Markowitz vs Black–Litterman: A Comparison of Two Portfolio Optimisation Models

by

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Abstract

Modern portfolio theory first gained its ground among researchers and academics, but has become increasingly popular among practitioners. This paper examines the two popular portfolio optimization models, Markowitz mean-variance model and Black-Litterman formula and compares their results on real data. In second chapter mean-variance model is derived step-by-step using Lagrange multipliers and matrices, whereas in third chapter Black-Litterman formula is proved by two different methods - by Maximum Likelihood method and Theil’s model.

Two portfolio optimization models are used on real data, monthly data from November 2007 to November 2017. In order to build the two models, Microsoft Excel is used. Swedish 30-day Treasury Bill is taken as risk-free asset and SIXPRX as a benchmark. Detailed results are presented in Chapter 4. In Black-Litterman model two different views are implemented to see if the model outperforms Markowitz mean-variance model. All in all there is a significant difference in the outcomes, Black-Litterman portfolio performs better than mean-variance portfolio.
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Chapter 1

Introduction

The ground for modern portfolio theory was laid when Harry Markowitz published his work *Portfolio Selection* in 1952. Markowitz’s paper tried to answer the important question: How to allocate wealth between different investment choices? [7] The main part of Markowitz’s idea was to build and use mathematical models to diversify portfolios. He used probability theory to quantify assets risk and return and proposed that they should viewed together as a risk-return trade-off. The idea is that portfolio’s risk depends on how correlated are the assets with one another in a portfolio. Furthermore, it will become a mean-variance optimization problem where investor should choose the portfolio that have the desired return, but the lowest level of risk. This area of finance has since then influenced modern thinking in both investments and in the academics world.

Years it has had a big impact on research in financial industry, but also it has influenced new way of thinking about investments. As of 2013 there have been almost 20 000 articles in Google Scholar that refer to Markowitz’s original paper [7]. Yet it was not implemented by investment managers for a long time. The major problem is many pitfalls with the model. For instance, very sensitive nature of changes in inputs to output. Therefore, it was considered as opaque and unstable.

Since Markowitz model is considered as unintuitive, in early 1990s Fischer Black and Robert Litterman developed a new model which aimed to deal with mean-variance model’s pitfalls and shortcomings. Mean-variance model’s starting portfolio is the null portfolio, whereas Black–Litterman model’s (B-L) initial point is the equilibrium portfolio. An investor then assigns views, i.e. his/her opinion that one or many assets outperform the others. These different viewpoints are are combined in calculation. Sometimes it is thought that B-L model is a totally new model, yet it differs only in expected returns [10]. The first part of thesis gives an overview and mathematical background of both models. The second part covers empirical findings where we compare the results from the two models. And finally, we draw a conclusion and suggest further possible research topics.
1.1 Aim and Purpose

The aim of this paper is to describe the background of models for portfolio optimization and diversification with important mathematical derivations and proofs. To make it more clear and perceivable, we analyze real data and compare the results from B-L and MV models. Furthermore, we want to know how much better performance has the B-L model compared to Markowitz model since B-L model addresses the pitfalls of the latter model and therefore, is assumed to outperform mean-variance optimization approach.

1.2 Methodology

The choice of possible range of equities that might be in portfolio is not easy. For the construction of Markowitz mean-variance and Black–Litterman model we use 29 equities that form OMX Stockholm 30 Index. The reason for 29 equities and not all 30 is the lack of data for Essity B stock (it has data available only from June 2017). The monthly data is taken over 10 year period — from 30th of November 2007 to 30th of November 2017. One drawback with this period is that it also covers one year of the very turbulent times during the global financial crisis. This means that the volatility could be higher than usual during that period of time. The data is downloaded from Thomson Reuters EIKON platform. The 30-day Swedish Treasury Bill over 10 years is used as risk-free interest rate in the models. The reasoning behind it is that Treasury Bills are considered to be the closest to risk-free asset because financial world does not have completely safe instrument. The benchmark is SIXPRX.

The data analyzing and portfolio constructions are done in Microsoft Excel.

1.3 Outline of the Thesis

Chapter 2 This chapter discusses the Markowitz mean-variance model with its assumptions and limitations. Furthermore, the formulas for minimum variance, orthogonal, and tangency portfolios are derived.

Chapter 3 Gives the two different proofs of Black-Litterman master formula. Including short discussion about the assumptions and limitations.

Chapter 4 In that chapter are showed and discussed the results of the model construction on real data. The comparison and analysis is conducted.

Chapter 5 This chapter concludes and gives some further research topics.

Chapter 6 The last chapter presents the objectives of thesis and their fulfillment.
Chapter 2

Markowitz Mean-Variance Model

Harry Markowitz’s paper from 1952 established some key concepts that laid the ground to Modern Portfolio Theory. He mathematically formulated the idea of diversifying the risk of portfolio and discussed the components that possibly have an impact on return and volatility. The conceptual framework was based on key concept of investors risk appetite. In other words, an investor targets at all times at maximal return on his/her investment while wanting to minimize the risk. Although his work did not draw much attention at the time, in the 60’s academics in finance started to pay attention. Nowadays his approach is used to construct a portfolio as well as measure its performance by many investors and portfolio managers. During the 65 years Markowitz model has been revised numerous ways [12]. This chapter is covering first the assumptions of the model, followed by important parts of the model, efficient frontier, global minimum variance and tangency portfolios and their mathematical derivations. We conclude the chapter with a brief discussion of the model’s limitations.

2.1 Assumptions

Markowitz model is based on several assumptions which have been questioned time and time again [9]. First, investors are rational — it means that they like to maximize return while minimizing risk. More risk is only accepted if there is a compensation by higher returns. Investors have limitless access to capital with risk-free rate. Furthermore, markets are held to be efficient and there are no transaction costs or taxes.

2.2 Symbols

The following notation is used in defining and deriving models:

\(\mathbf{w}\) \(n\)-column vector with components \(w_1, w_2, \ldots, w_n\) that denote weights of the assets in the portfolio where \(i = 1, 2, \ldots, n\).

\(\mathbf{R}\) \(n\)-column vector of expected returns \(R_1, R_2, \ldots, R_n\) where \(i = 1, 2, \ldots, n\).

