Feedback to encourage creative reasoning

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This paper presents the way pilot studies underpin a design for a future project investigating how formative feedback can be designed in order to support students’ creative reasoning when constructing solutions to mathematical tasks. It builds on the idea that creative reasoning is beneficial to students’ mathematical learning. Four pilot studies have been performed with the purpose of creating an empirical base for the preparation of formative feedback to students in mathematics classrooms. The results represent a development of general theoretical guidelines for formative feedback. Our specific and empirically based guidelines will act as a starting point for further intervention studies investigating the design of formative feedback aimed at supporting students’ creative reasoning.

Students, who are encouraged to construct their own solutions and create arguments when solving mathematical tasks, tend to, if they are successful, learn or remember more from such activities than students who are being guided by templates and prepared examples (Hiebert, 2003; Jonsson, Norqvist, Liljekvist & Lithner, 2014; Olsson, 2017). Despite the disadvantages, described in numerous research reports, of teaching mathematics by providing solution methods to tasks, such teaching is still prevalent in many classrooms, in Sweden, as well as around the world (Blomhøj, 2016; Boesen, Lithner & Palm, 2010; Hiebert & Grouws, 2007). Teaching where students create and justify their own solutions (i.e. engaging in creative reasoning) require other teacher-student interactions than the traditional teaching. Rather than explaining which method to use as well as how and why it works the teacher must encourage students, not only to construct own solutions, but also to challenge them to justify their choice of method (Hmelo-Silver, Duncan & Chinn, 2007).

A teacher-student interaction aimed at supporting students’ construction of solutions can be compared to feedback aimed at supporting the students’ learning processes and hence relies on the active involvement of the student (Hattie & Timperley, 2007; Nicol & Macfarlane-Dick, 2006). Teaching that incorporates

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feedback on process level requires the teacher to provide such feedback in very short cycles of interaction where they may have only seconds to decide on an appropriate response to a question (Wiliam & Thompson, 2008). Research on formative assessment and feedback often reports general guidelines on how to incorporate such practices in teaching but few studies present empirical results detailing how feedback can be prepared and designed in classroom situations (Hattie, 2012; Palm, Andersson, Boström & Vingsle, 2017). Hence there is need for a deeper understanding of how formative assessment/feedback is implemented on classroom level (Hirsh & Lindberg, 2015). This study is one in a series of four pilot studies with the aim of identifying what characterizes the kind of feedback that leads students to reason creatively. We plan to use the result of this study to support the design of a set of future interventions in an iterative cycle of design research with the ultimate goal of developing guidelines for formative feedback that will lead to creative reasoning.

**Background**

Teaching that encourages students to construct and justify solutions to mathematical tasks entails a focus on reasoning. Furthermore, it is reasonable to assume that teachers’ feedback may guide the character of students’ reasoning. The following paragraphs will outline distinctions between different types of reasoning and feedback relevant for the didactic design addressed in this paper.

**Imitative and creative reasoning**

If the teacher explains a definition of a mathematical concept and then demonstrates how to solve tasks associated with this particular concept, it is possible for students to solve similar tasks by remembering the procedure without understanding the definition. Lithner (2000, 2003) found that students trying to apply memorized procedures often had difficulties when solving tasks for which there had been no recent teaching. For example, calculating $2^3 \times 2^4$ using a memorized process could mean mixing up whether the numbers should be added or multiplied. The reasoning associated with such an approach is defined as imitative (IR) (Lithner, 2008). A variant of IR is AR, algorithmic reasoning which is relevant for this paper. AR entails recalling a memorized, stepwise, procedure or following procedural instructions from a teacher or textbook, that are supposed to solve a task (Lithner, 2008). AR is algorithmic in the sense that it solves the associated task, but it does not require an understanding of the mathematics on which the procedure is based.

An alternative approach to the example above, $2^3 \times 2^4$, may be to consider what the mathematical meaning behind the expression is, i.e. $2^3$ means $2 \times 2 \times 2$, and $2^4$ means $2 \times 2 \times 2 \times 2$. After realizing this, the next step is to put the two together, $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$, which is $2^7$. If the student can express
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mathematical arguments for the solution she is engaged in creative mathematical reasoning (CMR). CMR is characterized by the constructing or reconstructing of a solution method and the expressing of arguments for the solution method and the solution (Lithner, 2008).

