Asymptotic results for American option prices under extended Heston model

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Asymptotic results for American option prices under extended Heston model

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Abstract

In this thesis, we consider the pricing problem of an American put option. We introduce a new market model for the evolution of the underlying asset price. Our model adds a new parameter to the well known Heston model. Hence we name our model the extended Heston model. To solve the American put pricing problem we adapt the idea developed by Fouque et al. (2000) to derive the asymptotic formula. We then connect the idea developed by Medvedev and Scaillet (2010) to provide an asymptotic solution for the leading order term $P_0$. We do numerical analysis to gain insight into the accuracy and validity of our asymptotic approximation formula.

**Keywords:** American options; Stochastic Volatility; Extended Heston model; Fast mean–reversion volatility; Asymptotic expansion, Average Volatility
Dedication

I dedicate this thesis to my family members, for their support and encouragement, nothing would be possible without them.
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# Contents

List of Figures .............................................. ii
List of Tables ............................................... iii
List of Acronyms ............................................. iv

1 Introduction ............................................. 1
   1.1 Motivation and Context ............................... 1
   1.2 Thesis Contribution .................................. 3
   1.3 Overview and Outline ................................. 4

2 The Model ............................................... 5
   2.1 The Extended Heston Model (EH) ....................... 5
   2.2 The pricing method ................................... 7

3 Asymptotic results for American options ............ 9
   3.1 Fast–mean reverting asymptotic results ............... 9
   3.2 Third–order short–maturity asymptotic result for $P_0$ 14
   3.3 The main result ...................................... 17

4 Numerical Results ...................................... 19

5 Conclusion and Further Research .................... 24
   5.1 Review and Conclusion ............................... 24
   5.2 Further Research .................................... 24

Appendices .................................................. 26

Bibliography ................................................. 36
List of Figures

4.1 A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM with different strike prices. ................................................................. 21
4.2 A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM with $\eta = 0.7$, $\theta = 0.08$. 22
4.3 A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM with different strikes and varying $\eta$. ................................................................. 23
List of Tables

4.1 A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM. . . . . . . . . . . . . . 20
List of Acronyms

BOPM  Binomial Option Pricing Model.
BS   Black–Sholes.
CIR  Cox–Ingersoll–Ross.
EH   Extended Heston Model.
OU   Ornstein Uhlenbeck.
PDE  Partial Differential Equation.
SDE  stochastic differential equation.
Chapter 1

Introduction

This chapter provides an introduction to the topic. It starts by introducing and defining financial options and some related terms. Later, the chapter gives the fundamental models used to price options. The reader will be introduced to European and American options pricing models. The focus is the stochastic volatility models that capture the variability of the asset prices.

1.1 Motivation and Context

Options or contingent claims are one of the most profitable tools available to traders today. They offer traders the ability to leverage positions, manage risk, and enhance returns on existing portfolios. There are two types of option, call option and put option. The first one gives the holder the right to buy the underlying asset on or before a certain date for a certain price. The second one gives the holder the right to sell the underlying asset on or before a certain date for a certain price. The price in the contract is known as the exercise or strike price. The date in the contract is known as the expiration or maturity date. American options can be exercised at any time up to the expiration date. European options can be exercised only on the expiration date itself.

American call options on non-dividend-paying stocks have the same value with the European call options. When a call option is exercised early it discards its time value for money in the option. If the underlying pays dividend it’s worthy to exercise early to capture a dividend. When the underlying is non-dividend-paying stock there is no benefit to early exercise of the American call option. For the case of an American or European put option on a non-dividend paying stock the case is different in terms of options values. American put option is worth more than its European counterpart. In this research we will focus on pricing American put option on a non-dividend-paying stock.
Every option contract has two sides; on one side there is an investor who buys the option (long position), on the other side there is an investor who sells the option (short position). Moreover, for both option contract we have the following payoff functions:

- for a call option \((S_T - K)^+\)
- for a put option \((K - S_T)^+\)

where \(K\) is the strike price and \(S_T\) is the price at maturity of the underlying asset. \((S_T - K)^+\) denotes the greater value \(S_T - K\) and 0 while \((K - S_T)^+\) denotes the greater value of \(K - S_T\) and 0. We refer to Hull (2008) for more properties of options.

For American options the exercise time \(\tau\) can be represented as a stopping time; so American options are an example of optimal stopping time problems. Not knowing the exercise time \(\tau\) makes it much harder to evaluate these options. The holder of an American option is thus faced with the dilemma of deciding when, if at all, to exercise. If, at time \(t\), the option is out-of-the-money then it is clearly best not to exercise. However, if the option is in-the-money it may be beneficial to wait until a later time where the payoff might be even bigger.

The difficulty in pricing American options lies in its early exercise right that’s where we come up with a boundary value problem since the optimal early exercise price is time-dependent and became part of the solution. Since the unknown boundary is a part of the solution to the problem it makes American option to be a free boundary value problem. Valuation of such a problem is a difficult task. There is no closed-form solution to price American options in contrast to European options for which the closed-form formula in the classical Black Scholes model is available.

Generally, the complexity of computing American options prices push practitioners to base their work on mathematical methods. Black and Scholes (1973) developed a model to price options with an assumption of constant volatility. As recognized by other authors, this assumption is not consistent with observed market data. To overcome this inconsistency different authors put their efforts to develop pricing methods that capture the variability of the asset prices, i.e. models with stochastic volatility. For European options such methods have been developed by Fouque et al. (2000), Heston (1993), Gulisashvili and Stein (2006, 2010), Hull and White (1987) and Canhanga et al. (2016), among others. For pricing American options Clarke and Parrott (1999), Fouque et al. (2000), Medvedev and Scaillet (2010), Tzavalis and Wang (2003) and Zhang and Lim (2006), developed such models among others.

Heston (1993) proposes a stochastic volatility model, which allows the volatility to be correlated with the asset price, and derives closed-form solutions for vanilla options in terms of the characteristic functions. Medvedev and Scaillet (2010), among other contributions, provide an analysis of the impact of volatility mean–reversion on
the American put price. Fouque et al. (2000), developed and derive the option pricing framework with fast mean reverting stochastic volatility. The latter means that the volatility level rapidly returns to its mean value whilst also containing a random component.

The idea in Fouque et al. (2000) has been instrumental in solving the fast mean reverting stochastic volatility problems with asymptotic techniques. However, there has been very little numerical validation of the technique. According to them, the volatility process is modeled as a function of a mean reverting Ornstein Uhlenbeck (OU) process. In this thesis we adopt the same idea of solving the fast mean reverting stochastic volatility problems but with a different stochastic process. We propose a new model and adapt the asymptotic technique to it for finding American option prices. We name this new model the extended Heston model (EH) addressed in details in Chapter 2.

Our task is to price American options, more specifically American put options, under the EH model. We propose the EH model after realizing that the Heston (1993) as a single–factor stochastic volatility model can’t capture certain feature of the volatility surface observed from the option market. According to Christoffersen et al. (2009) the model is overly restrictive in the modeling of the relationship between the volatility level and the slope of the smirks (see Christoffersen et al. (2009) for more details on this). Moreover, Gatheral (2006), shows that the Heston (1993) model does not fit very well the observed implied volatilities for longer maturities. To overcome this in our model (EH) we introduce an extra parameter $\eta$ in the stochastic volatility component i.e. $V_t^{\eta}$ (see the first equation of 2.1) to the original single–factor Heston (1993) and thus it adds one more degree of freedom while fitting our model to the market data. Later, we will connect it with an idea developed by Medvedev and Scaillet (2010) to provide an asymptotic solution for the leading order term $P_0$ for which Fouque et al. (2000) does not give an explicit formula. Solving for $P_0$ (as shown in Section 3.2) is equivalent to solving for the American put price under the Black-Scholes model with an average volatility term for which we give an explicit formula. According to Medvedev and Scaillet (2010), their method is stable and accurate in the Black–Scholes (BS) case even for longer maturity. This means their approach is suitable for all maturities of practical interests for the Black-Scholes case, which motivates our adoption of their approach to our leading-order problem.

Later, we will explore the numerical analysis of the asymptotic solution basing on the American put options.

