BACHELOR THESIS IN MATHEMATICS / APPLIED MATHEMATICS

Implied Volatility Surface Approximation under a Two-Factor Stochastic Volatility Model

by

Nathaniel Ahy and Mikael Sierra

Kandidatarbete i matematik / tillämpad matematik

DIVISION OF APPLIED MATHEMATICS
MÄLARDALEN UNIVERSITY
SE-721 23 VÄSTERÅS, SWEDEN
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Abstract

Due to recent research disproving old claims in financial mathematics such as constant volatility in option prices, new approaches have been incurred to analyze the implied volatility, namely stochastic volatility models. The use of stochastic volatility in option pricing is a relatively new and unexplored field of research with a lot of unknowns, where new answers are of great interest to anyone practicing valuation of derivative instruments such as options.

With both single and two-factor stochastic volatility models containing various correlation structures with respect to the asset price and differing mean-reversions of variance the question arises as to how these values change their more observable counterpart: the implied volatility.

Using the semi-analytical formula derived by Chiarella and Ziveyi, we compute European call option prices. Then, through the Black–Scholes formula, we solve for the implied volatility by applying the bisection method. The implied volatilities obtained are then approximated using various models of regression where the models’ coefficients are determined through the Moore–Penrose pseudo-inverse to produce implied volatility surfaces for each selected pair of correlations and mean-reversion rates. Through these methods we discover that for different mean-reversions and correlations the overall implied volatility varies significantly and the relationship between the strike price, time to maturity, implied volatility are transformed.
Acknowledgements

We would like to thank everyone in the UKK department of Mälardalen University who has been a part of the Analytical Finance program and supported us as students. We especially thank our supervisor, Ying Ni, PhD in Mathematics/Applied mathematics, who patiently provided us with the tools and guidance necessary making it possible for us to complete this thesis. We have been fortunate to have a supervisor who cared so much about our work. Finally, we wish to thank our reviewer, Prof. Anatoliy Malyarenko, who contributed with valuable inputs allowing us to improve our thesis further.
Chapter 1

Introduction

1.1 Background and Literature Review

In the financial industry, derivative instruments such as options have been used widely for different purposes, for example hedging positions or just simply as bets on future prices of the underlying asset. Therefore, option valuation and future pricing have been deemed highly attractive by practitioners. In 1973, Fischer Black and Myron Scholes derived the famous Black–Scholes model for pricing options which assumes that stock prices follows a geometric Brownian motion \(^1\). This model however has been considered to flawed based on the premise that it assumes a constant volatility over the life of an option. This assumption has been proven to be incorrect due to the observation of volatility smiles obtained from real market data. The procedure for this proof simply consists of setting a market option price equal to the corresponding Black–Scholes (BS) option price and solving for the volatility component of the well-known BS model. Repeating this process for different options yields different implied volatilities thereby contradicting the constant volatility assumption.

To overcome the drawbacks of the classical Black–Scholes model, new extended models have been proposed. One line of research is on the family of stochastic volatility (inspired) models, which we will refer to as “SVI models”.

The family of SVI models include such models as the “CEV” model which is short for “The Constant Elasticity of Variance Model”, consisting of a stochastic differential equation (SDE) which leads to a diffusion process\(^1\) of one dimension and is considered to be one of the first alternative processes to model the movement of asset prices, instead of using geometric Brownian motion \(^7\).

Furthermore, there is the Stochastic Alpha Beta Rho model, also known as the “SABR model”. This model was derived in conjunction with studies of predicting volatility smile dynamics with Dupire local volatility models. Where the Dupire models regards volatility as a function of the asset price at a particular time and time itself, thereby making it a generalization of the famous Black–Scholes model. The reason for the creation of the model was that the predicted results of the local volatility models were essentially the opposite of real market

\(^1\)A diffusion process is a solution to a SDE; which is chosen to be a continuous Markov process - a process where the probability of each event is solely dependent on the event a single instance prior.
observations [3].

Then we have the “GARCH model” which stands for generalized autoregressive conditional heteroskedasticity model. In this model, the error variance assumes an autoregressive moving average, which is the condition for it to be classified as a generalized model. The GARCH model is originally a statistical model that has been extended with stochastic volatility, resulting in a more flexible lag-structure along with a longer memory 2 [9].

The last model we will mention is the model Heston [2] proposed, an SVI model based on the BS model, where the instantaneous variance follows a mean-reverting CIR process 3. In his work, a semi-analytical European option pricing formula has been obtained using a Fourier-based method.

As Heston [2] suggested, the BS model’s success depended on a specific feature, that is, the attachment of the spot return distribution to the option prices cross-sectional properties. A feature which is retained in the Heston [2] model.

As a natural extension to Heston [2], Christoffersen et. al [3] introduced a two-factor stochastic volatility model with two separate variance processes. In particular Christoffersen et al. [3] have illustrated how successful this more complex model is in capturing features of market implied volatility surfaces. Such features included the ability of capturing the slope of a volatility smile 4 whilst simultaneously explaining large and independent movements in the level and slope over time, which the single-factor model lacked.

Conducting studies with the two-factor model proposed by Christoffersen et al. [3], Chiarella and Ziveyi [5] observed that at short term horizons, returns on assets are not normally distributed and that volatility of the returns on assets is not constant, confirming the inaccuracy of the assumption regarding constant volatility in the BS model.

The model under consideration in this thesis is the two-factor model studied by Christoffersen et al. [3] and Chiarella & Ziveyi [5], which will be presented in Section 2.2.

While working on the American option price problem under the aforementioned two-factor model, Chiarella and Ziveyi [5] derived a semi-analytical formula for an American option which includes, as a by-product, a formula of the European option component. We will use this formula for our studies on the European option pricing problem.

1.2 Problem Formulation

The overall problem in which we are concerned with consists of a noteworthy number of subproblems. We will begin by defining the implied volatility, $\sigma^*$, as the volatility solved for by setting our two-factor stochastic volatility model option price equal to the Black–Scholes formula and solving for the $\sigma$ component within the BS formula. We will provide a more mathematical definition in Chapter 3.

2 Long memory in this sense means the dependence between a prior event and more present events decays at a rate slower than exponentially.

3 A Cox–Ingersoll–Ross process is a single-factor short rate model that originally describes the evolution of interest rates.

4 A volatility smile/smirk is the curve of the options volatility, plotted over the life of the option. The curve has, for the most part, a parabolic shape which looks as if it has positive second derivative.
The object of interest is the implied volatility surface under the two-factor stochastic volatility model. Among our goals is to find a favorable approximation of this surface as a function of the strike price and time to maturity, $K$ and $T$ respectively and investigate its behavior under changing selected model parameters. Such a simple closed-form approximation for the implied volatility surface provides a convenient basis for understanding the shift in the dynamic of the relationship between the main drivers of an option price, $K$ and $T$, through the change in the indirect drivers: the aforementioned parameters. Achieving this requires solving the following subproblems:

1. Acquire the European call option price under our two-factor stochastic volatility model.
2. Solving a non-linear equation for the implied volatility component.
3. Repeating the above tasks for different strike prices and maturities.
4. Acquiring an adequate surface approximation.
5. Repeating the above tasks for different sets of correlations and mean-reversions.

### 1.3 Description of Results

The contribution of this paper is threefold. The first consists of the results and conclusions from investigating what influence the correlation coefficients have on the implied volatility surface with respect to the strike price and the maturity. The second, similar to the first, is from examining the impact the differing mean-reversion rates have on the implied volatility surface. The insight gained from these two investigations paves the way for further research into the two factor stochastic volatility with deeper insight of the behavior. Additionally, this knowledge may increase the chance of new ideas and notions coming to life.

The third consists of the evaluation of the performance of different approximation models. Since there exists many methods and models for approximation we can, by our measurements, propose models that are more suitable to tasks related to ours and which ones we would discourage usage of.

### 1.4 Outline

The outline of this thesis is as follows. We begin by explaining the methodology used in this thesis. Then we introduce the reader to the stochastic model of Heston type with its two-factor counterpart and the option pricing formula we apply for our analysis in Chapter 2. In Chapter 3 we present all parameters with respective values and intervals. In the same chapter we will also explain the method for acquiring the implied volatility in detail and then finish the chapter by introducing the regression models used for approximating the surfaces along with the measuring method used for validating the regression model. In Chapter 4 we present the findings along with a rigorous analysis. We then end the paper by summarizing the work and state the conclusions in chapter five.
1.5 Methodology

Our first step is the implementation of the European option pricing formula into computational software. The formula we will make use of is the option pricing formula derived by Chiarella and Ziveyi [5]. This merely involves restating the formula in MATLAB syntax with parameters that comply with the necessary conditions thereby providing us with usable dividend-free call option prices. These option prices will then be used as input in the BS model, where we set the BS model equal to these values and solve for the volatility component $\sigma$, for all chosen strikes prices, time to maturities, mean-reversions, and correlations. This will be accomplished through the use of root finding function as the volatility component cannot be solved for algebraically. The final step consists of approximating the implied volatility surfaces obtained from simulations. When choosing a method for approximation our choice consisted of using variations of all the models presented in Dumas et al. [4] including an additional linear model, and finally an asymptotic expansion model for implied volatility [11]. As we will see in our models, the volatility will be determined by a threshold function, that is the maxima of 0.01 and a function of the strike price and time to maturity which then yields a smooth surface. The reason for the minimum value of 0.01 is because in practice it is highly unusual to find assets that have a volatility less than this value. Our method of approximation will involve the Moore–Penrose pseudo inverse which is a form of decomposition and a generalization of the more commonly used least square method.
Chapter 2

Theory

2.1 Heston Type Stochastic Volatility

When Heston [2] derived his closed form solution of the stochastic volatility, it was necessary to make the assumption that the spot asset under consideration, follows a diffusion process. The process takes the form of

$$dS(t) = \mu Sdt + \sqrt{\nu(t)}SdZ_1. \quad (2.1)$$

Here, we have that $S$ is the asset price, $Z_1$ is a Wiener process, $\nu(t)$ is the variance process driving $S$, $t$ is time, and $\mu$ is the instantaneous return per unit time. If the volatility follows an Ornstein–Uhlenbeck process [6], which is a diffusion process used for modeling the volatility of the price process of assets, then Heston [2] states the process as follows:

$$d\sqrt{\nu(t)} = -\beta \sqrt{\nu(t)}dt + \delta dZ_2. \quad (2.2)$$

Further, to write (2.2) as a variance process we let $u(t) = \sqrt{\nu(t)}$, $h(u(t)) = u^2(t) = \nu(t)$ and by Itô’s lemma we get that

$$d\nu(t) = \frac{1}{2} \frac{\partial^2 h}{\partial u^2}(du)^2 + 2 \frac{\partial h}{\partial u} du = (du)^2 + 2udu. \quad (2.3)$$

Substituting $du$ in (2.3) with (2.2) then gives the following stochastic differential equation:

$$d\nu(t) = \kappa \left[ \theta - \nu(t) \right] dt + \sigma \sqrt{\nu(t)}dZ_2. \quad (2.4)$$

Heston, by letting $\kappa = 2\beta$, $\sigma = 2\delta$, $\theta = \frac{\delta^2}{2\beta}$, then rewrites this process as

$$d\nu(t) = \kappa[\theta - \nu(t)]dt + \sigma \sqrt{\nu(t)}dZ_2.$$
This is the final model for $v(t)$ where the correlation coefficient $\rho$ describes the relationship between $Z_2$ in (2.4) and $Z_1$ in (2.1). In addition, we have the $\sigma$-value representing the instantaneous volatility of the variance process, which is also known as the volatility of volatility, $\kappa$ is the mean reversion speed, the $\theta$-value signifies the long run mean of the variance process, $\nu$. Ultimately we have that $\mu$ is the instantaneous return per unit time of the underlying asset.

