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WEAK CONVERGENCE OF FIRST-RARE-EVENT TIMES FOR SEMI-MARKOV PROCESSES

Myroslav Drozdenko

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Myroslav Drozdenko

Akademisk avhandling

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MÄLARDALEN UNIVERSITY
SWEDEN

Institutionen för matematik och fysik

Abstract

In this thesis we study necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes, we describe the class of all possible limit distributions, and give the applications of the results to risk theory and queueing systems.

In paper **A**, we consider first-rare-event times for semi-Markov processes with a finite set of states, and give a summary of our results concerning necessary and sufficient conditions for weak convergence of first-rare-event times and their actuarial applications.

In paper **B**, we present in detail results announced in paper **A** as well as their proofs. We give necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes with a finite set of states in non-triangular-array mode and describe the class of all possible limit distributions in terms of their Laplace transforms.

In paper **C**, we study the conditions for weak convergence for flows of rare events for semi-Markov processes with a finite set of states in non-triangular array mode. We formulate necessary and sufficient conditions of convergence and describe the class of all possible limit stochastic flows. In the second part of the paper, we apply our results to the asymptotical analysis of non-ruin probabilities for perturbed risk processes.

In paper **D**, we give necessary and sufficient conditions for the weak convergence of first-rare-event times for semi-Markov processes with a finite set of states in triangular array mode as well as describing the class of all possible limit distributions. The results of paper **D** extend results obtained in paper **B** to a general triangular array mode.

In paper **E**, we give the necessary and sufficient conditions for weak convergence for the flows of rare events for semi-Markov processes with a finite set of states in triangular array case. This paper generalizes results obtained in paper **C** to a general triangular array mode. In the second part of the paper, we present applications of our results to asymptotical problems of perturbed risk processes and to queueing systems with quick service

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TO MY RELATIVES

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Västerås, November 2007
Myroslav Drozdenko

List of papers

Thesis includes the following articles and internal reports:

Paper A: Silvestrov, D.S., Drozdenko, M.O. (2005) Necessary and sufficient conditions for weak convergence of the first-rare-event times for semi-Markov processes. *Dopov. Nats. Akad. Nauk Ukr.* Mat. Prirodozn. Tekh. Nauki, no. 11, 25–28.

Paper B: Silvestrov, D.S., Drozdenko M.O. (2006) Necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes. I, *Theory of Stochastic Processes*, **12(28)**, no. 3–4, 151–186.

Paper C: Silvestrov, D.S., Drozdenko, M.O. (2006) Necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes. II, *Theory of Stochastic Processes*. **12(28)**, no. 3–4, 187–202.

Paper D: Drozdenko, M. (2007) Weak convergence of first-rare-event times for semi-Markov processes. I. Research Report 2007-1, Department of Mathematics and Physics, Mälardalen University, 40 pages.

Paper E: Drozdenko, M. (2007) Weak convergence of first-rare-event times for semi-Markov processes. II. Research Report 2007-2, Department of Mathematics and Physics, Mälardalen University, 29 pages.

Other publications:

- Silvestrov, D., Drozdenko, M. (2005) Necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes with applications to risk theory. Research Report 2005-2, Department of Mathematics and Physics, Mälardalen University, 59 pages, (A report version of papers **B** and **C**).
- Drozdenko, M.O., Zhegrij, T.I., Yadrenko, M.I. (2001) On some generalizations of mixtures of exponential distributions. *Teor. Imovir. Mat. Stat.*, **65**, 39–45 (English translation in *Theory Probab. Math. Statist.*, **65**, (2002), 45–52).

Presentations

Parts of this thesis were presented by the author at the following conferences and workshops:

1. 4th Conference on Actuarial Science and Finance,
   Karlovassi, Samos, Greece, September 11–20, 2006 (communication);
2. International Conference Modern Stochastics: Theory and Applications,
   Kiev, Ukraine, June 19–23, 2006 (communication);
3. Workshop on Actuarial and Financial Mathematics,
   Västerås – Stockholm, Sweden, October 26–27, 2005 (communication);
4. 25th European Meeting of Statisticians,
   Oslo, Norway, July 24–28, 2005 (poster presentation);
5. 6th World Congress of Bernoulli Society,
   Barcelona, Spain, July 26–31, 2004 (communication);
6. Conference Dedicated to the 100th Anniversary of A.N. Kolmogorov,
   Moscow, Russia, June 16–21, 2003 (poster presentation);
7. Conference Dedicated to the 90th Anniversary of B.V. Gnedenko,
   Kiev, Ukraine, June 3–7, 2002 (communication);
8. 4th International School on Mathematical and Statistical Applications in Economics,
   Västerås, Sweden, January 15–19, 2001 (communication).

Abstract

In this thesis we study necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes, we describe the class of all possible limit distributions, and give the applications of the results to risk theory and queueing systems.

In the introduction, we give basic definitions and describe the models which are studied in the thesis as well as giving examples of applied models where the techniques of first-rare-event times can be used. We also give a survey of publications which contain results closely related to the results of the thesis.

In paper **A**, we consider first-rare-event times for semi-Markov processes with a finite set of states, and give a summary of our results concerning necessary and sufficient conditions for weak convergence of first-rare-event times and their actuarial applications.

In paper **B**, we present in detail results announced in paper **A** as well as their proofs. We give necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes with a finite set of states in non-triangular-array mode and describe the class of all possible limit distributions in terms of their Laplace transforms.

In paper **C**, we study the conditions for weak convergence for flows of rare events for semi-Markov processes with a finite set of states in non-triangular array mode. We formulate necessary and sufficient conditions of convergence and describe the class of all possible limit stochastic flows. In the second part of the paper, we apply our results to the asymptotical analysis of non-ruin probabilities for perturbed risk processes in the case of asymptotically large values of initial capitals and, simultaneously, values of safety loading coefficients which are close to critical value 1.

In paper **D**, we give necessary and sufficient conditions for the weak convergence of first-rare-event times for semi-Markov processes with a finite set of states in triangular array mode as well as describing the class of all possible limit distributions. The results of paper **D** extend results obtained in paper **B** to a general triangular array mode.

In paper **E**, we give the necessary and sufficient conditions for weak convergence for the flows of rare events for semi-Markov processes with a finite set of states in triangular array case. This paper generalizes results obtained in paper **C** to a general triangular array mode. In the second part of the paper, we present applications of our results to asymptotical problems of perturbed risk processes and to queueing systems with quick service.

Introduction

1 Asymptotics of fist-rare-event times

In this section we introduce the subject of our research to the reader, give definitions and explanations of main notions as well as giving examples that indicate possible areas of application of our results.

1.1 Formulation of the problem

First-rare-event times often appear in probabilistic research under such names as life time, ruin time, stopping time, death time, catastrophic time, extension time, epidemic time, record time, failure time, killing time, passage time, hitting time, absorption time, transition time, etc.