1.2 Thesis Contribution

In this thesis we focus on pricing American put options. The report has the following contributions. Firstly, we propose a new model the EH model. Secondly, we provide the solution by adapting Fouque et al. (2000) approach under the EH model and then adopt Medvedev and Scaillet (2010) idea to solve the leading order term.
1.3 Overview and Outline

The thesis is structured as follows:

Chapter 1 gives a brief introduction of the research topic. In Chapter 2, we introduce our model EH with fast-mean reverting stochastic volatility and then provide a description of the pricing method. Chapter 3 presents asymptotic results for American options. Chapter 4 contains numerical results. Finally, Chapter 5 provides a conclusion and areas for further research. Appendix A, Appendix B and Appendix C provides more detailed mathematical derivations.
Chapter 2
The Model

2.1 The Extended Heston Model (EH)

The extended Heston model is built upon the assumption that the option’s underlying asset’s price process is modeled as a diffusion process. Mathematically, let \( \{S_t : t \geq 0\} \) be a stochastic process defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration continuous on the right, where \( \Omega \) is the sample space, \( \mathcal{F} \) the \( \sigma \)-algebra/filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions and \( \mathbb{P} \) is the physical probability measure. Then, \( \{S_t : t \geq 0\} \) as a stochastic process satisfies the following stochastic differential equation (SDE) under fast mean reverting.

\[
\begin{align*}
    dS_t &= (\mu - q)S_t dt + V_t^{\eta} S_t dB_t^1, \\
    dV_t &= \frac{1}{\epsilon} (\theta - V_t) dt + \sigma_V \sqrt{V_t} dB_t^2, \\
    \mathbb{E}[dB_t^1 dB_t^2] &= \rho dt,
\end{align*}
\]

(2.1)

where, \( S_t, \mu, q, V_t, \sigma_V \), represent the price of stock at time \( t \), the expected rate of return, the dividend yield, the volatility of stock price and the volatility of the volatility process respectively. Here \( \theta \) is the long run mean of the square root mean reverting process \( V_t \), \( \frac{1}{\epsilon} \) is the fast rate of reversion (0 < \( \epsilon \) << 1), and \( B_t^1 \) and \( B_t^2 \) are correlated Wiener processes (with correlation parameter \( \rho \)).

The Cox–Ingersoll–Ross (CIR) processes \( V_t \) on each finite time interval \( t \in [0, +\infty) \) remain strictly positive if the Feller’s condition, (i.e. \( 2\frac{1}{\epsilon}\theta > \sigma_V^2 \)), is satisfied (Feller (1951)). We assume that the Feller condition is satisfied.

Note that we have introduced an extra parameter \( \eta \). For simplicity at this stage we consider \( \eta \in [0.5, 1] \). If \( \eta = 0.5 \), our model reduces to the well-known Heston’s stochastic volatility model. So we may consider our model as an extended version of the Heston model hence name it the EH model hereafter. Note that when \( \eta \neq 0.5 \), the
process \( \{ V_t, t \geq 0 \} \) is no longer precisely the variance process but a volatility driving process. With one more parameter our EH model has the potential to offer more flexibility in fitting market data. This is one of the motivations of considering the pricing problem under EH model. Another motivation is that our combined approach of Fouque et al. (2000) chapter 9 and Medvedev and Scaillet (2010) for American option pricing can be conveniently adapted to EH model. Our main results are therefore applicable to the special case of Heston stochastic volatility model under fast-mean reverting, which is a contribution to the literature by itself.

The EH model uses the familiar CIR process as the volatility driving process. We would like to mention another market model with a similar name studied in Altmayer and Neuenkirch (2015). This model is called the generalized Heston model. In this generalized Heston model, \( \eta \) as in (2.1) is equal to 0.5 as standard but the variance process is known as the mean-reverting CEV process,

\[
dv_t = \kappa (\lambda - v_t) \, dt + \theta v_t^\gamma \, dW^1_t,
\]

In this paper, the authors study a different problem of finding expectation of option payoff functionals using multilevel Monte Carlo methods.

We can do a Cholesky decomposition of the correlated Wiener processes (see Kijima (2003) chapter 10 page 164 and Appendix C) and rewrite Equation (2.1) in the form of

\[
dS_t = (\mu - q) S_t \, dt + V_t^n S_t \, dW^1_t,
\]

\[
dV_t = \frac{1}{\varepsilon} (\theta - V_t) \, dt + \sigma V_t \sqrt{V_t} \left( \rho \, dW^1_t + \sqrt{1 - \rho^2} \, dW^2_t \right),
\]

(2.2)

where \( W^1_t \) and \( W^2_t \) are independent Wiener processes.

Moreover, we can further simplify Equation (2.2) into the following form

\[
dS_t = (\mu - q) S_t \, dt + V_t^n S_t \, dW^1_t,
\]

\[
dV_t = \frac{1}{\varepsilon} (\theta - V_t) \, dt + \sigma V_t \sqrt{V_t} \left( \rho \, dW^1_t + \sqrt{1 - \rho^2} \, dW^2_t \right).
\]

(2.3)

Using the same idea by Canhanga et al. (2016) we make the volatility of the volatility process to depend on the rate of mean reversion i.e. \( \sigma V = \frac{1}{\sqrt{\varepsilon}} \) and transform Equation (2.3) into

\[
dS_t = (\mu - q) S_t \, dt + V_t^n S_t \, dW^1_t,
\]

\[
dV_t = \frac{1}{\varepsilon} (\theta - V_t) \, dt + \frac{1}{\sqrt{\varepsilon}} \rho \sqrt{V_t} \, dW^1_t + \frac{1}{\sqrt{\varepsilon}} \sqrt{(1 - \rho^2)} V_t \, dW^2_t.
\]

(2.4)

In pricing a contingent claim there must be no arbitrage in the market. In order to eliminate arbitrage opportunities we use Girsanov’s theorem and transform the system
of equations (2.4) into another system of SDEs under risk neutral probability measure, to incorporate market price of volatility risk. The change of measure is accomplished by applying the Girsanov’s theorem for Wiener processes for which there exist

$$dW_t^j = \Lambda_t^{(j)}dt + dW_t^j,$$  \hspace{1cm} (2.5)

where $W_t^j$ are Wiener processes under the risk neutral measure and $\Lambda_t^{(j)}$ are the market prices of risk associated with the Wiener instantaneous shocks $dW_t^j$ for $j = 1, 2$. From (2.5) we substitute $dW_t^j$ s in (2.4) and collecting similar terms and considering that $\mu = r - q$ under risk neutral measure, we obtain the following system of SDEs

$$dS_t = (r - q)S_t dt + V_t \eta S_t dW_t^1,$$

$$dV_t = \left( \frac{1}{\varepsilon} (\theta - V_t) - \frac{1}{\sqrt{\varepsilon}} \xi \Lambda_t^{(2)} \sqrt{(1 - \rho^2)V_t} \right) dt + \frac{1}{\sqrt{\varepsilon}} \xi \sqrt{V_t} \rho dW_t^1 + \frac{1}{\sqrt{\varepsilon}} \xi \sqrt{(1 - \rho^2)V_t} dW_t^2.$$

(2.6)

Here, $\Lambda_t^{(1)} = \frac{\mu - (r - q)}{2\varepsilon}$ and $\Lambda_t^{(2)}$ is the so called market price of volatility risk which is an unknown function of $V_t$. According to Chiarella and Ziveyi (2013), $\Lambda_t^{(2)}$ is assumed to have the following form

$$\Lambda_t^{(2)} = \frac{\lambda \sqrt{V_t \varepsilon}}{\xi \sqrt{1 - \rho^2}},$$

where, $\lambda$ is a new constant which should be estimated from market data. We adopt the same form of $\Lambda_t^{(2)}$ and substitute into Equation (2.6), after simplifying we end–up with the following form of SDE

$$dS_t = (r - q)S_t dt + V_t \eta S_t dW_t^1,$$

$$dV_t = \left( \frac{1}{\varepsilon} (\theta - V_t) - \lambda V_t \right) dt + \frac{1}{\sqrt{\varepsilon}} \xi \sqrt{V_t} \rho dW_t^1 + \frac{1}{\sqrt{\varepsilon}} \xi \sqrt{(1 - \rho^2)V_t} dW_t^2.$$

(2.7)

After having our system under risk neutral probability measure we then find the price of an American put option.

### 2.2 The pricing method

Now, consider the payoff function of an American put option with strike price $K$ maturing at time $T$ given by $h(S_T) = (K - S_T)^+$. The option is exercised only when $K > S_T$. Let $P(t, s, v)$ be the price of the option at time $t < T$ when the underlying derivative has the price $s$ and volatility $v$. By no arbitrage arguments financial derivatives must be priced under the risk-neutral probability measure.

By the risk–neutral valuation theory, the price $P(t, s, v)$ of an American put option under the risk-neutral probability measure is the supremum of the expected discounted payoff over all stopping times $\tau \in [t, T]$, i.e.
The function $P(t,s,v)$ satisfies a free boundary problem. This free boundary is a surface that can be written as $s = s_{fb}(t,v)$. In the exercise region $s < s_{fb}(t,v)$ we have the following problem

$$P(t,s,v) = K - s \quad \text{for} \quad s < s_{fb}(t,v) \quad (2.8)$$

By Fouque et al. (2000) chapter 9 Equation 9.2 and Heston (1993) adapted to the EH model, we can apply Feynman–Kac theorem and obtain the price of an American contingent claim in the hold region $s > s_{fb}(t,v)$ is expressed as the solution of the following boundary value problem

$$\frac{\partial P}{\partial t} + (r - q)s \frac{\partial P}{\partial s} + \left[ \frac{1}{\varepsilon} (\theta - v) - \lambda v \right] \frac{\partial P}{\partial v} + \frac{1}{2} \varepsilon^{2} s^{2} \frac{\partial^{2} P}{\partial s^{2}} + \frac{1}{2} \varepsilon^{2} v \frac{\partial^{2} P}{\partial v^{2}} + \frac{\xi}{\sqrt{\varepsilon}} \rho \left( \eta + \frac{1}{2} \right) s \frac{\partial^{2} P}{\partial s \partial v} - rP = 0,$$

(2.9)

with boundary condition

$$P(T,s,v) = (K - s)^{+},$$

$$s_{fb}(T,v) = K. \quad (2.10)$$

It is known from Fouque et al. (2000) chapter 9 that $P$, $\frac{\partial P}{\partial s}$, and $\frac{\partial P}{\partial v}$ are continuous across the boundary $s_{fb}(t,v)$ so that

$$P(t,s_{fb}(t,v),v) = (K - s_{fb}(t,v))^{+},$$

$$\frac{\partial P}{\partial s}(t,s_{fb}(t,v),v) = -1, \quad (2.11)$$

$$\frac{\partial P}{\partial v}(t,s_{fb}(t,v),v) = 0.$$

The first equation in (2.11) shows that the option price is the intrinsic value of the option or the payoff when the optimal exercise price is attained. The second and third equations are known as smooth pasting conditions which means that the option price is smoothly connected to the payoff function at $s = s_{fb}(t,v)$. The first and second conditions are well–known in the literature on American option pricing.
Chapter 3

Asymptotic results for American options

3.1 Fast–mean reverting asymptotic results

We adapt Fouque et al. (2000) chapter 9 Equation 9.2 to derive the asymptotic approximation for American option pricing problem under the EH model.