**Option Pricing for No-Arbitrage Arguments** We briefly review how the pricing PDE of this European call option can be obtained using no arbitrage arguments [2]. To begin, we introduce the following risk-free portfolio:

$$S - \frac{C}{\partial C/\partial S}. \quad (2.5)$$

We consider European call options with strike price $K$ and time to maturity $T$. For simplicity, we assume that the underlying asset is a non-dividend paying stock, where $C$ is the price of our European call option. To see why this portfolio is risk free, consider that when the value of asset $S$ changes by $\Delta S$ it then follows that the price of the corresponding European call option changes by a factor of $\Delta S \frac{\partial C}{\partial S}$. The portfolio described in (2.5) then changes with respect to the asset as:

$$\Delta S - \frac{\Delta C}{\partial C/\partial S}, \quad (2.6)$$

which, in order to satisfy the standard arbitrage arguments, must in turn satisfy:

$$\Delta S - \frac{\Delta C}{\partial C/\partial S} = \left( S - C/\frac{\partial C}{\partial S} \right) r\Delta t \quad (2.7)$$

where $r$ is the risk-free interest rate. To understand why an inequality in (2.7) would eliminate arbitrage opportunities consider the case where (2.6) is less than the RHS of (2.7), a market participant could then short-sell the hedging portfolio and purchase $S - C/\frac{\partial C}{\partial S}$ units of the risk-free asset, $r$, and make an entirely risk-free profit. When the converse is true one can perform the reversed version of the aforementioned process which then yields the same outcome. Furthermore, for the sake of acquiring a more explicit identity for no-arbitrage opportunities we will use the identity that $\Delta C \overset{def}{=} C(S + \Delta S, t + \Delta t) - C(S, t)$. Proceeding, by using arguments similar to those used by Black and Scholes, as well as using arguments from Heston we get that:

$$\Delta C = \left( \frac{1}{2} \nu S^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma \nu S \frac{\partial^2 C}{\partial S \partial \nu} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 C}{\partial \nu^2} \right) \Delta t + \frac{\partial C}{\partial S} \Delta S. \quad (2.8)$$

Here we have that $\lambda$ corresponds to the market price of risk associated with the instantaneous shock in $Z_1$ and is required to be independent of the asset under consideration. By plugging
(2.8) into (2.7), subtracting both sides by \((S - C/\partial S) r \Delta t\), and multiplying by \(\partial C/\partial S\) we then get the following partial differential equation (PDE) as written by Heston:

\[
\frac{1}{2} \nu S^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma S \frac{\partial^2 C}{\partial S \partial \nu} + \frac{1}{2} \sigma^2 \nu \frac{\partial^2 C}{\partial \nu^2} + rS \frac{\partial C}{\partial S} \]

\[
+ \{ \kappa [\theta - \nu(t)] - \lambda \} \frac{\partial C}{\partial \nu} - rC + \frac{\partial C}{\partial t} = 0. \tag{2.9}
\]

Moreover, the above PDE must satisfy the following boundary conditions:

\[
C(T) = \begin{cases} 
S(T) - K, & \text{for } S(T) > K \\
0, & \text{for } S(T) \leq K.
\end{cases}
\]

Having an option price that satisfies the above equation ensures that no arbitrage opportunities can occur under Heston’s model.

Heston’s model is the system of stochastic differential equations comprising of (2.1) and (2.4) with assumptions stated in Section 1 of his paper [2]. The model also includes a set of initial conditions which ensure that the variance process is positive, the conditions of which will be discussed in the input parameter section. Christofferson et al. subsequently extended this model into a system of three stochastic differential equations that includes two stochastic volatility factors. The PDE satisfying no-arbitrage arguments (2.9) is thereby extended with regard to the new volatility factor and solved by both Christofferson et al. [3] and Chiarella & Ziveyi [5]. The extension of (2.9) is straightforward and will therefore not be derived. However, it will be stated in the following section. Additionally, we will refer to Chiarella and Ziveyi [5] for a method of solving this PDE but will state the solution in the next section as it is an essential component for demonstrating our means of obtaining our implied volatility surfaces.

### 2.2 Two-Factor Stochastic Volatility Model of Heston Type

The model we apply in our analysis is a two-factor stochastic volatility model of Heston type. As we have mentioned earlier, it describes the system of equations representing the corresponding asset price movements with stochastic volatility. The volatility of which is captured by two factors associated with two stochastic processes known as Wiener processes, instead of a single process as in the single-factor model. The two-factor extension consists of simply adding a second variance process that drives the asset price \(S\), along with an additional correlation coefficient. After adding the second variance process we obtain a system of stochastic differential equations (SDEs) as follows:

**Definition 1.** The two-factor Heston model

\[
\begin{align*}
    dS &= \mu S dt + \sqrt{\nu_1} S dZ_1 + \sqrt{\nu_2} S dZ_2, \\
    d\nu_1 &= \kappa_1 (\theta_1 - \nu_1) dt + \sigma_1 \sqrt{\nu_1} dZ_3, \\
    d\nu_2 &= \kappa_2 (\theta_2 - \nu_2) dt + \sigma_2 \sqrt{\nu_2} dZ_4, \tag{2.10}
\end{align*}
\]
where $S$ is the asset price, $v_1$ and $v_2$ are the variance processes driving $S$. The Zs represent Wiener processes such that under the real-world probability measure $\mathbb{P}$ we have that $E^\mathbb{P}[dZ_t dZ_s] = \rho_{13}$ and $E^\mathbb{P}[dZ_t dZ_4] = \rho_{24}$ where the $\rho_{13}$ and $\rho_{24}$ are correlation coefficients. Furthermore $\rho_{12}, \rho_{34}, \rho_{23}$ are 0 under the probability measure $\mathbb{P}$. The $\sigma$-values represent the instantaneous volatility of their respective volatility factors, $\kappa$-values symbolize the mean-reversion speeds, our $\theta$-values correspond to the long run means of their respective processes. Finally $\mu$ is the momentary return per unit time of the underlying asset. Since the derivation of the option pricing formula which we will use is beyond the scope of this paper it has been omitted, however, we will mention the sequence of operations leading to the final formula. For the reader interested in more depth, we refer to the appendices of the article by Chiarella and Ziveyi [5]. As Chiarella and Ziveyi show deriving the final formula consists applying Girsanov’s theorem[3] onto the Wiener processes before using Itô’s lemma to derive the option pricing PDE that satisfies no-arbitrage arguments, which takes the following form

$$
\frac{\partial}{\partial T} C_2(T, S, v_1, v_2) = \mathcal{L} C_2(T, S, v_1, v_2) - r C_2, 
$$

(2.11)

where

$$
\mathcal{L} = r S \frac{\partial}{\partial S} + \left[ \kappa_1 (\theta_1 - v_1) - \lambda_1 v_1 \right] \frac{\partial}{\partial v_1} + \left[ \kappa_2 (\theta_2 - v_2) - \lambda_2 v_2 \right] \frac{\partial}{\partial v_2} + \frac{1}{2} v_1^2 S^2 \frac{\partial^2}{\partial S^2} \rho_{13} \frac{\partial^2}{\partial S \partial v_1} + \frac{1}{2} v_2^2 S^2 \frac{\partial^2}{\partial S^2} \rho_{24} \frac{\partial^2}{\partial S \partial v_2},
$$

where $\lambda_1, \lambda_2$ are constants included in the market price of volatility risk [5]. Thus, it is clear that this PDE (2.11) is the two-factor version of (2.9) where $C_2$ is the coupled volatility factor version of $C$, i.e. the option price. We refer to Chiarella and Ziveyi’s paper for the remaining derivation of the pricing formula through solving (2.11) which we use for our European call options. With this we then acquire a full integral representation of the European call option price with the transition density in place, which in turn, we will define with Proposition 1.

**Proposition 1.** The integral form of the dividend-free European call option price derived by Chiarella and Ziveyi is expressed as:

$$
C_{CZ}(\mathbb{P}) \overset{\text{def}}{=} C_{CZ}(S, v_1, v_2, \kappa_1, \kappa_2, \rho_{13}, \rho_{24}; K, T, v_1(0), v_2(0), \lambda_1, \lambda_2, \sigma_1, \sigma_2, \theta_1, \theta_2) = SP_1(S; K, T, v_1(0), v_2(0)) - e^{-rT} KP_2(S; K, T, v_1(0), v_2(0)),
$$

(2.12)

where

$$
P_j(S; K, T, v_1(0), v_2(0)) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{g_j(S; \eta, T, v_1(0), v_2(0)) e^{-i \eta \ln K}}{i \eta} \right) d\eta
$$

(2.13)

for $j = 1, 2$ with the description of $g_j$ lying in Appendix A.

---

[1] Girsanov’s Theorem is a means of relating a continuous probability measure to another continuous probability measure on the space of continuous paths through providing a formula for the likelihood ratios between them.
Thus, what we have in (2.12) is the mathematical representation of the option price, where the price is determined by the two major components: the asset price and strike price. As we can see we have that the first components drives the option price higher while the second component lessens the price. Naturally, as can be seen in (2.12), the strike price component is discounted by the interest rate scaled by time. The sense of this is based on the avoided consequence having to spend the exercise price at the birth of the option.

It is also worth noting that the constants $\nu_1(0)$ and $\nu_2(0)$ are the initial values of the processes $\nu_1$ and $\nu_2$. Here, we have that $\lambda_1$ and $\lambda_2$ are constants forming the market prices of volatility risk associated with the Wiener instantaneous shocks of the variance processes driving the asset price $S$. All the variables and/or parameters which remain unexplained exist merely for the sake of compressing the finalized option pricing formula (2.13) and can be found in Appendix A.
Chapter 3

Implied Volatility Surface Approximation

3.1 The Input Parameters

Despite the substantial number of the input parameters not laying within the aim of this thesis they still play a noteworthy role in this paper. Nevertheless, since we are only concerned about the effects of the correlation coefficients and the mean-reversion rates have on the volatility surfaces, less effort will be put into choosing the remaining parameters. An aspect which is of utmost importance for the leftover parameters is their compliance with the conditions imposed on them for various reasons. The rationale behind us investigating the mean-reversion and correlation coefficients above all other parameters is that they are the main drivers of the implied volatility in this respect. This is so on account of their role in the variance’s influence on the asset price (due to correlations) and the predictability of the variance processes, i.e. the certainty of the uncertainty component of the asset (due to the mean-reversions).
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value/Interval</th>
<th>Verbal Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>110</td>
<td>Asset price</td>
</tr>
<tr>
<td>$T$</td>
<td>$[1/12, 1]$</td>
<td>Time to maturity</td>
</tr>
<tr>
<td>$K$</td>
<td>$[79, 165]$</td>
<td>Strike price</td>
</tr>
<tr>
<td>$r$</td>
<td>0.03</td>
<td>Interest rate</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0</td>
<td>Market price of volatility risk for shocks in $Z_1$</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0</td>
<td>Market price of volatility risk for shocks in $Z_2$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$\sqrt{\kappa_1 \theta_1}$</td>
<td>Instantaneous volatility for the first variance process</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>$\sqrt{\kappa_2 \theta_2}$</td>
<td>Instantaneous volatility for the second variance process</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.04</td>
<td>Long-term mean for the first variance process</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.04</td>
<td>Long-term mean for the second variance process</td>
</tr>
<tr>
<td>$\nu_1(0)$</td>
<td>0.04</td>
<td>Initial value for the first variance process driving $S$</td>
</tr>
<tr>
<td>$\nu_2(0)$</td>
<td>0.04</td>
<td>Initial value for the second variance process driving $S$</td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>$2^{-\frac{\theta}{z}}$, $z = 1, ..., 5$</td>
<td>Mean reversion rate for the first variance process</td>
</tr>
<tr>
<td>$\kappa_2$</td>
<td>$2^{-\frac{\theta}{z}}$, $z = 1, ..., 5$</td>
<td>Mean reversion rate for the second variance process</td>
</tr>
<tr>
<td>$\rho_{13}$</td>
<td>$-1 + 0.5(w - 1)$, $w = 1, ..., 5$</td>
<td>Correlation between $Z_3$ and $Z_1$</td>
</tr>
<tr>
<td>$\rho_{24}$</td>
<td>$-1 + 0.5(k - 1)$, $k = 1, ..., 5$</td>
<td>Correlation between $Z_4$ and $Z_2$</td>
</tr>
</tbody>
</table>

Table 3.1: The input parameters for our algorithm.

Moreover, by the above parameters we can see that the moneyness, i.e. one of the variables we use to approximate our implied volatility surface is $0.667 \leq S_0/K \leq 1.392$. This is an interval we derived through viewing open interest on the biopharmaceutical company, AstraZeneca’s call options. The reason for all parameters with subscripts of one and two being equivalent is that they then yield more predictable characteristics of our implied volatility surfaces which in turn makes any trends for varying correlation coefficients, $\rho_{13}, \rho_{24}$ and the mean-reversions, $\kappa_1, \kappa_2$ easier to detect. The exception of this rule aside from our parameters of investigation are the instantaneous volatilities $\sigma_1, \sigma_2$ of the variance processes $\nu_1, \nu_2$ respectively. The reason why we vary these parameters is because we discovered through computing the option prices that under certain circumstances the option prices become too high for any implied volatilities to be found. This is easiest explained when we discuss in detail the definition of implied volatility in Section [3.3]. Moreover, for $\nu_1, \nu_2$ to be positive processes and finite, which by their definition they must, it is necessary that they satisfy the following inequalities, thence:

$$2\kappa_i \theta_i \geq \sigma_i^2 \quad (3.1)$$
and
\[ -1 \leq \rho_{i,i+2} \leq \min\left(\frac{\kappa_i}{\sigma_i}, 1\right), \quad \text{for} \quad i = 1, 2. \]  
(3.2)

Since we want to observe as much of the correlation coefficients as possible we set \( \sigma_i = \sqrt{\kappa_i \theta_i} \) which satisfies both the above inequalities - the Feller conditions \( (3.1) \) and the Cheang, Chiarella, & Ziogas conditions \( (3.2) \) for all correlations. The intuitive translation of this identity is that the instantaneous volatility is described by the geometric mean of the mean-reversion of the long-term volatility.

### 3.2 Option Pricing

With our parameters selected we intend on, as previously stated, pricing our underlying European call options through the semi-analytical formula derived by Chiarella and Ziveyi. This formula will be implemented into MATLAB along with its parameters providing us with a simple means of computing our option price. With the output of this formula for the parameter values of interest we can then find our implied volatility.