The notion of first-rare-event times is usually used to describe the first occurrence of events which have small probability of appearance. One of the fundamental questions is a question of the asymptotic analysis of distributions of first-rare-event times taking into account possible changes of model characteristics. Semi-Markov processes are often used for this kind of modeling.

A semi-Markov process with finite phase space can be described with the use of a two-component Markov chain $(\eta_n, \varkappa_n)$, $n = 0, 1, 2, \cdots$, with the phase space $X \times [0, +\infty)$, where $X = \{1, 2, \cdots, m\}$, and transition probabilities

$$\begin{aligned} &\mathsf{P}\{\eta_{n+1} = j, \varkappa_{n+1} \le t / \eta_n = i, \varkappa_n = s\} \\ &\quad = \mathsf{P}\{\eta_{n+1} = j, \varkappa_{n+1} \le t / \eta_n = i\} \\ &\quad = Q_{ij}(t),\ i, j \in X,\ s, t \ge 0. \end{aligned} \tag{1}$$

The characteristic property of this Markov chain, which is usually referred as a Markov renewal process, is that its transition probabilities depend only on the values of the first component η_n. In this case, the first component η_n is itself a homogeneous Markov chain which is referred as the embedded Markov chain.

Using the two-component process $(\eta_n, \varkappa_n)$, $n = 0, 1, 2, \cdots$, the semi-Markov process $\eta(t)$ can be defined in the following way

$$\eta(t) = \eta_n \quad \text{for} \quad \tau_n \le t < \tau_{n+1}, \tag{2}$$

where

$$\tau_0 = 0, \quad \text{and} \quad \tau_n = \varkappa_1 + \cdots + \varkappa_n, \quad n = 1, 2, \cdots .$$

Figure 1 illustrates the structure of the trajectories for the semi-Markov process $\eta(t)$ and its embedded Markov chain η_n.

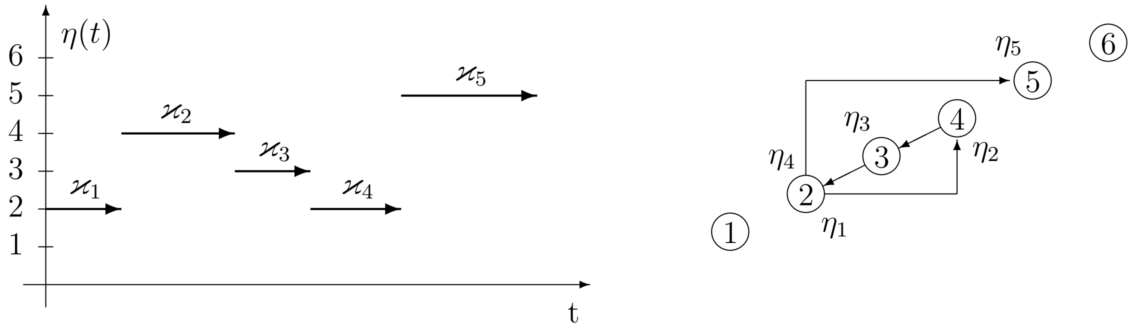

Figure 1: Semi-Markov process $\eta(t)$ and its embedded Markov chain η_n.

To model rare events related to the semi-Markov process $\eta(t)$, the third component ζ_n, which is often referred as the flag variable, is added to the two-component process $(\eta_n, \varkappa_n)$, $n = 0, 1, \cdots$. Thus, we introduce a three component Markov chain $(\eta_n, \varkappa_n, \zeta_n)$, $n = 0, 1, \cdots$ with phase space $X \times [0, +\infty) \times Y$, (here Y is some set) and transition probabilities

$$\begin{aligned} &\mathsf{P}\{\eta_{n+1} = j, \varkappa_{n+1} \le t, \zeta_{n+1} \in A/\eta_n = i, \varkappa_n = s, \zeta_n = y\} \\ &= \mathsf{P}\{\eta_{n+1} = j, \varkappa_{n+1} \le t, \zeta_{n+1} \in A/\eta_n = i\} \\ &= Q_{ij}(t, A),\ i, j \in X,\ y \in Y, A \subseteq Y\ s, t \ge 0. \end{aligned} \tag{3}$$

Let also D_ε be some measurable subsets from Y depending on the parameter $\varepsilon > 0$. Then, the events $\{\zeta_n \in D_\varepsilon\}$ can be considered as asymptotically rare if the following condition holds

$$0 < \max_{i \in X} \mathsf{P}\{\zeta_1 \in D_\varepsilon | \eta_0 = i\} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4}$$

The object of our studies is the time of running of the semi-Markov process $\eta(t)$ until the first appearance of the rare event, or in other words, the *first-rare-event time* ξ_ε, which can be formally defined as:

$$\xi_\varepsilon = \sum_{k=1}^{\nu_\varepsilon} \varkappa_n, \quad \text{where} \quad \nu_\varepsilon = \min\{n \ge 1 : \zeta_n \in D_\varepsilon\}. \tag{5}$$

Figure 2 gives an illustration of the first-rare-event time for a semi-Markov process. In this figure, flag "Go" is identified with the event $\{\zeta_n \notin D_\varepsilon\}$, while flag "Stop" is identified with the rare event $\{\zeta_n \in D_\varepsilon\}$.

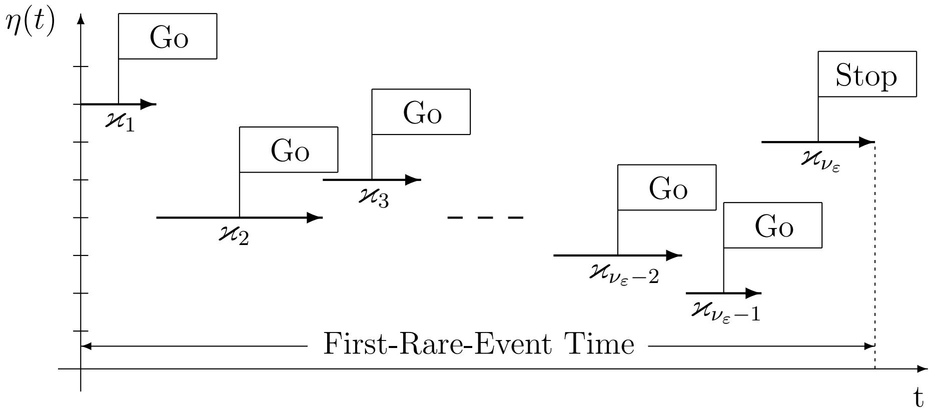

Figure 2: First-rare-event time for a semi-Markov process.