The PDE equation (2.9) can be rewritten in a more compact form as

\[
\left( \frac{1}{\varepsilon} L_0 + \frac{1}{\sqrt{\varepsilon}} L_1 + L_2 \right) P^\varepsilon = 0, \tag{3.1}
\]

\( P^\varepsilon(t,s,v) = P(t,s,v) \) is the price of the option. The superscript \( \varepsilon \) is added to emphasize that the solution depends on \( \varepsilon \). The operators are denoted as

\[
L_0 = (\theta - v) \frac{\partial}{\partial v} + \frac{1}{2} \hat{\sigma}^2 v \frac{\partial^2}{\partial v^2},
\]

\[
L_1 = \xi \rho v^{(\eta + \frac{1}{2})} s \frac{\partial^2}{\partial s \partial v},
\]

\[
L_2 = \frac{\partial}{\partial t} + (r - q)s \frac{\partial}{\partial s} + \frac{1}{2} v^2 s^2 \frac{\partial^2}{\partial s^2} - \lambda v \frac{\partial}{\partial v} - r = L_{BS}(\hat{\sigma}),
\]

where,

- \( \frac{1}{\varepsilon} L_0 \) is the infinitesimal generator of the square root mean-reverting process.
- \( L_1 \) contains mixed partial derivatives due to the correlation \( \rho \) between asset price \( s \) and volatility \( v \).
- \( L_2 \) also denoted by \( L_{BS}(\hat{\sigma}) \) is the Black-Scholes operator at the average volatility level \( (\hat{\sigma}) \). A explicit formula for the average volatility \( (\hat{\sigma}) \) is given in Equation (3.12).
Moreover, $1/\varepsilon$, $1/\sqrt{\varepsilon}$ and 1 are the term of orders for $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$ respectively.

Again, from (3.1) denote

$$\mathcal{L}^\varepsilon = \left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right),$$

and finally, we have,

$$\mathcal{L}^\varepsilon P^\varepsilon(t,s,v) = 0 \quad \text{in} \quad s > s_{fb}^\varepsilon(t,v) \quad (\text{i.e. the hold region}),$$

(3.2)

where the indexation to $\varepsilon$ denotes the dependence on it.

As suggested by Fouque et al. (2000) chapter 9 page 134, we assume the following asymptotic expansion for the option price

$$P^\varepsilon(t,s,v) = P_0(t,s,v) + \sqrt{\varepsilon} P_1(t,s,v) + \varepsilon P_2(t,s,v) + \cdots ,$$

(3.3)

where the $P_i^\varepsilon$s are the coefficient functions to be determined. Moreover, we assume the following asymptotic expansion for the free boundary,

$$s_{fb}^\varepsilon(t,v) = s_0(t,v) + \sqrt{\varepsilon} s_1(t,v) + \varepsilon s_2(t,v) + \cdots ,$$

(3.4)

Substituting Equation (3.3) into equation (3.2) and then collecting terms of up to order $\sqrt{\varepsilon}$ we obtain,

$$\frac{1}{\varepsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}} (\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0)$$

$$+ \sqrt{\varepsilon} (\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \cdots = 0.$$

(3.5)

Meanwhile, from (3.4) we keep terms of order up to $\sqrt{\varepsilon}$ and expand the boundary condition (2.11) as follows;

$$P_0(t,s_0(t,v),v) + \sqrt{\varepsilon} \left( s_1(t,v) \frac{\partial P_0}{\partial s}(t,s_0(t,v),v) + P_1(t,s_0(t,v),v) \right)$$

$$= K - s_0(t,v) - \sqrt{\varepsilon} s_1(t,v),$$

(3.6)

$$\frac{\partial P_0}{\partial s}(t,s_0(t,v),v) + \sqrt{\varepsilon} \left( s_1(t,v) \frac{\partial^2 P_0}{\partial s^2}(t,s_0(t,v),v) + \frac{\partial P_1}{\partial s}(t,s_0(t,v),v) \right)$$

$$= -1,$$

(3.7)

$$\frac{\partial P_0}{\partial v}(t,s_0(t,v),v) + \sqrt{\varepsilon} \left( s_1(t,v) \frac{\partial^2 P_0}{\partial v^2}(t,s_0(t,v),v) + \frac{\partial P_1}{\partial v}(t,s_0(t,v),v) \right)$$

$$= 0,$$

(3.8)
The terminal conditions in the hold region are $P_0(T, s, v) = (K - s)^+$ and $P_1(T, s, v) = 0$, and in the exercise region are $P^e = (K - s)^+$ as $P_0(t, s, v) = 0$ and $P_1(t, s, v) = 0$.

Now to determine $P_i's$, we use Equation (3.5) and equate various orders of $\varepsilon$ to zero.

1. **Term of order $1/\varepsilon$:** we have the following problem:

   \[
   L_0 P_0(t, s, v) = 0 \quad \text{in} \quad s > s_0(t, v),
   \]
   \[
   P_0(t, s, v) = (K - s)^+ \quad \text{in} \quad s < s_0(t, v),
   \]
   \[
   P_0(t, s_0(t, v), v) = (K - s_0(t, v))^+,
   \]
   \[
   \frac{\partial P_0}{\partial s}(t, s_0(t, v), v) = -1.
   \]

   As $L_0$ is the generator of an ergodic Markov process acting on variable $v$ it contains only partial derivatives with respect to $v$ in each of its components, it means that $P_0$ is independent of $v$ on each side of $s_0$. It cannot depend on $v$ on the surface $s_0$ also, therefore, $s_0 = s_0(t)$ does not depend on $v$.

2. **Term of order $1/\sqrt{\varepsilon}$:**

   \[L_0 P_1 + L_1 P_0 = 0\]

   The operator $L_1$ contains only the mixed partial derivatives with respect to cross term of $s$ and $v$ and since $P_0$ does not depend on $v$, therefore $L_1 P_0 = 0$ which gives

   \[
   L_0 P_1(t, s, v) = 0 \quad \text{in} \quad s > s_0(t),
   \]
   \[
   P_1(t, s, v) = 0 \quad \text{in} \quad s < s_0(t),
   \]
   \[
   P_1(t, s_0(t), v) = 0,
   \]
   \[
   s_1(t, v) \frac{\partial^2 P_0}{\partial s^2}(t, s_0(t)) + \frac{\partial P_1}{\partial s}(t, s_0(t), v) = 0.
   \]

   Therefore, $P_1$ also does not depend on $v$: $P_1 = P_1(t, s)$.

3. **Term of order $1 (\varepsilon^0)$:**

   \[L_0 P_2 + L_1 P_1 + L_2 P_0 = 0\]

   As showed before $L_1 P_1 = 0$, then we have

   \[
   L_0 P_2(t, s, v) + L_2 P_0(t, s, v) = 0 \quad \text{in} \quad s > s_0(t)
   \]
   \[
   P_2(t, s, v) = 0 \quad \text{in} \quad s < s_0(t)
   \]

   which is a Poisson equation for $P_2$. By Fouque et al. (2000) chapter 9 there is no solution unless averaging the source term $L_2 P_0$ with respect to the invariant distribution of the Cox–Ingersoll–Ross (CIR) process $V_t$ must be zero.
In other words,

$$\langle \mathcal{L}_2 P_0 \rangle = 0,$$

where $\langle \cdot \rangle$ is the averaging / expectation operator with respect to invariant distribution of the $V_t$ process. Since the variance process $V_t$ is a CIR process its invariant distribution is a gamma distribution, (see Feller (1957)), with probability density given by

$$\pi(v) = \left( \frac{\alpha}{\theta} \right)^{\alpha} \frac{1}{\Gamma(\alpha)} v^{\alpha - 1} e^{-\alpha v/\theta}, \quad \text{where} \quad \alpha = \frac{2\theta}{\xi^2}$$

with shape parameter $\alpha$ and rate parameter $\alpha / \theta$.

The averaging operator $\langle g \rangle$ for any function $g$ is defined as

$$\langle g \rangle = \int g(v) \pi(v) dv.$$ 

We set now $\langle g \rangle = 0$.