\(\mathbf{V}\) \(n \times n\) variance-covariance matrix with components \(\sigma_{ij}\) where \(i, j = 1, 2, \ldots, n\).
\[ n \text{-column vector of ones.} \]
\[ r \text{ the vector of expected excess returns.} \]
\[ \top \text{ denotes the transpose of vector or matrix.} \]
\[ R_p \text{ portfolio’s return.} \]
\[ \sigma^2_p \text{ portfolio’s variance.} \]
\[ r_f \text{ risk-free asset.} \]

### 2.3 Expected Return of Portfolio

Markowitz model uses the expected return as a measure of central tendency \([5]\). The individual assets’ returns are observed as the historical performances. The expected return of a portfolio is the weighted average of the expected returns on the individual assets \([3]\):

\[
R_p = \sum_{i=1}^{N} w_i R_i = w^\top R. \tag{2.1}
\]

### 2.4 Variance, Covariance, and Correlation

An investor prefers an asset with lower variance if two assets have the same expected return. The variance of portfolio (measure of dispersion) is the expected value of the squared deviations of the return on the portfolio from the mean return on the portfolio:

\[
\sigma^2_p = E[(r_p - R_p)^2] = E \left( \sum_{i=1}^{N} (w_i(r_i - R_i) \right)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = w^\top V w. \tag{2.2}
\]

Diversification cannot eliminate all risk, since there is unsystematic risk that can be reduced significantly and systematic risk that is seen as market risk and therefore, cannot be get rid of.

Covariance expresses how assets move together \([5]\). This is an important aspect. It is useful to standardize the covariance by dividing the covariance term with the standard deviations of assets:

\[
\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \tag{2.3}
\]

This is called correlation and is between -1 and +1. Negative correlation coefficient means that assets’ returns move in opposite directions and although perfectly negative correlation does not happen in real life due to the presence of systematic risk, it would be possible to construct a portfolio with zero risk. Positive correlation means that assets’ returns move in same direction and zero correlation means that they are independent given that they are jointly normally distributed.
2.5 Efficient Frontier

I derive the formulas by using matrices and Lagrange multipliers [3]. Efficient frontier is a curve or in some cases a line that represents all the portfolios of highest level of return with given level of risk. The points inside of the curve are inefficient, since an investor can have more return with the same level of risk or vice versa. We would like to minimize risk given the two constraints - asset weights are summing up to one and second, portfolio earns expected rate of return at a given level, i.e. the problem formulation for attainable portfolios is:

\[
\text{minimize } w^\top Vw \\
\text{subject to } w^\top 1 = 1 \\
R_p = w^\top R
\]

To solve this, we set up the Lagrange function, \(L\), with multipliers \(\lambda_1\) and \(\lambda_2\):

\[
L = w^\top Vw - \lambda_1 (w^\top R - R_p) - \lambda_2 (w^\top 1 - 1).
\]  
(2.4)

Next, we take the partial derivatives with respect to \(w\), \(\lambda_1\), and \(\lambda_2\). The First Order Conditions become:

\[
\frac{\delta L}{\delta w} = 2Vw - \lambda_1 R - \lambda_2 1 = 0 \quad \Rightarrow \quad 2Vw = \lambda_1 R + \lambda_2 1,
\]  
(2.5)

where \(0\) is the \(n\)-vector of zeros. From (2.5) we get the weights, \(w\):

\[
w = \frac{1}{2} V^{-1} (\lambda_1 R + \lambda_2 1).
\]  
(2.6)

The other two Lagrange equations become:

\[
\frac{\delta L}{\delta \lambda_1} = R_p - w^\top R = 0 \quad \Rightarrow \quad R_p = w^\top R,
\]

\[
\frac{\delta L}{\delta \lambda_2} = 1 - w^\top 1 = 0 \quad \Rightarrow \quad 1 = w^\top 1.
\]  
(2.7)

Writing the \((\lambda_1 R + \lambda_2 1)\) as matrix form in (2.6):

\[
w = \frac{1}{2} V^{-1} (\lambda_1 R + \lambda_2 1) = \frac{1}{2} V^{-1} \begin{bmatrix} R & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.
\]  
(2.8)

Next, we want to solve for \(\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}\) using the other two F. O. C rewritten as:

\[
\begin{bmatrix} R & 1 \end{bmatrix}^\top w = \begin{bmatrix} R & 1 \end{bmatrix}^\top \frac{1}{2} V^{-1} \begin{bmatrix} R & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} R_p \\ 1 \end{bmatrix}.
\]  
(2.9)

Multiplying both side of Equation (2.8) by \(\begin{bmatrix} R & 1 \end{bmatrix}^\top\), we get:

\[
\begin{bmatrix} R & 1 \end{bmatrix}^\top w = \frac{1}{2} \begin{bmatrix} R & 1 \end{bmatrix}^\top V^{-1} \begin{bmatrix} R & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} R_p \\ 1 \end{bmatrix}
\]  
(2.10)
For convenience, let’s introduce a new symmetric matrix $A$ that is called also information matrix:

$$A = [R \ 1]^\top V^{-1} [R \ 1], \quad (2.11)$$

where the entries are

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = [R^\top V^{-1} R \ 1]^\top V^{-1} [R \ 1]. \quad (2.12)$$

Given that, we need to show that the matrix $A$ is positive definite. If we have any $y_1, y_2$ where at least one is nonzero, we see that

$$[R \ 1] [y_1 \ y_2] = [y_1 R + y_2 1]. \quad (2.13)$$

This is a nonzero vector with $n$ elements and by assumption the variables in $R$ are not all equal. Given that, the $A$ is positive definite, since

$$[y_1 \ y_2] A [y_1 \ y_2] = [y_1 R + y_2 1]^\top V^{-1} [y_1 R + y_2 1] > 0 \quad (2.14)$$

by the positive definiteness of $V^{-1}$. Next, we substitute the $A$ in (2.11) and yield to the result:

$$\frac{1}{2} A [\lambda_1 \ \lambda_2] = [R_p] \quad (2.15)$$

Since $A$ is non-singular and there is an inverse, we can solve for multipliers:

$$\frac{1}{2} [\lambda_1 \ \lambda_2] = A^{-1} [R_p] \quad (2.16)$$

From Equations (2.16) and (2.8) we obtain the $n$-vector of portfolio weights that minimizes the portfolio variance for a given return:

$$w = \frac{1}{2} V^{-1} [R \ 1] [\lambda_1 \ \lambda_2] = V^{-1} [R \ 1] A^{-1} [R_p] \quad (2.17)$$

Given mean $R_p$, definitions of variance, derived previous solutions and matrix $A$, we can express the variance of minimum variance portfolio:

$$\sigma_p^2 = w^\top V w = [R_p \ 1] A^{-1} [R \ 1]^\top V^{-1} V V^{-1} [R \ 1] A^{-1} [R_p]$$

$$= [R_p \ 1] A^{-1} [R_p]$$

$$= [R_p \ 1] \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} [R_p]$$

$$= a - 2bR_p + cR_p^2 \quad (2.18)$$
2.6 Global Minimum Variance Portfolio

In the previous section, we discussed what the efficient frontier is and how to derive it. In this section, we go through the derivation of Global Minimum Variance Portfolio (GMV), i.e. the portfolio that has the smallest amount of risk. This is the absolute minimum point on efficient frontier. We denote the mean of GMV as $R_G$ and get it when we minimize equation (2.18) with respect to $R_p$, which yields to:

$$R_G = \frac{b}{c}. \quad (2.19)$$

By inserting this into the variance formula (2.18) we get the variance of GMV:

$$\sigma_p^2 = \frac{a - 2bR_G + cR_G^2}{(ac - b^2)} = \frac{a - 2b\left(\frac{b}{c}\right) + c\left(\frac{b}{c}\right)^2}{(ac - b^2)} = \frac{1}{c}. \quad (2.20)$$

For GMV weights, we insert $R_G$ into Equation (2.17) and yield:

$$w_G = V^{-1} \begin{bmatrix} R \\ 1 \end{bmatrix} A^{-1} \begin{bmatrix} R_G \\ 1 \end{bmatrix} = \frac{V^{-1} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \begin{bmatrix} b/c \\ 1 \end{bmatrix}}{(ac - b^2)} = V^{-1} \begin{bmatrix} 1 \end{bmatrix}. \quad (2.21)$$

Note that the parameters for $a, b$ and $c$ are in information matrix $A$ in Equation (2.12).

