vertex v_j . This can be seen as the expected number of visits to v_j if we do multiple random walks, starting in each vertex once and weighting each of these random walks by vector $\vec{w}$ [33].

1.6.1 PageRank as a solution to a linear system

We describe briefly some classical techniques in solving a linear system of equations. PageRank algorithm can be formulated as a linear system, and a resourceful textbook is by Langville and Meyer [76]. We also recommend textbooks [63, 87, 120–122] for further treatments of numerical techniques associated with Markov chains and linear system of equations.

We will focus on four classical techniques of solving large linear system of equations in ranking problems, namely, power series, Jacobi, power method and successive-over relaxation methods. The first, second and third methods are closely related, that is, power method can be rewritten as the Jacobi method and power series. On the other hand, the formulation of power series of PageRank is related to Definition 1.6.1, which we will mainly use in Chapter 2 of this thesis.

The stochastic matrix $\mathbf{G}$ in (1.10) is irreducible and aperiodic, therefore by Perron–Frobenius theorem, it has a unique stationary distribution (PageRank vector) $\vec{\pi}$ such that the following equality holds.

$$\vec{\pi} = \mathbf{G}\vec{\pi}, \tag{1.13}$$

Clearly, (1.13) is an eigenvector problem which can be rewritten as a linear system of equations. Therefore, substituting (1.10) into (1.13), we get

$$\begin{aligned} \vec{\pi} &= \left(c(\mathbf{P} + \vec{g}\vec{u}^{\top})^{\top} + (1-c)\mathbf{D}\right)\vec{\pi} && (1.14)\\ &= c\mathbf{P}^{\top}\vec{\pi} + \vec{u}\left(c\vec{g}^{\top} + (1-c)\vec{e}^{\top}\right)\vec{\pi} \\ &= c\mathbf{P}^{\top}\vec{\pi} + \eta\vec{u}, && (1.15) \end{aligned}$$

where $\eta = 1 - \|c\mathbf{P}^{\top}\|_1$ and $\vec{e}^{\top}$ is a row vector with all entries equal to 1 [5]. Note that $\|\vec{y}\|_1 = \sum_{k=1}^{n} |y_k|$.

Set $\vec{v} = \eta\vec{u}$, then (1.15) becomes

$$(\mathbf{I} - c\mathbf{P}^{\top})\vec{\pi} = \vec{v}, \tag{1.16}$$

where $\mathbf{I}$ is an identity matrix of order m and $\mathbf{P}^{\top}$ is an information (a hyperlink) matrix. Relation (1.16) can be solved by any numerical methods. However the matrix $\mathbf{P}^{\top}$ is usually very large such that applying a direct method is practically impossible. A better approach is to use iterative methods for a linear system of equations. Note that the solution $\vec{\pi}$ in (1.16) is not necessarily normalised PageRank, but this can easily be done by dividing the solution by $\|\vec{\pi}\|_1$.

Let us assume that a network corresponding to matrix $\mathbf{P}$ has no self loop, then by splitting technique, the matrix $(\mathbf{I} - c\mathbf{P}^{\top})$ takes the form

$$\mathbf{I} - c\mathbf{P}^{\top} = \mathbf{N} - \mathbf{M}, \tag{1.17}$$

where $\mathbf{N}$ is a non-singular matrix of same order as $\mathbf{P}$. The conditions for the choice of $\mathbf{N}$ can be found in [63]. With arbitrary initial vector $\vec{\pi}^{(0)}$, the

iterative scheme of the linear system (1.16) takes the form

$$\vec{\pi}^{(k)} = \mathbf{N}^{-1}\mathbf{M}\vec{\pi}^{(k-1)} + \mathbf{N}^{-1}\vec{v}. \tag{1.18}$$

The convergence of relation (1.18) depends on the spectral radius of $\mathbf{N^{-1}M}$. The advantage of using (1.18) is the sparsity of the matrix $\mathbf{P}^\top$ can be exploited and it is not necessary to adjust for dangling nodes or sinks.