Since $P_0$ does not depend on $v$ and the $\mathcal{L}_2$ depends on $v$ only through the $v$ coefficient, we have $\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0$ and it follows that

$$\left( \frac{\partial P_0}{\partial t} + (r - q) s \frac{\partial P_0}{\partial s} + \frac{1}{2} v^2 \eta^2 s^2 \frac{\partial^2 P_0}{\partial s^2} - r P_0 \right) = 0.$$ 

Then we have

$$\left( \frac{\partial}{\partial t} + (r - q) s \frac{\partial}{\partial s} + \frac{1}{2} \hat{\sigma}^2 s^2 \frac{\partial^2}{\partial s^2} - r \right) P_0 = 0,$$

where $\hat{\sigma}^2 = \langle v^2 \eta \rangle$.

We shall call $\hat{\sigma}$ average volatility hereafter. The explicit formula for average...
volatility $\hat{\sigma}$ is derived below.

$$
\hat{\sigma}^2 = \langle v^{2\eta} \rangle = \int v^{2\eta} \pi(v) dv \\
= \int_0^\infty v^{2\eta} \frac{\alpha^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-\alpha \frac{v^\alpha}{\theta}} dv \\
= \int_0^\infty v^{2\eta+\alpha-1} \frac{\alpha^\alpha}{\Gamma(\alpha)} e^{-\frac{v^\alpha}{\theta}} dv \\
= \frac{1}{\Gamma(\alpha)} \int_0^\infty v^{2\eta+\alpha-1} e^{-\frac{v^\alpha}{\theta}} dv \\
= \frac{\Gamma(\alpha+2\eta) \alpha^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{\Gamma^{2\eta+a}}{\Gamma(\alpha+2\eta)} e^{-\frac{v^\alpha}{\theta}} dv \\
= \frac{\Gamma(\alpha+2\eta) \alpha^\alpha e^{-\frac{v^\alpha}{\theta}}}{\Gamma(\alpha)} \\
(3.12)
$$

We define $\Gamma(x) = \int_0^\infty y^{x-1} e^y dy$. Hence the last equality follows because the integrand of $\int_0^\infty \frac{\Gamma^{2\eta+a}}{\Gamma(\alpha+2\eta)} e^{-\frac{v^\alpha}{\theta}} dv$ is a density function of a gamma distribution with shape parameter $\tilde{\alpha} = \alpha + 2\eta$ and rate parameter $\alpha \theta$.

Thus $P_0(t,s)$ and $s_0(t)$ satisfy the Black–Scholes (BS) American put problem with average volatility $\hat{\sigma}$ expressed in Equation 3.12.

We have now the following problem for $P_0(t,s)$ (see section 3.2 for detailed discussion) which is exactly the BS American put problem with average volatility $\hat{\sigma}$

$$
P_0 = K - s \quad \text{in} \quad s < s_0(t) \quad \text{– The exercise region,} \\
\langle L_2 \rangle P_0 = 0 \quad \text{in} \quad s > s_0(t) \quad \text{– The hold region,}
$$

with boundary conditions,

$$
P_0(T,s) = (K - s)^+, \\
P_0(t,s_0(t)) = (K - s_0(t))^+, \\
\frac{\partial P_0}{\partial s}(t,s_0(t)) = -1,
$$

where,

$$
\langle L_2 \rangle = \mathcal{L}_{BS}(\hat{\sigma}) = \frac{\partial}{\partial t} + (r - q)s \frac{\partial}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2}{\partial s^2} - r 
$$

Lemma 3.1. Consider an American put option with strike price $K$ and time-to-maturity $\tau$. Under the EH model (2.1) for the underlying process and under the assumption (3.3), the zeroth-order asymptotic expansion with respect to the mean-reversion parameter $\epsilon$ for the option’s price is given by

$$
P^\epsilon(t,s,v) = P_0 + O(\sqrt{\epsilon}) \quad \text{as} \quad \epsilon \to 0
$$
where $P_0$ is the solution to American pricing problem (3.13) under a modified Black–Scholes model with average volatility ($\hat{\sigma}$) given in equation (3.12).

Remark 1. Coefficient $P_0$ is the leading-order term in expansion (3.3). It is possible to determine the coefficient $P_1$ by solving numerically a fixed boundary problem, which we don’t discuss in this paper but refer to Fouque et al. (2000) chapter 9 for more details.

### 3.2 Third–order short–maturity asymptotic result for $P_0$

In this section we adopt the approach proposed by Medvedev and Scaillet (2010) to solve (3.13) – the BS American put problem. Note that, once $\hat{\sigma}$ is determined the problem is identical to the BS American option pricing problem in Medvedev and Scaillet (2010). To be self contained we review the derivation on how to obtain third–order short–maturity asymptotic expansion for pricing American put options by Medvedev and Scaillet (2010). However we provide more computation details.

We start by introducing the normalized moneyness ratio as

$$\theta = \frac{\ln(K/s)}{\hat{\sigma}\sqrt{\tau}}, \quad (3.14)$$

Here $\tau = T - t$ refers to the time–to–maturity of the option and $s$ is the underlying stock price. The normalized moneyness ratio measures the distance between logarithm of stock price and that of strike price in terms of standard deviation.

We consider the same idea of suboptimal exercise strategy to replace the optimal exercise rule, which involves exercising an option when its moneyness reaches some specified level. We choose to exercise the option when it is in the money and has large moneyness. We set,

$$\bar{y}(\theta, \tau) = \arg\max_{y \geq \theta} \{P(\theta, \tau, v)\}. \quad (3.15)$$

The decision to exercise American put option early depends on the comparison of $\theta$ and the early exercise level of normalised moneynes $\bar{\theta}(\tau)$.

$$\bar{\theta}(\tau) = \frac{\ln(K/s_0(t))}{\hat{\sigma}\sqrt{\tau}} \sim \sqrt{\ln(1/\tau)}. \quad (3.16)$$

$\bar{\theta}(\tau)$ can be approximated in the following way

$$\bar{\theta}(\tau) = \arg\min_{\theta} \{\bar{y}(\theta, \tau) = \theta\}. \quad (3.17)$$
From (3.16) we observe that when \( \tau \) goes to zero no matter how deep in the money the put option is, it is suboptimal to exercise the option before maturity (see Barles et al. (1995)).

We consider Equation (3.13) with the last boundary condition replaced by explicit rule, that is to exercise as soon as its moneyness reaches some specified level. The new problem is the same PDE addressed in (3.13); i.e.,

\[
\frac{\partial P_0}{\partial t} + (r - q)s \frac{\partial P_0}{\partial s} + \frac{1}{2} \hat{\sigma}^2 s^2 \frac{\partial^2 P_0}{\partial s^2} - r P_0 = 0 \quad \text{in} \quad s > s_0(t) \tag{3.18}
\]

with boundary conditions:

\[
P_0(T, s) = (K - s)^+ \tag{3.19}
\]

\[
P_0(t, s_0(t)) = (K - s_0(t))^+ \tag{3.20}
\]

where \( s_0(t) \) satisfies \( s_0(t) = Ke^{-\hat{\sigma} \sqrt{\tau}} \). Here \( y \) is the specified level of moneyness in which the option can be exercised as soon as it hits it.

We denote \( P_0(\theta, \tau; y) \) the price of an American put option and that we choose to exercise as soon as the normalised moneyness reaches some barrier level \( y \).

We rewrite (3.18) in terms of \((\theta, \tau)\) instead of \((s, t)\). Using the definition of \( \theta \) in (3.14), we set \( P_0(\theta, \tau) = P_0(Ke^{-\hat{\sigma} \sqrt{\tau}}, T - \tau) \) and use the chain rule to obtain the following derivatives

\[
\frac{\partial P_0}{\partial t} = -P_{0\tau} + \frac{\theta}{2\tau} P_{0\theta}
\]

\[
\frac{\partial P_0}{\partial s} = -\frac{1}{\hat{\sigma} \sqrt{\tau}} P_{0\theta}
\]

\[
\frac{\partial^2 P_0}{\partial s^2} = \frac{1}{\hat{\sigma}^2 \tau^2} P_{0\theta} + \frac{1}{\hat{\sigma}^2 \tau} P_{0\theta}
\]

Substituting the above derivatives into Equation (3.18) we obtain the following equation

\[
-P_{0\tau} + \frac{\theta}{2\tau} P_{0\theta} + (r - q) s \left( \frac{-1}{\hat{\sigma} \sqrt{\tau}} P_{0\theta} \right) + \frac{1}{2} \hat{\sigma}^2 s^2 \left( \frac{1}{\hat{\sigma}^2 \tau^2} P_{0\theta} + \frac{1}{\hat{\sigma}^2 \tau} P_{0\theta} \right) - r P_0 = 0,
\]

\[
-P_{0\tau} + \frac{\theta}{2\tau} P_{0\theta} - \frac{(r - q)}{\hat{\sigma} \sqrt{\tau}} P_{0\theta} + \frac{1}{2\tau} P_{0\theta} + \frac{\hat{\sigma}}{2\sqrt{\tau}} P_{0\theta} - r P_0 = 0,
\]

\[
-2\tau P_{0\tau} + \theta P_{0\theta} - \frac{2\sqrt{\tau}(r - q)}{\hat{\sigma}} P_{0\theta} + P_{0\theta} + \hat{\sigma} \sqrt{\tau} P_{0\theta} - 2\tau r P_0 = 0,
\]

then we have

\[
\theta P_{0\theta} + P_{0\theta} + \frac{1}{\hat{\sigma}} \left[ \hat{\sigma}^2 + 2(q - r) \right] P_{0\theta} \sqrt{\tau} - 2 \left( P_{0\tau} + r P_0 \right) \tau = 0. \tag{3.21}
\]
with the following boundary conditions
\[ P_0(T, s) = (K - s)^+, \quad (3.22) \]
\[ P_0(y, \tau) = K \left( 1 - e^{-\hat{\sigma} \sqrt{\tau}} \right)^+ = K \left( 1 - e^{-\hat{\sigma} \sqrt{\tau}} \right). \quad (3.23) \]