Formative feedback

As stated above it is possible for students to reach correct answers to tasks without understanding the mathematical concepts involved (Brousseau, 2002). If a student should fail in his or her attempts to solve a mathematical task, the most obvious feedback from the teacher may be an explanation regarding how to solve the task, not to explain the mathematics it is based on. Should the student, however, be responsible for the construction of the solution method, she is helped by understanding the mathematics required by the task. In such cases, if the student encounter difficulties, it is appropriate for the teacher to inquire into the student’s thinking. By knowing something about the student’s thinking the teacher has a base on which formative feedback that supports the student’s solving process, can be built (Hattie & Timperley, 2007).

Formative feedback can address several different dimensions of learning. Hattie and Timperley (2007) defines four levels of feedback, task level, process level, self-regulation level and self-level. In this paper we are interested in feedback on task level and process level. Feedback at task-level is about how well a task is being accomplished or performed and 90% of feedback in classroom is on task-level (Airasian, 1997). Such feedback typically distinguishes correct answers from incorrect ones and suggests methods for solving a task, which often entails a focus on surface understanding (Hattie & Timperley, 2007). Feedback on process-level turns the focus to the underlying process that will solve a task. This involves relations between e.g. the mathematical content and students’ perceptions of a task. Such feedback supports a deeper understanding and construction of meaning, which are cognitive processes (ibid). Feedback on process-level is typically given in dialogue.

Method

In this study we focus on teachers’ feedback to students in situations where they need help with their problem solving and/or their explaining or presenting of solution methods or solutions. Feedback will be considered as both a response to students’ actions and guiding their continued reasoning. The chain students’ action – teacher’s feedback – students’ continued reasoning will be the unit of analysis. That chain is possible to capture through voice-recordings which may be transcribed into written text. The feedback and reasoning of interest for this framework may be described as informal and as the first visible (audible) result of human thinking. If such data would be captured later in the process,
for example through interviews, it would be more or less modified and possibly more different from the thinking processes that created it. Therefore, to be as close as possible to the feedback and reasoning, we choose to capture them in direct relation to the thinking processes that create them.

Sample
The method has been developed in 4 pilot-studies with 11–12-year-old students in classroom activities. However, it is reasonable to use it in other situations with students of different ages.

Didactic situation
The teaching environment is set up in line with ideas of Brousseau (2002), i.e. students learn mathematics when they construct solutions to mathematical problems prepared by the teacher. Learning takes place when students create meaning of the mathematics included in the problem, in the sense that they understand the way it contributes to the solution. The role of the teacher is to support students’ construction of solutions and the main principles for the didactic situation are:

- The task must constitute a problem in the sense that students must not know a solution method in advance.
- Students are responsible to construct solution methods.
- Students have responsibility to justify solution methods and solutions.
- The teacher’s interaction aims to support students’ construction of solutions without revealing a solution method.

Tasks
The tasks are designed in line with Lithner’s (2017) guidelines for tasks requiring CMR. That (1) no complete solution method is available from the start to a particular student, and (2) it is reasonable for students to justify the construction and implementation of a solution. Figure 1 shows an example of one of the tasks used in the pilot studies.

Figure 1. The task Flowers and tiles

To the right is a picture of flowers and tiles.

a. How many tiles would you need if there were 5 flowers in the same arrangement?

b. How many tiles would you need for 10 flowers?

c. How many tiles would you need for 100 flowers?

d. How would you calculate the number of tiles for $n$ flowers?
Prepared feedback

The idea of prepared feedback used in our pilot studies builds on three main principles: asking students to explain their thinking, challenging students to justify why their solution method will work and challenging students to justify why their solution is correct (Olsson, 2017).

To prepare feedback a hypothetical path to solve a specific task is established. In relation to the mathematical ability of the students who will work on the task reasonable solution methods are foreseen and expected difficulties noted. Furthermore, acceptable justifications for both solution methods and solutions are formulated. For each of these items specific feedback in line with the main principles is prepared. In table 1 is an example of prepared feedback on an expected difficulty when solving the task flowers and tiles (figure 1). The student has solved a) by counting tiles. When solving b) she doubles the result from a).