2.7 Orthogonal Portfolio

Before we can move on to tangency portfolio, we have to establish a concept of orthogonal portfolio. If the first portfolio has the mean $R_p$, then the orthogonal portfolio has mean $R_h$ with [3]:

$$R_h = \frac{a - bR_p}{b - cR_p} \quad (2.22)$$

To set up Equation (2.22), we denote two arbitrary minimum variance portfolios, $p$ and $h$, with weights $w_p$ given by Equation (2.17) and $w_h$ by:

$$w_h = V^{-1} \begin{bmatrix} R \\ 1 \end{bmatrix} A^{-1} \begin{bmatrix} R_h \\ 1 \end{bmatrix}. \quad (2.23)$$

Two portfolios are orthogonal when their covariance is zero, this implies that

$$0 = w_h^T V w_p = \begin{bmatrix} R_h & 1 \end{bmatrix} A^{-1} \begin{bmatrix} R_p \\ 1 \end{bmatrix}. \quad (2.24)$$

Rewriting (2.24) we get (2.22).
2.8 Tangency Portfolio

"Risk-averse" investor is defined as an investor who prefers safer investments for riskier investments with higher returns [5]. Most investors are not so risk-averse that they would choose GMV, but would pick the tangency portfolio. Here we introduce the riskless asset \( r_f \), i.e. a security that has a certain future return. Although even this asset carries very small amount of risk, we assume the degree of risk for \( r_f \) is so small that \( \sigma_f = 0 \).

Let \( R_i \) where \( i = 1, 2, 3, \ldots, n \) present the expected returns in gross terms. The expected return in net terms is denoted by \( r \) and includes elements \( r_i = R_i - r_f \) where \( i = 1, 2, 3, \ldots, n \). An investor is interested in growing assets in relation to net of future outflows. He/she is concerned that his/her net worth may change, so here the term "net" is defined in terms of existing liabilities [5].

So the weight vector for risky assets is same as before, \( w \) and we denote the weight of riskless asset as \( w_f = 1 - w^\top 1 \). [3] The portfolio \( p \) mean excess return that an investor is interested in is given by

\[
R_p = w^\top R + (1 - w^\top 1)r_f - r_f = w^\top r
\]  
(2.25)

and variance by

\[
\sigma_p^2 = w^\top V w
\]  
(2.26)

We have at hand another optimisation problem

\[
\text{minimize } w^\top V w\quad \text{subject to } w^\top r = R_p.
\]

Solving this problem as in Section 2.5 by using Lagrange multipliers:

\[
L = w^\top V w - \lambda (R_p - w^\top r).
\]  
(2.27)

By taking the partial derivatives with respect to \( w \) and \( \lambda \), the First Order Conditions are:

\[
\frac{\delta L}{\delta w} = 2Vw - \lambda r = 0 \quad \Rightarrow \quad 2Vw = \lambda r \quad \Rightarrow \quad w = \frac{\lambda}{2} V^{-1} r, \]  
(2.28a)

\[
\frac{\delta L}{\delta \lambda} = R_p - w^\top r = 0 \quad \Rightarrow \quad R_p = w^\top r
\]  
(2.28b)

Next, we multiply both sides by \( r^\top \) in Equation (2.28a),

\[
r^\top w = r^\top \frac{\lambda}{2} V^{-1} r.
\]  
(2.29)

Given the Equation (2.28b), we get

\[
\frac{\lambda}{2} = \frac{R_p}{r^\top V^{-1} r}.
\]  
(2.30)
Next we substitute Equation (2.30) into first order condition’s Equation (2.28)a and we get the weight-vector \( w_{TP} \) of tangency portfolio,

\[
w_{TP} = \frac{R_p}{r^\top V^{-1} r} V^{-1} r.
\]  

(2.31)

For obtaining the expected return of tangency portfolio, we multiply both sides of (2.31) by \( 1^\top \) and use that the tangency portfolio consists only of risky assets, i.e. \( 1^\top w_{TP} = 1 \)

\[
R_{TP} = \frac{r^\top V^{-1} r}{1^\top V^{-1} r}.
\]  

(2.32)

And finally, we have the variance of portfolio \( \sigma^2_{TP} \) where we plug in the obtained values,

\[
\sigma^2_{TP} = w^\top V w = \left( \frac{R_p}{r^\top V^{-1} r} \right)^2 r^\top V^{-1} V V^{-1} r
\]  

\[
= \frac{R_p^2}{r^\top V^{-1} r}.
\]  

(2.33)

Next we find the Sharpe Ratio. It is given by formula \( SR = \frac{\mu_p - r_f}{\sigma_p} \) where \( \mu_p \) is expected return of a portfolio that contains only risky assets. We use the result in (2.33) to derive Sharpe Ratio \( SR \),

\[
SR^2_{TP} = \left( \frac{R_{TP}}{\sigma_{TP}} \right)^2 = \frac{R_p^2}{\frac{R_p}{r^\top V^{-1} r}} = r^\top V^{-1} r
\]  

(2.34)

We take a square root of Equation (2.34) and have the Sharpe Ratio as \( SR = \sqrt{r^\top V^{-1} r} \).

### 2.9 Capital Market Line

The Capital Market Line (CML) is a straight line that begins at the initial point that is equal to risk-free rate of return and touches one point on efficient frontier [5]. While tangency portfolio consists only of risky assets, the portfolios on CML include the borrowing and lending of risk-free asset (except tangency portfolio). We have initial inputs to CML as \( r_f = r_0 \) and \( \sigma_f = \sigma_0 \). The line equation is mathematically presented as \( y = m + kx \) where \( r \) lies on y-axis and \( \sigma \) on x-axis. Let \( k \) be the slope of CML where it is Sharpe Ratio by tangency portfolio. We have

\[
k = \frac{R_{TP} - r_0}{\sigma_{TP} - \sigma_0} = \frac{R_{TP} - r_f}{\sigma_{TP} - \sigma_f} \text{ where } \sigma_f = 0,
\]  

so the CML formula becomes:

\[
CML = r_f + \frac{R_{TP} - r_f}{\sigma_{TP}} \sigma
\]  

(2.35)
2.10 Limitations

In theory mean-variance model is reasonable, but problems arise in practice. Different researchers over the years have discussed the limitations of Markowitz mean-variance model.