The solution to equation (3.21) has the following regular asymptotic expansion near maturity,
\[ P_0(\theta, \tau) = \sum_{n=1}^{\infty} P_n(\theta) \tau^{\frac{n}{2}}, \quad (3.24) \]
where \( P_n(\theta), \) are the coefficients of short maturity asymptotic expansion in \( \tau \) and \( n = 1, 2, \ldots \). The condition (3.20) is implicated in (3.24). When \( \tau = 0 \) and \( \theta \) is held fixed we obtain \( s = K \) and \( (K - s)^+ = 0 \), thus \( P_0(\theta, \tau) = 0 \).

From equation (3.24) we take first and second order derivatives with respect to \( \theta \) and the first order derivative with respect to \( \tau \) and substitute in (3.21) then we obtain:
\[ -nP_n + \theta P_n + P_n + \frac{1}{\theta} \left[ \hat{\sigma}^2 + 2(q - r) \right] P_{n-1} - 2r P_{n-2} = 0, \quad n = 1, 2 \ldots, \quad (3.25) \]

We see that Equation (3.25) comprises two terms; the homogeneous part which consists of the first three terms on the left hand side and the rest is the non-homogeneous part. Now, by Proposition 1 in Medvedev and Scaillet (2010) we have the general solution in the following form,
\[ P_n(\theta) = C_n \left[ p_n^0(\theta) \Phi(\theta) + q_n^0(\theta) \phi(\theta) \right] + p_n^1(\theta) \Phi(\theta) + q_n^1(\theta) \phi(\theta). \quad (3.26) \]

where,
\[ \triangleright \] \( C_n \) is a constant coefficient with \( n = 1, 2, \ldots \)
\[ \triangleright \] \( p_n^0, q_n^0, p_n^1 \) and \( q_n^1 \) are polynomial solutions to be determined (see Appendix A).
\[ \triangleright \] \( \Phi(\theta) \) is the cumulative distribution function denoted by
\[ \Phi(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-\frac{s^2}{2}} ds, \quad (3.27) \]
\[ \triangleright \] \( \phi(\theta) \) is the standard normal density function denoted by
\[ \phi(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}. \quad (3.28) \]
Our next task is to solve a unique \( n \)th order expansion by determining \( n \) constants \( C_n \) in Equation (3.26). We do this using a third order expansion of Equation (3.24). When we substitute \( n = 1, 2, 3 \) (refer to Appendix B) we have,

\[
P_0 = C_1 \left[ \theta \Phi(\theta) + \phi(\theta) \right] \sqrt{\tau} + \left[ C_2 \left( (\theta^2 + 1)\Phi(\theta) + \theta \phi(\theta) \right) + \frac{C_1 \hat{\sigma}}{2} \Phi(\theta) - \frac{2\mu C_1}{\hat{\sigma}} \Phi(\theta) \right] \tau
\]

\[
+ \left[ C_3 \left( (\theta^3 + 3\theta)\Phi(\theta) + (\theta^2 + 2)\phi(\theta) \right) + \left( \frac{\hat{\sigma} C_2}{\hat{\sigma}} - \frac{2C_2 \mu}{\hat{\sigma}} - rC_1 \right) \theta \right] \Phi(\theta)
\]

\[
+ \left( \frac{\hat{\sigma} C_2}{\hat{\sigma}} - \frac{2C_2 \mu}{\hat{\sigma}} - rC_1 - \frac{1}{2} C_1 \mu + \frac{C_1 \hat{\sigma}}{8} + \frac{1}{2} \frac{C_1 \mu^2}{\hat{\sigma}^2} \right) \phi(\theta) \right] \tau \sqrt{\tau} + O(\tau^2).
\]

(3.29)

where, \( \mu = r - q \).

The coefficients \( C_1 \) and \( C_2 \) are obtained by imposing early exercise condition (3.23). Moreover, short maturity expansion for the payoff function is given by,

\[
P_0(y, \tau; y) = K[1 - \exp(-\hat{\sigma}y\sqrt{\tau})]
\]

\[
= \hat{\sigma}yK\sqrt{\tau} - \frac{\hat{\sigma}^2 y^2 K}{2} \tau + \frac{\hat{\sigma}^3 y^3 K}{6} \tau \sqrt{\tau} + O(\tau^2).
\]

(3.30)

We compare the missing coefficients by equating equation (3.29) at \( \theta = y \) to expansion (3.30) for the same order of \( \tau \).

For example \( C_1 \) is obtained as follows;

\[
C_1[\theta \Phi(\theta) + \phi(\theta)] \sqrt{\tau} = \hat{\sigma}yK\sqrt{\tau},
\]

\[
C_1 = \frac{\hat{\sigma}yK}{\theta \Phi(\theta) + \phi(\theta)},
\]

at \( \theta = y \)

\[
C_1 = \frac{\hat{\sigma}yK}{y \Phi_0 + \phi_0},
\]

where

\[
\Phi_0 = \Phi(y), \quad \phi_0 = \phi(y).
\]

We refer to Appendix B for the proof of short maturity expansion \( P_0(\theta, \tau; y) \) up to 3rd order.

3.3 The main result

We summarize the discussions above into the following theorem which is our main result.
Theorem 3.1. Consider an American put option with strike price $K$ and time-to-maturity $\tau$. Under the EH model (2.1) for the underlying process and under the assumption (3.3), the zeroth-order asymptotic expansion with respect to the mean-reversion parameter $\epsilon$ for the option’s price is given by

$$P^0(t,s,v) = P_0 + O(\sqrt{\epsilon}) \quad \text{as} \quad \epsilon \to 0.$$  \hspace{1cm} (3.31)

Here the leading-order term $P_0$ has the following third-order asymptotic expansion with respect to time-to-maturity $\tau$,

\begin{align*}
P_0 &= C_1 [\theta \Phi(\theta) + \phi(\theta)] \sqrt{\tau} + \left[ C_2 \left[ (\theta^2 + 1) \Phi(\theta) + \theta \phi(\theta) + \frac{C_1 \hat{\sigma}}{2} \phi(\theta) - \frac{2\mu C_1}{\sigma^2} \Phi(\theta) \right] \right] \tau \\
&\quad + \left[ C_3 \left[ (\theta^3 + 3\theta) \Phi(\theta) + (\theta^2 + 2) \phi(\theta) + \left( \hat{\sigma} C_2 - \frac{2C_2 \mu}{\sigma} - r C_1 \right) \theta \right] \phi(\theta) \right] \tau \sqrt{\tau} + O(\tau^2) \quad \text{as} \quad \tau \to 0.
\end{align*}

Here $\hat{\sigma} = \frac{\Gamma(\alpha + 2\eta)}{\Gamma(\alpha)} \frac{\alpha}{\theta}^{2\eta}$, where $\alpha = \frac{2\theta}{\xi \epsilon}$,

\begin{align*}
C_1 &= (K \hat{\gamma}) (\Phi_0 y + \phi_0)^{-1}, \\
C_2 &= -\left( \Phi_0 C_1 \hat{\sigma}^2 - 2\Phi_0 C_1 \mu + K \hat{y}^2 \hat{\sigma}^3 \right) \left[ \hat{\sigma} \left( \Phi_0 y^2 + \Phi_0 + \phi_0 y \right) \right]^{-1}, \\
C_3 &= \left[ 24\hat{\sigma}^2 \left( \Phi_0 y^3 + 3\Phi_0 y \Phi_0 + \Phi_0 y^2 + 2\phi_0 \right) \right]^{-1} \times \left( -24\Phi_0 y \hat{\sigma}^3 C_2 + 48\Phi_0 y \hat{\gamma} C_2 \mu + 24\Phi_0 y \hat{\gamma} \hat{\sigma}^2 r C_1 \\
&\quad - 24\Phi_0 C_2 \hat{\sigma}^3 + 48\Phi_0 C_2 \hat{\sigma} \mu + 24\Phi_0 r C_1 \hat{\sigma}^2 + 12\Phi_0 C_1 \hat{\sigma}^2 \mu - 3\Phi_0 C_1 \hat{\sigma}^4 - 12\Phi_0 C_1 \mu^2 + 4K \hat{y}^3 \hat{\sigma}^5 \right).
\end{align*}

