Mälardalen University Licentiate Thesis

N:o 57

A Collection of Games in Mathematics

Hillevi Gavel
Department of Mathematics and Physics
Mälardalen University
Västerås, Sweden

20th November 2005

ISSN 1651-9256
ISBN 91-85485-01-2
Printed by Arkitektkopia, Västerås, Sweden
Distribution: Mälardalen University Press

Contents

Summary in English

This thesis is a collection of three papers in mathematics/applied mathematics.

The first one (written together with Pontus Strimling) is in the area of *combinatorial game theory*, and analyses a variation on the well-known game Nim. In Nim, the participants alternate picking sticks, and the one who takes the last one is the winner. In this version, the one making the move also has a restriction (a "Muller Twist") on which numbers that may be taken imposed by the other player.

The second one, in the area of *discrete mathematics*, investigates certain properties of matrices used for sorting permutations. The contents of these matrices form interesting patterns, and the causes of these patterns are analysed in detail.

The third one (written together with professor Kimmo Eriksson), in the area of *mathematical education*, describes the development and evaluation of a computer game intended for use in math teaching at elementary school level.

Summary in Swedish

Denna avhandling är en samling av tre artiklar i matematik/tillämpad matematik.

Den första (skriven tillsammans med Pontus Strimling) tillhör området *kombinatorisk spelteori*, och analyserar en variant av det välkända spelet Nim. I Nim turas spelarna om med att plocka stickor, och den som tar den sista vinner. I denna version blir den som plockar också av den andra spelaren belagd med en begränsning (en "Muller-Twist") beträffande vilka antal som är tillåtna.

Den andra, inom området *diskret matematik*, undersöker vissa egenskaper hos matriser använda vid sortering av permutationer. Innehållet i dessa matriser bildar intressanta mönster, och orsakerna till dessa mönster analyseras i detalj.

Den tredje (skriven tillsammans med professor Kimmo Eriksson), inom området *matematisk didaktik*, beskriver utvecklingen och utvärderingen av ett datorspel avsett för användning inom matematikundervisningen på grundskolenivå.

Acknowledgements

I would like to thank my primary supervisor Kimmo Eriksson for encouragement and support (which among other things made me start these PhD studies), and my assistant supervisor Peter Gustafsson for his interest. I would also like to thank Pontus Strimling for our very entertaining cooperation in game theory. And last but not least: My family, Kai-Mikael, Alvin and Vanda, for surviving rambling monologues about permutations and similar things.

1 Introduction

I started my PhD studies after working for several years as a lecturer in mathematics at the Royal Institute of Technology and at Mälardalen University. Being a computer science major, my main subject has been discrete mathematics. Together with professor Kimmo Eriksson, I have written two textbooks in discrete mathematics [5, 6] for use at basic and more advanced university level. My PhD studies have been mainly in the areas of discrete mathematics, game theory and mathematical didactics [7], and in the border areas between these subjects. This thesis is a collection of three of the finished projects this far. Finding a common denominator for them has not been easy, but it seems as if they can all be included under the heading "games".

1.1 The Meaning of the Word *Game*

When trying to define *game*, one finds that it is quite a wide concept. Still, given a concrete example, most persons will agree as to whether it is a game or not. My supervisor, Kimmo Eriksson, remarked that the philosopher Wittgenstein [18] has many interesting points to make concerning the many meanings of the word. Not being of a philosophical mind myself, I checked a dictionary.

The Compact Oxford English Dictionary [17] gives the following definition of the noun:

1. An activity engaged in for amusement.
2. A form of competitive activity or sport played according to rules.
3. A complete episode or period of play, ending in a final result.
4. A single portion of play, forming a scoring unit within a game.
5. (Games) a meeting for sporting contests.
6. The equipment used in playing a board game, computer game, etc.
7. A type of activity or business regarded as a game.
8. A secret plan or trick.
9. Wild mammals or birds hunted for sport or food.

This thesis will be about games in sense 1, 2, 3, 6 and 7.