A natural way to analyze the random variable ξ_ε is to look at its conditional distribution functions

$$F_i^{(\varepsilon)}(x) = \mathsf{P}\{\xi_\varepsilon \le x \,|\, \eta_0 = i\}, \quad i \in X. \tag{6}$$

Due to relation (4), the first-rare-event time ξ_ε tends in probability to $+\infty$ as $\varepsilon \to 0$. That is why, instead of analyzing ξ_ε, it is often more suitable to look at the normalized first-rare-event time $\xi_\varepsilon/u_\varepsilon$ under some balance conditions for the speed of convergence to zero of the probability $\mathsf{P}\{\zeta_1 \in D_\varepsilon | \eta_0 = i\}$, $i \in X$ and the speed of growing to $+\infty$ for some normalizing function $0 < u_\varepsilon \to \infty$ as $\varepsilon \to 0$. The distribution function of the normalized random variable $\xi_\varepsilon/u_\varepsilon$ obviously has the following form $F_i^{(\varepsilon)}(xu_\varepsilon) = \mathsf{P}\{\xi_\varepsilon/u_\varepsilon \le x \,|\, \eta_0 = i\}$.

The main problem studied in the thesis is the problem of finding conditions for weak convergence of the distribution functions $F_i^{(\varepsilon)}(xu_\varepsilon)$ to some limit distribution function $F(x)$ as $\varepsilon \to 0$, i.e. conditions of pointwise convergence of $F_i^{(\varepsilon)}(xu_\varepsilon)$ to $F(x)$ as $\varepsilon \to 0$ for all points of continuity of the limit distribution function.

In the papers **A** and **B** we describe the class of all possible limit distributions and give necessary and sufficient conditions for the weak convergence of the distribution $F_i^{(\varepsilon)}(xu_\varepsilon)$ to a given distribution from this class.

Another problem which is studied in the thesis is connected with the convergence of the flows of "rare" events described by the following counting process

$$N_\varepsilon(tu_\varepsilon) = \max(n : \sum_{k=1}^{n} \varkappa_\varepsilon(k) \leq tu_\varepsilon), \quad t \geq 0, \tag{7}$$

where

$$\varkappa_\varepsilon(k) = \sum_{n=\nu_\varepsilon(k-1)+1}^{\nu_\varepsilon(k)} \varkappa_k \quad k = 1, 2, \cdots$$

and

$$\nu_\varepsilon(0) = 0, \quad \nu_\varepsilon(k) = \min\{n > \nu_\varepsilon(k-1) : \ \zeta_n \in D_\varepsilon\}.$$

In paper **C**, we describe the class of all limit (in the sense of weak convergence) flows of rare events and give necessary and sufficient conditions of convergence to a given flow from this class.

The counting process defined in (7) can be linked, in the obvious way, with the concept of thinning of stochastic flow, when some events, that have occurred, are removed from the register that may happen due to the selection of the occurred events.

A graphical illustration for the flow or rare events described above is given in Figure 3.

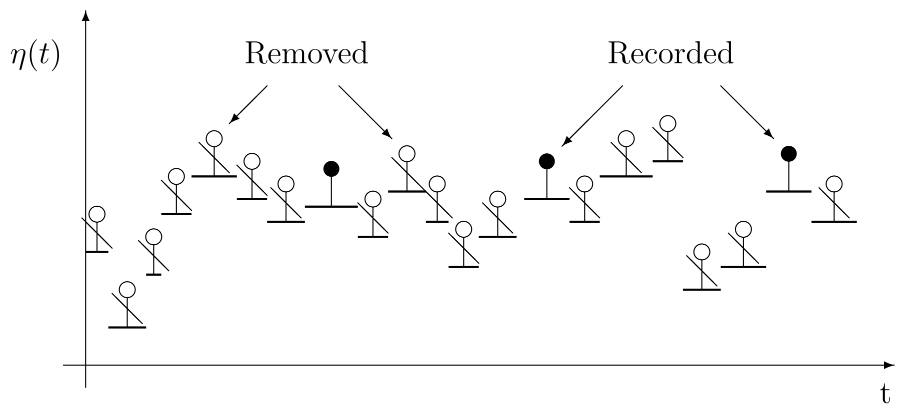

Figure 3: The flow of rare events.

A usual assumption for the models based on semi-Markov processes $\eta(t)$, is that the embedded Markov chain η_n, $n \geq 0$ is ergodic. This makes possible a formulation of conditions for weak convergence of first-rare-event times in terms of characteristics of the semi-Markov process $\eta(t)$ averaged by the stationary distribution π_i of the embedded

Markov chain η_n. This approach allows us to get necessary and sufficient conditions for weak convergence of first-rare-event times weaker than usual sufficient conditions obtained in preceding works.

The advantage of an approach based on the use of averaged characteristics for the semi-Markov process with respect to the stationary distribution of embedded Markov chain is commented on and several examples are given in paper **E** of the present thesis.

In many applications, one should study models where all three components of the Markov renewal process $(\eta_n, \varkappa_n, \zeta_n)$, $n = 0, 1, 2, \cdots$, depend on a perturbation parameter ε. In this case, one may consider a Markov chain $(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)}, \zeta_n^{(\varepsilon)})$, $n = 0, 1, 2, \cdots$. with a phase space $X \times [0, +\infty) \times Y$ (here $X = \{1, \cdots, m\}$, and Y is some set) and transition probabilities

$$\begin{aligned} &\mathsf{P}\left\{\eta_{n+1}^{(\varepsilon)} = j, \varkappa_{n+1}^{(\varepsilon)} \le t, \zeta_{n+1}^{(\varepsilon)} \in A / \eta_n^{(\varepsilon)} = i, \varkappa_n^{(\varepsilon)} = s, \zeta_n^{(\varepsilon)} = y\right\} \\ &\quad = \mathsf{P}\left\{\eta_{n+1}^{(\varepsilon)} = j, \varkappa_{n+1}^{(\varepsilon)} \le t, \zeta_{n+1}^{(\varepsilon)} \in A / \eta_n^{(\varepsilon)} = i\right\} \\ &\quad = Q_{ij}^{(\varepsilon)}(t, A),\ i, j \in X,\ y \in Y,\ A \subseteq Y,\ s, t \ge 0. \end{aligned} \tag{8}$$

Analogously to the model described above, we define the first-rare-event time

$$\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}, \quad \text{where} \quad \nu_\varepsilon = \min\left\{n \ge 1 \,:\, \zeta_n^{(\varepsilon)} \in D_\varepsilon\right\}. \tag{9}$$

In a similar way to those described above, we also introduce the flows of rare events for perturbed semi-Markov processes.

In paper **D** we give necessary and sufficient conditions of weak convergence for the distribution function of first-rare-event times defined in (9), subject to normalization, as well as describe the class of all possible limit distributions.

The first-rare-event times for stochastic processes have many applications in queueing and reliability theories, risk processes etc.