Table 1. Example of prepared feedback

<table>
<thead>
<tr>
<th>Difficulty</th>
<th>Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student has not identified the &quot;unit&quot; through which the pattern is extended in order to add a new flower</td>
<td>Ask the student to verify her answer (if she is not able to verify the answer, ask her to draw and count), if (when) the student realizes that the answer is not correct, ask her why her solution method does not work (if needed, ask her to draw the extra tiles necessary to be able to include another flower). If the student manages to reach a correct answer, ask her if and why she is certain she’s right, ask her what happens when one more flower is added (5 tiles must be added).</td>
</tr>
</tbody>
</table>

Procedure

The students work in pairs. The task is presented in written text and the students are asked to present written solutions. The students are encouraged to collaborate and ask questions if they don't understand the instructions.

During the lesson the teacher interacts with the students and provides the prepared feedback when appropriate. The recordings are transcribed into written text with a focus on spoken language.

Analysis method

The analysis will focus on the chain students’ action – teacher’s feedback – students’ continued reasoning and will be performed through the following steps:

1. Parts of transcripts where the teacher interacts with students will be identified.

2. The relation between teachers’ feedback and students’ reasoning is described.
3 Determine how feedback supports or does not support students’ CMR.

4 Guidelines for feedback are revised.

The revised guidelines for feedback to support students’ CMR are used to design specific feedback for the next intervention, in which it will be tested and subject to further revision.

Results from pilot studies

The results from the four pilot studies underpin a didactical design that is planned to be tested and developed in a future series of intervention studies. The ambition is to develop the general principles for feedback (see method) into more specific guidelines. When analysing transcripts from teacher interactions in the pilot studies some examples have been perceived as interesting. We have observed that students’ oral and written solutions are often fragmented and many of the elements are understood implicitly. In those situations, the teacher should not participate in the implicit shared understanding but should encourage students to be explicit and structured in their written and oral presentation of solutions. Feedback in this situation, when the students were engaged in understanding that 5 tiles must be added to every flower (see figure 1), was thus designed with the guideline “encourage students to explicitly articulate their understanding”. The following transcript is an example when the teacher does not accept implicit understanding:

Teacher: What did you do to figure out this for example? The first one?
Student: We counted
Teacher: You counted. OK. And what did you say that ...?
Student: That ... that I cannot ... because ... if ... well this goes to 2
Teacher: Uhu
Student: So I added five more
Teacher: Uhu
Student: ... and then there were five flowers ...
Teacher: OK and if you had added another flower, how many tiles would you have added then?
Student: Well then I would have added five more
Teacher: Five more. And if you had added one more flower, how many would you have added then?
Student: Five. Five, five, five, five ...
Teacher: OK. So here you have concluded something. What did you conclude?
Student: ... that there are five
Teacher: ... [five] what?
Student: ... five tiles
Teacher: For what?
Student: For each flower
Teacher: OK. Five tiles for each flower.

Another example of a designed feedback was considered after the observation that a great number of students did not take advantage of the easier parts of tasks like *Flowers and tiles*, when solving the more difficult parts of the task. The teacher should thus encourage the student/s to revisit an earlier part of the task and reflect on the similarities between the earlier and present parts in order to verify solutions or discover faulty conclusions. The following extract from a transcript where students claimed that $5 \times 100 + 28$ would calculate the number of tiles for 100 flowers (see figure 1):

Teacher: Uhu ... could you try your theory on this one [pointing at the subtask with 5 flowers]
Student: Uhu ... then we count 5 times 5 ... which makes ... but this isn’t correct ...
Teacher: At what point did you go wrong ... or do you have to rethink this ...