First, the estimates of risk and return are prone to estimation error. For instance, it puts more weight on securities that have large estimated returns, small dispersion and negative correlation [11].

It is also argued that the substitution of the expected return with sample average does not work well. Moreover, small changes in inputs (especially expected return) can lead to large changes in output which makes the mean-variance model very unstable. Markowitz model does not make a distinction if there is a possible uncertainty in estimated inputs of model [10].
Chapter 3

Black-Litterman Model

In section 2.10 we discussed the limitations of Markowitz mean-variance model. Fischer Black and Robert Litterman [1] tried to solve two problems of mean-variance model, namely, the difficulty of estimating expected returns and their extreme sensitivity of return assumptions. Black and Litterman’s key is to combine mean-variance optimization and the capital asset pricing model (CAPM) by Sharpe and Lintner. Their neutral starting point is market equilibrium where investor views and the level of confidence are added, i.e views are combined using Bayesian mixed estimation techniques. This creates the B-L expected returns which are optimized mean-variance way and finally, an optimal portfolio is constructed [10, 6].

3.1 Assumptions

There are several assumptions for every model. The list of assumptions has many levels, for instance, typical for quantitative models are normally distributed returns, no arbitrage, capital markets are efficient etc.

For portfolio models, only risk and return are used for making decisions, no transaction costs or taxes or risk-averse investor. Specifically, for B-L model first assumption is that investors have views which create better portfolios, but they are not hundred percent certain about the views. So, for every belief we have a level of confidence. Inefficiency in markets is assumed [10].

3.2 Symbols

The following notation is used in Black–Litterman master model:

- $P$ the matrix of asset weights within each view, size $k \times N$
- $q$ the vector of expected returns with investor views
- $\hat{\mu}$ the posterior expected return vector, size $N \times 1$
\( \hat{\mathbf{\Sigma}} \) posterior covariance matrix containing variances and covariances of the model’s assets, size \( N \times N \)

\( \mathbf{\Pi} \) the column vector of expected returns for market estimates

\( \tau \) the measure of uncertainty coefficient

\( \bar{\mathbf{q}} \) the vector of the returns including views, size \( k \times 1 \)

\( \mathbf{\Omega} \) the covariance matrix with diagonal elements \( \omega_j^2 \), size \( k \times k \)

\( \mathbf{\Sigma} \) the known covariance matrix, size \( N \times N \)

\( k \) the number of views in view matrix

### 3.3 Proof of Black–Litterman Model

In this section, we prove the Black–Litterman model using first the Maximum Likelihood method and second the proof by Theil’s model. First, let’s make an introduction to overall notation and remarks about the variables that are used in the model and in proofs [10], [14].

There are \( d \) assets and \( m \) market observations \( \mathbf{r}_1, \ldots, \mathbf{r}_m \in \mathbb{R}^d \) of normally distributed with mean value \( \mathbf{\mu} \in \mathbb{R}^d \) and covariance matrix \( \mathbf{\Sigma} \).

Let \( \mathbf{P} \) be the matrix of weights and \( \mathbf{q} \in \mathbb{R}^k \) the vector of expected returns of the \( k \) portfolios which contains the investor views where \( k \leq d \). The returns are calculated by the formula

\[
\mathbf{q} = \mathbf{P}\mathbf{r}',
\]

where \( \mathbf{r}' \in \mathbb{R}^d \) is the vector of expected returns estimated by the investor.

Furthermore, \( \mathbf{\Omega} \) is the \( k \times k \) diagonal matrix with diagonal elements \( \omega_j^2 = \text{Var}[q_j] \). By Equation (3.1),

\[
q_j = \mathbf{P}\mathbf{r}_j, \quad m + 1 \leq j \leq m + n,
\]

where \( n \) is the number for investor observations \( q_{m+1}, \ldots, q_{m+n} \).

In the case where we only have market observations, the expected return for market estimate is

\[
\hat{\mathbf{r}}^M := \frac{1}{m} \sum_{i=1}^m \mathbf{r}_i.
\]

The random vector \( \hat{\mathbf{r}}^M \) has normal distribution with mean \( \mathbf{\mu} \) and covariance matrix \( \frac{1}{m} \mathbf{\Sigma} \). The standard notation:

\[
\mathbf{\Pi} = \hat{\mathbf{r}}^M, \quad \bar{\mathbf{q}} = \bar{\mathbf{q}}' = \frac{1}{n} \sum_{j=m+1}^{m+n} q_j,
\]

where \( \bar{\mathbf{q}} \) is the investor’s estimate of the expected return with normal distribution \( N(\mathbf{P}\mathbf{\mu}, \frac{1}{n} \mathbf{\Omega}) \).

**Theorem 1.** Combining estimates from both investors and market, we yield to Black–Litterman master formula:

\[
\hat{\mathbf{\mu}} = [(\tau \mathbf{\Sigma})^{-1} + \mathbf{P}^\top \mathbf{\Omega}^{-1} \mathbf{P}]^{-1}[(\tau \mathbf{\Sigma})^{-1} \mathbf{\Pi} + \mathbf{P}^\top \mathbf{\Omega}^{-1} \bar{\mathbf{q}}],
\]

\[
\hat{\mathbf{\Sigma}} = [(\tau \mathbf{\Sigma})^{-1} + \mathbf{P}^\top \mathbf{\Omega}^{-1} \mathbf{P}]^{-1}.
\]