In paper **E** we give necessary and sufficient conditions for the weak convergence of flows of rare events for perturbed semi-Markov processes. We also present applications for asymptotic analysis of non-ruin probabilities for perturbed risk processes and asymptotic analysis of distributions of failure times in queueing systems with quick service.

1.2 Examples

In this section we present several examples of models of stochastic systems where first-rare-event times may be applied.

Hitting type problems constitute a class of problems where first-rare-event times appear in a natural way. These deal with times of crossing of certain levels by stochastic processes.

As an example of insurance applications, we demonstrate possible applications of our results to the stochastic analysis of ruin probabilities for an insurance company. We consider the *classical risk model*, which was first studied thoroughly in Lundberg (1903), and which is still an object of intense research study. The model is based on the following assumptions: an insurance company starts its activity with an initial capital u; premiums come to the company with a constant intensity c_ε (in our case, dependent on the perturbation parameter $\varepsilon > 0$), i.e., the amount of premiums collected by the company on the time interval $[0, t]$ is equal to $c_\varepsilon t$; arrivals of claims are modeled by the Poisson process $N(t)$, $t \geq 0$ with intensity λ; all claims Z_k, $k = 1, 2, \cdots$ are independent and identically distributed non-negative random variables with distribution function $H(x)$ which have a finite expectation μ; the claims Z_k, $k = 1, 2, \cdots$, and the Poisson counting process $N(t)$ are mutually independent. The capital of the company at time t can be represented as

$$U_\varepsilon(t) = u + c_\varepsilon t - \sum_{k=1}^{N(t)} Z_k, \quad t \geq 0.$$

The structure of the classical risk process $U_\varepsilon(t)$ is shown in Figure 4.

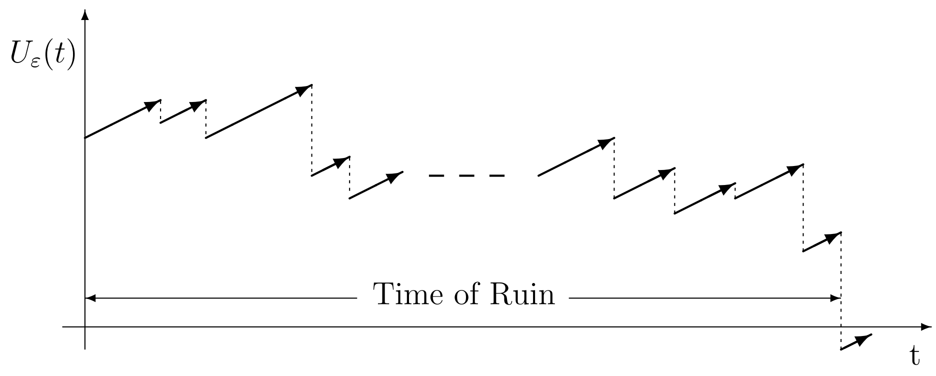

Figure 4: The classical risk process.

One of the important quantities in the risk theory is the *probability of ruin* on the infinite time interval

$$\psi_\varepsilon(u) = \mathsf{P}\left\{U_\varepsilon(t) \le 0, \text{ for some } t > 0\right\}.$$

The stochastic dynamics of the risk process is determined by the *safety loading coefficient* which is defined as $\alpha_\varepsilon = \lambda\mu/c_\varepsilon > 0$, for $\varepsilon > 0$. It is supposed to be strictly less than 1, otherwise the risk process goes to ruin with probability 1 whatever is the value u of the initial capital.

The non-ruin probability, namely $\overline{\psi}_\varepsilon(u) = 1 - \psi_\varepsilon(u)$, satisfies all the requirements for being a distribution function. Due to the well-known Pollaczek–Khinchine formula, it can be represented as a distribution function of the geometric random sum:

$$\overline{\psi}_\varepsilon(u) = \mathsf{P}\{\sum_{n=1}^{\nu_\varepsilon - 1} \varkappa_n \le u\}, \quad u > 0, \tag{10}$$

where: $\varkappa_n$, $n = 1, 2, \cdots$ is a sequence of non-negative i.i.d. random variables with distribution function $G(x) = \int_0^x (1 - H(s))ds / \int_0^\infty sH(ds)$, $u > 0$ (it is the so-called steady transformation of the distribution $H(x)$); $\nu_\varepsilon = \min(n \ge 1 : \zeta_n \in D_\varepsilon)$ where ζ_n, $n = 1, 2, \cdots$ are uniformly distributed on $[0, 1]$ random variables; $D_\varepsilon = [0, p_\varepsilon)$; $p_\varepsilon = 1 - \alpha_\varepsilon$; the random variables $\varkappa_n$, $n = 1, 2, \cdots$ and ζ_n, $n = 1, 2, \cdots$ are mutually independent.

First-rare-event time $\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon - 1} \varkappa_n$, defined in (10), is a geometric random sum that can be interpreted as the first-rare-event time for a special semi-Markov process. In this case the leading component η_n of the three component Markov renewal process $(\eta_n, \varkappa_n, \zeta_n)$ is degenerate and has only one possible state and the second component $\varkappa_n$, $n = 1, 2, \cdots$ is independent with the third component ζ_n, $n = 1, 2, \cdots$.

The Pollaczek–Khinchine formula (10) makes possible an application of the results concerning weak convergence of first-rare-event times for semi-Markov processes to the analysis of the asymptotic behaviour of the non-ruin probability $\overline{\psi}_\varepsilon(u)$ in the case of large values of initial capital and, simultaneously, values of safety loading coefficients which are close to critical.

For a more detailed description of applications of first-rare-event times to risk processes, as well as an analysis of weak convergence of non-ruin probabilities $\overline{\psi}_\varepsilon(u)$ and a description of the class of possible limit distributions, see papers **C** and **E** of the thesis. In these papers we give necessary and sufficient conditions for stable and infinitely divisible approximations of non-ruin probabilities.

Let us now give an example of financial applications. Semi-Markov processes can be used to model credit ratings for financial institutions. For example, the Standard & Poor's proposes the following set of rates $X = \{AAA; AA; A; BBB; BB; B; CCC; D; N.R.\}$. The state $N.R.$ is supposed to be absorbing, i.e., a company that enters that state loses the ability to conduct its business. The rate dynamics can be described by the following two-component Markov chain $(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)})$, $n = 0, 1, \cdots$, where $\eta_n^{(\varepsilon)} \in X$ is the rating of the company at the moment of $n-$th change of rating, and $\varkappa_n^{(\varepsilon)} > 0$ is the time between $(n-1)$-th and n-th moment of rating changes. Let us also define the flag variables $\zeta_n^{(\varepsilon)} = \chi(\eta_n^{(\varepsilon)} = N.R.)$, $n = 0, 1, \cdots$. Then, the "lifetime" of the company can be defined as

$$\xi_\varepsilon = \sum_{k=1}^{\nu_\varepsilon - 1} \varkappa_k^{(\varepsilon)}, \qquad \text{where } \nu_\varepsilon = \min\{n \geq 1 : \ \eta_n^{(\varepsilon)} = N.R.\}$$

An illustration of the model is given in the Figure 5.