2 Game Theory

For a mathematician, the most obvious connection between games and mathematics is *game theory*. Game theory [14, 6] is a fairly new branch of applied mathematics, created in the 1940s by Oskar Morgenstern and John von Neumann. The Compact Oxford English Dictionary gives this definition of the subject:

> The mathematical study of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants.

The word *game* in *game theory* can mean any kind of interaction (possibly a game, but more probably a more serious activity). Its connection to economics has made it possible for mathematicians to win Nobel Prizes[1] (such as John Nash 1996, and Robert J. Aumann 2005).

To evaluate the mathematical models of human behaviour, game theorists make experiments, using voluntary participants and carefully designed games (supposed to model real situations), usually in a computer laboratory [2]. Mälardalen University has got one such laboratory, which we used in the study described in section 4.3.

2.1 Combinatorial Game Theory

Combinatorial games are games in the traditional sense, played for amusement according to rules to reach a final result, and *combinatorial game theory* is the study of such games. To be classified as a *combinatorial* game, the game has to be *full information* and the number of possible *moves limited*. The game also has to *terminate*, either as a draw or declaring one of the players the winner. Bridge is not a combinatorial game, since the players do not know what cards the other players hold. Ludo is not a combinatorial game, since nobody knows how the dice will roll. Croquet is not a combinatorial game, since the balls can be in infinitely many positions. Chess *is* a combinatorial game, since you can get complete information on the state of the game by looking on the board and being told whose turn it is to move. The fact that the games are full information means that they can (in theory, at least) be completely analysed, and the optimal strategy for playing them determined (thus making the games pointless to play).

An entertaining book on the subject is *Winning Ways for your Mathematical Plays* [1].

2.2 The Game Nim

One of the most important combinatorial games is the game *Nim*, which is played with piles of sticks. The two participants alternate making moves. A move consists of choosing one pile and picking a positive number of sticks. The one who takes the last stick is the winner.

To find the winning strategy, you have to identify what kind of position you should try to hand over to your opponent (and which you hope your opponent will not hand to you). Such a position is called a $\mathcal{P}$-position, and in Nim, the initially surprising property of the $\mathcal{P}$-positions is this:

Theorem 1 *In Nim, a position is a $\mathcal{P}$-position if and only if the sum of the pile sizes, computed in binary and without carry (that is: by bit-wise Xor), is zero. (A sum computed in this way is called a* Nimsum.*)* ■

[1] Strictly speaking not Nobel prizes, but *The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel*.

Proofs of this theorem can be found in [1] and in [6]. In brief: The opponent's counter-move will destroy the zero. In your next move you restore it. And so on, until you hand over a position where all the piles have size zero (where the sum of course is zero), which means that you have won.

The real importance of the game Nim is that all two-person impartial[2] combinatorial games are equivalent to Nim [1]. The challenge is to find out how.

2.3 Paper I: *Nim with a Modular Muller Twist*

Since the original version of the game Nim is completely analysed, it is interesting to study different variations on the game and how they affect the strategy. Many variants have been tried, and my colleague Pontus Strimling (PhD student in game theory at Mälardalen University [16]) and I were inspired by the article *Comply/Constrain Games or Games with a Muller Twist* [15] by F. Smith and P. Stănică. This article describes (among other games) a variation on Nim where the player handing over the position also decides if the next player should take an odd or an even number of sticks. (Putting a restriction on the other players' options is called a *Muller Twist*.) The authors prove the following result:

Theorem 2 (F. Smith and P. Stănică) *In odd-or-even-Nim, a position is a $\mathcal{P}$-position if and only if it satisfies one of these properties:*

- *Nimsum 0, restriction even*
- *Nimsum 1, restriction even*
- *Nimsum 0, all pile sizes even, restriction odd* ■

We thought that the proof of Smith and Stănică, although perfectly correct, did not convey any deep understanding of why this result holds. We tried to gain a better understanding by extending the idea "odd-or-even" to any modulus[3]. We named our game "k-blocking modular Nim", and in it the player who hands over the position includes a restriction of the form "the number of sticks taken must not be equal to $n_1, n_2, \dots$ or n_k modulo n". We found the following generalisation (although it is not immediately evident that $k = 1$ implies the previous theorem):