(3.2)
3.3.1 Proof by Maximum Likelihood Method

Proof. [10] Let the probability density of the random vector \( r_i \) be
\[
f(r_i, \nu) = \frac{1}{(2\pi)^{d/2} \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2} (r_i - \nu)^\top \Sigma^{-1} (r_i - \nu) \right),
\]
The probability density function of the random vector \( q_j \) is
\[
g(q_j, \nu) = \frac{1}{(2\pi)^{d/2} \sqrt{\det \Omega}} \exp \left( -\frac{1}{2} (q_j - P \nu)^\top \Omega^{-1} (q_j - P \nu) \right).
\]
By definition, the likelihood function is the product of both densities:
\[
L(\nu) = \prod_{i=1}^{m} f(r_i, \nu) \prod_{j=m+1}^{m+n} g(q_j, \nu).
\]
The maximum likelihood estimate, \( \hat{\mu} \), is the value of \( \nu \) where the function \( L(\nu) \) attains its maximal value. It is difficult to take a derivative from this likelihood function, therefore, we first take the logarithm and get the logarithmic likelihood function
\[
\ell(\nu) = \ln L(\nu)
\]
We obtain
\[
\ell(\nu) = \sum_{i=1}^{m} \ln f(r_i, \nu) + \sum_{j=m+1}^{m+n} \ln g(q_j, \nu)
\]
\[
= m \ln \frac{1}{(2\pi)^{d/2} \sqrt{\det \Sigma}} + n \ln \frac{1}{(2\pi)^{d/2} \sqrt{\det \Omega}}
\]
\[
- \frac{1}{2} \sum_{i=1}^{m} (r_i - \nu)^\top \Sigma^{-1} (r_i - \nu) - \frac{1}{2} \sum_{j=m+1}^{m+n} (q_j - P \nu)^\top \Omega^{-1} (q_j - P \nu).
\]
Next we evaluate the gradient of \( \ell(\nu) \). For that, we let \( e_k \) be the column vector with \( d \) components, where the \( k \)th component is equal to 1, and the remaining components are equal to 0. We get
\[
(\nabla \ell(\nu))_k = \frac{\partial \ell(\nu)}{\partial v_k}
\]
\[
= -\frac{1}{2} \sum_{i=1}^{m} (-e_k^\top \Sigma^{-1} (r_i - \nu) + (r_i - \nu)^\top \Sigma^{-1} (-e_k))
\]
\[-\frac{1}{2} \sum_{j=m+1}^{m+n} (-e_k^\top P^\top \Omega^{-1} (q_j - P \nu) + (q_j - P \nu)^\top \Omega^{-1} P (-e_k))
\]
\[
= e_k^\top \Sigma^{-1} \left[ m \frac{1}{m} \sum_{i=1}^{m} r_i - m \nu + \frac{1}{n} \sum_{j=m+1}^{m+n} q_j - n P \nu \right]
\]
\[
= e_k^\top \left[ m \Sigma^{-1} (\Pi - \nu) + n P^\top \Omega^{-1} (\bar{q} - P \nu) \right],
\]
By definition, the points where $\nabla \ell(\nu) = 0$ are critical points of $\ell(\nu)$. Then the equation is

$$m\Sigma^{-1}(\Pi - \hat{\mu}) +nP^\top\Omega^{-1}(\bar{q} - P\hat{\mu}) = 0.$$ 

which can be written as

$$\tau^{-1}\Sigma^{-1}(\Pi - \hat{\mu}) + P^\top\Omega^{-1}(\bar{q} - P\hat{\mu}) = 0,$$

As we can see, we have introduced the variable $\tau$ by

$$\tau = \frac{n}{m}.$$ 

After moving the terms with $\hat{\mu}$ to left and others to right, we get

$$[(\tau\Sigma)^{-1} + P^\top\Omega^{-1}P]\hat{\mu} = (\tau\Sigma)^{-1}\Pi + P^\top\Omega^{-1}\bar{q}.$$ 

Next we multiply both hand sides by $[(\tau\Sigma)^{-1} + P^\top\Omega^{-1}P]^{-1}$. We get the equation in (3.2). As last we prove that the critical point $\hat{\mu}$ is a maximum. We calculate the Hessian of the function $\ell(\nu)$, i.e. the matrix of its second partial derivatives. We yield

$$\left(\nabla^2 \ell(\nu)\right)_{kl} = \frac{\partial(\nabla \ell(\nu))_k}{\partial \nu_l}$$

$$= \frac{\partial}{\partial \nu_l} e_k^\top \left[ m\Sigma^{-1}(\Pi - \nu) +nP^\top\Omega^{-1}(\bar{q} - P\nu) \right]$$

$$= e_k^\top \left[ m\Sigma^{-1}(-e_l) +nP^\top\Omega^{-1}(-Pe_l) \right]$$

$$= -m\Sigma^{-1} - n(P^\top\Omega^{-1}P)_{kl}.$$ 

Since both matrices $\Sigma^{-1}$ and $P^\top\Omega^{-1}P$ are positive-definite. We can see that the Hessian is negative-definite, which means that the critical point $\hat{\mu}$ is really a maximum. For calculating $\hat{\Sigma}$,

$$\hat{\Sigma} := E[(\hat{\mu} - \mu)(\hat{\mu}^\top - \mu^\top)]$$

$$= E[\hat{\mu}\hat{\mu}^\top] - E[\hat{\mu}]\mu^\top - \mu E[\hat{\mu}^\top] + \mu\mu^\top.$$ 

We must evaluate $E[\hat{\mu}]$ and $E[\hat{\mu}\hat{\mu}^\top]$.

$$E[\hat{\mu}] = (\tau\Sigma)^{-1} + P^\top\Omega^{-1}P^{-1}[(\tau\Sigma)^{-1}E[\Pi] + P^\top\Omega^{-1}E[\bar{q}]]$$

$$= (\tau\Sigma)^{-1} + P^\top\Omega^{-1}P^{-1}[(\tau\Sigma)^{-1} + P^\top\Omega^{-1}P]\mu$$

$$= \mu.$$ 

This means that the estimates of the expected returns from both market and investor are unbiased.