Remark 2. By formula (3.31), $P_0$ serves as an approximation solution to the American put price under our EH model for small values of parameter $\epsilon$ (hence for fast mean-reverting). Note that, the error of this approximation is of order $O(\sqrt{\epsilon})$ if $P_0$ is given exactly. We point it out also that, for asymptotic formula (3.31) to be valid, the time-to-maturity for the option should not be too small (e.g. a few days), otherwise there is not enough time for the fast-mean reverting effect to come out (see also Fouque et al. (2000) chapter 9). However, the formula (3.32) is not an exact formula but a short-maturity asymptotic expansion. A natural question that arises is whether the asymptotic expansion formula (3.32), truncated to a third-order expansion, provides a good approximation to $P_0$ for moderate or longer maturities, for example, maturities of 0.5 years or 1.0 years. We address this question in numerical studies in the next chapter.
Chapter 4

Numerical Results

In this chapter, we study the accuracy of third-order short-maturity asymptotic expansion (3.32) as an approximation to $P_0$ in the asymptotic formula (3.31). It is worth reiterating that $P_0$ is the solution to the subproblem (3.13) of an American put pricing under the BS model. Under the BS model, it is well known that the binomial tree approach with correctly chosen up and down factors ($u = e^{\hat{\sigma} \sqrt{\Delta t}}, d = e^{-\hat{\sigma} \sqrt{\Delta t}}, \Delta t$ is the time step) gives a very accurate numerical approximation to the American put price. We use therefore the binomial option pricing model (BOPM) with a sufficiently small time step to get a benchmark price. We compare to our third-order asymptotic expansion in (3.32). Our numerical studies suggest that, for the parameters under consideration, the third-order short-maturity asymptotic expansion is a plausible approximation to $P_0$ for maturities up to 1 year. For the maturity of 1.5 years, the option needs to be deep in-the-money to have a decent approximation. The performance is also good and robust for a variety of choices of the new parameter $\eta = 0.5, 0.6, 0.7$. All numerical studies were performed in Python 3.6.

Our numerical experiment assume the following parameters, $S_0 = 40, r = 0.05, q = 0, \eta = \{0.5, 0.6, 0.7\}, \theta = \{0.04, 0.08\}, \bar{\sigma} = 0.5, \text{three different strike } K = \{35, 40, 45\}, \text{five different maturity } T = \{\frac{1}{12}, \frac{4}{12}, \frac{7}{12}, \frac{11}{12}, \frac{18}{12}\} \text{ and one more parameter, } M = \left[ \frac{T}{\Delta t} \right] \text{ for the BOPM, where } \left[ . \right] \text{ denotes the integer part and } \Delta t = \frac{1}{(2 \times 252)} \text{ i.e. time step of half of a day.}$

Furthermore, we also have two addition quantities i.e., the average volatility ($\hat{\sigma}$) and optimal boundary $y$. For each value of $\eta$, we compute $\hat{\sigma}$ using equation (3.12). To find the optimal price of an American put option, we iterate the optimal boundary $y$ from 0.5 to 3, with 200 steps. Here we find the optimal boundary $y$ that gives the maximum price. These optimal boundaries $y$ together with optimal prices are presented in Table 4.1.

Table 4.1 presents results with $\theta = 0.08, \eta = 0.7$, while keeping other parameter values
constant. The table shows prices of third–order short–maturity asymptotic approximation, BOPM, the relative deviation and optimal boundary which gives optimal American put price. The results show that the third–order short–maturity asymptotic approximation gives results that are very close to those obtained by the BOPM and hence we confirm that our approximation gives plausible results for a variety of maturities up to 1 year. For time to maturity of 1.5 years the option needs to be deep in–the–money to have a decent approximation, for example, when $K = 45$, $\tau = 1.5$ the relative error is -0.017.

We define the relative deviation between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM as follows

$$\text{Relative deviation} = \frac{P_0^{(3)} - P_{BOPM}}{P_{BOPM}}$$

where, $P_0^{(3)}$ is (3.32) truncated to the third order and $P_{BOPM}$ is the reference price by BOPM.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\tau = 0.08$</th>
<th>$\tau = 0.33$</th>
<th>$\tau = 1$</th>
<th>$\tau = 1.5$</th>
<th>$\tau = 0.08$</th>
<th>$\tau = 0.33$</th>
<th>$\tau = 1$</th>
<th>$\tau = 1.5$</th>
</tr>
</thead>
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<tr>
<td>Opt. boundary (y)</td>
<td>2.463</td>
<td>1.942</td>
<td>1.724</td>
<td>1.520</td>
<td>1.363</td>
<td>2.050</td>
<td>1.640</td>
<td>1.473</td>
</tr>
<tr>
<td>3rd order</td>
<td>0.005</td>
<td>-0.179</td>
<td>0.304</td>
<td>0.663</td>
<td>0.900</td>
<td>0.800</td>
<td>1.498</td>
<td>1.888</td>
</tr>
<tr>
<td>BOPM</td>
<td>0.005</td>
<td>-0.196</td>
<td>0.424</td>
<td>0.740</td>
<td>1.048</td>
<td>0.842</td>
<td>1.563</td>
<td>1.969</td>
</tr>
<tr>
<td>Rel deviation</td>
<td>-0.048</td>
<td>-0.085</td>
<td>-0.091</td>
<td>-0.189</td>
<td>-0.138</td>
<td>-0.050</td>
<td>0.042</td>
<td>-0.041</td>
</tr>
</tbody>
</table>

Table 4.1: A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM.

Note: For the purpose of plotting we use strike, $K$ range from 35 to 45.

Figure 4.1 shows the price of the 3rd order short–maturity asymptotic approximation and results obtained by BOPM, with $\theta = 0.04$, $\xi = 0.5$, $\eta = 0.6$ and five different maturities $T = \{ \frac{1}{12}, \frac{4}{12}, \frac{7}{12}, \frac{12}{12}, \frac{18}{12} \}$. 

20
Figure 4.1: A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM with different strike prices.

Figure 4.2 shows the price of the 3rd order short–maturity asymptotic approximation and results obtained by BOPM, with $\theta = 0.08$, $\eta = 0.7$. $\xi$ and time to maturity remain the same.
(a) $\tau = 0.083$ year, $\theta = 0.08$, $\eta = 0.7$

(b) $\tau = 0.33$ year, $\theta = 0.08$, $\eta = 0.7$

(c) $\tau = 0.583$ year, $\theta = 0.08$, $\eta = 0.7$

(d) $\tau = 1$ year, $\theta = 0.08$, $\eta = 0.7$

(e) $\tau = 1.5$ year, $\theta = 0.08$, $\eta = 0.7$

Figure 4.2: A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM with $\eta = 0.7$, $\theta = 0.08$.

Figure 4.3 shows the price of the 3rd order short–maturity asymptotic approximation and results obtained by BOPM, with $\theta = 0.08$, $\xi = 0.5$, $\eta \in \{0.5, 0.6, 0.7\}$ and two different maturities $T = \{\frac{1}{12}, \frac{18}{12}\}$.
Figure 4.3: A comparison between the price by our third–order short–maturity asymptotic expansion and the reference price by BOPM with different strikes and varying $\eta$.

Figures 4.1, 4.2 and 4.3 show that when the time to maturity is short, the price of asymptotic approximation is very close to the reference model BOPM for the variety of parameters used. Therefore, our finding on the good performance of the short-maturity expansion for moderate and longer maturities under the BS model is consistent with Medvedev and Scaillet (2010).
Chapter 5

Conclusion and Further Research

This chapter presents the conclusion to the analysis and provides some recommendations for further research.

5.1 Review and Conclusion

In this thesis, we find an analytical approximation formula for American put option prices under a new model namely the extended Heston model. The derivation of analytical approximation is based on the Fouque et al. (2000) chapter 9 combined with Medvedev and Scaillet (2010).

In chapter 4, we explore the numerical analysis of the accuracy of our asymptotic approximation formula. Specifically, we test the accuracy by changing the value of $\theta$ and a new parameter $\eta$, while keeping other parameter values constant. We compare our results with the results from the binomial option pricing method as our benchmark price. Our experiments show good and robust results for a variety of choices (i.e. $\eta = 0.5, 0.6, 0.7$ and $\theta = 0.04, 0.08$).

5.2 Further Research

Future research can conduct a numerical study on the full problem that is to check the accuracy of fast–mean reverting asymptotic formula (3.31) with small values of parameter $\epsilon$, where $P_0$ is considered as an approximate solution to extended Heston model (2.1).

Future research can also conduct more studies on the calibration of the extended Heston model to extract effective parameters particularly the range of a new introduced parameter $\eta$. 
Moreover, future research can extend an analytical solution to the higher–order term i.e. $P_1$ in price expansion formula.
Appendix A

Proof of formula (3.26)

We review the proof given in Medvedev and Scaillet (2010) but provide more computational details.

From equation (3.24) we take the first and second order derivatives with respect to \( \theta \) and the first order derivative with respect to \( \tau \) and substitute in (3.21) obtaining

\[
-nP_n + \theta P_{n\theta} + P_{n\theta \theta} + \frac{1}{\hat{\sigma}^2} \left[ \theta^2 + 2(q - r) \right] P_{n-1\theta} - 2r P_{n-2} = 0, \quad n = 1, 2, \ldots \quad (A.1)
\]

with \( P_0 = P_{-1} = 0 \). The first three terms to the right of equation (A.1) comprise the homogeneous part and the other terms comprise the non-homogeneous part.