Theorem 3 (H. Gavel and P. Strimling) *In k-blocking modular Nim the $\mathcal{P}$-positions are the positions where*

- *the Nimsum of the quotients of the pile sizes when divided by $k + 1$ is zero*

and

- *all remainders are smaller than the smallest number of sticks that can be taken according to the given constraint.* ■

[2] In an impartial game, both participants can make exactly the same moves. Chess is not impartial, since white can not move black pieces, and vice-versa.

[3] Odd or even is determined by equivalences modulo 2.

The resulting article [9] (in the writing of which we participated equally), was published in *Integers* 2004.

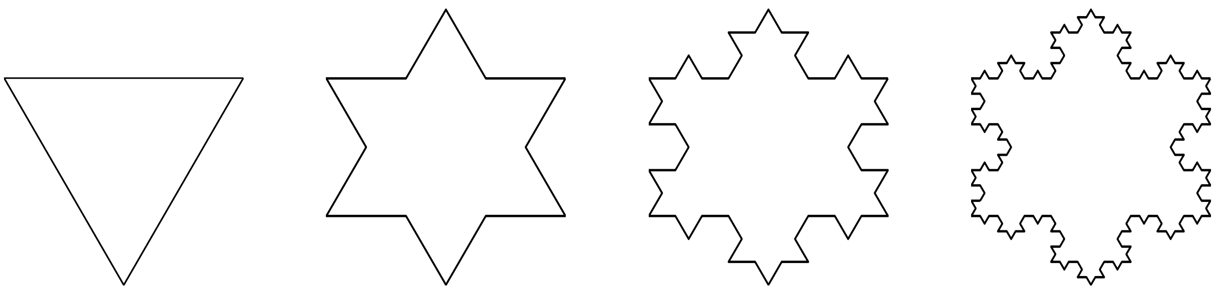
Pontus Strimling and I have several other Muller Twist variations on well-known games under investigation. Our hope is that in time it will be possible for someone to find the "general theory of Muller Twists", and we think that an adequate supply of solved examples is necessary for that work.

3 Games of Patterns

One thing that tends to fascinate mathematicians, and where it is possible to explain to others wherein the fascination lays, is *patterns*. Many mathematical objects and algorithms generate interesting patterns, and the activity of generating fascinating patterns is surely a game in sense 1 of the word.

3.1 The von Koch Curve

One of the more well-known recursively generated figures is the *von Koch curve*, perhaps better known as the *snow-flake curve*. (Actually, the snow-flake consists of three von Koch curves, put together to form a closed figure.)



(a) Start **(b)** After one iteration **(c)** After two iterations **(d)** After three iterations

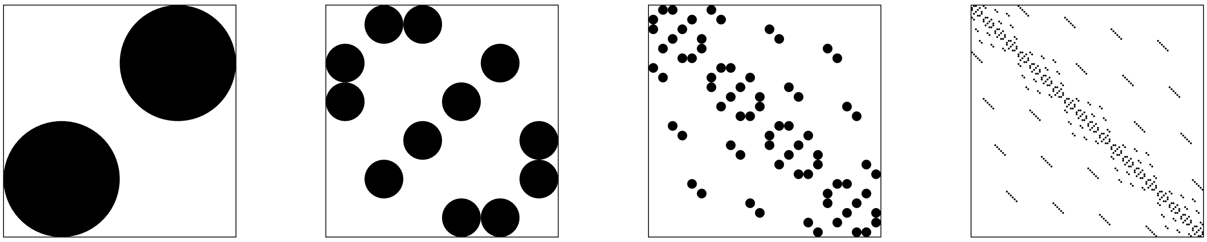
Figure 1: The first four snow-flake curves.