$$\hat{\Sigma} = E[\hat{\mu}\hat{\mu}^\top] - \mu\mu^\top$$

For simplification we denote,

$$A = (\tau\Sigma)^{-1} + P^\top\Omega^{-1}P.$$ 

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Since we have
\[
\mathbf{\mu}^T = [\Pi^T (\tau \Sigma)^{-1} + \mathbf{q}^T \Omega^{-1} \mathbf{P}] A^{-1}
\]
and
\[
E[\mathbf{\hat{\mu}}^T] = A^{-1} E[(\tau \Sigma)^{-1} \Pi + \mathbf{P}^T \Omega^{-1} \mathbf{q}] [\Pi^T (\tau \Sigma)^{-1} + \mathbf{q}^T \Omega^{-1} \mathbf{P}] A^{-1},
\]
then expected value is
\[
E[(\tau \Sigma)^{-1} \Pi + \mathbf{P}^T \Omega^{-1} \mathbf{q}] [\Pi^T (\tau \Sigma)^{-1} + \mathbf{q}^T \Omega^{-1} \mathbf{P}] = (\tau \Sigma)^{-1} E[\Pi \Pi^T] (\tau \Sigma)^{-1} + (\tau \Sigma)^{-1} E[\Pi \mathbf{q}^T] \Omega^{-1} \mathbf{P} + \mathbf{P}^T \Omega^{-1} E[\mathbf{q} \Pi^T] (\tau \Sigma)^{-1} + \mathbf{P}^T \Omega^{-1} E[\mathbf{q} \mathbf{q}^T] \Omega^{-1} \mathbf{P}.
\]
We have
\[
E[\Pi \Pi^T] = \Sigma + \mathbf{\mu} \mathbf{\mu}^T, \quad E[\Pi \mathbf{q}^T] = \mathbf{\mu} \mathbf{\mu}^T \mathbf{P},
\]
\[
E[\mathbf{q} \Pi^T] = \mathbf{P} \mathbf{\mu} \mathbf{\mu}^T, \quad E[\mathbf{q} \mathbf{q}^T] = \Omega + \mathbf{P} \mathbf{\mu} \mathbf{\mu}^T \mathbf{P}.
\]
Since the random vectors \( \Pi \) and \( \mathbf{q} \) are independent,
\[
E[(\tau \Sigma)^{-1} \Pi + \mathbf{P}^T \Omega^{-1} \mathbf{q}] [\Pi^T (\tau \Sigma)^{-1} + \mathbf{q}^T \Omega^{-1} \mathbf{P}] = (\tau \Sigma)^{-1} (\Sigma + \mathbf{\mu} \mathbf{\mu}^T) (\tau \Sigma)^{-1} + (\tau \Sigma)^{-1} \mathbf{\mu} \mathbf{\mu}^T \mathbf{P}^T \Omega^{-1} \mathbf{P} + \mathbf{P}^T \Omega^{-1} \mathbf{P} \mathbf{\mu} \mathbf{\mu}^T (\tau \Sigma)^{-1} + \mathbf{P}^T \Omega^{-1} (\Omega + \mathbf{P} \mathbf{\mu} \mathbf{\mu}^T \mathbf{P}) \Omega^{-1} \mathbf{P} = A + A \mathbf{\mu} \mathbf{\mu}^T A.
\]
It follows that
\[
E[\mathbf{\hat{\mu}}^T] = A^{-1} (A + A \mathbf{\mu} \mathbf{\mu}^T A) A^{-1} = A^{-1} + \mathbf{\mu} \mathbf{\mu}^T.
\]
and
\[
\hat{\Sigma} = A^{-1} + \mathbf{\mu} \mathbf{\mu}^T - \mathbf{\mu} \mathbf{\mu}^T = A^{-1}.
\]

### 3.3.2 Proof by Theil’s model

**Proof.** [14] We have
\[
\Pi = X \mathbf{\mu} + \mathbf{u}, \quad \mathbf{q} = \mathbf{P} \mathbf{\mu} + \mathbf{v},
\]
where \( X \) is the identity matrix, \( \mathbf{u} \) is the normally distributed residual with mean \( \mathbf{0} \) and covariance matrix \( \Phi = \tau \Sigma \), \( \mathbf{v} \) is the normally distributed residual with mean \( \mathbf{0} \) and covariance matrix \( \Omega \). The residuals are assumed to be independent. Notating it in matrix form,
\[
\begin{bmatrix}
\Pi \\
\mathbf{q}
\end{bmatrix} = \begin{bmatrix} X & \mathbf{u} \\
\mathbf{P} & \mathbf{v}
\end{bmatrix}, \quad \text{(3.3)}
\]
The covariance matrix of the random vector \( [\mathbf{u}^T \mathbf{v}^T]^T \) is
\[
\begin{bmatrix}
\Phi & 0 \\
0 & \Omega
\end{bmatrix}.
\]
The least square estimate of \( \mathbf{\mu} \) is
\[
\mathbf{\hat{\mu}} = \begin{bmatrix} X & \mathbf{P} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\
0 & \Omega
\end{bmatrix}^{-1} \begin{bmatrix} X^T \\
\mathbf{P}^T
\end{bmatrix}^{-1} \begin{bmatrix} X^T & \mathbf{P}^T \end{bmatrix} \begin{bmatrix} \Phi & 0 \\
0 & \Omega
\end{bmatrix}^{-1} \begin{bmatrix} \Pi \\
\mathbf{q}
\end{bmatrix}.
\]
In ordinary notation, this estimate gives
\[ \hat{\mu} = [X\Phi^{-1}X^T + P\Omega^{-1}P^T]^{-1}[X^T\Phi^{-1}(X\mu + u) + P^T\Omega^{-1}(P\mu + v)]. \] (3.4)

Since \( X \) is the identity matrix and \( \Phi = \tau\Sigma \), we get the same equation as the first one in (3.2). Next, we prove the second equation in (3.2), we substitute (3.3) to (3.4). This gives
\[
\hat{\mu} = [X\Phi^{-1}X^T + P\Omega^{-1}P^T]^{-1}[X^T\Phi^{-1}(X\mu + u) + P^T\Omega^{-1}(P\mu + v)]
= [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}[(\tau\Sigma)^{-1}(\mu + u) + P^T\Omega^{-1}(P\mu + v)]
= [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}[(\tau\Sigma)^{-1}\mu + (\tau\Sigma)^{-1}u + P^T\Omega^{-1}P\mu + P^T\Omega^{-1}v]
= [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}[(\tau\Sigma)^{-1}\mu + P^T\Omega^{-1}P\mu]
+ [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}[(\tau\Sigma)^{-1}u + P^T\Omega^{-1}v]
= \mu + [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}[(\tau\Sigma)^{-1}u + P^T\Omega^{-1}v].
\]

It follows that
\[ \hat{\mu} - \mu = [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}[(\tau\Sigma)^{-1}u + P^T\Omega^{-1}v]. \]

The covariance matrix of the left hand side is
\[
\hat{\Sigma} = \mathbb{E}[(\hat{\mu} - \mu)(\hat{\mu} - \mu)^T]
= [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}\mathbb{E}[(\tau\Sigma)^{-1}u + P^T\Omega^{-1}v][((\tau\Sigma)^{-1}u + P^T\Omega^{-1}v)^T]
\times [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}
\]
and the expected value inside the right hand side is
\[
\mathbb{E}[(\tau\Sigma)^{-1}u + P^T\Omega^{-1}v][((\tau\Sigma)^{-1}u + P^T\Omega^{-1}v)^T] = (\tau\Sigma)^{-1}\mathbb{E}[uu^T](\tau\Sigma)^{-1}
+ (\tau\Sigma)^{-1}\mathbb{E}[uv^T]\Omega^{-1}P + P^T\Omega^{-1}\mathbb{E}[vu^T] + P^T\Omega^{-1}\mathbb{E}[vv^T]\Omega^{-1}P.
\]

The expected values of residuals are
\[ \mathbb{E}[uu^T] = \tau\Sigma, \quad \mathbb{E}[uv^T] = \mathbb{E}[vu^T] = 0, \quad \mathbb{E}[vv^T] = \Omega. \]

Then
\[
\mathbb{E}[(\tau\Sigma)^{-1}u + P^T\Omega^{-1}v][((\tau\Sigma)^{-1}u + P^T\Omega^{-1}v)^T] = (\tau\Sigma)^{-1}\tau\Sigma(\tau\Sigma)^{-1}
+ P^T\Omega^{-1}\Omega\Omega^{-1}P = (\tau\Sigma)^{-1} + P^T\Omega^{-1}P.
\]
\[ \hat{\Sigma} = [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}[(\tau\Sigma)^{-1} + P\Omega^{-1}P^T][((\tau\Sigma)^{-1} + P\Omega^{-1}P^T)^{-1}]
= [(\tau\Sigma)^{-1} + P\Omega^{-1}P^T]^{-1}. \]
3.4 Limitations

Black–Litterman model has obtained increasing popularity since its first publication. Black and Litterman saw two strengths in their model - investor views can be easily added to the portfolio optimization process and B-L model does give more reasonable portfolios compared to standard mean-variance optimization [2].