Observing that the matrices $\mathbf{N} = \mathbf{I} - \mathbf{L}$ and $\mathbf{M} = \mathbf{U}$, where $\mathbf{L}$ and $\mathbf{U}$ are strictly lower and upper triangular matrices of $c\mathbf{P}^\top$, respectively. The matrix splitting of $(\mathbf{I} - c\mathbf{P}^\top)$ becomes

$$(\mathbf{I} - c\mathbf{P}^\top) = \mathbf{I} - \mathbf{L} - \mathbf{U}. \tag{1.19}$$

We remark that if $\mathbf{P}^\top$ has no self-loop, then the matrices $\mathbf{N}$ and $\mathbf{M}$ in relation (1.18) correspond to $\mathbf{I}$ and $(\mathbf{L} + \mathbf{U})^\top$, respectively. It turns out that for the Jacobi method, $\mathbf{L} + \mathbf{U} = c\mathbf{P}^\top$. Therefore, the iterative scheme becomes

$$\vec{\pi}^{(k)} = c\mathbf{P}^\top\vec{\pi}^{(k-1)} + \vec{v}. \tag{1.20}$$

In the case where self loops are not ignored, (1.20) becomes

$$\vec{\pi}^{(k)} = c\mathbf{D}_0^{-1}(\mathbf{L} + \mathbf{U})^\top\vec{\pi}^{(k-1)} + \mathbf{D}_0^{-1}\vec{v}, \tag{1.21}$$

where $\mathbf{D}_0 = \mathbf{N}$ is the diagonal of $(\mathbf{I} - c\mathbf{P}^\top)$ and $(\mathbf{L} + \mathbf{U})^\top = \mathbf{M}$.

From relation (1.16), we see that the power series formulation of PageRank of a graph can be expressed as

$$\vec{\pi} = (1 - c)\sum_{l=0}^{\infty}\left(c\mathbf{P}^\top\right)^l\vec{v}. \tag{1.22}$$

We are aware that different interpretations of (1.22) are given by several authors, for example [5] and [33]. It is also natural to think of it in a different way. We give an alternative description of the series using probability ideas, that is, regenerative and renewal theory as we will see in Chapter 4.

To calculate non-normalised PageRank using (1.22), we replace the infinite sum in (1.22) by step k (the number of iterations), which is large enough. Therefore, a recursive formula of the non-normalised PageRank $\vec{\pi}$

can be expressed as

$$\begin{aligned}\vec{\pi}^{\,(k)} &= \vec{v} + \sum_{l=1}^{k} \left(c\mathbf{P}^{\top}\right)^{l} \vec{v} \\ &= \vec{v} + \left(c\mathbf{P}^{\top}\right) \sum_{l=0}^{k-1} \left(c\mathbf{P}^{\top}\right)^{l} \vec{v} \\ &= \vec{v} + c\mathbf{P}^{\top}\vec{\pi}^{\,(k-1)}. \end{aligned} \tag{1.23}$$

Clearly, (1.20) and (1.23) show that the Jacobi and power series methods have the same expression for an information network with no dangling vertices and same initial guess $\vec{\pi}^{\,(0)}$.

In Successive over-relaxation (SOR) method, relaxation parameter ω' is required and this has to be chosen in such a way that residual errors are reduced. The splitting matrices $\mathbf{N}$ and $\mathbf{M}$ are obtained as described in [29]. If we maintain that self loops are ignored, then the iterative scheme associated to SOR takes the form

$$\vec{\pi}^{\,(k+1)} = (\mathbf{I} - \omega'\mathbf{L})^{-1} \left((1-\omega')\mathbf{I} + \omega'\mathbf{U}\right) \vec{\pi}^{\,(k)} + (\mathbf{I} - \omega'\mathbf{L})^{-1}\vec{v}. \tag{1.24}$$

However, if we allow for self loops, then we replace the identity matrix $\mathbf{I}$ in (1.24) by the diagonal matrix $\mathbf{D}_0$ of $(\mathbf{I} - c\mathbf{P}^{\top})$.

To guarantee convergence of relations (1.20)–(1.24) for arbitrary initial guess, the following result holds [29].

Theorem 1.6.1. *A necessary and sufficient condition for a stationary iterative method* $\vec{z}^{(k+1)} = \mathbf{T}\vec{z}^{(k)} + \vec{b}$ *to converge for an arbitrary approximation* $\vec{z}^{0}$ *is that*

$$\rho(\mathbf{T}) = \max_{1 \le i \le m} |\lambda_i(\mathbf{T})| < 1, \tag{1.25}$$

where $\rho(\mathbf{T})$ *is the spectral radius of* $\mathbf{T}$.