**Definition 2.** The (model) implied volatility, \( \sigma^* \), is the value of the volatility component of the Black–Scholes call option pricing formula that matches the output of the BS formula, \( C_{BS} \), and the Chiarella–Ziveyi formula \( (2.12) \). That is, using \( \bar{p} \) from \( (2.13) \), the solution of the non-linear equation;
\[ C_{BS}(S_0, K, r, T, \sigma^*) = C_{CZ}(\bar{p}). \]

**Remark 1.** If one matches \( C_{BS}(S_0, K, r, T, \sigma^*) \) to observed market option prices then \( \sigma^* \) is called the market implied volatility. In this paper we focus solely on the model implied volatility. Thus, for simplicity we will omit the prefix ‘model’ in terms like ‘model’ implied volatility or ‘model’ implied volatility surface.

**Definition 3.** To maintain consistency with the above remark we will use the term ‘model option price’ interchangeably with \( C_{CZ}(\bar{p}) \).

### 3.3 Bisection Method

The Bisection Method is a root-finding algorithm for equations. We wish to have a function of our volatility which we submit as an input argument to our algorithm to then find a volatility component which sets our input function to some small value close to zero. This method finds the root of a function through two additional input parameters one of which produces a negative value and the other a positive value for the aforementioned function. Thereafter the algorithm takes the mean of these two values and keeps the input value that yields a sign in our function different from the previously stated mean. That is, if e.g. our function evaluated at the point of this mean is negative then the value that provides a positive output of our function will be kept. This process is repeated until either a sufficient substitute for a root is found or until the number of iterations have exceeded their upper bound. To put this in mathematical
terms, say we have a function $f$ (in this case an increasing function in the interval $[a_1, b_1]$) and input values $a_1$ and $b_1$ such that $f(a_1) < 0$, $f(b_1) > 0$. Thereafter we have the next value, $c = \frac{a_1 + b_1}{2}$ which will yield the next assignment:

$$
\begin{align*}
    b_2 &:= c, & \text{if } f(a) f(c) < 0 \\
    a_2 &:= c, & \text{if } f(b) f(c) < 0.
\end{align*}
$$

The root is found if $f(a) f(b) = 0$. Through such an assignment we can always be sure that the two guesses will always approach the function’s root since:

$$
\frac{b_{i+1} - a_{i+1}}{2} = \frac{b_i - \frac{b_i + a_i}{2}}{2} = \frac{b_i - a_i}{4}, \quad \text{for } i = 1, \ldots, n - 1, \quad (3.3)
$$

where $n$ is the maximum number of iterations for our algorithm, $f(a_i) f(b_i) \leq 0$, and (3.3) applies to the converse case as well. Naturally, the algorithm will perform the above command until a reasonable root is reached or the maximum number of iterations have been performed.

**Bisection function input** Once we have acquired the option price we wish to solve for the Black–Scholes implied volatility in the above manner. Meaning that we will set the call option, $C_{CZ}$ computed according to (2.12) and its formidable list of parameters equal to the corresponding Black–Scholes option price and solve for the volatility parameter in the BS-formula. The equation is as follows:

$$
C_{CZ}(\vec{p}) = \frac{1}{\sqrt{2\pi}} \left(S_0 \int_{-\infty}^{d_1(\sigma^*)} e^{-\frac{x^2}{2}} dx - Ke^{-rT} \int_{-\infty}^{d_2(\sigma^*)} e^{-\frac{x^2}{2}} dx\right) = C_{BS}(S_0, K, r, T, \sigma^*) \quad (3.4)
$$

where

$$
\begin{align*}
    d_1(\sigma^*) &= \frac{\ln(S_0/K) + T\left(\frac{(\sigma^*)^2}{2} + r\right)}{\sigma^* \sqrt{T}} \\
    d_2(\sigma^*) &= \frac{\ln(S_0/K) + T\left(r - \frac{(\sigma^*)^2}{2}\right)}{\sigma^* \sqrt{T}}.
\end{align*}
$$

We can then take the right-hand side of (3.4), subtract it by its left-hand side and submit it as a function of its volatility component into our bisection algorithm. The algorithm will then find a root or a value which is sufficiently close to one. Otherwise, in more explicit terms, we wish to find the implied volatility which is a $\sigma$, call it $\sigma^*$, such that $f(\sigma^*) \approx 0$ where

$$
f(\sigma) = \frac{1}{\sqrt{2\pi}} \left(S_0 \int_{-\infty}^{d_1(\sigma)} e^{-\frac{x^2}{2}} dx - Ke^{-rT} \int_{-\infty}^{d_2(\sigma)} e^{-\frac{x^2}{2}} dx\right) - C_{CZ}(\vec{p}). \quad (3.5)
$$

**Remark 2.** We can also rewrite (3.5) as $f(\sigma) = C_{BS} - C_{CZ}$ where we use the following shortened notation $C_{BS} \overset{\text{def}}{=} C_{BS}(S_0, r, T, \sigma)$ and $C_{CZ} \overset{\text{def}}{=} C_{CZ}(\vec{p})$. 

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As one can see if $C_{CZ}$ becomes too large it is possible that $f$ will always be negative which would mean that no $\sigma^*$ could be found - no implied volatility. This is what occurred in the case mentioned in Section 3.1 where the instantaneous volatilities, $\sigma_1$ and $\sigma_2$, were held constant. We will also reach a singularity at $\sigma = 0$. This is chiefly a problem in acquiring implied volatilities when $\sigma^*$ is very close to 0. Thus, to counteract this we set these low volatilities to 0.01 since in practice the implied volatility is typically much higher. Moreover, an alternative approach is to compute the implied volatility using the built in MATLAB function \textit{blsimpv} which we will do as well to verify our results. Due to the 12 different strike prices, time periods as well as the 5 different correlations: $-1, -0.5, ..., 1$ for each correlation coefficient and 5 different sets of mean-reversion rates our matrix containing the $\sigma^*$ values is $12 \times 12 \times 5 \times 5 \times 5$ that is, a five dimensional array which can be considered a collection of matrices which we will further define in Definition 4.

\textbf{Definition 4.} The five dimensional array containing all of the implied volatilities which we have computed is our \textit{implied volatility matrix}, which we will denote as $\Sigma^*$.

Plotting a subset of $\Sigma^*$ where the correlations and mean-reversions are held constant then provides us with the \textit{true} implied volatility surface.

\textbf{Definition 5.} The \textit{true} implied volatility surface is $\sigma^*$ computed in (3.5) plotted as a graph with the strike $K$ and time to maturity $T$ on the dependent variable axes.

We will denote the \textit{true} implied volatility surface as $\sigma^* = g(K, T)$, note that due to the vastness of variables the option pricing function by Chiarella and Ziveyi (2.12) we do not have a closed-form expression for $g(K, T)$.

### 3.4 Surface Approximation

After having acquired the implied volatilities we then regress them into a surface inspired by methods from Dumas, Fleming, Whaley \cite{4}, among other methods.

\textbf{Definition 6.} The \textit{implied volatility surface} that approximates $g(K, T)$ is the function $\sigma^* = \Gamma(K, T)$.
We investigate the following variants of the implied volatility surface approximations.

Model 1: $\Gamma_1(K, T) = \max [0.01, a_0] = \sigma^*$

Model 2: $\Gamma_2(K, T) = \max \left[ 0.01, a_0 + a_1 \left( \frac{S_0}{K} \right) + a_2 \left( \frac{S_0}{K} \right)^2 \right] = \sigma^*$

Model 3: $\Gamma_3(K, T) = \max \left[ 0.01, a_0 + a_1 \left( \frac{S_0}{K} \right) + a_2 T \right] = \sigma^*$

Model 4: $\Gamma_4(K, T) = \max \left[ 0.01, a_0 + a_1 \left( \frac{S_0}{K} \right) + a_2 \left( \frac{S_0}{K} \right)^2 + a_3 T + a_4 T^2 \right] = \sigma^*$

Model 5: $\Gamma_5(K, T) = \max \left[ 0.01, a_0 + a_1 \left( \frac{S_0}{K} \right) + a_2 \left( \frac{S_0}{K} \right)^2 + a_3 T + a_4 T^2 + a_5 \left( \frac{S_0}{K} \right) T \right] = \sigma^*$

Model 6: $\Gamma_6(K, T) = \max \left[ 0.01, a_0 + a_1 T + (a_2 T + a_3) \frac{1}{T} \ln \left( \frac{K}{S_0} \right) \right] = \sigma^*$.

Remark 3. The 6th model is rewritten and modified version of Canhanga [11]. The model is a first order asymptotic expansion for the implied volatility under a specification of our two-factor stochastic volatility model. It is verified to be an asymptotic model since it holds for $\kappa_1$ approaching 0 whilst $\kappa_2$ approaches infinity. Nevertheless, we do not investigate the ideal circumstances under which this model would be a good approximation. Instead we use Model 6 as an extra candidate for our surface approximation as we do at least have one large and one small mean-reversion rate.

We wish to select $a_i$ coefficients ($0 \leq i \leq 5, i \in \mathbb{Z}$) from the above equations which most accurately describe the corresponding volatility surface. Since we will plot different surfaces for different correlations and mean-reversion rates for the variance processes we need only be concerned with the strike price and time to maturity variables. Consequently for the twelve different strikes and maturities of our options we can set up the equation for our $3 \times 3$ parameters of $\Gamma_5$ in matrix form as

\[
\begin{pmatrix}
1 & K_1 & K_1^2 & T_1 & T_1^2 & K_1T_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & K_{12} & K_{12}^2 & T_{12} & T_{12}^2 & K_{12}T_{12}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{pmatrix}
= 
\begin{pmatrix}
\sigma_{1,1}^* \\
\sigma_{12,1}^* \\
\sigma_{12,2}^* \\
\sigma_{12,12}^*
\end{pmatrix}
\]

or

\[
Ka = \sigma^*.
\]

It is straightforward to see how the matrix equation above can be modified for the alternative approximations, except for the constant volatility model 1 where only the average implied volatility is used. The $\sigma_{i,j}^*$ component denotes the element of the $i$th row and the $j$th column of the implied volatility matrix, i.e. the RHS of (3.7) is an implied volatility matrix within our 5-dimensional array reshaped in the form of a vector for constant $\rho_{13}, \rho_{24}, \kappa_1,$ and $\kappa_2$. 

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Cross-Validation Since we will be performing an 8-fold cross-validation \( \frac{1}{8} \)th of \( K \) and \( \sigma^* \) will be removed and thereafter used for testing. That is, we will solve the system of equations (3.6) using \( \frac{7}{8} \)ths of \( K \) and \( \sigma^* \) and test the acquired coefficients scaled by their respective variables in the 8th (testing) fold. This can be visualized by slicing a fraction of the respective matrices forming a new and smaller version of the matrix equation (3.6). Subsequently, one solves for \( a \), acquiring the approximation function and measuring its error with respect to the corresponding \( \sigma^* \)-values of that set. We will repeat this process for all models and compute the relative error, which will be defined in the following section, for each fold.

Remark 4. The purpose of performing a \( k \)-fold cross-validation with \( k > 1 \) is that it reduces the risk of overfitting. This is due to the fact that the relative error produced by a fitted model and actual data, can be reduced to zero by overfitting. Thereby making the model performance error measurement in that case useless.

Solving the Matrix Equation Naturally, since \( K \) in the case considered in (3.6) is a 144\( \times \)6 matrix it cannot be inverted. Nonetheless, we can multiply both sides with the transpose of \( K \) and thereby acquire a 6\( \times \)6 matrix. But if this matrix has a small reciprocal condition number, \( rcond \), meaning it is poorly conditioned, inverting it can yield highly problematic results since small \( rcond \) causes relatively small changes in the input to yield high changes in the output. This is especially alarming due to the frequent occurrence of round off errors: errors that occur when approximating a value - something very common in programming. If we denote \( K^\top K \) as \( \tilde{K} \) \( rcond \) is computed as:

\[
\text{rcond}(\tilde{K}) = \frac{1}{\|\tilde{K}^{-1}\|_1\|\tilde{K}\|_1},
\]

where letting \( \tilde{k}_{ij} \) denote the element located in the \( i \)th row of the \( j \)th column of the respective matrix, \( \|\tilde{K}\|_1 \) denotes:

\[
\|\tilde{K}\|_1 = \max_{1 \leq j \leq 6} \left\{ \sum_{i=1}^{6} |\tilde{k}_{ij}| \right\}.
\]

Using this means of computation we get \( \text{rcond}(\tilde{K}) = 10^{-22} \) - this matrix is poorly conditioned.