Over these years some researchers and managers have brought out some possible misuses and problems of the model. For instance, if we consider the active portfolio management where the main objective is to maximize active alpha for the same level of active risk, then the difference from the mean-variance portfolio efficiency (B-L is derived under this) may result in unintentional trades which may cause losses [4].

As for any model, all the assumptions that the models have make it more sensitive, e.g. the assumption of independence of views. Moreover, since investor includes his/her views, it is not necessarily the best possible portfolio, but the best portfolio given the views.
In this chapter we present the main findings of the constructed portfolios and discuss the obtained results.

4.1 Markowitz Mean-Variance Portfolio

The mean-variance portfolio was constructed in Microsoft Excel using the matrix derivations from Chapter 2. All calculations are done on monthly data and the results are presented also on monthly frequency. Figure 4.1 illustrates the efficient frontier and underlying assets. We can see that one asset (FING.ST) lies very much apart from the others.

Figure 4.1: Efficient Frontier and Underlying Assets
This is because of the very volatile nature of this stock. It had very small stock value (and return) in the beginning of our sample period, but the company experienced very large stock price change starting from 2015. Therefore, much larger expected return, but also very high volatility.

Global minimum variance, tangency, and portfolio with equal weights were calculated using the derivations from Chapter 2. The Table 4.1 presents the results of expected return and standard deviation (risk) computed on monthly data over 10-year period.

Table 4.1: Mean-Variance Portfolios (on Monthly Data)

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Expected Return</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMV</td>
<td>0.3144%</td>
<td>3.0372%</td>
</tr>
<tr>
<td>Equally weighted</td>
<td>0.8665%</td>
<td>5.2577%</td>
</tr>
<tr>
<td>Tangency Portfolio</td>
<td>5.1581%</td>
<td>10.4141%</td>
</tr>
</tbody>
</table>

As we can observe, the GMV portfolio has very low expected return with low risk. In equally weighted portfolio we do not have the short selling, i.e. all assets get the weight $1/N$ which is in our analysis is $1/29$. This portfolio does not have a good performance either. The tangency portfolio gives us 5.1581% of return with 10.4141% of risk.

Figure 4.2: Portfolio Weights of GMV and Optimal Portfolio
When calculating the optimal portfolio, we take into account the risk-free asset which in our case is a 1-month Swedish Treasury bill, $r_f$, with average rate of 0.8232%. The portfolio weights of GMV and equally weighted do not have the short selling constraint, but when calculating the tangency portfolio, we used the Lintner short-selling definition\(^1\).

The Figure 4.2 shows the clearly one flaw of mean-variance optimization, it recommends to heavily short-sell some stocks and buy the others. One of the extreme short-selling suggestions is for TELIA.ST, followed by NDA.ST, ATCO.ST and SHBa.ST. get a strong buy recommendation.

The Figure 4.3 presents all the assets individually on the return-standard deviation space with efficient frontier, capital market line (CML), and portfolios mentioned before.

We can see a quite steep efficient frontier, that means that the investor does not have to take on much risk to increase the expected return. On the other hand, if the efficient frontier would be flatter, then an investor should take a lot of risk in order to have more return\([8]\).

---

\(^1\)Short sales are a use of investor’s funds, but investor gets the riskless rate of the funds that are used in short sales\([5]\)
4.2 Black-Litterman Portfolio

As mentioned before, the main difference between Markowitz mean-variance and Black-Litterman portfolio construction is that the investor can express his/her views into the optimized portfolio. We are going to apply two different views and calculate expected return and risk for both portfolios.

Before we do that, we discuss some important parts in B-L portfolio model that we have to use in computation part. The Black-Litterman formula includes the parameter $\tau$ that can be confusing. There has been several discussions how to calibrate it. Charlotta Mankert in her paper [10] proposes that $\tau$ should be calibrated by using Maximum Likelihood Estimator (MLE). Then $\tau = \frac{1}{T}$ is the biased MLE estimator and $\tau = \frac{1}{T - N}$ is the unbiased MLE estimator, where $T$ is sample size and $N$ is the number of assets included in portfolio analysis (in this paper it would be 29). In this work we use the $\tau = 1$ as Satchell and Scowcroft suggested in their paper [13]. We also need risk aversion coefficient. This is connected to tangency portfolio and is calculated by $\delta = \frac{(R_T - r_f)}{\sigma_T}$ and is $\delta = 3.997$.

The first views that are implied are rather strong and are as follows $^2$:

- ATCOa.ST will outperform TELIA.ST by 5%.
- SCABBb.ST will outperform ABB.ST by 2%.
- SHBa.ST will outperform NDA.ST by 3%.

The second set of views are as follows:

- ALIVsdb.ST will outperform AZN.ST by 1%.
- FINGb.ST will outperform ERICb.ST by 1.5%.
- VOLVb.ST will outperform KINVb.ST by 2.5%.

In other words, let’s present these views in matrix form. The matrix $P$ contains views given above,

$$P = \begin{bmatrix}
P_{1,1} & \cdots & P_{1,N} \\
\vdots & \ddots & \vdots \\
P_{k,1} & \cdots & P_{k,N}
\end{bmatrix}$$

where $k$ is number of different views, i.e. $k = 3$ and $N$ is number of assets in portfolio construction, i.e. $N = 29$. The vector $q$ is the vector of expected returns with investor views,

$$q = \begin{bmatrix}
q_1 \\
\vdots \\
q_k
\end{bmatrix}$$

$^2$Views are selected randomly.
In other words, vector $q$ presents the percentages from the given views. The last part we need for portfolio construction with the B-L model (3.2) in Chapter 3.3 is $\Omega$.

$$\Omega = P(\tau \Sigma)P^T \tag{4.1}$$

Since we have $\tau = 1$ the (4.1) becomes,

$$\Omega = P\Sigma P^T \tag{4.2}$$

The Figure 4.4 presents the weights of Black-Litterman portfolios with different views. We can observe that the B-L portfolio 1 with first set of views has very large volume in those assets we chose into our view matrix. This is expected since we chose $\tau = 1$ that means that we did not apply uncertainty coefficient. Moreover, the randomly selected percentages are rather larger for being sure of one asset’s outperforming nature of other asset. When observing the weights of B-L portfolio 2, we can see that the larger percentage we have applied, the more it differentiates from previous weights. The model is sensitive to inputs. When investor does not have views, then the equilibrium portfolio is selected.