The von Koch curve starts with a straight line. In each iteration, you add detail by expanding each line segment ___ into a crooked line _/_. If you look at the curves, you note that each one consists of several instances of the previous one. If the process is carried on indefinitely, you get a curve of infinite length that encloses a limited area.

This curve was initially studied by the Swedish mathematician Helge von Koch [11] in the beginning of the 20th century. At that time, it was mainly regarded as a nice-looking example of an everywhere continuous, nowhere differentiable curve, but it gained a renewed interest in the 1970s, when Benoit Mandelbrot popularised the concept *fractal* [13].

3.2 Paper II: *Patterns in Matrices Describing Permutation Orders*

When studying a variation on sorting, I made a print-out of some data matrices describing the available moves, and found that they had an intriguing recursive nature. The matrices are shown in Figure 2, and I guess that the recursive properties are obvious. Each matrix contains several instances of the previous matrix, just as each von Koch curve contains the previous curve.



(a) 2 elements **(b)** 3 elements **(c)** 4 elements **(d)** 5 elements

Figure 2: The matrices that were the starting point for the research. They are binary matrices, the dots representing 1:s. They describe the relationship between permutations of 2, 3, 4 and 5 elements, where two permutations are considered related if you can obtain one of them by transposing two adjacent elements in the other. The matrices were to be used in a brute-force sorting program, and printed for debugging purposes.

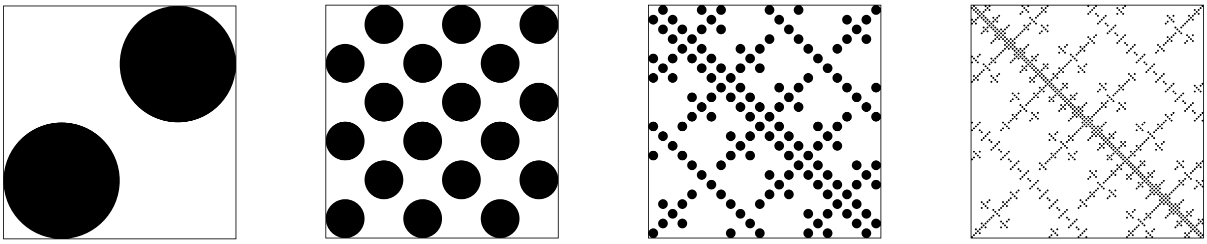
When sorting, you usually have a set of available operations, and you can order unsorted sequences according to how many operations that are required to sort them. An unsorted sequence can be represented by a permutation, the identity permutation representing the sorted state. The 1:s in the matrices represent pairs of permutations that differ by one operation. For example, the first dot in the first row of the matrix in Figure 2(b) indicates that you can get from the first permutation of 3 objects, 123, to the second[4] one, 132, using one operation when the allowed set of operations is transposition of adjacent objects.

I started a separate investigation on the patterns in this kind of matrices, since I got curious about what it was that I was seeing. By taking the permutations in a different order, I got another set of matrices where each matrix was a more detailed version of the previous one, another property of the von Koch curve. A sample of this kind of matrices is shown in Figure 3.

The properties of the von Koch curve follow directly from its definition as a geometrical object. In contrast, the matrices are based on the algebraic properties of permutations, and it is rather surprising to find the same kind of structure here.

The paper describing my study was presented at the peer-reviewed conference *Permutation Patterns* in Canada July 2004. Even if it is not about "permutation patterns"

[4]The permutations are numbered according to their order when interpreted as numbers.



(a) 2 elements **(b)** 3 elements **(c)** 4 elements **(d)** 5 elements

Figure 3: One series of matrices of the second kind mentioned. Here, each dot in the previous matrix "dissolves" into a cross of dots in the next one, just as each line segment in the von Koch curve grows into several line segments. In this case, two permutations are considered related if you can get one of them by reversing a segment in the other.

in the usual sense of the word [4] it is about permutations and patterns, and an audience of specialists in these areas could appreciate this variation on the theme.

Because of the intended audience, the reader of the paper is presumed to have basic knowledge of discrete mathematics in general [5] and particularly about the algebra of permutations [6]. A reader not having this special knowledge may still enjoy looking at the pictures.