The homogeneous solution of equation (A.1) form a two dimensional space. We have independent solution of the form

\[
P_n^0(\theta) = p_n^0(\theta) \Phi(\theta) + q_n^0(\theta) \phi(\theta). \quad (A.2)
\]

We differentiate equation (A.2) and substitute the derivatives in homogeneous part of (A.1). Simplifying and re-arranging the terms we obtain

\[
\left( \frac{d^2 p_n^0}{d\theta^2} + \theta \frac{d p_n^0}{d\theta} - np_n^0 \right) \Phi(\theta) + \left( -(n + 1)q_n^0 - \theta \frac{d q_n^0}{d\theta} + \frac{d^2 q_n^0}{d\theta^2} + 2 \frac{d p_n^0}{d\theta} \right) \phi(\theta) = 0. \quad (A.3)
\]

The PDE

\[
\frac{d^2 p_n^0}{d\theta^2} + \theta \frac{d p_n^0}{d\theta} - np_n^0 = 0,
\]

has a polynomial solution

\[
p_n^0(\theta) = \pi_{n0}^0 \theta^n + \pi_{n1}^0 \theta^{n-2} + \pi_{n2}^0 \theta^{n-4} + \ldots \quad (A.4)
\]
\[
\frac{dp_n^0(\theta)}{d\theta} = n \pi_n^0 \theta^{n-1} + (n-2) \pi_n^1 \theta^{n-3} + (n-4) \pi_n^2 \theta^{n-5} + \ldots
\]
\[
\frac{d^2 p_n^0(\theta)}{d\theta^2} = n(n-1) \pi_n^0 \theta^{n-2} + (n-2)(n-3) \pi_n^1 \theta^{n-4} + (n-4)(n-5) \pi_n^2 \theta^{n-6} + \ldots
\]

where
\[
\pi_n^0 = 1,
\]
\[
\pi_{n,i+1}^0 = \frac{(n-2i)(n-2i-1)}{2i+2} \pi_{n,i}^0.
\]

The solution to polynomial,
\[
-(n+1)q_n^0 - \theta \frac{dq_n^0}{d\theta} + \frac{d^2 q_n^0}{d\theta^2} + 2 \frac{dp_n^0}{d\theta} = 0,
\]
has the form
\[
q_n^0(\theta) = \chi_n^0 \theta^{n-1} + \chi_n^1 \theta^{n-3} + \chi_n^2 \theta^{n-5} + \ldots, \quad (A.5)
\]

where
\[
\chi_{n,0}^0 = 1.
\]
\[
\chi_{n,i+1}^0 = \frac{\chi_{n,i}^0 (n-1-2i)(n-2-2i) + 2 \pi_{n,i+1}^0 (n-2i-2)}{2n-2i-2}.
\]

We then find a particular solution for \( P_n^1 \) of (A.1) which satisfy the boundary condition at infinity. Any solution to (A.1) with appropriate behavior at the boundary is given by
\[
P_n(\theta) = C_n P_n^0(\theta) + P_n^1(\theta),
\]
\( P_n^1 \) is of the form
\[
P_n^1(\theta) = p_n^1(\theta) \Phi(\theta) + q_n^1(\theta) \phi(\theta)
\].

Therefore, the general solution is given by
\[
P_n(\theta) = C_n \left[ p_n^0(\theta) \Phi(\theta) + q_n^0(\theta) \phi(\theta) \right] + p_n^1(\theta) \Phi(\theta) + q_n^1(\theta) \phi(\theta). \quad (A.6)
\]

We guess the polynomials \( p_n^1 \) and \( q_n^1 \) are as follows
\[
p_n^1(\theta) = \pi_n^1 \theta^n + \pi_{n,1}^1 \theta^{n-2} + \pi_{n,2}^1 \theta^{n-4} + \ldots,
\]
\[
q_n^1(\theta) = \chi_n^1 \theta^{n-1} + \chi_{n,1}^1 \theta^{n-3} + \chi_{n,2}^1 \theta^{n-5} + \ldots. \quad (A.7)
\]
where \( \pi_{n_0}^{1} = x_{n_0}^{1} = 0 \).

If \( P_{n}^{1} \) is a solution to \( P_{n} \) it is also a solution to equation (A.1). Hence, we rewrite \( P_{n} \) in terms of \( P_{n}^{1} \)

Consider the homogenous part we have the following equation

\[
P_{n_\theta}^{1} + \theta P_{n_\theta}^{1} - n P_{n}^{1} = 0.
\]

We take first and second derivatives of \( P_{n}^{1} \) with respect to \( \theta \)

\[
p_{n_\theta}^{1} = p_{n_\theta}^{1} \Phi(\theta) + p_{n_\theta}^{1} \Phi(\theta) + q_{n_\theta}^{1} (-\theta \phi(\theta)) + q_{n_\theta}^{1} \phi(\theta),
\]

\[
p_{n_\theta}^{1} = p_{n_\theta}^{1} \Phi(\theta) + \left[ 2 p_{n_\theta}^{1} - \theta p_{n_\theta}^{1} + q_{n_\theta}^{1} - 2 \theta q_{n_\theta}^{1} - q_{n_\theta}^{1} + \theta^{2} q_{n_\theta}^{1} \right] \phi(\theta).
\] (A.8)

Substituting \( P_{n}^{1}, P_{n_\theta}^{1} \) and \( P_{n_\theta}^{1} \) in the homogeneous part we obtain

\[
p_{n_\theta}^{1} \Phi(\theta) + \left[ 2 p_{n_\theta}^{1} - \theta p_{n_\theta}^{1} + q_{n_\theta}^{1} - 2 \theta q_{n_\theta}^{1} - q_{n_\theta}^{1} + \theta^{2} q_{n_\theta}^{1} \right] \phi(\theta)
\]

\[
+ \theta p_{n_\theta}^{1} \Phi(\theta) + (\theta p_{n_\theta}^{1} + \theta q_{n_\theta}^{1} - \theta^{2} q_{n_\theta}^{1}) \phi(\theta) - n p_{n_\theta}^{1} \Phi(\theta) - n q_{n_\theta}^{1} \phi(\theta) = 0,
\]

(A.9)

collecting common terms in \( \Phi(\theta) \) and \( \phi(\theta) \) we get

\[
\left[ p_{n_\theta}^{1} + n p_{n_\theta}^{1} - n p_{n}^{1} \right] \Phi(\theta) + \left[ q_{n_\theta}^{1} - \theta q_{n_\theta}^{1} - (n + 1) q_{n_\theta}^{1} + 2 p_{n_\theta}^{1} \right] \phi(\theta) = 0.
\] (A.10)

Consider the term that contain \( P_{n-1}, P_{n-2} \) the non homogenous part of equation (A.1) for \( P_{1} \):

\[
\frac{1}{\sigma} \left[ \hat{\phi}^{2} + 2(q - r) \right] P_{n-1_\theta} - 2 r P_{n-2_\theta},
\]

\[
P_{n-1} = C_{n-1} P_{n-1}^{0} + P_{n-1}^{1},
\]

\[
P_{n-2} = C_{n-2} P_{n-2}^{0} + P_{n-2}^{1},
\]

\[
P_{n-1_\theta} = C_{n-1} P_{n-1_\theta}^{0} + P_{n-1_\theta}^{1},
\]

substituting in the non homogenous part we obtain

\[
\frac{1}{\sigma} \left[ \hat{\phi}^{2} + 2(q - r) \right] \left[ C_{n-1} P_{n-1_\theta}^{0} + P_{n-1_\theta}^{1} \right] - 2 r \left[ C_{n-2} P_{n-2}^{0} + P_{n-2}^{1} \right].
\] (A.11)

From (A.2) we have

\[
P_{n-1}^{0} = P_{n-1}^{0}(\theta) \Phi(\theta) + q_{n-1}^{0}(\theta) \phi(\theta),
\]

\[
P_{n-2}^{0} = P_{n-2}(\theta) \Phi(\theta) + q_{n-2}^{0}(\theta) \phi(\theta),
\]

28
and from
\[ P_n^1 = p_n^1(\theta) \Phi(\theta) + q_n^1(\theta) \phi(\theta), \]
we have
\[ P_{n-1}^1 = p_{n-1}^1(\theta) \Phi(\theta) + q_{n-1}^1(\theta) \phi(\theta), \]
\[ P_{n-2}^1 = p_{n-2}^1(\theta) \Phi(\theta) + q_{n-2}^1(\theta) \phi(\theta), \]