The Moore–Penrose pseudo inverse Since the matrix we intended on inverting, \( K^\top K \), yielded an apprehensive reciprocal condition number we decided to evaluate the Moore–Penrose pseudo inverse (MPPI). An important property of this pseudo inverse is that if the aforementioned reciprocal condition number is not sufficiently close to zero then evaluating MPPI will be equivalent to using the multivariate least squares method. The method for acquiring the MPPI is easiest described by the singular value decomposition (SVD). We can perform the SVD on \( K \) to acquire the following identity

\[
K = U\Pi_{\parallel}V^\top,
\]

where \( U \) and \( V \) are orthogonal matrices\(^1\) and \( \Pi_{\parallel} \) is a diagonal rectangular matrix containing the singular values of \( K \). We will let \( P \) be an approximation of \( \Pi_{\parallel} \) where \( P \)'s diagonal elements

\(^1\)A matrix \( A \) is orthogonal if and only if \( A^\top A = I \) and \( AA^\top = I \) where \( I \) is the identity matrix.
are set to 0 if the they are less than some pre-specified tolerance. In our case we will use
the default tolerance which is: \( \max(\text{size}(K)) \times \text{eps}(\text{norm}(K)) \). That is, the largest
dimension of \( K \) scaled by the floating point accuracy of the norm of our matrix\(^2\). Thus, if we
let \( \lambda_i(\tilde{K}) \) denote the \( i \)th eigenvalue of \( \tilde{K} \) and \( \text{eps}(\lambda_i(\tilde{K})) \), the relative floating point accuracy
of the corresponding eigenvalue then our tolerance, \( \text{tol} \), can be expressed as:

\[
\text{tol} = \max_{1 \leq i \leq 6} \left[ \{144, 6\} \times \text{eps} \left( \sqrt{\lambda_i(\tilde{K})} \right) \right].
\]

Remark 5. The dimensions above exclude the correlation and mean-reversion dimensions since we approximate our surfaces for all \( \rho_{13}, \rho_{24} \) and pairs of \( \kappa_1, \kappa_2 \). The set of values scaling the \( \text{eps} \) term are the dimensions of \( K \).

\[
\text{tol} = 144 \times \max_{1 \leq i \leq 6} \left[ \text{eps} \left( \sqrt{\lambda_i(\tilde{K})} \right) \right]. \tag{3.9}
\]

Having understood the necessary properties of our principal components we can now carry on with our derivation. We will now multiply both sides of (3.8) by \( K^\top \), which provides us with a square matrix:

\[
K^\top K = V \Pi^\top \Pi \Pi^\top U^\top U V^\top \Pi \Pi^\top V^\top = V \Pi^\top \Pi \Pi^\top V^\top, \tag{3.10}
\]

where the third equality in (3.10) is due to the fact that \( U \) is an orthogonal matrix. Given
the composition of the matrix in RHS of (3.10) we can now invert it without yielding any problematic results. Thus, we can solve (3.7) by multiplying both sides by our alternative representation of \( K^\top \) and solve for \( \mathbf{a} \):

\[
\mathbf{a} = (V \Pi^\top \Pi \Pi^\top)^{-1} (U \Pi^\top V^\top)^\top \sigma^*,
\]

which is exactly what the MATLAB function \textit{pinv} does. The essence of this programming
function is if \( K^\top K \) is poorly conditioned then \textit{pinv} approximates \( \mathbf{a} \) as:

\[
\mathbf{a} \approx (V \Pi^\top P \Pi^\top)^{-1} (U \Pi^\top V^\top)^\top \sigma^*.
\]

Through this method we can now approximate our implied volatility surfaces without receiv-
ing warnings from MATLAB regarding singularities.

---

\(^2\)The MATLAB matrix norm command, \textit{norm}, with the sole input of matrix \( A \) is the square root of the largest eigenvalue of \( A^\top A \).
Chapter 4

Results

Through the methods above we have produced 125 volatility surfaces for five different sets of mean-reversion rates and five different values of each correlation coefficient.

4.1 Implied Volatility Surface

Before delving into the influence the $\kappa$ and $\rho$ values had on the implied volatility surfaces we wish to see how well our approximations and our method of acquiring the implied volatilities performed. We will let $\hat{\alpha}_{KT}$ denote the implied volatility evaluated using the method described in the bisection portion of this paper for strike price $K$ and time $T$. In addition $\alpha_{KT}$ will denote the implied volatility by taking the maximum of 0.01 and the output of the MATLAB function `blsimpv` we will compute the relative error as:

$$\epsilon_\sigma = \left| \frac{\hat{\alpha}_{KT} - \alpha_{KT}}{\alpha_{KT}} \right|.$$  

We perform this same computation for all mean-reversions and correlation coefficients and thereafter find the average. The result is then an average relative error over our 18 000 implied volatilities calculated of $7.178 \times 10^{-5}$. Since $\epsilon_\sigma$ even in the scale of relativity is minuscule a consensus is established - we can proceed using these implied volatilities with confidence.

4.2 Model Performances

The models which we mentioned in Section 3.4 had greatly varying performances. It is to be expected that these performances differ for various mean-reversions and correlation coefficients, however, we will begin with the overall performance. Using the 8-fold cross-validation method as also mentioned in Section 3.4 we acquired the values displayed in Table 4.1.

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Table 4.1: Average relative errors for surface approximation models.

<table>
<thead>
<tr>
<th>Model Number</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2860</td>
</tr>
<tr>
<td>2</td>
<td>0.1221</td>
</tr>
<tr>
<td>3</td>
<td>0.1129</td>
</tr>
<tr>
<td>4</td>
<td>0.1088</td>
</tr>
<tr>
<td>5</td>
<td>0.0949</td>
</tr>
<tr>
<td>6</td>
<td>0.1013</td>
</tr>
</tbody>
</table>

It may come as a surprise to the reader that Model 6 which was acquired through far more analysis than the other models was only the next best performing model after Model 5. This can be attributed to our circumstances not being optimal for this model’s accuracy. Due to Model 5’s superior performance we computed the percentage of instances in which it was outperformed by the other models and acquired the following results:

<table>
<thead>
<tr>
<th>Model</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6.4%</td>
</tr>
<tr>
<td>3</td>
<td>20.8%</td>
</tr>
<tr>
<td>4</td>
<td>24%</td>
</tr>
<tr>
<td>6</td>
<td>46.4%</td>
</tr>
</tbody>
</table>

Table 4.2: Share in percent where the other models outperformed the overall superior Model 5.

We notice in Table 4.2 that clearly Model 6 is still the only noteworthy competitor for Model 5. This is at the least what one would expect when knowing the background of such a model. Hence, we will proceed with our analysis by finding the instances in which Model 6 outperforms Model 5. By ‘outperform’ we mean for what correlations and mean-reversion values Model 6 produced a smaller error in our cross-validation scheme. Subsequently we computed the share for the particular instances in which Model 5 was outperformed for each correlation value and mean-reversion values separately. The results are presented in Table 4.3.
Table 4.3: Share of correlation and mean-reversion values where Model 6 had a lower relative error than Model 5.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = )</td>
<td>( \rho_{24} )</td>
</tr>
<tr>
<td>-1</td>
<td>0.0240</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.1120</td>
</tr>
<tr>
<td>0</td>
<td>0.0720</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1520</td>
</tr>
<tr>
<td>1</td>
<td>0.1040</td>
</tr>
</tbody>
</table>

The distribution of Model 6’s triumphs with respect to varying mean-reversion is relatively uniform which indicates that variance processes with more extreme mean-reversion coefficients and instantaneous volatilities is not necessarily better or worse captured by Model 6. Additionally, we notice that the correlations have no seemingly detectable trend for these models. Here it is most clear that Model 5 is vastly better than Model 6 when both variance processes are perfectly negatively correlated with their asset price. Yet when the variance processes aren’t correlated or have a lower positive correlation Model 6 performs notably better than Model 5.

4.3 Mean Reversion and Implied Volatility

Average Implied Volatility for Different Mean Reversions Here we wish to see how the implied volatility values in general change with respect to the varying mean-reversion coefficients, \( \kappa \). This will be done by computing the average implied volatility through using the formula (4.1). We will average the implied volatilities over all \( K, T, \rho_{13}, \) and \( \rho_{24} \) thereby acquiring a generalized plot. Performing these computations on our implied volatility matrix produces Figure 4.1. Moreover, the mean-reversions only influence the implied volatility through the model option price. Its influence on the model option price is, however, unclear given that it contributes to both the positive and negative components of the option pricing formula (2.12) similar to \( T \). Therefore, to comprehend the \( \kappa \)-values’ influence on implied volatility it is necessary observe their impact on the model option price for our given set of parameters.

Definition 7. The average implied volatility for different mean-reversions is computed in the following manner:

\[
\mu_z = \frac{1}{5 \times 5 \times 12 \times 12} \sum_{w=1}^{5} \sum_{k=1}^{5} \sum_{j=1}^{12} \sum_{i=1}^{12} \sigma_{i,j,k,w,z},
\]  

(4.1)

Definition 8. Here, \( \sigma_{i,j,k,w,z} \) is the implied volatility derived using the \( i \)th element of the \( K \) vector, the \( j \)th element of the \( T \) vector, where \( \rho_{24}(k) = -1 + 0.5(k-1), \rho_{13}(w) = -1 + 0.5(w-1), \) and \((\kappa_1, \kappa_2)(z) = (2^{-z/2}, 2^{z/2}) \) for \((k, w, z) = 1, \ldots, 5\).
In our study we compute implied volatility surfaces for \( z = 1, \ldots, 5 \) and therefore \( \mu \) is a vector of 5 elements. Using Equation (4.1) we then compute the figure on the left of Figure 4.1 and moreover substituting \( \sigma \) with \( C_{CZ} \) in this context produces the figure on the right.

**Figure 4.1**: Implied volatility and option price development for lower (higher) \( \kappa_1 \) (\( \kappa_2 \)).

It is clear by the above figures that as \( z \) increases the implied volatility increases. We attribute this increase to the sum of mean-reversion coefficients since both mean-reversions would have the same bearing on the model option price had they possessed the same values. This influence can be seen in the two-factor stochastic volatility model section when viewing the \( \kappa \)-values’ role in the option pricing function (2.12). Their apparently positive drive on the model option price shifts \( f \) from (3.5) down thereby leading to a later \( \sigma \) intercept causing a higher implied volatility. That is, letting ‘\( \Rightarrow \)’ denote a causal effect, ‘\( \uparrow \)’ a positive effect, ‘\( \downarrow \)’ a negative effect, ‘\( \leftrightarrow \)’ something that is deduced, and ‘\( \land \)’ how these factors work in conjunction with one another, we have, under our setting of model parameters, the following relationship:

\[
z \uparrow \Rightarrow \sum_{i=1}^{2} \kappa_i \uparrow \Rightarrow C_{CZ} \uparrow \Rightarrow \sigma^* \uparrow .
\]

**Remark 6.** For every situation in which we use our arrow symbols we are always referring to our observations. As a result, we wish to emphasize that the relationships we present are only for our setting of model parameters.

Furthermore, we can see that the curve of the model option price resembles the behaviour of the curve of the implied volatility. This is also caused by the relationship we discussed above, that is, the only effect the mean-reversion has on the implied volatility is its impact on the model option price. Of course, the reason as to why the rates aren’t exactly the same nor the changes one-to-one is due to \( f \) (3.5), the function in which we derive our implied volatility from is in itself not a linear function.

**Implied Volatility Surface Shape**

**Remark 7.** Since the relationship between \( C_{CZ} \) and \( C_{BS} \) have with \( \sigma^* \) are not, in themselves, perfectly negatively correlated we will view them separately. For instance, when we say
our understanding of this relationship. Denoting evaluated it for our different time to maturities strike prices and implied volatilities to further (3.5), with respect to time and using the shortened notation differentiating is also unclear relationship with the model option price, moreover, its relationship with As previously mentioned the time to maturity has an unclear relationship with implied volatilities. It can be seen that \( K \) drives down the model option price when observing (2.12) thus \( K \uparrow \Rightarrow C_{\text{BS}} \downarrow \Rightarrow \sigma^* \downarrow \). The strike price also drives down the function value \( C_{\text{BS}} \) (3.5) which in turn drives up the implied volatility. Due to this bipolar relationship no conclusions in the absolute sense can be drawn. This then leads to the question of whether \( K \)'s negative influence on the model option price is outweighed by its negative influence on the BS option price \( C_{\text{BS}} \) rendering it a positive driver of implied volatility. In order to find the solution to this question we must investigate the observed influence \( K \) has had on the implied volatility surface. Moreover, the time to maturity scales both the positive and negative components of the option pricing formula and therefore shares this concern. To investigate the mean-reversion effects on the implied volatility surface we will compute the change of the implied volatility by varying both \( K \) and \( T \) to see what impact the mean-reversions have on the strike prices’ and time to maturities’ bearing on the surface. Letting \( \sigma^*_{i,j,k,w,z} \) denote the element on the \( i \)th row and \( j \)th column with the \( k,w,z \) coordinates in the first, second, and third dimension respectively of the implied volatility matrix, \( \Sigma^* \), we will use Definition 9 to determine the relationship our variables have with respect to the implied volatility for fixed \( \rho_{13} \) and \( \rho_{24} \).