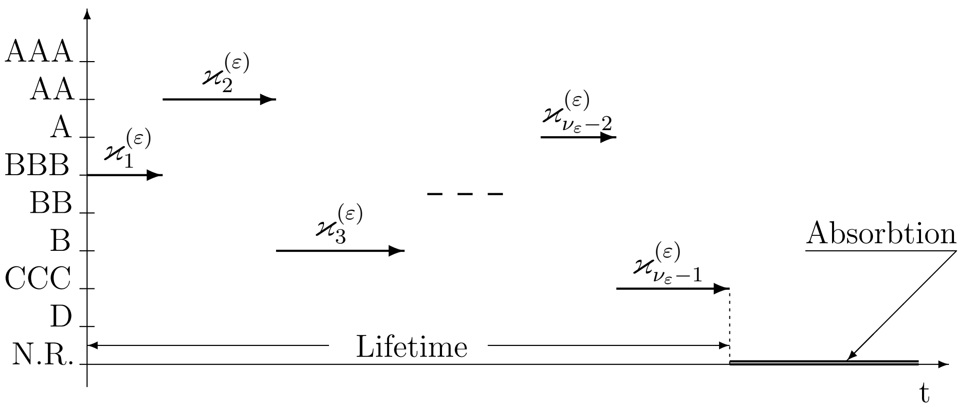

Figure 5: Dynamics of credit rating.

Usually the event $\{\eta_n^{(\varepsilon)} = N.R.\}$ happens rarely, and, hence, the random variable ξ_ε can be interpreted as a first-rare-event time for the corresponding semi-Markov process, and our results can be used.

Other types of models, where hitting times are often in use, are reliability and engineering models. We illustrate this by the example of model of water level dynamics in rivers, lakes and storage pools.

Let the variable $\alpha_\varepsilon(t)$ represent the height of water in a reservoir at the time moment t. One of the models often used in applications is where the process $\alpha_\varepsilon(t)$ is assumed to be a diffusion process.

Let us assume that there exists a maximum possible value $h_{\max}$, and introduce the set of intermediate levels $h_1 = 0 < h_2 < \cdots < h_m = h_{\max}$. For simplicity, let us assume that $\alpha_\varepsilon(0) = h_k$ for some k. Let $0 = \tau_0^{(\varepsilon)} < \tau_1^{(\varepsilon)} < \cdots$ be sequential time moments when the process $\alpha_\varepsilon(t)$ hits some of the levels $h_1, \cdots, h_m$. Let us also define random variables $\eta_n^{(\varepsilon)} = r$ if $\alpha_\varepsilon(\tau_n^{(\varepsilon)}) = r$, and let $\varkappa_n^{(\varepsilon)} = \tau_n^{(\varepsilon)} - \tau_{n-1}^{(\varepsilon)}$. Finally let $D = \{h_m\}$ and the flag variable $\zeta_n^{(\varepsilon)} = \chi(\eta_n^{(\varepsilon)} \in D)$.

Due to the diffusion assumption about the structure of the process $\alpha_\varepsilon(t)$, the triple random sequence $(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)}, \zeta_n^{(\varepsilon)})$, $n = 0, 1, \cdots$ is a renewal Markov process.

When parameters of the water level process $\alpha_\varepsilon(t)$ are such that probabilities of transitions $h_k \to h_{r\pm 1}$ are separated from zero for $r < m-1$ but the probability of transition $h_{m-1} \to h_m = h_{\max}$, is small, then the event $\{\zeta_n^{(\varepsilon)} = 1\}$ can be interpreted as a rare event. In this case

$$\xi_\varepsilon = \inf\{s : \ \alpha_\varepsilon(s) = h_{\max}\} = \sum_{k=1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}, \quad \text{where } \nu_\varepsilon = \min\{n : \ \eta_n^{(\varepsilon)} \in D\}.$$

Now, let us give an example of possible applications of our results in queueing theory. Let us consider the M/G type queueing system with m different types of customers and one server. If the customer of type i is coming to the system, its service time has the distribution function $H_{i\varepsilon}(x)$ and the corresponding interarrival time (difference in arrival moments for the next and the current customers) is exponentially distributed with parameter $\lambda_{i\varepsilon}$. The appearance of a customer of a certain type is modeled by an ergodic Markov chain η_n, with phase space $X = \{1, 2, \cdots, m\}$, and transition probabilities $p_{ij}^{(\varepsilon)}$. The flow of the customers in the system is modeled by the semi-Markov process with transition probabilities:

$$Q_{ij}^{(\varepsilon)}(t) = p_{ij}^{(\varepsilon)} \left(1 - e^{-\lambda_{i\varepsilon} t}\right), \quad t \geq 0, \ i, j \in X.$$

Let random variables $\varkappa'_{n\varepsilon}$ and $\varkappa''_{n\varepsilon}$ be, respectively, interarrival and service time for the n-th customer which are assumed to be independent. The flag variables $\zeta_n^{(\varepsilon)} = \chi(\varkappa'_{n\varepsilon} < \varkappa''_{n\varepsilon})$ are indicators of the event: "the next customer arrives before the service for previous customer is finished".

Then the total working time of the system can be defined as

$$\xi_\varepsilon = \sum_{k=1}^{\nu_\varepsilon} \varkappa'_{k\varepsilon},$$

where,

$$\nu_\varepsilon = \min\left\{n \geq 1 : \chi(\varkappa'_{n\varepsilon} < \varkappa''_{n\varepsilon}) = 1\right\}$$

represents the number of service cycles before a failure in the system occurs.

The illustration of the model is given in the Figure 6.