4 Games in Math Education

Most games (in the traditional sense) have some mathematical contents. At the least, they usually involve adding the numbers of the dices, scoring, moving the right number of steps, or assessing the probability of something.

Since playing games is a popular activity, using games for educational purposes is an old tradition. The idea can either be that since children are playing anyway, they may as well gain something worthwhile from it, or that since the activity is enjoyable, they may carry on doing it for longer (and with greater enthusiasm) than they would when doing ordinary school work.

Since the games tend to involve mathematics, using them in math education in particular seems sensible. The article [10] reviews games that are used in practice in math education.

4.1 Games in Education

Using games in education is an idea that has grown in importance. During the 19th century, the growing middle classes, who set great store by their children's education, provided an increasingly larger market for games with (alleged) educational contents. In the 1960s, when a lot of educational reforms took place, games started to be more commonly used in school. Still, educational games are often used *outside* schools; in homes or in day-cares. Using games as an alternative teaching aid in schools can lead to complaints from groups of parents, who do not think that anything except textbooks is valid teaching. (This happens to almost anything that is not classical textbooks.)

Educational games can be of almost any kind (computerised or "old-fashioned"), and evaluating them can be tricky. Some games have the educational contents deeply integrated, in others they are more "glued on" to a normal game. The latter type are usually bad, regarded as *games*; they are not anything anybody would choose to play, given a choice. An example could be a classical board game where you have to solve a math problem to be allowed to make your move. The math distracts from the strategy in the game, and the game distracts from the math.

Games with their educational contents well integrated are usually better (regarded as *games*), but they sometimes develop the children's intellects without affecting their results in school, since children often do not realise that things learnt in one context can be used in another. (In many cases, they even think that they are not *allowed* to use anything except the things taught at lessons.)

Educational games differ from ordinary games in that they are usually not bought by the persons intended to use them. Persons buying games for themselves buy them for their entertainment value. Persons buying educational games try to go by the educational value. That can either be estimated from reviews in the papers or from what is written on the box. Games with their educational components well integrated can *look* less educational than games the educational part is more superficial, since in the latter case the educational contents are more easily seen.

In short: Educational games can be useful, but determining their usefulness is not a trivial matter.

4.2 Some Examples of Educational Games

Educational games can be anything from an activity that does not require any equipment at all to the most expensive computer game. Here, I give three very different examples of what they can consist of.

From my own school, I remember "multiplication bingo", played like ordinary bingo, except that the teacher called out "seven times six" instead of "42". As far as I can recall, it was an enjoyable way of practising the times tables, and caught the attention of the students more than a classical oral examination would have. (And it made the rather dull game *bingo* more challenging too.) The teacher had made the equipment himself out of some cardboard.

Another example of a game (in senses 1 and 7) is *Cuisinaire rods* [8]. Here, you need to invest in the equipment, consisting of coloured rods of different lengths, that can be used for representing numbers. With teacher prompting, children playing with the rods can develop an understanding of the relationships between numbers. (Without teacher prompting, I suspect that the rods would be used for building houses or throwing at other children[5]. Educational games need the same teacher participation as other educational activities.) A very good teachers manual is available.

Chefrens pyramid [3] is a computer game for math teaching, aimed at children from age 10 and upwards. Here, the story is that you are lost in a pyramid, and to get out, you have to solve the problems you encounter in the different rooms. If you know what you are doing, this takes about 4 hours. There is a very varied mix of problems, which is entertaining but also means that you do not have a choice on what you want to practice at this moment. I do not think that this game works as an integrated part of the school-work, but it is popular and funny enough to be played outside school-time, and may make the whole subject math more attractive. The game is quite expensive, comparable to equipping a form with a new set of textbooks.