**Definition 9.** The overall effect coefficient of \( K \) for different mean-reversions, \( \bar{z}^{K}_i \), is computed as:

\[
\bar{z}^{K}_i = \frac{1}{11 \times 12 \times 5 \times 5} \sum_{w=1}^{5} \sum_{k=1}^{5} \sum_{j=1}^{12} \sum_{i=1}^{12} \left( \sigma^*_{i,j,k,w,z} - \sigma^*_{i-1,j,k,w,z} \right)
\]

and the effect \( T \) has on implied volatility:

\[
\bar{z}^{T}_i = \frac{1}{12 \times 11 \times 5 \times 5} \sum_{w=1}^{5} \sum_{k=1}^{5} \sum_{j=1}^{12} \sum_{i=1}^{12} \left( \sigma^*_{i,j,k,w,z} - \sigma^*_{i-1,j,k,w,z} \right)
\]

**Remark 8.** We refer to the change with respect to the strike price, \( K \), even though our functions rely on moneyness, \( S_0/K \), since \( K \) is the only one of these parameters that changes. The \( \bar{z}^{K}_i \) values produced will be vectors of 5 elements. Each of which telling us the overall influence the strike price and time to maturity have on the implied volatility for different mean-reversion rates.

**Reversion and Time to Maturity** As previously mentioned the time to maturity has an unclear relationship with the model option price, moreover, its relationship with \( f \) is also unclear. In order to understand this relationship we computed the differential of \( C_{\text{BS}} \) and evaluated it for our different time to maturities strike prices and implied volatilities to further our understanding of this relationship. Denoting \( N \) as the cumulative normal distribution and differentiating \( C_{\text{BS}}(S_0,K,r,T,\sigma) \), the Black–Scholes call option, the first term on the LHS of (3.5), with respect to time and using the shortened notation \( d_i(\sigma) \overset{\text{def}}{=} d_i \), for \( i = 1,2 \), gives us:

\[
\frac{\partial}{\partial T} C_{\text{BS}}(S_0,K,r,T,\sigma) = \frac{\partial}{\partial T} \left[ (S_0N(d_1) - Ke^{-rT}N(d_2)) \right],
\]

22
where by the chain-rule:
\[
\frac{\partial}{\partial T} N(d_1) = \frac{1}{2} N'(d_1) \left( -\ln\left( \frac{S_0}{K} \right) + \frac{(r + \sigma^2 T^3/2)}{\sigma T^{3/2}} \right).
\]

Noting that \( d_2 = d_1 - \sigma \sqrt{T} \), performing the same operation as above, and applying the product rule to \( Ke^{-rT} N(d_2) \) then yields the following formula for the differential of the BS-formula:
\[
\frac{1}{2} N'(d_1) \left( -\ln\left( \frac{S_0}{K} \right) + \frac{(r + \sigma^2 T^3/2)}{\sigma T^{3/2}} \right) \left( S_0 N'(d_1) - Ke^{-rT} N'(d_2) \right) + \frac{\sigma}{2\sqrt{T}} Ke^{-rT} N'(d_2) + r Ke^{-rT} N(d_2).
\]

Remark 9. Here, \( N' \) denotes the probability density function of a standard normal distribution.

The easiest way to show that the derivative of the aforementioned formula is positive for our given/acquired \( K, T \), in Table 3.1 and \( \sigma^* \) values is to write a function in MATLAB, evaluate the function for all aforementioned variable values and see what the minimum value acquired is. Doing this then provides us with a minimum value of zero. Therefore, we can conclude with a high degree of certainty that time positively influences \( C_{BS} \) for our given set of parameters.

On the other hand, \( T \)'s relationship with the model option price is far less clear particularly since we have no closed form expression. To observe this we need to compute the change in the model option price value with respect to different times to maturity. Particularly we will compute the percentage of instances where \( T \) increases \( C_{CZ} \). This will be computed as follows:
\[
\frac{1}{11 \times 12 \times 5 \times 5 \times 5} \sum_{\xi_{i,j,k,w,z}=1}^{5} \sum_{w=1}^{5} \sum_{k=1}^{5} \sum_{i=1}^{12} \sum_{j=2}^{12} \xi_{i,j,k,w,z},
\]
where
\[
\xi_{i,j,k,w,z} = \begin{cases} 1, & \text{if } C_{CZ,i,j,k,w,z} - C_{CZ,i,j-1,k,w,z} \geq 0 \\ 0, & \text{otherwise.} \end{cases}
\]

Through this we discover that 80.68% of time to maturities under all the investigated circumstances have a positive influence on the model option price. As a result we cannot confidently state the causal effects that \( T \) has with respect to the model option price. Nevertheless, we can still state with certainty the causal effect time has with respect to \( C_{BS} \) (which we know is always positive) and moreover deduce its overall effect on the model option price for different mean-reversions and by that answer the question whether its influence on implied volatility is overall positive or negative.

Initially the change in mean-reversion doesn’t do much to the effect the time to maturity has on implied volatility. On the other hand, as seen in Table 4.4, for \( z \geq 4 \) the time to maturity’s relationship with implied volatility is reversed, becoming positive as opposed to negative and it continues towards this path to the final mean-reversions which we observed; \( \kappa_1 = 2^{-5/2}, \kappa_2 = 2^{5/2} \). As the table presents and moreover, as can be seen in \( f \) defined in equation (3.5) the time to maturity does not have as significant of a bearing on implied volatility
as the strike price. Therefore, the slope of the surface with respect to the $T$ axis will not be as steep as the $K$-axis. We will denote $\updownarrow$ as an unclear relationship and with this information we get that:

$$T \uparrow \Rightarrow \begin{cases} (C_{BS} \uparrow \Rightarrow \sigma^* \downarrow) \land (C_{CZ} \downarrow \Rightarrow \sigma^* \uparrow) = \sigma^*, & \text{for } z = 1, 2, 3 \\ (C_{BS} \uparrow \Rightarrow \sigma^* \downarrow) \land (C_{CZ} \downarrow \Rightarrow \sigma^* \uparrow) = \sigma^* \leftrightarrow C_{CZ} \uparrow, & \text{for } z = 4, 5. \end{cases}$$

Remark 10. A later $\sigma$ intercept for $f$ means a higher implied volatility, therefore $f \downarrow \Rightarrow \sigma^* \uparrow$. We wish to emphasize that the above symbolic representations are not true for all conditions which follow their statements. They are only true in the majority of scenarios, it is necessary in this case to sacrifice a certain degree of rigor for the sake of conveying the bigger picture. For instance, as can be seen in the last step of the second case, we know that the time to maturity always positively influences $f$ and can therefore only have a positive impact on $\sigma^*$ through its effect on the model option price. Therefore, we could deduce that $T$, for the most part, drove up the model option price under the forenamed circumstances. This is not to say that it is always the case for mean-reversions $2^{-4/2}, 2^{4/2}$ and $2^{-5/2}, 2^{5/2}$. Nevertheless, the most crucial element of the family of relationships above is that they indicate the true implied volatility surface flips with respect to the $T$-axis.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$\xi_K$</th>
<th>$\xi_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0052</td>
<td>-0.0003</td>
</tr>
<tr>
<td>2</td>
<td>-0.0062</td>
<td>-0.0003</td>
</tr>
<tr>
<td>3</td>
<td>-0.0060</td>
<td>-0.0004</td>
</tr>
<tr>
<td>4</td>
<td>-0.0069</td>
<td>0.0015</td>
</tr>
<tr>
<td>5</td>
<td>-0.0086</td>
<td>0.0048</td>
</tr>
</tbody>
</table>

Table 4.4: The overall change of $\sigma^*$ with respect to the increments $K$ and $T$ defined by $\xi_K$ and $\xi_T$ respectively.

Mean Reversion and Strike Price Since it is clear that the strike price drives down the model option price (why pay more for less when you can pay less for more?) we will not give the same rigorous attention to this driver of implied volatility as we did with the time to maturity. In Table 4.4 we can see that as the sum of the mean-reversions increase, the strike price has, for the most part, a more powerful negative influence on the implied volatilities. This means that, geometrically, the volatility surfaces for higher $z$ will in general be more steeply downward sloping with respect to the $K$ axis. Intuitively we can say that greatly mean-reverting assets viewed from the scope of this observation are considered less volatile for higher strike prices. Thus, using our arrow indicators we can state:

$$K \uparrow \Rightarrow (C_{BS} \downarrow \Rightarrow \sigma^* \uparrow) \land (C_{CZ} \downarrow \Rightarrow \sigma^* \downarrow) = \sigma^* \downarrow, \quad \text{for } z = 1, \ldots, 5.$$ 

This relationship type is unconditional with respect to all mean-reversions observed as opposed to the time to maturity. We noticed upon direct inspection of the surfaces that our surface approximations’ errors are primarily due to the spikes in the original volatility surfaces.
Therefore, we deem our approximations as better describers of the implied volatility trends which we have discovered. Through using our best performing model in this circumstance, Model 5, we plot $\Gamma_5(K,T)$ for fully negative correlations between the variance processes and the asset price processes in (2.10) and for the lowest and highest $z$-values to illustrate all we have discussed above.

In Figure 4.2 we notice that, as previously mentioned, the time to maturity for the underlying option has a more positive influence on implied volatility for higher aggregate mean-reversion, however, our approximations were not able to capture times’ shift from an adverse to positive driver of implied volatility. In addition, as the strike increases, i.e. as moneyness decreases, the implied volatility decreases, this case is even more noteworthy for larger $z$ as Table 4.4 indicated. It is also clear that in this case that $z$ increases the average implied volatilities since the surface corresponding to the higher $z$-value is, for the most part, significantly higher in this space than the darker surface. Thereby confirming all the aspects which we shed light on earlier in this section excluding the shift in relationship between $T$ and $\sigma^*$.

### 4.4 Correlations and Implied Volatility

In this section we will investigate the impact of the correlation coefficients $\rho_{13}$ and $\rho_{24}$ on the implied volatility surface. This will be done by using fixed mean-reversion rates and by using
the computational methods presented in the previous section. Concerning the κ-values, we will take the average over all sets of pairs. We will begin with the influence the correlation coefficients have on the overall implied volatility. This can simply yet effectively be seen through observing the averages of the implied volatility surfaces for all varying correlations.

If we let $\mu_{w,k}$ denote the average implied volatility for the implied volatility surface of combinations $\rho_{24}(k)$ and $\rho_{13}(w)$ which are equal to $-1 + 0.5(k-1)$ and $-1 + 0.5(w-1)$ respectively. Then let $\sigma_{i,j,k,w,z}$ denote its representation in the previous section we then have that the average implied volatility for all implied volatility surfaces is given by

$$
\mu_{k,w} = \frac{1}{5 \times 12 \times 12} \sum_{z=1}^{12} \sum_{j=1}^{12} \sum_{i=1}^{12} \sigma_{i,j,k,w,z}.
$$

(4.2)

Using this formula, we acquired an average implied volatility surface based on all combinations of $\rho_{13}(w), \rho_{24}(k)$ with respect to $K$ and $T$. For example, $\mu_{1,2}$ is the average value of the implied volatility surface where $\rho_{24} = -1$ and $\rho_{13} = -0.5$. Performing these calculations on each individual surface then produces the results illustrated by the surface to the left in Figure 4.3. In addition we will also plot the surface representing the average option prices with the same line of reasoning used in (4.2).

![Figure 4.3: Relationship between $\rho_{13}, \rho_{24}$, the implied volatility, and the option price, based on the parameters of Table 3.1.](image)

As one can easily see, the overall trend is that the correlation coefficients have a strictly positive influence on the implied volatility with respect to $K$ and $T$. Signifying that the implied volatility for the most part peaks when $|\rho_{13}| = |\rho_{24}| = 1$. In addition, for all cases, the implied volatility tends to decrease as the correlation coefficients move away from values of the aforementioned pairs of correlations. This is consistent with the correlations’ influence on the model option price as can be observed on the right hand plot in Figure 4.3.

Upon observing the pairs of correlations closer to zero, we notice that the implied volatility decreases to its lowest points when the correlation pairs takes on one of the two combinations of $[-0.5, 0]$. Thus, we cannot conclude anything in the absolute sense since there is no detectable entirely consistent trend. Nonetheless, we can make the conditional conclusion that the implied volatility increases as the correlations approach fully negative or fully positive values.
The Surface Shape  Now that we have a distinct picture showing the general implied volatilities we can gain a deeper understanding of typical values in which the implied volatility tends to acquire. We can now for example see how the shape of the implied volatility surface changes for various correlations. This is synonymous to investigating the relationship the strike price and time to maturity have with implied volatility.

Correlations and Strike Price  After inspecting (A.1) and \( f \) in (3.5), it is easy to deduce that the influence the correlation coefficients have on the implied volatility surface is, as the mean-reversions, entirely through the model option price.

In the previous section we learned that:

\[ K \text{ drives down the model option price (2.12)} \]

\[ \text{delaying the } \sigma \text{ intercept (3.5), and decreases the value of } C_{BS}. \]

Thus, an increase in the strike price has both an indirect negative influence on the implied volatility:

\[ K \uparrow \Rightarrow C_{CZ} \downarrow \Rightarrow \sigma^* \downarrow \]

and a direct positive influence on implied volatility through its decrease in \( C_{BS} \). Thereby again raising the question as to which impact outweighs the other for different correlations. Understanding the dynamic of this relationship is easiest done by observing the change of the implied volatility for varying correlations with respect to the strike price. Therefore, we will adapt Definition 9 for the case of constant mean-reversion and varying correlations to get the average effect of \( K \) on \( \sigma^* \).