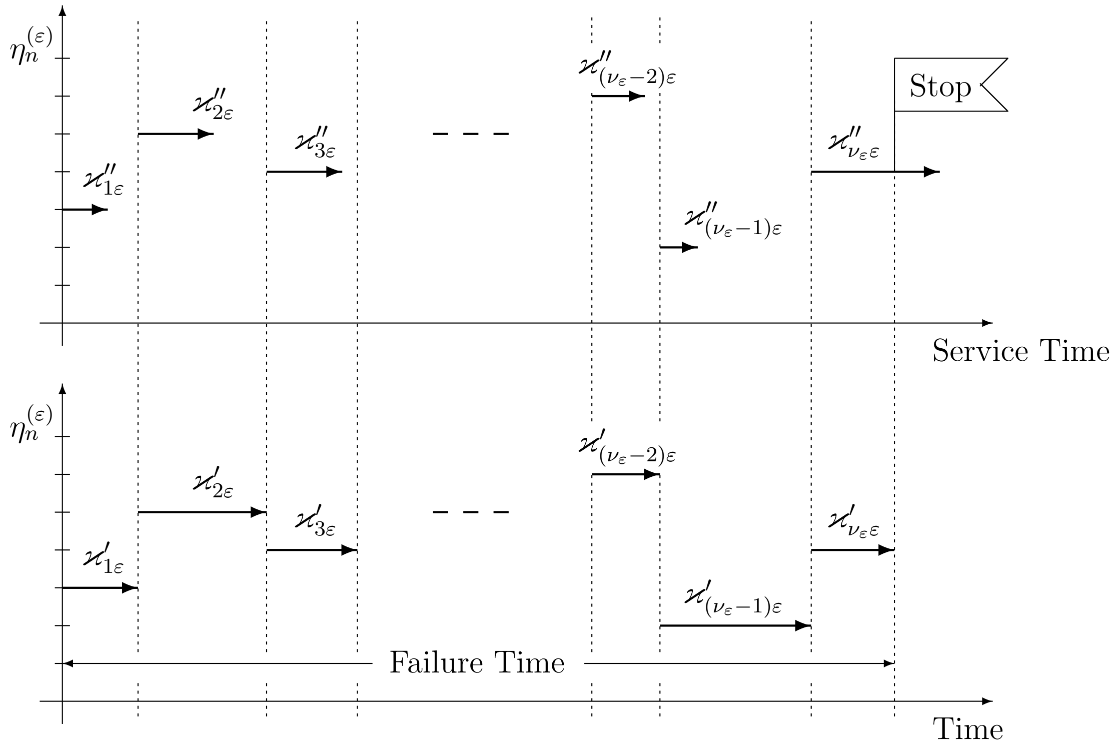

Figure 6: Failure time for the queueing system.

A more detailed description of the model and applications of our results to asymptotic analysis of failure time for the above described queueing system is given in paper **E** of the current thesis.

First-rare-event times are also used in some biological applications. Let us consider a population consisting of m individuals. Number of infected individuals at time t is modeled by a semi-Markov process $\eta^{(\varepsilon)}(t)$, $t \geq 0$ with phase space $X = \{1, \cdots, m\}$. Let $(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)})$, $n = 0, 1, \cdots$, be the corresponding Markov renewal process associated with the process $\eta^{(\varepsilon)}(t)$, i.e. $\eta_n^{(\varepsilon)}$ is a number of infected individuals at sequential moments of transitions for the process $\eta^{(\varepsilon)}(t)$, while $\varkappa_n^{(\varepsilon)}$ are the corresponding occupation times.

In applications, the hitting of some critical domain $D = \{i \in X : i \geq c\}$ by the process $\eta^{(\varepsilon)}(t)$, for some "large" c, is often interpreted as epidemic. In this case, the flag variable can be defined as $\zeta_n^{(\varepsilon)} = \chi(\eta_n^{(\varepsilon)} \in D)$ and the time of appearance of the epidemic in the population can be defined as

$$\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon - 1} \varkappa_n^{(\varepsilon)}, \qquad \text{where } \nu_\varepsilon = \min(n \geq 1 : \ \zeta_n^{(\varepsilon)} \in D).$$

When the probabilities of transition $i \to j$, where $i, j \in X \setminus D$, are separated from zero, and the probabilities of transition $i \to j$, where $i \in X \setminus D$ and $j \in D$ are small, then time ξ_ε can be interpreted as first-rare-event time for the corresponding semi-Markov process. An illustration of the model is given in Figure 7.

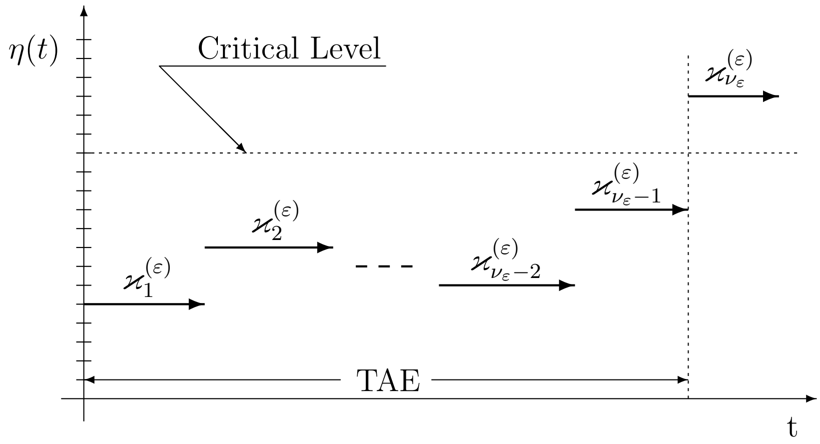

Figure 7: Time of appearance of an epidemic (TAE).

The list of examples presented above can be continued. A detailed analysis of two examples, namely, related to asymptotics of ruin probabilities and lifetimes of queueing systems with quick service is presented in papers **C** and **E**.

2 Review of closely related publications

A detailed survey of the results related to asymptotic problems for first-rare-event times is given in the introductions to papers **B** and **C** of the thesis.

In this section we give an overview of publications containing results directly related to the results presented in the thesis.

First-rare-event times for semi-Markov processes reduce to random geometric sums in the case of the degenerated embedded Markov chain.

In this way, our results are connected with asymptotical results for geometric sums. Here, we would like to mention the original paper by Rényi (1956), who first formulated conditions for the weak convergence of distributions of geometric sums to the exponential law.

The class of possible limit laws for standard geometric random sums, with summands independent of geometric random indexes and possessing regularly varying tail probabilities, was described by Kovalenko (1965), who is also credited with originating necessary and sufficient conditions for weak convergence of such sums. Gnedenko and Fraier (1969) described the domains of attraction for some distributions from the class of limit distributions discovered by Kovalenko (1965). These results were generalized by Kruglov and Korolev (1990), in particular to the case of triangular array mode, and by Solov'ev (1971), who gave general conditions for the weak convergence of distributions of rare-event-times for regenerative processes.

Our results given in papers **C** and **E** yield necessary and sufficient conditions for weak convergence of more general geometric sums with summands that can depend on geometric random indexes via indicators of rare events. Also, our conditions have a different and, in our view, more convenient form than in the works mentioned above.

First-rare-event times for semi-Markov processes are natural generalizations for the model of geometric random sums.

Limit theorems for first-rare-event times, known under such names as first hitting times, first passage times, first record times, etc. for Markov and semi-Markov processes, have been studied by many authors.

Sufficient conditions of convergence to exponential law for distributions of such functionals were first given for finite Markov chains by Korolyuk (1969).