Here, iterative matrix $\mathbf{T}$ is equivalent to the matrix $\mathbf{N}^{-1}\mathbf{M}$ in relation (1.18). The quantities $\rho(\mathbf{T}_J)$ and $\rho(\mathbf{T}_{SOR})$ represent spectral radii for Jacobi and SOR iterative matrices, respectively. $\rho(\mathbf{T}_J) = \rho(c\mathbf{D}_0^{-1}(\mathbf{L}+\mathbf{U})^{\top}) < 1$, since sum of the columns of $c\mathbf{D}_0^{-1}(\mathbf{L}+\mathbf{U})^{\top}$ is less than 1, see (1.21) and similarly for $\rho(\mathbf{T}_{SOR})$ in (1.24).

Another iterative scheme of importance is the Power method. Here, we use iterative matrix $\mathbf{G}$ in (1.10) or (1.11), depending on the graph connectivity, and choose initial guess $\vec{\pi}^{(0)}$. Then the approximate solution $\vec{\pi}^{(k)}$ at the k^{th} iterates as follows

$$\vec{\pi}^{(k)} = \frac{\mathbf{G}\vec{\pi}^{(k-1)}}{\|\mathbf{G}\vec{\pi}^{(k-1)}\|_1}. \tag{1.26}$$

Since the stochastic matrix $\mathbf{G}$ is irreducible and aperiodic, (1.26) reduces to $\vec{\pi}^{(k)} = \mathbf{G}\vec{\pi}^{(k-1)}$. Although, $\mathbf{G}$ is not sparse, this is essentially a matrix-vector product which does require storing the vector $\vec{\pi}^{(k)}$ at every iteration. Adjusting the matrix $\mathbf{P}^\top$ to $\mathbf{G}$ has a positive bearing on the quality of PageRank, since it reduces the effect of dangling vertices or dead ends (loops). Moreover, convergence to stationary distribution is guaranteed. Computing the stationary distribution with high degree of accuracy can be challenging or may not be beneficial at all in some applications where the actual order of the stationary distribution is more preferred. We will look into this problem in Chapter 3.

In summary, two types of PageRank representation will be considered, the normalised and non-normalised. The latter fits in the framework of determining ranks as a solution to a linear system of equations, and it will be used in Chapter 2. We would like to remark that normalised PageRank has greater potential if one needs to investigate PageRank and its associated characteristics using probabilistic ideas as we will see in Chapter 4.

1.7 Summaries of the Chapters

1.7.1 Chapter 2

Information networks take several structures, such as those with several strongly connected components or tree graphs. Authors citation networks, networks of integers and ecological networks are the typical examples of directed tree graphs. In such a network model, the corresponding weighted adjacency matrix (matrix of transition probabilities) is nilpotent. The motivation of studying tree graphs arises from various applications in the real world and the importance of updating PageRank of such evolving networks. In this chapter, we describe and discuss common changes in evolving tree

graphs and analyse the time complexity of the changes, as well as give algorithms for recalculating PageRank of evolving tree graphs.

1.7.2 Chapter 3

PageRank is a popular tool in web information retrieval. It is a solution to a large sparse linear system derived from an information network such as social, communication, biological and financial networks. It is an attractive area of research in the present time. Several iterative schemes aim at finding an accurate solution to a linear system, making the ranking of nodes of information network challenging. In many instances, it is the relative order (rank) of vertices for PageRank, that is interesting to users. Thus, paying much attention to the accuracy of the solution of such a system underscores the practical significance of the PageRank algorithm. This chapter presents a survey of some stopping criteria used in solving a linear system of equations. Also, we evaluate a clustering algorithm as an alternative stopping criterion for ranks.

1.7.3 Chapter 4

This chapter focuses on studies of perturbed Markov chain models commonly used for describing information networks. In such models, stationary distribution and other related characteristics of information Markov chains serve as basic tools for ranking of nodes in information networks. Usually, a matrix of transition probabilities of an information network is regularised by adding a special damping matrix multiplied by small damping (perturbation) parameter ε. We describe the procedure of stochastic modeling of Markov chains with damping component, and the procedure of embedding such Markov chains in the model of discrete time regenerative processes with special damping regenerative times. Furthermore, we give explicit coupling type upper bounds for the rate of convergence in ergodic theorems for Markov chains with damping component. We also give ergodic theorems for Markov chains with damping component in triangular array mode. We illustrate with numerical examples some of the results.

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