**Definition 10.** The overall effect coefficient for different correlations, \( \bar{\xi}^{k,w} \), is computed as:

\[
\bar{\xi}^{k,w} = \frac{1}{11 \times 12 \times 5} \sum_{z=1}^{5} \sum_{j=1}^{12} \sum_{i=1}^{12} \left( \sigma_{i,j,k,w,z}^* - \sigma_{i-1,j,k,w,z}^* \right)
\]

and the average effect \( T \) has on \( \sigma^* \):

\[
\bar{\xi}^{k,w}_T = \frac{1}{12 \times 11 \times 5} \sum_{z=1}^{5} \sum_{i=1}^{12} \sum_{j=1}^{12} \left( \sigma_{i,j,k,w,z}^* - \sigma_{i-1,j,k,w,z}^* \right).
\]

The values are then presented in Table (4.5), showing the average change in implied volatility with respect to \( K \). As can be seen on the table, \( K \) becomes a positive driver of \( \sigma^* \) (and increasingly so) as the correlations increase. This is more evident for the correlation coefficient of the greater mean reverting volatility factor \( \rho_{24} \). To be more cogent on the matter, using Remark 11, \( \bar{\xi}^{k,w}_K \geq 0 \) if \( 3 \rho_{24} + \rho_{13} \geq 0.5 \) for the correlations observed.

**Remark 11.** The bounds that will be displayed in (4.3) were acquired through direct observation of Table (4.5) and therefore do not necessarily hold for the \( \rho \)-values which we have not considered in this paper, i.e. they do not unquestionably hold for \(-1 \leq (\rho_{13}, \rho_{24}) \leq 1 : (\rho_{13}, \rho_{24}) \notin \{-1,-0.5,0,0.5,1\} \).

The only thing differentiating the two correlation coefficients is the underlying processes which they dictate and moreover the sole inherent differentiator of those processes is the mean-reversion rate, \( \kappa \). Also, as we notice in the previous section as the aggregate mean-reversion
increases we see an increase in implied volatility thereby signifying why $K$’s relationship with $\sigma^*$ hinges more on the correlation between the more mean-reverting process: $\rho_{24}$ as opposed to $\rho_{13}$. With the information provided by Table 4.5 we can now answer the question as to which of $K$’s different types of impacts on implied volatility outweighs the other. We can see that the answer to this question is conditional; where $\bar{\zeta}_{k,w} K \geq 0$, $K$’s negative influence on $C_{BS}$ is more significant than its negative influence on the stochastic volatility option price. A brief illustration of both potential cases is as follows:

$$K \uparrow \Rightarrow \begin{cases} (C_{BS} \downarrow \Rightarrow \sigma^* \uparrow) \land (C_{CZ} \downarrow \Rightarrow \sigma^* \downarrow) = \sigma^* \uparrow, & \text{for } 3\rho_{24} + \rho_{13} \geq 0.5 \\ (C_{BS} \downarrow \Rightarrow \sigma^* \uparrow) \land (C_{CZ} \downarrow \Rightarrow \sigma^* \downarrow) = \sigma^* \downarrow, & \text{otherwise.} \end{cases} (4.3)$$

Since this relationship is contingent on the values of the correlation coefficients it is clear that the correlation coefficients, in general, increase the strike price’s influence on implied volatility. The geometric interpretation of this is that as the correlation coefficients increase the implied volatility surface flips with respect to the $K$ axis.

**Correlations and Time to Maturity**  Upon inspecting (2.12) we see that the maturity has both a negative and positive bearing on the model option price, making the relationship somewhat unclear. Observing the option pricing formula (2.12), its vast set of parameters, and $C_{BS}$ renders one to conclude that the influence the time to maturity has on implied volatility is considerably weaker in comparison to the strike price. Based on the premise that the relationship between $T$ and $\sigma^*$, particularly for varying correlations is unclear it is necessary to compute the $\bar{\zeta}_{k,w}$-values.

Using the same method as for the relationship between the correlations and the strike price, we computed Table 4.6 containing the following values.

Upon inspection of Table 4.6 we notice a somewhat similar behavior as in the case with the strike price. Yet the general behavior is characterized by a negative and unstable trend which is detectable when looking closely at each respective value and the magnitude of the difference with respect to each other. Fortunately the change of sign of the $\bar{\zeta}_{k,w}$-values is rather consistent. To give particular bounds with respect to the correlations we get that $\bar{\zeta}_{k,w} \geq 0$ for $\rho_{13} + 2\rho_{24} \leq 0$ for our correlations. Using our by now well-known arrow symbols and the fact

<table>
<thead>
<tr>
<th>$\rho_{24}$</th>
<th>$-1$</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{13}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>-0.1068</td>
<td>-0.0957</td>
<td>-0.0857</td>
<td>-0.0753</td>
<td>-0.0581</td>
</tr>
<tr>
<td>-0.5</td>
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<td>-0.0689</td>
<td>-0.0594</td>
<td>-0.0485</td>
<td>-0.0212</td>
</tr>
<tr>
<td>0</td>
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<td>-0.0423</td>
<td>-0.0022</td>
<td>0.0404</td>
<td>0.0428</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0070</td>
<td>0.0437</td>
<td>0.0532</td>
<td>0.0580</td>
<td>0.0628</td>
</tr>
<tr>
<td>1</td>
<td>0.0447</td>
<td>0.0610</td>
<td>0.0674</td>
<td>0.0718</td>
<td>0.0753</td>
</tr>
</tbody>
</table>

Table 4.5: The overall change of $\sigma^*$ with respect to $K$ for different correlations, $\bar{\zeta}_{K}$. 

28
\[ \rho_{13}^{24} \rho_{13}^{13} - 0.5 0.5 1 \]

\[ \begin{array}{cccccc}
\rho_{24} & -1 & 0.0264 & 0.0194 & 0.0133 & 0.0083 & 0.0100 \\
& -0.5 & 0.0058 & 0.0050 & 0.0032 & 0.0051 & 0.0094 \\
& 0 & 0.0098 & 0.0059 & 0.0036 & -0.0051 & -0.0054 \\
& 0.5 & 0.0012 & -0.0079 & -0.0105 & -0.0110 & -0.0122 \\
& 1 & -0.0113 & -0.0107 & -0.0091 & -0.0077 & -0.0092 \\
\end{array} \]

Table 4.6: The overall change of \( \sigma^* \) with respect to \( T \) for different correlations, \( \tilde{\xi}_T \).

derived in the last section that: \( T \uparrow \Rightarrow C_{BS} \uparrow \) we acquire the following relationship identity:

\[ T \uparrow \Rightarrow \begin{cases} (C_{BS} \uparrow \Rightarrow \sigma^* \downarrow) \land (C_{CZ} \uparrow \downarrow \Rightarrow \sigma^* \uparrow \downarrow) = \sigma^* \uparrow \Leftrightarrow C_{CZ} \uparrow, & \text{for } \rho_{13} + 2\rho_{24} \leq 0 \\
(C_{BS} \uparrow \Rightarrow \sigma^* \downarrow) \land (C_{CZ} \uparrow \downarrow \Rightarrow \sigma^* \uparrow \downarrow) = \sigma^* \downarrow, & \text{otherwise.} \end{cases} \]  \( (4.4) \)

The representations (4.4) and (4.3) then indicate that when \( 2\rho_{24} + \rho_{13} > 0 \) the implied volatility surfaces flip with respect to both axes. This can be seen in Figure 4.4, where we select the combination of surfaces in which this observation is most clear, excluding the case \( \rho_{13} = \rho_{24} = -1 \) which has already been displayed in Figure 4.2.

**Figure 4.4:** Implied volatility surface approximations for \( \rho_{13}, \rho_{24} = (-0.5, -1) \) (black) and \( \rho_{13}, \rho_{24} = (1, 0.5) \) and \( z = 1 \).
As suspected, the moneyness has a heavier bearing on the asset price than the time to maturity and the volatility surfaces are flipped by merely changing the correlation coefficients. The reason for this is clearly seen mathematically since \( \rho_{13} \) and \( \rho_{24} \) scale a large set of values in (A.1) and (2.12) and therefore have the power of changing those values’ influence on the model option price.

Something one notices is for a change such as this to occur and be continuous, which, based on our tables, it seems to be, there must be a break-even point for our correlations, e.g. a point where neither \( K \) nor \( T \) influence \( \sigma^* \). This closest thing to such a point is where \( \rho_{13} = \rho_{24} = 0 \). We plot all five implied volatility surfaces for these correlations for different mean-reversions within the same space and acquire Figure 4.5. We will plot the true implied volatility surface, \( g(K, T) \), since we don’t believe any of our surface approximations would serve as a better illustration.

![Figure 4.5: True implied volatility surfaces for \( \rho_{13} = \rho_{24} = 0 \) for \( z = 1, \ldots, 5 \).](image)

Admitting it may be difficult to see, all the plots are merely stacked upon one another. The only aspect which differentiates these plots is that each surface contains values roughly \( 10^{-4} \) area units larger for higher \( z \). The plummet for the highest moneyness and time to maturity is due to the implied volatility not being found in that instance through our bisection algorithm, this
Table 4.7: Absolute option error coefficient values for all correlations.

<table>
<thead>
<tr>
<th>$\rho_{24}$</th>
<th>$\rho_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1.0000</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.9999</td>
</tr>
<tr>
<td>0</td>
<td>-0.9998</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.9997</td>
</tr>
<tr>
<td>1</td>
<td>-0.9996</td>
</tr>
</tbody>
</table>

plummet also exists for the surfaces acquired through the MATLAB built-in function `blsimpv` leading us to believe that there was no visible way around it. The cause of this borderline constant volatility is that when the Wiener processes dictating the variance factors’ movement are not correlated with those dictating the asset price movement, the Black–Scholes option price, $C_{BS}(S_0, K, r, T, \tilde{\sigma})$ and the Chiarella, Ziveyi option $C_{CZ}$ change in virtually the same way with respect to $K$ and $T$ therefore leading the $\sigma$ intercept of $f$ to constantly occur in a similar position. In order to verify this we will use the following definition.

**Definition 11.** The option price absolute difference coefficient, $\varepsilon_{C,k,w}$ for correlations $\rho_{13} = \rho_{24} = 0$ is evaluated as:

$$\varepsilon_{C,3,3} = \sum_{z=1}^{5} \sum_{j=1}^{12} \sum_{i=1}^{12} \left| C_{BS,i,j,3,3,z}(S_0, K, r, T, \tilde{\sigma}_{k,w,z}) - C_{CZ,i,j,3,3,z} \right|.$$ 

**Remark 12.** $C_{BS,i,j,k,w,z}(S_0, K, r, T, \tilde{\sigma}_{k,w,z})$ is the Black–Scholes option price for $i$th $K$-value, $j$th $T$-value, $k$th $\rho_{24}$-value, $w$th $\rho_{13}$-value, and $z$th pair of mean-reversions.

**Remark 13.** Here the $\tilde{\sigma}_{k,w,z}$ input of the BS-option price is the median implied volatility of the surface comprised of the $k, w, z$ correlations and mean-reversions. We select the median to omit the effect of the plummet.

This definition is intended to capture to what extent our function component of implied volatility, $C_{BS}$, and the option price component $C_{CZ}$ differ from one-another. A low $\varepsilon_{C,k,w}$ indicates that the two do not differ much therefore leading to a considerably flat volatility surface. The outcome of this is a value of 0.0285, comparing this to the computation for other correlations displayed in Table 4.7, we notice that this is a relatively small value thereby justifying this claim. Moreover, the same could be said had we used the mean as opposed to the median.
Chapter 5

Conclusion

5.1 Conclusion

In this paper we priced a set of options through a two-factor stochastic volatility model as opposed to the more traditional Black–Scholes pricing formula. The reason for this is because the BS-formula assumes constant volatility which is flawed in the sense that even volatility is volatile. Based on the premise of the above statement we used the more sensible Chiarella and Ziveyi option pricing formula based off of Heston’s work. Through this stochastic volatility model we priced options for different correlation coefficients, time to maturities, and strike prices. Subsequently, we set these prices equal to the Black–Scholes formula with the corresponding inputs and solved for the volatility component via the bisection method due to the equation being impossible to solve through algebraic manipulation and thereby acquired the implied volatility. Thereafter we approximated several implied volatility surfaces using different methods, the majority of which were based on intuition and one was constructed off of mathematical rigor. We thereafter discovered that for our set of parameters the more rigorously derived Model 6 was only the second best performing model overall.