To obtain \( P_{n-1}^0 \) we take derivative of \( P_{n-1}^0 \) with respect to \( \theta \) and to obtain \( P_{n-1}^1 \) we take derivative of \( P_{n-1}^1 \) with respect to \( \theta \). Substituting for \( P_{n-1}^0 \), \( P_{n-1}^1 \), \( P_{n-2}^0 \) and \( P_{n-2}^1 \) in equation (A.11) we obtain
\[
\left[ Z C_{n-1} \frac{dp_{n-1}^0}{d\theta} + Z C_{n-1} \frac{dp_{n-1}^1}{d\theta} - 2r C_{n-2} p_{n-2}^0 - 2r p_{n-2}^1 \right] \Phi(\theta) \\
+ \left[ Z C_{n-1} p_{n-1}^0 + Z C_{n-1} \frac{dq_{n-1}^0}{d\theta} - Z C_{n-1} q_{n-1}^0 \frac{dq_{n-1}^0}{d\theta} + Z p_{n-1}^1 + Z \frac{dq_{n-1}^1}{d\theta} - Z \theta q_{n-1}^1 \\
- 2r C_{n-2} q_{n-2}^0 - 2r C_{n-2} q_{n-2}^1 \right] \phi(\theta) = 0. \tag{A.12}
\]

Adding equation (A.10) and (A.12) we obtain the system of two equations
\[
\frac{d^2 p_n^1}{d\theta^2} + \theta \frac{dp_n^1}{d\theta} - np_n^1 + Z C_{n-1} \frac{dp_{n-1}^0}{d\theta} + Z \frac{dp_{n-1}^1}{d\theta} - 2r C_{n-2} p_{n-2}^0 - 2r p_{n-2}^1 = 0, \tag{A.13}
\]
\[-(n + 1)q_n^1 - \theta \frac{dq_n^1}{d\theta} + \frac{d^2 q_n^1}{d\theta^2} + 2 \frac{dp_n^1}{d\theta} + Z C_{n-1} p_{n-1}^0 + Z C_{n-1} \frac{dq_{n-1}^0}{d\theta} - Z C_{n-1} q_{n-1}^0 \\
+ Z p_{n-1}^1 + Z \frac{dq_{n-1}^1}{d\theta} - Z \theta q_{n-1}^1 - 2r C_{n-2} q_{n-2}^0 - 2r C_{n-2} q_{n-2}^1 = 0. \tag{A.14}
\]

where \( Z = \frac{1}{\sigma} [\sigma^2 - 2\mu] \) and \( \mu = r - q \).
Appendix B

The 3rd order expansion of the solution for $P_0$

The 3rd order short–maturity expansion for $P_0$ has the form

$$P_0(\theta, \tau) = \sum_{n=1}^{3} \tau^n \{ C_n \left[ p_n^0(\theta) \Phi(\theta) + q_n^0(\theta) \phi(\theta) \right] + p_n^1(\theta) \Phi(\theta) + q_n^1 \phi(\theta) \}, \quad (B.1)$$

where, from Appendix A we use Equations (A.4), (A.5), (A.7), (A.13) and (A.14) to obtain the following,

$$p_1^1(\theta) = q_1^1(\theta) = q_2^1(\theta) = 0, \quad p_0^0(\theta) = q_2^0(\theta) = \theta, \quad p_0^1(\theta) = \theta^2 + 1,$$

$$\begin{align*}
p_2^1(\theta) &= \frac{1}{2\hat{\sigma}} C_1 \left( \hat{\sigma}^2 - 2\mu \right), \\
p_0^2(\theta) &= \theta^3 + 3\theta, \quad q_0^3 = \theta^2 + 2, \\
p_1^2(\theta) &= \frac{1}{\hat{\sigma}} \left[ C_2 \hat{\sigma}^2 - 2C_2 \mu - rC_1 \hat{\sigma} \right] \theta, \\
q_1^3(\theta) &= \frac{1}{8\hat{\sigma}^2} \left( \left[ 8C_2 \hat{\sigma}^3 - 16C_2 \hat{\sigma} \mu - 8rC_1 \hat{\sigma}^2 - 4C_1 \hat{\sigma}^2 \mu + C_1 \hat{\sigma}^4 + 4C_1 \mu^2 \right] \right),
\end{align*}$$

and

$$C_1 = (Ky\hat{\sigma}) \left( \Phi_0 y + \phi_0 \right)^{-1},$$

$$C_2 = - \left( \Phi_0 C_1 \hat{\sigma}^2 - 2\Phi_0 C_1 \mu + Ky\hat{\sigma}^3 \right) \left[ 2\hat{\sigma} \left( \Phi_0 y^2 + \Phi_0 + \phi_0 y \right) \right]^{-1},$$

$$C_3 = \left[ 24\hat{\sigma}^2 \left( \Phi_0 y^2 + 3\Phi_0 y + \phi_0 y^2 + 2\phi_0 \right) \right]^{-1} \times \left( -24\Phi_0 y \hat{\sigma}^3 C_2 + 48\Phi_0 y \hat{\sigma} C_2 \mu + 24\Phi_0 y \hat{\sigma}^2 rC_1 \
- 24\phi_0 C_2 \hat{\sigma}^3 + 48\phi_0 C_2 \hat{\sigma} \mu + 24\phi_0 rC_1 \hat{\sigma}^2 + 12\phi_0 C_1 \hat{\sigma}^2 \mu - 3\phi_0 C_1 \hat{\sigma}^4 - 12\phi_0 C_1 \mu^2 + 4Ky^3 \hat{\sigma}^5 \right).$$
with

$$\mu = r - q, \quad \Phi_0 = \Phi(y), \quad \phi_0 = \phi(y).$$
Appendix C

Constructing two-correlated Wiener processes

We show in detail how we construct two correlated Wiener processes using Cholesky decomposition.

Consider a correlation matrix

\[
A = \begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix},
\]

We can perform a Cholesky factorization in which every positive definite matrix \(A\) has a unique factorization such that \(A = LL^T\) where \(L\) is a lower triangular matrix and \(L^T\) is its transpose. For a real positive definite \(2 \times 2\) matrix we have a lower triangular matrix of the form

\[
L = \begin{bmatrix}
L_{1,1} & 0 \\
L_{2,1} & L_{2,2}
\end{bmatrix},
\]

\[
A = LL^T
\]

\[
A = \begin{bmatrix}
L_{1,1} & 0 \\
L_{2,1} & L_{2,2}
\end{bmatrix}
\begin{bmatrix}
L_{1,1} & L_{2,1} \\
0 & L_{2,2}
\end{bmatrix},
\]

\[
A = \begin{bmatrix}
L_{1,1}^2 & L_{1,1}L_{2,1} \\
L_{2,1}L_{1,1} & L_{2,1}^2 + L_{2,2}^2
\end{bmatrix},
\]

then we have

\[
\begin{bmatrix}
1 & \rho \\
\rho & 1
\end{bmatrix} = \begin{bmatrix}
L_{1,1}^2 & L_{1,1}L_{2,1} \\
L_{2,1}L_{1,1} & L_{2,1}^2 + L_{2,2}^2
\end{bmatrix},
\]
which gives $L_{1,1} = 1$, $L_{2,1} = \rho$, $L_{2,2} = \sqrt{1 - \rho^2}$ and finally we have the following lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{bmatrix}.$$  

Then we generate correlated Wiener processes by

$$\begin{pmatrix} dZ_1^1 \\ dZ_2^1 \end{pmatrix} = L \begin{pmatrix} dW_1^1 \\ dW_2^1 \end{pmatrix}$$

$$\begin{pmatrix} dZ_1^2 \\ dZ_2^2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{bmatrix} \begin{pmatrix} dW_1^1 \\ dW_2^2 \end{pmatrix}$$

which leads to

$$dZ_1^1 = dW_1^1$$

$$dZ_2^1 = \rho dW_1^1 + \sqrt{1 - \rho^2} dW_2^2$$

where $W_1^1$ and $W_2^2$ are independent Wiener processes.
Appendix D

Criteria for a Master thesis

In this section, we discuss how thesis objectives as requirements set by the Swedish National Agency for Higher Education for 2 years Master theses have been fulfilled.

Objective 1: Knowledge and understanding.
In this thesis, we started with an introduction of options and different models that are used in pricing American and European options. We reviewed the literature on pricing American options. We proposed a new model and present an analytical approximation formula by combining and adapting the idea by Fouque et al. (2000) chapter 9 and Medvedev and Scaillet (2010).

Objective 2: Methodological knowledge.
In this thesis, we formulated the model as a mathematical problem. We described clearly all the details on different parts of our model. We presented the fast–mean-reverting asymptotic results and third-order short maturity asymptotic results for the leading order term. Moreover, we include the mathematical proofs in the appendices.

Objective 3: Critically and Systematically Integrate Knowledge.
The thesis uses information from different sources to develop the main concept. Many sources were suggested by the thesis supervisor to extensively elaborate on the particular concept of the project.

Objective 4: Ability to Critically, Independently and Creatively Identify and Carry out Advanced Tasks.
In our model formulation chapter, we proposed a new model, the EH, and described the model and defined each parameter. Moreover, the author has shown a significant ability to identify and formulate questions, within a given time frame.

Objective 5: Ability in both national and international contexts, Present and Discuss Conclusions and Knowledge.
In asymptotic results for American option pricing section and third–order short maturity asymptotic results for the leading order term, we have described in a way that any reader with financial mathematics background can understand.

**Objective 6: Scientific, Social and Ethical Aspects.**
This report will help the reader to understand research areas in financial mathematics through the formulation of a different model. Our numerical results shows that our third–order short–maturity asymptotic results are accurate when we compare with the binomial option pricing method.
Bibliography