These results were generalized in different ways by Silvestrov (1970, 1971, 1980), Anisimov (1971a, 1971b, 1988), Kovalenko (1973), and Korolyuk and Turbin (1976). The most general conditions for weak convergence in triangular array mode of first-rare-event times for finite semi-Markov processes with ergodic embedded Markov chains were given in Silvestrov (1974).

The main features for the most previous results is that they give only sufficient conditions of convergence for such functionals. As a rule, those conditions involve assumptions, which imply convergence of distributions for sums of i.i.d. random variables distributed as sojourn times for the semi-Markov process (for every state) to some infinitely divisible laws plus some ergodicity condition for the embedded Markov chain plus the condition of vanishing for the probability that a rare event occurs during one transition step for the semi-Markov process.

Our results are related to the model of semi-Markov processes with a finite set of states. In papers **A** and **B**, we consider the case of stable type asymptotics for distributions of sojourn times, while in papers **D**, the case of infinitely divisible asymptotics is treated. Instead of conditions based on "individual" distributions of sojourn times, we use more general and weaker conditions imposed on distributions (of sojourn times) averaged by the stationary distribution of the embedded Markov chain. Moreover, we show that these conditions are not only sufficient but also necessary conditions for weak convergence of first-rare-event times, and describe the class of all possible limiting laws not concentrated in zero. The results presented in the paper give some kind of a "final solution" for limit theorems for first-rare-event times for finite semi-Markov processes with ergodic embedded Markov chains.

In the case of semi-Markov processes with countable and general phase spaces, the majority of known results give sufficient conditions of convergence. The most general results known so far are given in Kaplan (1979), Silvestrov (1974, 1980, 1981), Anisimov (1988, 2007), Kartashov (1996), and Koroliuk and Limnios (2005).

As far as the necessary and sufficient conditions are concerned, we can mention the paper by Silvestrov and Velikii (1988), where such conditions were given for the case of convergence to the exponential law.

The question about general necessary and sufficient conditions of the weak convergence of first-rare-event times for the case of countable and general phase space of semi-Markov processes remains open.

We also apply the results related to necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes given in papers **A**, **B**, **D** to counting processes generated by flows of rare events controlled by semi-Markov processes.

Here we again refer to the paper by Rényi (1956) who first gave conditions for weak convergence of thinned renewal flows of rare events to a Poisson flow. These results were generalized in different ways by Belyaev (1963), Kovalenko (1965), Mogyoródi (1972a, 1972b), Råde (1972a, 1972b), Jagers (1974), and Jagers and Lindvall (1974). In particular, Kovalenko (1965) gave necessary and sufficient conditions for weak convergence of renewal flows and described the class of all possible flows in the case of the stable approximation for interarrival times. Sufficient conditions for the weak convergence of thinned flows of rare events controlled by a finite semi-Markov process were given by Zakusilo (1972a, 1972b), where the conditions were expressed in terms of individual distributions of sojourn times. References to the latest works on thinning of renewal and semi-Markov type flows can be found in Manor (1998), Bening and Korolev (2002), and Anisimov (2007).

In papers **C** and **E**, we describe the class of all possible limit flows and give not only sufficient but necessary and sufficient conditions for the weak convergence of flows of rare events controlled by semi-Markov processes with finite sets of states. The case of stable approximation for sojourn times is treated in paper **C** and the case of infinitely divisible approximation is treated in paper **D**.

As in the case of limit theorems for first-rare-event times, we give the conditions of convergence in terms of distributions of sojourn times averaged by the stationary distribution of the embedded Markov chain.

As we show, limit distributions of the first-rare-event times under introduced assumptions do not depend on the initial state of the embedded Markov chain. This makes our results relevant to the results about so-called solidarity properties for functionals defined on Markov chains, i.e., properties which hold for all states in an irreducible class of a Markov chain if and only if the property holds at list for one state in this class. One of the first solidarity results was given in Chung (1954). Namely it was shown that if the l-th moment of recurrence time for one state is finite, then it is also finite for all other states. Various solidarity theorems were given in Pyke and Schaufele (1964), Teugels (1970), and Silvestrov (1974). A survey of the results related to solidarity properties for Markov chains can be found in Pyke (1999).

In papers **B** and **D**, we give a number of new solidarity results connecting them with asymptotics of averaged sojourn time distributions.

The results obtained in the thesis, can be used for the analysis of diffusion type and heavy tailed approximations for risk processes and queueing systems. The first results on diffusion approximations for risk processes were obtained by Iglehart (1969), who showed (using the fact that risk processes, subjected to the proper normalization, converge to Wiener process with a shift) that non-ruin probability for risk process converges to an exponential distribution as the safety loading coefficient tends to 1 from below and, simultaneously, the initial capital tends to infinity, under some balancing condition between the safety loading coefficient and the initial capital.

Another originating results, which we would like to mention, is due to von Bahr (1975), who derived the asymptotic of non-ruin probability for heavy tail claim distribution in the subcritical case. The results mentioned above were further improved by many authors. The corresponding references can be found in paper **C**. We only refer to the latest works in the area, Schmidli (1992), Embrechts, Klüppelberg, and Mikosch (1997), Asmussen (2000, 2003), Gyllenberg and Silvestrov (2000), Silvestrov (2000), and Kartashov and Stroēv (2005).

In papers **C** and **E**, we illustrate our general results for first-rare-event times by giving necessary and sufficient conditions for stable and infinitely divisible approximations of non-ruin probabilities in the critical case, i.e., where the safety loading coefficient tends to 1 and, simultaneously, the initial capital of an insurance company tends to infinity.

It is not out of place to mention that asymptotical relations related to diffusion approximation for ruin probabilities can also be interpreted in terms of queueing theory as a variant of so-called heavy traffic approximation for waiting times (see, for instance, Asmussen (2003)).

As other possible areas of application, we can point out limit theorems for different lifetime functionals such as occupation times or waiting times in queueing systems, lifetimes in reliability models, extinction times in population dynamic models, etc.

We mention here some initial works by Gnedenko (1964a, 1964b), Gnedenko and Kovalenko (1964), Solov'ev (1983), and the extensive survey by Kovalenko (1994). Additional references can be found in the introduction to paper **C**.

In paper **E**, we give an example, which shows how our results concerning asymptotics of first-rare-event times for semi-Markov processes can yield necessary and sufficient conditions for the weak convergence of lifetimes for queueing systems with quick service.

Finally, we would like to mention that reviews of the latest results related to semi-Markov processes, in particular, to limit theorems for such processes, can be found in the recent books by Silvestrov (2004), Koroliuk and Limnios (2005), Janssen and Manca (2006), Anisimov (2007), and Harlamov (2007).