We subsequently inspected the bearing of which the mean-reversion and the correlations of the volatility processes had on the implied volatility surfaces. The mean-reversions rendered the time to maturity’s effect on the implied volatility more positive whereas it amplified the strike prices negative influence on implied volatility. These effects were however, not major, the more noteworthy of which was the mean-reversions increase in the average volatility. Moreover, the correlations decreased the implied volatility as they approached zero and increased it as they reached their more extreme values of -1 or 1. The impact varying correlations between the variance processes and the underlying asset’s price processes had differed in the sense that they reversed the relationship the strike price and the time to maturity had on the implied volatility surface. Signifying that the correlation coefficients eventually flipped the implied volatility surfaces with respect to both the independent variables’ axes. This impact was consistent, even rendering the volatility surface to be flat for variance processes uncorrelated with respect to the underlying asset.
Bibliography


Appendix A

Composition of $g_j$

The composition of a part of $P_j$'s integrand \( 2.13 \) is as follows:

$$g_j(S, \eta; T, v_1(0), v_2(0)) = \exp \left\{ i \eta \ln S + B_j(T, \eta) + D_{1,j}(T, \eta)v_1(0) + D_{2,j}(T, \eta)v_2(0) \right\}. $$

We let $\exp \{ \bullet \} = e^\bullet$ then $g_j$ will consist of the following components:

$$B_j(\eta; T) = i\eta \ln r T + \frac{\Phi_1}{\sigma_1^2} \left\{ (\Theta_{1,j} + \Omega_{1,j}) T - 2 \ln \left( \frac{1 - Q_{1,j} e^{\Omega_{1,j}T}}{1 - Q_{1,j}} \right) \right\} + \frac{\Phi_2}{\sigma_2^2} \left\{ (\Theta_{2,j} + \Omega_{2,j}) T - 2 \ln \left( \frac{1 - Q_{2,j} e^{\Omega_{2,j}T}}{1 - Q_{2,j}} \right) \right\};$$

$$D_{1,j}(\eta; T) = \frac{\Theta_{1,j} + \Omega_{1,j}}{\sigma_1^2} \left\{ 1 - e^{\Omega_{1,j}T} \right\} \left\{ 1 - \frac{1}{1 - Q_{1,j} e^{\Omega_{1,j}T}} \right\},$$

$$D_{2,j}(\eta; T) = \frac{\Theta_{2,j} + \Omega_{2,j}}{\sigma_2^2} \left\{ 1 - e^{\Omega_{2,j}T} \right\} \left\{ 1 - \frac{1}{1 - Q_{2,j} e^{\Omega_{2,j}T}} \right\}.$$ 

Here $Q_{m,j} = \frac{(\Theta_{m,j} + \Omega_{m,j})}{(\Theta_{m,j} - \Omega_{m,j})}$ for $m = 1, 2$ and $j = 1, 2$ where $\Theta_{1,1} = \Theta_1(i - \eta)$

$$\Theta_{1,2} = \Theta_1(-\eta), \quad \Theta_{2,1} = \Theta_2(i - \eta), \quad \Theta_{2,2} = \Theta_2(-\eta), \quad \Omega_{1,1} = \Omega(i - \eta), \quad \Omega_{1,2} = \Omega_1(-\eta), \quad \Omega_{2,1} = \Omega_2(i - \eta) \text{ and } \Omega_{2,2} = \Omega_2(-\eta).$$

$$\Phi_1 = \kappa_1 \theta_1, \quad \Phi_2 = \kappa_2 \theta_2, \quad \beta_1' = \kappa_1 + \lambda_1 \quad \text{and} \quad \beta_2' = \kappa_2 + \lambda_2,$$

$$\Theta_1 = \Theta_1(\eta) \equiv \beta_1 + i\eta \rho_{13} \sigma_1, \quad \Theta_2 = \Theta_2(\eta) \equiv \beta_2 + i\eta \rho_{24} \sigma_2,$$

$$\Lambda(\eta) = i\eta - \eta^2, \quad \Omega_1 = \sqrt{\Theta_1^2 - \Lambda(\eta) \sigma_1^2}, \quad \Omega_2 = \sqrt{\Theta_2^2 - \Lambda(\eta) \sigma_2^2}. \quad (A.1)$$

Remark 14. The majority of the parameters and functions stated above have little or no intuition behind them since they serve solely as a means of compressing the option pricing formula \( 2.12 \).
Appendix B

MATLAB Codes

In this part of the appendix we display all the codes with a brief description stated prior to the display and a slightly more thorough description in the beginning of the code as comments.

B.1 Chiarella and Ziveyi Option Pricing

Prices a European call option using Chiarella and Ziveyi’s option pricing method described in Appendix A.

function [ Opt, p1, p2] = ChiEur1( S, K, r, T, q1, par )

k1=par(1); % mean reversion
k2=par(2);
lambda1=par(3); % market price of volatility risk
lambda2=par(4);
rho13=par(5); % correlation
rho24=par(6);
sigma1=par(7); % volvol
sigma2=par(8);
theta1= par(9); % long-run average
theta2=par(10);
V1=par(11); % eta(0)
V2=par(12);
% theta~=Theta

Phi1=k1.*theta1;
Phi2=k2.*theta2;
Lambda=@(eta) 1i.*eta-eta.^2;
beta1=k1+lambda1;
beta2=k2+lambda2;
% thetas and omegas are functions of eta.
Theta1=@(eta) beta1+1i.*eta.*rho13.*sigma1;
Theta2=@(eta) beta2+1i.*eta.*rho24.*sigma2;
Theta11=@(eta) beta1+1i.*(1i-eta).*rho13.*sigma1;
Theta12=@(eta) beta1+1i.*(-eta).*rho13.*sigma1;
Theta21=@(eta) beta2+1i.*(1i-eta).*rho24.*sigma2;
Theta22=@(eta) beta2+1i.*(-eta).*rho24.*sigma2;
Omega11=@(eta) sqrt(Theta1(eta).^2-Lambda(1i-eta).*sigma1.^2);
Omega12=@(eta) sqrt(Theta1(eta).^2-Lambda(-eta).*sigma1.^2);
Omega21=@(eta) sqrt(Theta2(eta).^2-Lambda(1i-eta).*sigma2.^2);
Omega22=@(eta) sqrt(Theta2(eta).^2-Lambda(-eta).*sigma2.^2);
Q11=@(eta) (Theta11(eta)+Omega11(eta))./(Theta11(eta)-Omega11(eta));
Q12=@(eta) (Theta12(eta)+Omega12(eta))./(Theta12(eta)-Omega12(eta));
Q21=@(eta) (Theta21(eta)+Omega21(eta))./(Theta21(eta)-Omega21(eta));
Q22=@(eta) (Theta22(eta)+Omega22(eta))./(Theta22(eta)-Omega22(eta));
B1=@(eta) 1i.*eta.*(r-q1).*T+Phi1./sigma1.^2.*((Theta11(eta)+Omega11(eta)).*T-2.*log((1-Q11(eta).*exp(Omega11(eta).*T))./(1-Q11(eta))));
B1=@(eta) B1(eta)+Phi2./sigma2.^2.*((Theta21(eta)+Omega21(eta)).*T-2.*log((1-Q21(eta).*exp(Omega21(eta).*T))./(1-Q21(eta))));
B2=@(eta) 1i.*eta.*(r-q1).*T+Phi1/sigma1.^2.*((Theta12(eta)+Omega12(eta)).*T-2.*log((1-Q12(eta).*exp(Omega12(eta).*T))./(1-Q12(eta))));
B2=@(eta) B2(eta)+Phi2./sigma2.^2.*((Theta22(eta)+Omega22(eta)).*T-2.*log((1-Q22(eta).*exp(Omega22(eta).*T))./(1-Q22(eta))));
D11=@(eta) (Theta11(eta)+Omega11(eta))./sigma1.^2.*(1-exp(Omega11(eta).*T))./(1-Q11(eta).*exp(Omega11(eta).*T)));
D12=@(eta) (Theta12(eta)+Omega12(eta))./sigma1.^2.*(1-exp(Omega12(eta).*T))./(1-Q12(eta).*exp(Omega12(eta).*T)));
D21=@(eta) (Theta21(eta)+Omega21(eta))./sigma2.^2.*(1-exp(Omega21(eta).*T))./(1-Q21(eta).*exp(Omega21(eta).*T)));
D22=@(eta) (Theta22(eta)+Omega22(eta))./sigma2.^2.*(1-exp(Omega22(eta).*T))./(1-Q22(eta).*exp(Omega22(eta).*T)));
%Vi=nui~=sigmai^2
g1=@(eta) exp(1i.*eta.*log(S)+B1(eta)+D11(eta).*V1+D21(eta).*V2);
g2=@(eta) exp(1i.*eta.*log(S)+B2(eta)+D12(eta).*V1+D22(eta).*V2);

p1=@(eta) real((g1(eta).*exp(-1i.*eta.*log(K)))./(1i.*eta));
p2=@(eta) real((g2(eta).*exp(-1i.*eta.*log(K)))./(1i.*eta));

p11=integral(p1,0,50);\%Chiarella and Ziveyi state that 50 is an adequate upper bound.
p22=integral(p2,0,50);

P(1)=1/2+1./pi.*p11;
P(2)=1/2+1./pi.*p22;

Opt=exp(-q1.*T).*S.*P(1)-exp(-r.*T).*K.*P(2);
if Opt<0
    Opt=0;
end

end\%function end

\section*{B.2 Bisection Method}

Finds the root of an equation through the bisection method described in Section 3.3.

function [est, fc] = bisect4(f,a,b,tol)
    \% Compute the root of equation f(\sigma) = 0 using the \% bisection method
    \% Inputs:
    \% f - function handle
    \% a,b-initial guesses such that f(a)f(b)<0
    \% tol - tolerance
    \% Outputs:
    \% est- root of f(x) = 0
% fc- f(c) function handle at point of updated guess

% since we are using this for implied volatilites we should always get % positive values.
fa = f(a);
fb = f(b);
if fa*fb >= 0 % a root won’t be found with these guesses
    error('Bad initial guesses!')
end
imax = (1+round((log(b-a)-log(tol))/log(2)))*3;

i = 1; % iteration counter
% since if the mean of b-a is sufficiently small we know that the x % intercept is relatively close.
while ((b-a)/2>tol) & (i<imax)
    c = (a+b)/2;
est=c;
    fc = f(c);
    if fc==0 % c is solution
        break;
    end
    if sign(fb)*sign(fc)>0 % this would mean that c*a<0
        b = c;
        fb = fc;
    else
        a = c;
        fa = fc;
    end
    i = i+1;
    % this line is only necessary for the while loop.
end
if (b-a)/2>tol
    disp('Desired tolerance not achieved!')
est(i,1)=0.01; % implied volatility can’t be found when it’s intended to be 0 (or close)
else
    est(i,1) = (a+b)/2;
end
fc = f(est(i,1));
end
B.3 Implied Volatility Surface Approximation

Finds the coefficients for surface approximations using the Moore–Penrose pseudo inverse described in Section 3.4 using only the ‘training’ set.

```matlab
function [Isigma1, a1, Isigma2, a2, Isigma3, a3, Isigma4, a4, Isigma5, a5] = VolParPI3(KT1, KT2, KT3, KT4, KT5, Ivol, S, test, k)
% KT1-5-Matrix for model 2-6
% Ivol-Reshaped implied volatility matrix
% k- Number of folds
% test-Fold which we are testing
% S- Asset price at time 0

n2 = length(Ivol); %ivol is 144x1
Ivol(1+(test-1)*n2/k:test*n2/k) = []; %removing the entire test set.
KT1(1+(test-1)*n2/k:test*n2/k,:) = []; %we will test on this
KT2(1+(test-1)*n2/k:test*n2/k,:) = []; %divided into k sets of size n2/k
KT3(1+(test-1)*n2/k:test*n2/k,:) = [];
KT4(1+(test-1)*n2/k:test*n2/k,:) = [];
KT5(1+(test-1)*n2/k:test*n2/k,:) = [];

% k fold cross-validation, k-1 training k test
a1 = pinv(KT1) * Ivol; %acquiring coefficients using the MP-pseudo inverse
a2 = pinv(KT2) * Ivol;
a3 = pinv(KT3) * Ivol;
a4 = pinv(KT4) * Ivol;
a5 = pinv(KT5) * Ivol;

La1 = pinv(log(KT1)) * Ivol;
La2 = pinv(log(KT2)) * Ivol;
La3 = pinv(log(KT3)) * Ivol;
La4 = pinv(log(KT4)) * Ivol;

Isigma1 = @(K, T) max(0.01, a1(1) + a1(2) * S ./ K + a1(3) .* (S ./ K) .^ 2);
Isigma2 = @(K, T) max(0.01, a2(1) + a2(2) .* S ./ K + a2(3) .* T);
Isigma3 = @(K, T) max(0.01, a3(1) + a3(2) .* S ./ K + a3(3) .* (S ./ K) .^ 2 + a3(4) .* T + a3(5) .* T .^ 2);
Isigma4 = @(K, T) max(0.01, a4(1) + a4(2) .* S ./ K + a4(3) .* (S ./ K) .^ 2 + a4(4) .* T + a4(5) .* T .^ 2 + a4(6) .* S ./ K .* T); %model 4
Isigma5 = @(K, T) max(0.01, a5(1) + a5(2) .* T + a5(3) .* log(K ./ S) + a5(4) .* T .* log(K ./ S));
```