4.3 Paper III: *Exploring the Educational Possibilities of Computer Games for 7th Grade Math*

Kimmo Eriksson and I got the opportunity to try our hands at developing an (as we hope) educational computer game. It is aimed at the age category 10–15 years, and supposed to make drill practice more fun. The article [10] deprecatingly calls this kind of games "dressed-up work sheets", but on the other hand the article [12] claims that using computer games for exactly this purpose has been proven effective. (I believe them, since I remember spending most of the math lessons in my second year at school staring out of the window, because the long list of problems seemed to boring to even begin considering starting[6].)

The "gaming" parts of the game were designed and implemented by a group of students at the Royal Institute of Technology, as part of a course in programming. We have added the mathematical parts: around 20 000 problems in the arithmetic of fractions. The methods used for generating these problems are described in detail in the paper, since we think that they may be useful in other contexts.

The game is of the "Super Mario"-type, where a player runs through an obstacle course (with traps and monsters) and in the same time hunts for treasures. In this case, most of the "treasures" consist of math problems. The problems are multiple choice questions, designed to be worked out in the head. A correct answer gives bonus points, an incorrect one an irritating break in the flow of the game, when the correct answer is displayed. If the player cannot answer the question, a helptext is available.

The game can be further extended. An end-user can edit courses or design new ones, load new questions in the game or change the sets of questions used at each course, and add new helptexts.

During the development of the game, we have cooperated with teachers at

[5] According to my daughter (who used them in kinder-garten), they did pretend that the rods were families, but they did *not* throw them.

[6] Strangely enough, this did not seem to affect my studies the next year in any way at all.

Viksängsskolan in Västerås. We have made three trials of the game at our game laboratory, with pupils at the school as study participants. After each trial, we have redesigned features that seemed to need it.

In Paper III we describe and evaluate this game. Although this paper is co-authored by my supervisor, I am the main author and principal researcher. Please observe that the manuscript is not yet in its final form; I plan to make more extensive testing of the educational effectiveness of the game in order to verify the positive effects we have found so far.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays*, volume 1. A K Peters, Ltd., 2 edition, 2001.

[2] C. F. Camerer. *Behavioral Game Theory*. Princeton University Press, 2003.

[3] G. Celén. Chefrens pyramid: Mattespel som håller länge. *Datorn i Utbildningen*, December 2001.

[4] *The Electronics Journal of Combinatorics*, 9(2), 2002–2003.

[5] K. Eriksson and H. Gavel. *Diskret matematik och diskreta modeller*. Studentlitteratur, 2002.

[6] K. Eriksson and H. Gavel. *Diskret matematik, fördjupning*. Studentlitteratur, 2003.

[7] K. Eriksson and H. Gavel. Goal-based reform of first-year discrete mathematics: A Swedish case study. *Teaching Mathematics and its Applications*, 2003.

[8] C. Gattegno. *Modern Mathematics with 'Numbers in Colour'*. Educ. Explorers, 1963.

[9] H. Gavel and P. Strimling. Nim with a modular Muller twist. *Integers*, 4, 2004.

[10] J. Gough. Mathematics games that really teach mathematics, 1995. `http://education.deakin.edu.au/resources/maths_ed/StaffPages/Gough/articles/mathsgamesreallyteach.pdf`.

[11] B. G. Johansson. *Matematikens historia*. Studentlitteratur, 2004.

[12] M. M. Klawe. When does the use of computer games and other interactive multimedia software help students learn mathematics?, 1998. `http://mathforum.org/technology/papers/papers/klawe.html`.

[13] B. B. Mandelbrot. *Fractals and Chaos*. Springer, 2004.

[14] M. J. Osborne. *An Introduction to Game Theory*. Oxford University Press, 2003.

[15] F. Smith and P. Stănică. Comply/constrain games or games with a Muller twist. *Integers*, 3, 2002.

[16] P. Strimling. *Trying to find what you want*. Licenciate thesis, Mälardalen University, 2005.

[17] E. S. Weiner and J. Simpson, editors. *The Compact Oxford English Dictionary*. Oxford University Press, 1991.

[18] L. Wittgenstein. *Philosophical Investigations*. Prentice Hall, 3rd edition, 1999.