This interaction proceeded and in the end the students examined if their theory worked on one flower surrounded by tiles. Then they could formulate a general calculation:

Student: Now I got it ... $5 \times$ number of flowers $+ 3 = $ numbers of tiles
Teacher: Why is that working ...
Student: Because every flower means 5 tiles ... except for the last one ... or the first one ... you must add 3

**Summary of results**

The results presented above are examples of how general guidelines on how to provide formative feedback on process level to students are played out in actual classroom situations. Both examples show students whose reasoning is CMR before interacting with the teacher and that feedback guided to continuing CMR. The two examples of feedback; encouraging students to formulate all their justifications and conclusions explicitly through questions and encouraging students to look back at previous steps in their process, are based on empirical observations, which are used to connect what is happening in actual mathematical teaching to general guidelines of feedback. Furthermore, the expected parts of students’ path to solutions could be identified and the prepared feedback to support students CMR could be delivered.
Discussion

Creative reasoning is a powerful tool in the learning of mathematics. In mathematical problem solving creative reasoning, where conjecturing and justifying are viewed as important parts, leads students to construct their own solution to mathematical task, something that previous studies have found beneficial for their learning (Jonsson et al., 2014; Olsson, 2017). Justifying and conjecturing are cognitive challenges that support understanding of the mathematical content. If students are provided with the solution method and are informed about the correctness of their answer, such challenges are removed. For example, the task Flowers and tiles (see figure 1) could be introduced by the teacher showing the class how to solve a similar task, explaining how to look for the change depending on numbers and how to identify the constant-term. This kind of teaching has the potential to guide students towards trying to remember and apply the strategies rather than understanding when and why they are appropriate. If teaching however is focused on guiding the students to reason creatively, they will have to formulate mathematical support for their solutions. If students’ justifications are indeed based on the mathematics of the particular task, it is reasonable to assume that this entails a deeper and more sustainable learning compared to solutions based on remembering and applying strategies correctly.

Teaching where students are constructing solutions while engaging in CMR does not entail a passive teacher (Hmelo-Silver et al., 2007). To foster CMR a teacher must consider how to provide feedback that supports students’ problem solving without revealing solution methods. The first example presented in the results shows how a teacher instead of explaining how to solve a task poses questions addressing the process of solving the task rather than the numbers or answers involved. Furthermore, the teacher challenges students to justify both the solution method and the correctness of the solution. The teacher also refrains from formulating or interpreting implicit ideas and continuously encourages the student to articulate her thoughts. The teacher refuses to finish any of her student’s sentences or to provide clues as to whether the student is right or wrong. To ask students about their thinking, and to challenge them to justify their conclusions, may be seen as part of general guidelines on feedback. The contribution of these pilot studies is that they originate in empirical observations, e.g. that students are not using their previous experience of similar tasks or that they are often not explicit when they present their solutions. Based on these observations, appropriate formative feedback could be prepared. In literature, there are numerous reviews and effect studies reporting the great effectiveness of formative feedback for learning (Hattie & Timpeley, 2007; Hattie, 2012; Palm et al., 2017). Furthermore, they often provide general guidelines for how to design such feedback. In our opinion, research seldom offers concrete, and empirically supported, examples of how feedback specifically designed for mathematical teaching should be prepared and provided.
In the classroom, the teacher deals with the complexity of interacting with 20+ individuals. Providing feedback on process level or self-regulation level, which is typically given in dialogues, requires the teacher to, within seconds, decide on appropriate feedback, in the particular situation, with this particular student and with this particular mathematical task (Wiliam & Thompson, 2008). In such situations is may be easy direct feedback to task level and explain how to solve the task at hand, rather than challenging the student. The results from the pilot studies indicates it is possible to prepare for interactions with students through the design of feedback on process-level associated with expected paths through solutions of tasks. The two examples of feedback presented in results; using questions to encourage students to formulate all their justifications and conclusions explicitly and encouraging students to look back at previous steps in their process, are possibly parts of many teachers’ natural feedback repertoire, but taken together they form part of a set of concrete examples of feedback that have the potential to support students’ creative reasoning, something that we believe will lead to learning.

The formative feedback we suggest as supporting CMR builds on principles such as asking students to explain their thinking, challenging students to justify why their solution method will work and challenging students to justify why their solution is correct. To these general guidelines we have added two specific strategies: a) refraining from interpreting implicit ideas in order to give the student opportunities to not only think but also to articulate her thinking and b) asking the students to revisit earlier stages of the task. These guidelines for formative feedback are based on empirical evidence and our intention is to develop these guidelines over time. Our aim is to organise a series of interventions in order to further investigate the issue of how formative feedback can be designed to support students’ CMR in mathematical problem solving.

References
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