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B.4 Cross-Validation

Performs the k-fold cross-validation on Models 2-6 in Section 3.4.

```matlab
function [ err, Isigma1, a1, Isigma2,a2,Isigma3,Isigma4,a4,Isigma5,a5 ]
    = d2crossV( Ivol, fold, KT1, KT2, KT3, KT4, KT5, S)
%Input:
%K-Fold Cross-validation
%Ivol- Reshaped implied volatility matrix for fixed rho and kappa
%fold- Number of folds
%KT1, KT2, KT3, KT4, KT5 Matrices for models 2-6
%S - Asset price

%Output:
%err- average errors for all models
%Isigma1-5, Approximations for implied volatility surfaces in the form of
%models 2-6

%a1,a2,a4,a5-vector of coefficients for functions
n2=round((length(Ivol)));
err=zeros(n2/fold, fold, 5);%preallocating
KT=KT4(:,2).^(-1)*S;%for accurate k coordinates

for i=1:fold
    %i is the test set, fold is the number of folds
    [Isigma1, a1, Isigma2,a2,Isigma3,Isigma4,a4,Isigma5,a5] =
        VolParPI3( KT1, KT2, KT3, KT4,KT5, Ivol, S, i, fold );
    %testing the model
    for j=1:n2/fold
        %j allows us to cover a full fold (i-1)*n2/fold gives us
        %the right fold.
        err(j,i,1)=relativeError(Isigma1(KT(j+(i-1)*n2/fold), KT4(j+(i-1)*n2/fold,4)), Ivol(j+(i-1)*n2/fold));
    end
end
```

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err(j,i,2)=relativeError(Isigma2(KT(j+(i-1)*n2/fold), KT4(j+(i-1)*n2/fold,4)), Ivol(j+(i-1)*n2/fold));
err(j,i,3)=relativeError(Isigma3(KT(j+(i-1)*n2/fold), KT4(j+(i-1)*n2/fold,4)), Ivol(j+(i-1)*n2/fold));
err(j,i,4)=relativeError(Isigma4(KT(j+(i-1)*n2/fold), KT4(j+(i-1)*n2/fold,4)), Ivol(j+(i-1)*n2/fold));
err(j,i,5)=relativeError(Isigma5(KT(j+(i-1)*n2/fold), KT4(j+(i-1)*n2/fold,4)), Ivol(j+(i-1)*n2/fold));

end
end
err=reshape(mean(mean(err)),5,1);% average errors for all 9 models.

B.5 Script
Combines all the aforementioned functions in this section of the appendix and provides us with the means necessary for analyzing our implied volatility surfaces.

%Thesis script
clear
clc
V=[0.04 0.04]; theta=[0.04 0.04];
daynum =365; lambda=[0 0];
%preallocating our implied volatility matrix
incr=12;%incr=52 was too much for my pc’s memory to handle.
ImpVol=zeros(incr,incr,5,5,5);
ImpVol2=ImpVol;
fold=incr*2/3;
Opt=ImpVol;
w=1;%counting variable
k=1;%counting variable
S=110;%stock price at time 0
r=0.03;%risk free rate
q=0;%dividend yield

% K=[80:5:135];%strike
N=@(x) exp(-x.^2./2)/sqrt(2*pi);%normal distribution

KT4=ones(incr^2,6);
K=linspace(79,165,incr)’;
kspace=K(2)-K(1);
T=[1/incr:1/incr:incr]';
for i=1:incr
    KT4(:,2)=repmat(S./K,incr,1);
    KT4(:,3)=KT4(:,2).^2;
    KT4(1+incr*(i-1):incr*i,4)=T(i);
    KT4(1+incr*(i-1):incr*i,5)=T(i).^2;
end
KT4(:,6)=KT4(:,2).*KT4(:,4);%matrix for model 5
KT1=horzcat(KT4(:,1), KT4(:,2), KT4(:,3));%model 2, check to see if this works
KT2=horzcat(KT4(:,1), KT4(:,2), KT4(:,4));%model 3
KT3=KT4(:,1:5);%model 4
KT5=horzcat(KT4(:,1), KT4(:,4), log(KT4(:,2).^-1), log(KT4(:,2).^-1)./KT4(:,4));%asymptotic model 6
%5x5x5x12x12=18 000 iterations
z=1;
for z=1:5
    kappa(1)=2^(-z/2);
    kappa(2)=2^(z/2);
    sigma=[sqrt(kappa(1)*theta(1)) sqrt(kappa(2)*theta(2))];%consider %this as a substitute for your current sigma
    par=[kappa lambda -1 -1 sigma theta V];
    for rho13=-1:0.5:1
        for rho24=-1:0.5:1
            par(5:6)=[rho13 rho24];
            for T=1/incr:1/incr:1
                for K=79:kspace:165
                    i=round((K-(79-kspace))/kspace);%scaling matrix indices
                    j=round(T*incr);

                    %Chiarella Ziveyi option price
                    [ Opt(i,j,k,w,z)]=ChiEur1(S, K, r, T, q, par);

                    f=@(sigma) (S*integral(N,-inf,(log(S/K)+(r
                    +sigma.^2/2)*T))./(sigma*sqrt(T))));
                    f=@(sigma) f(sigma)-(K*exp(-r*T)*integral(N,-inf,
                    (log(S/K)+(r-sigma.^2/2)*T)./(sigma*sqrt(T))));
                    f=@(sigma) f(sigma)-Opt(i,j,k,w,z);
                    %function for bisection method.
                    fv{i,j,k,w,z}= @(sigma) f(sigma);

                    [est] = bisect4(f,-K,K,1e-4);%K>Opt

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\[ \text{ImpVol}(i,j,k,w,z) = \max(\text{est}(\text{end}), 0.01); \% \text{since it is uncommon to find implied volatilities even as low as } 0.01 \% \text{implied volatility with built-in matlab function} \]

\[ \text{ImpVol2}(i,j,k,w,z) = \max(\text{blsimpv}(S,K,r,T,\text{Opt}(i,j,k,w,z), 0.01)); \]

\[ \text{nnz(ImVol)} \]

\[ \text{end} \]

\[ \text{end} \]

\[ \text{Ivol} = \text{reshape}(\text{ImpVol}(:,:,k,w,z), \text{incr}^2,1); \% \text{see matrix equation} \]

\[ \% \text{surface approximation} \]

\[ \% 8 \text{ fold cross-validation.} \]

\[ [\text{err}(:,:,k,w,z), \text{Isigma1}(k,w,z), a(:,:,k,w,z), \text{Isigma2}(k,w,z), a2(:,:,k,w,z), \text{Isigma3}(k,w,z), \text{Isigma4}(k,w,z), a4(:,:,k,w,z), \text{Isigma5}(k,w,z), a5(:,:,k,w,z)] = \]

\[ \text{d2crossV} (\text{Ivol}, \text{fold}, \text{KT1}, \text{KT2}, \text{KT3}, \text{KT4}, \text{KT5}, S); \]

\[ k = k + 1; \% \text{new rho24} \]

\[ \text{end} \]

\[ k = 1; \]

\[ w = w + 1; \% \text{new rho13} \]

\[ \text{end} \]

\[ w = 1; \]

\[ \text{end} \% \text{new kappas} \]

\[ \text{mean} (\text{mean} (\text{mean} (\text{err}, 2), 3), 4) \% \text{errors} \]

### B.6 BS Time Differential

Evaluates the Black–Scholes time differential for all the strike prices, time to maturities, and a uniformly distributed set of the implied volatilities under consideration.

%Computes the value of the derivative of the BS-pricing formula with respect to time for all strike prices, time to maturities and (almost) all implied volatilities. If the minimum value within the set of
%is 0 we can infer that time positively contributes to the option price for all parameters we explore.
%This code is compiled after the script.
K=linspace(79, 165, incr);
T=[1/incr:1/incr:1];
vol=linspace(0.01,max(reshape(ImpVol,18000,1)),100);
%function for the differential of the black scholes option price with respect to T
Nprime=@(x) exp(-x.^2/2)/sqrt(2*pi);
d1=@(sigma, T, K)(log(S/K)+T*(r+sigma.^2/2))/(sigma*sqrt(T));
d2=@(sigma, T, K) d1(sigma,T,K)-sigma*sqrt(T);
%call black scholes time
CBST=@(sigma, T, K) (1/(2*T^(3/2))*((r+sigma.^2/2)*T-log(S/K)))*(S*Nprime(d1(sigma, T, K))-K*exp(-r*T)*Nprime(d2(sigma, T, K)));
CBST=@(sigma, T, K) CBST(sigma, T, K) +sigma/2/sqrt(T)*K*exp(-r*T)*Nprime(d2(sigma, T, K))+r*K*exp(-r*T)*integral(Nprime, -inf, d2(sigma, T, K));

%checking if there are any negative values (if the derivative is positive then T positively influences the BS-option price.)
for k1=1:100
    for j=1:12
        for i=1:12
            time(i,j,k1)=CBST(vol(k1), T(j), K(i));
        end
    end
    nnz(time)%lets you know how long it will take
end

B.7 Cross-Validation on Model 1

Performs an 8-fold cross-validation on the constant implied volatility model.
%Cross-validation for constant implied volatility (performed after the script has compiled)
n2=round((length(Ivol)));
err2=zeros(n2/fold, fold,1);
fold=8;
for z=1:5
    for w=1:5
        for k=1:5
            for i=1:fold
                Ivol2=reshape(ImpVol(:,:,k,w,z),144,1);
                Ivol=Ivol2;
                % i is the test set, fold is the number of folds
                Ivol2(1+(i-1)*n2/k:i*n2/fold)=[];
                model1=mean(Ivol2);
                % testing the model
                for j=1:n2/fold
                    err2(j,i,k,w,z)=relativeError(model1, Ivol(j+(i-1)*n2/fold));% KT is a vector (see assignment)
                end
            end
        end
    end
end
err2=reshape(mean(mean(err2)),5,5,5);

B.8 Overall Effect Coefficient

Computes the different overall effect coefficients, \( \bar{\zeta} \).

% Computes epsilonC (compiled after script)
K=linspace(79,165,incr);
T=[1/incr:1/incr:1];
for z=1:5
    for w=1:5
        for k=1:5
            for j=1:12
                for i=2:12
                    % computing the change in implied volatility
                    deltaKimp(i-1,j,k,w,z)=ImpVol(i,j,k,w,z)-ImpVol(i-1,j,k,w,z);
                    deltaTimp(i-1,j,k,w,z)=ImpVol(j,i,k,w,z)-ImpVol(j,i-1,k,w,z);
                end
            end
        end
    end
end

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%zeta values for different kappa
zetaK=reshape(mean(mean(mean(mean(deltaKimp,2),3),4),1),5,1);
zetaT=reshape(mean(mean(mean(mean(deltaTimp,1),3),4),2),5,1);
%zeta values for different rho
zetacorrK=reshape(mean(mean(mean((deltaKimp))),5),5,5);
zetaCorrT=reshape(mean(mean(mean((deltaTimp))),5),5,5);
mcorrVol=reshape(mean(mean(mean(ImpVol,5))),5,5);

B.9 Option Price Absolute Difference

Computes all option price absolute differences $\epsilon_{c,k,w}$.

%Computes the option price absolute difference coefficient

%normal distribution
N=@(x) 1/sqrt(2*pi)*exp(-x.^2./2);
%Black-Scholes option pricing formula
qq=@(sigma, K,T) (S*integral(N,-inf,(log(S/K)+(r+sigma.^2/2)*T)./(sigma*sqrt(T))));
qq=@(sigma,K,T) qq(sigma,K,T)-(K*exp(-r*T)*integral(N,-inf,(log(S/K)+(r-sigma.^2/2)*T)./(sigma*sqrt(T))));
%for these set of correlations $f$ increases at the same rate as $opt$
KK=linspace(79,165,12);
TT=[1/12:1/12:1];
g2=zeros(12,12,5,5,5);
for w=1:5
    for k=1:5
        vol=reshape(median(median(ImpVol(:,:,k,w,:))),5,1);
        for z=1:5
            for j=1:12
                for i=1:12
                    %Observing the difference for all $K,T,rho,kappa$
g2(i,j,z,k,w)=qq(vol(z),KK(i),TT(j))-Opt(i,j,k,w,z);
nz(g2)
                end
            end
        end
    end
end
%overall absolute error for different correlations
reshape(mean(mean(mean(abs(g2)))),5,5)

%same process as above but with averages

g3=zeros(12,12,5,5,5);
for w=1:5
  for k=1:5
    for z=1:5
      vol=reshape(mean(mean(ImpVol(:,:,k,w,:))),5,1);
      for j=1:12
        for i=1:12
          g3(i,j,z,k,w)=qq(vol(z),KK(i),TT(j))-Opt(i,j,k,w,z);
        end
      end
    end
  end
end
reshape(mean(mean(mean(abs(g3)))),5,5)
Appendix C

UKÄ Requirements

In this paper, we demonstrated knowledge and understanding in the field of financial engineering through employing mathematical and computational methods into a major topic in finance. We then used these methods to produce large data sets yielding our results which we coherently described through explaining the components of the relationships between all variables under inspection. All problems mentioned in the problem formulation section were clearly identified and solved within the specified time frame of this thesis. Finally, we were able to proceed our evaluations through systematic observations leading us to identify interesting characteristics of implied volatility surfaces and moreover associate our stochastic volatility model (Christoffersen, Chiarella et al.) to our constant volatility model (Black and Scholes) under certain circumstances.