Using these weights, we calculate the Black-Litterman portfolio’s risk and return that are presented in Table 4.2. The stronger the investor views are, the more the portfolio’s outcome is affected. The difference in expected returns is very large.

![Figure 4.4: Portfolio Weights of Black-Litterman Optimization](image-url)
Table 4.2: Black-Litterman Portfolio (on Monthly Data)

<table>
<thead>
<tr>
<th></th>
<th>B-L Portfolio 1</th>
<th>B-L Portfolio 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Return</td>
<td>9.0328%</td>
<td>1.6335%</td>
</tr>
<tr>
<td>Risk</td>
<td>11.0652%</td>
<td>5.5374%</td>
</tr>
</tbody>
</table>

4.3 Comparison

Table 4.3 presents all the performances of different portfolios. We can observe very diverse results. Global minimum variance portfolio has very low return, only 0.3144%. Sharpe Ratio expresses the risk-return relationship. GMV portfolio has the lowest ratio from all the portfolios. Tangency portfolio has the highest Sharpe ratio out of Markowitz mean-variance portfolio optimization. B-L portfolio 1 outperforms all other portfolios, but as mentioned before the views incorporated into the modelling are rather too strong, especially considering that the uncertainty coefficient $\tau$ was equal to 1. B-L portfolio 2 is performing worse than tangency portfolio. One of the reasons could be that we used views on stocks that historically were not performing that well.

This once again highlights the sensitivity of the model. In Markowitz mean-variance model the small changes in historical returns can give very different portfolio composition and therefore, changing the expected rate of return and risk.

Table 4.3: Different Portfolios (on Monthly Data)

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Expected Return</th>
<th>Risk</th>
<th>Sharpe Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMV</td>
<td>0.3144%</td>
<td>3.0372%</td>
<td>0.1035</td>
</tr>
<tr>
<td>Equally Weighted</td>
<td>0.8665%</td>
<td>5.2577%</td>
<td>0.1648</td>
</tr>
<tr>
<td>Tangency Portfolio</td>
<td>5.1581%</td>
<td>10.4141%</td>
<td>0.4953</td>
</tr>
<tr>
<td>Black-Litterman 1</td>
<td>9.0328%</td>
<td>11.0652%</td>
<td>0.8163</td>
</tr>
<tr>
<td>Black-Litterman 2</td>
<td>1.6335%</td>
<td>5.5374%</td>
<td>0.2950</td>
</tr>
</tbody>
</table>
On Figure 4.5 we can see that the Black-Litterman portfolio 1 is outside of the efficient frontier. This means that we have reached the point which was unattainable before. By accepting a bit more of risk, B-L portfolio 1 has much higher expected rate of return. However, the weights consists of heavy short-selling or buying recommendations. The Black-Litterman portfolio 2 is inside of efficient frontier. Although the portfolio performs better than underlying assets or equally weighted portfolio, it is not efficient. Although B-L portfolio 1 can be the most desirable out of others, an investor should be cautious because it is easy to be overconfident.

![Figure 4.5: Different Portfolios](image-url)
Chapter 5

Conclusion

In this paper we derived and proved two popular portfolio optimization models and discussed their application. The main difference in these two models is that an investor can express his/her views and these are taken into account.

In Chapter 2 we discussed thoroughly Markowitz mean-variance model. We derived global minimum variance, orthogonal, and tangency portfolio using the Lagrange multipliers and matrices. This laid the base and helped to apply the model in Microsoft Excel. The empirical results clearly demonstrated that while global minimum variance portfolio has very small amount of risk, then tangency portfolio generates significantly higher expected return with a little more risk.

In Chapter 3 we proved Black-Litterman master formula by two different methods, Maximum Likelihood and Theil’s model. In the following chapter we applied real data on the model and showcased that the portfolio outcomes can be very different depending on the inputs. Global minimum variance portfolio had expectedly the least amount of risk and also less expected return compared to other portfolios. Two different portfolios created by Black-Litterman model had contrasting results. When we applied very strong views with uncertainty coefficient $\tau = 1$, the expected return was much higher than the one of tangency portfolio whereas the risk was only slightly larger. The second B-L portfolio performed worse than tangency portfolio.

All in all the we can conclude that although both models are very useful tools for making decisions about portfolio construction, they are very sensitive to changes in inputs (historical returns or views) and can therefore have very different outcomes.

5.1 Further Research Proposal

Since the Markowitz publish his paper in 1952, there have been several papers addressing the flaws and limitations of mean-variance portfolio optimization model. There have been improvements on this model and investors could use the updated Markowitz 2.0 model. It would be interesting to make a comparison between BL and the latter model.

Moreover, it would be interesting to do a research on the different aspects of view vector in B-L model. The investor’s behaviour affects a lot the views he/she may have. Therefore,
the behavioural side of finance is an interesting topic to combine with Black-Litterman model.

## 5.2 Objectives of the Thesis

**Objective 1** *For Bachelor degree, student should demonstrate knowledge and understanding in the major field of study, including knowledge of the field’s scientific basis, knowledge of applicable methods in the field, specialization in some part of the field and orientation in current research questions.*

Author has demonstrated knowledge and understanding in mathematics by applying and presenting detailed mathematical methods and proofs. For instance, proof by Theil’s method and Maximum Likelihood. Moreover, the step by step demonstration of Markowitz mean-variance portfolio optimization is done using the matrices. Two different optimization models are applied to real data using Microsoft Excel. Furthermore, the limitations of Markowitz and Black-Litterman models are discussed.

**Objective 2** *For Bachelor degree, the student should demonstrate the ability to search, collect, evaluate and critically interpret relevant information in a problem formulation and to critically discuss phenomena, problem formulations and situations.*

Author has read numerous academic articles on related topic as well as books and lecture notes from different courses. This had helped to come up with study idea for this work. All the significant part from these sources are interpreted and referred in this thesis. Moreover, real data is used to highlight the theoretical parts.

**Objective 3** *For Bachelor degree, the student should demonstrate the ability to independently identify, formulate and solve problems and to perform tasks within specified time frames.*

Author has gotten the amount of help offered by the course outline. Otherwise the student has demonstrated the ability to identify, formulate, and solve problems within specified time frames.

**Objective 4** *For Bachelor degree, the student should demonstrate the ability to present orally and in writing and discuss information, problems and solutions in dialogue with different groups.*

This objective will be demonstrated on oral presentation that is planned on 1st of June 2018.

**Objective 5** *For Bachelor degree, student should demonstrate ability in the major field of study make judgments with respect to scientific, societal and ethical aspects.*

Author has tried to demonstrate ability in applied mathematics to make judgments with respect to different scientific, societal and ethical aspects.