3 Summary of papers

The thesis is based on the results summarized in the following five papers: **A:** Silvestrov and Drozdenko (2005a), **B:** Silvestrov and Drozdenko (2006a), **C:** Silvestrov and Drozdenko (2006b), **D:** Drozdenko (2007a), **E:** Drozdenko (2007b).

3.1 Paper A

In paper **A**, we consider first-rare-event times for a semi-Markov process with a finite set of states, and give a brief survey of our results related to studies of necessary and sufficient conditions for the weak convergence of first-rare-event times. We describe the class of all possible limit distributions and formulate necessary and sufficient conditions under which first-rare-event times weakly converge to a given distribution from the limit class. Conditions are formulated in terms of the corresponding semi-Markov transition characteristics averaged by the stationary distribution of its embedded Markov chain. Applications for the asymptotic analysis for non-ruin probabilities for the classical risk process are also discussed.

3.2 Paper B

In paper **B**, we present in detail results announced in paper **A**. In Theorem 1 we give necessary and sufficient conditions for the weak convergence of first-rare-event times for a semi-Markov process with a finite set of states. In this paper we consider a non-triangular-array case of stable type asymptotics for sojourn times. Instead of conditions based on "individual" distributions of sojourn times, we use more general and weaker conditions imposed on distributions of sojourn times averaged by the stationary distribution of the embedded Markov chain. Moreover, we show that these conditions are not only sufficient but also necessary for the weak convergence of first-rare-event times. We describe the class of all possible limit distributions as well as comment on the structure of this class. In fact, it is the class of distributions concentrated on the positive half-line with Laplace transform of the form $(1 + as^{\gamma})^{-1}$, where $a > 0$ and $0 < \gamma \leq 1$. The results presented in the paper give some kind of "final solution" for limit theorems for first-rare-event for semi-Markov processes with a finite set of states in non-triangular-array mode. Several lemmas, which describe asymptotic solidarity cyclic properties of semi-Markov processes and are used in the proof of the main Theorem 1, are also interesting in themselves.

3.3 Paper C

In Paper **C**, we study the conditions of weak convergence for flows of rare events in the model considered in paper **B**. We formulate necessary and sufficient conditions of convergence and describe the class of all possible limit stochastic flows.

In the second part of paper **C**, we apply our results to the asymptotical analysis of non-ruin probabilities for risk processes. We study asymptotics for non-ruin probabilities in the critical case when the initial capital of the insurance company tends to infinity and simultaneously the safety loading coefficient tends to 1. Two cases are considered, where the claim distribution belong to the domain of attraction of a degenerated or stable law. The first case corresponds to the classical model of diffusion approximation, the second one can be referred to as the case of stable approximation. We give necessary and sufficient conditions for both diffusion and stable asymptotics.

3.4 Paper D

In paper **D**, we give necessary and sufficient conditions for the weak convergence of first-rare-event times for semi-Markov processes with a finite set of states in triangular array mode. These results generalize the results of paper **B** to the general triangular array case. We also describe the class of all possible limit distributions that is, in fact, the class of distributions concentrated on the non-negative half-line with the Laplace transform of the form $(1 + a(s))^{-1}$, where $a(s)$ is a cumulant of some non-negative infinitely divisible low. The main features of the majority of the previous results are that they give sufficient conditions of convergence for such functionals. As a rule, these conditions involve assumptions, which imply convergence of distributions for sums of i.i.d. random variables distributed as sojourn times for every state of the semi-Markov process to some infinitely divisible laws plus some triangular array ergodicity condition for the embedded Markov chain. Our results are based on conditions for distributions of individual sojourn types averaged by the stationary distribution of the limit embedded Markov chain. As in paper **B**, we also prove several asymptotic solidarity statements which may have research interest of their own.

3.5 Paper E

In paper **E**, we give necessary and sufficient conditions for weak convergence for the flows of rare events in the model described in paper **D**. The results generalize results obtained in paper **C** to a general triangular array mode. A special section of the paper deals with examples and shows the advantages of conditions formulated in terms of characteristics averaged by the stationary distribution of the limit embedded Markov chain compared with conditions imposed on individual sojourn distributions. In the second part of the paper we discuss possible applications of our results to asymptotical problems of perturbed risk processes as well as give application of our results for the asymptotic analysis of queueing systems with quick service.

Sammanfattning på svenska

I denna avhandling studerar vi nödvändiga och tillräckliga villkor för svag konvergens av första-sällan-händelsetider för semi-Markovska processer.

I introduktionen ger vi nödvändiga grundläggande definitioner och beskrivningar av modeller som betraktas i avhandlingen, samt ger några exempel på situationer i vilka metoder av första-sällan-händelsetider kan vara lämpliga att använda. Dessutom analyserar vi publicerade resultat om asymptotiska problem för stokastiska funktionaler som definieras på semi-Markovska processer.

I artikel **A** betraktar vi första-sällan-händelsetider för semi-Markovska processer med en ändlig mängd av lägen. Vi ger också en sammanfattning av våra resultat om nödvändiga och tillräckliga villkor för svag konvergens, samt diskuterar möjliga tillämpningar inom aktuarie-området.

I artikel **B** redovisar vi i detalj de resultat som annonseras i artikel **A** och bevisen för dem. Vi ger också nödvändiga och tillräckliga villkor för svag konvergens av första-sällan-händelsetider för semi-Markovska processer med en ändlig mängd av lägen i ett icke-triangulärt tillstånd. Dessutom beskriver vi med hjälp av Laplacetransformationen klassen av alla möjliga gränsfördelningar.

I artikel **C** studerar vi villkor av svag konvergens av flöden av sällan-händelser i ett icke-triangulärt tillstånd. Vi formulerar nödvändiga och tillräckliga villkor för konvergens, och beskriver klassen av alla möjliga gränsflöden. Vi tillämpar också våra resultat i asymptotisk analys av icke-ruin-sannolikheten för störda riskprocesser i fallet av asymptotiskt stora värden på initial-kapitalet, och samtidigt, för värden av säkerhetskoefficienten som är nära till det kritiska värdet 1.

I artikel **D** ger vi nödvändiga och tillräckliga villkor för svag konvergens av första-sällan-händelsetider för semi-Markovska processer med en ändlig mängd av lägen i ett triangulärt tillstånd, samt beskriver klassen av alla möjliga gränsfördelningar. Resultaten utvidgar slutsatser från artikel **B** till att gälla för ett allmänt triangulärt tillstånd.

I artikel **E** ger vi nödvändiga och tillräckliga villkor för svag konvergens av flöden av sällan-händelser för semi-Markovska processer i ett triangulärt tillstånd. Detta generaliserar resultaten från artikel **C** till att beskriva ett allmänt triangulärt tillstånd. Vidare ger vi tillämpningar av våra resultat på asymptotiska problem av störda riskprocesser och till kösystemen med snabb service.

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