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# Twisted derivations, quasi-hom-Lie algebras and their quasi-deformations

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## Abstract

Quasi-hom-Lie algebras (qhl-algebras) were introduced by Larsson and Silvestrov (2004b) as a generalisation of hom-Lie algebras, which are a deformation of Lie algebras. Lie algebras are defined by an operation called bracket, $[\cdot,\cdot]$, and a three-term Jacobi identity. By the theorem from Hartwig, Larsson, and Silvestrov (2003), this bracket and the three-term Jacobi identity are deformed into a new bracket operation, $\langle\cdot,\cdot\rangle$, and a six-term Jacobi identity, making it a quasi-hom-Lie algebra.

Throughout this thesis we deform the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0. We examine the quasi-deformed relations and six-term Jacobi identities of the following polynomial algebras: $\mathbb{F}[t]$, $\mathbb{F}[t]/(t^2)$, $\mathbb{F}[t]/(t^3)$, $\mathbb{F}[t]/(t^4)$, $\mathbb{F}[t]/(t^5)$, $\mathbb{F}[t]/(t^n)$, where $n$ is a positive integer $\geq 2$, and $\mathbb{F}[t]/((t-t_0)^3)$. Larsson and Silvestrov (2005) and Larsson, Sigurdsson, and Silvestrov (2008) have already examined some of these cases, which we repeat for the reader's convenience.

We further investigate the following $\sigma$-twisted derivations, and how they act in the different cases of mentioned polynomial algebras: the ordinary differential operator, the shifted difference operator, the Jackson q-derivation operator, the continuous q-difference operator, the Eulerian operator, the divided difference operator, and the nilpotent imaginary derivative operator. We also introduce a new, general, $\sigma$-twisted derivation operator, which is $\sigma(t)$ as a polynomial of degree $k$.

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# Contents

# Chapter 1

# Introduction

Lie algebras have widespread use in mathematics and physics. Named after the 19th century Norwegian mathematician Sophus Lie, it has been under vast development throughout the 20th century. A Lie algebra is a vector space together with an operation called bracket $[\cdot,\cdot]$. A renowned introduction to Lie algebras is given by Humphreys (1972), please refer to this book for further presentation of the subject.

A deformation of Lie algebras was performed by Hartwig, Larsson, and Silvestrov (2003), who twists Lie algebras using an algebra homomorphism, creating a hom-Lie algebra. For a certain value of this homomorphism, we get the definition of a Lie algebra, meaning that Lie algebras are a subclass of hom-Lie algebras. This deformation was generalised by Larsson and Silvestrov (2004b) into a quasi-hom-Lie algebra. Quasi-hom-Lie algebras also include Lie superalgebras, colour Lie algebras and some other more exotic types of algebras. A quasi-Lie algebra was described by Larsson and Silvestrov (2004a), which for some conditions gives a class of quasi-hom-Lie algebras. For formal definitions, see Chapter 2.

A major development of quantum (q-)deformations of Lie algebras was sparked when Drinfel'd (1985) and Jimbo (1985) independently and simultaneously considered deformations of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Since then multiple deformed and q-deformed Lie algebras have appeared in, for example, string theory, and it is interesting to study deformations of Lie algebras to see if these algebras obey deformed versions of Jacobi identites or skew-symmetries. This brings us to the content of this thesis.

The main task of this thesis is to investigate different cases and a generalisation of quasi-deformations on polynomial rings. The deformations are "quasi" because sometimes one cannot retrieve the undeformed object via limit transitions of parameters in the commutation relations.

The quasi-deformations on $\mathbb{F}[t]/(t^3)$ and $\mathbb{F}[t]/(t^n)$ have been investigated by Larsson and Silvestrov (2005) and Larsson, Sigurdsson, and Silvestrov (2008), respectively, which we re-examine in this thesis for the reader's convenience.

Furthermore, we also investigate how several known $\sigma$-twisted derivations and a generalised $\sigma$-twisted derivation act in the different cases of quasi-deformations on polynomial rings. $\sigma$-twisted derivations are operators satisfying a $\sigma$-twisted Leibniz rule:

$$\partial(ab) = \partial(a)b + \sigma(a)\partial(b),$$

where $\sigma$ is an algebra endomorphism. Derivation operators on algebras are a very important class of operators in mathematics, and the $\sigma$-twisted Leibniz rule is a general case of a derivation, which allows for many different well-known difference and differential operators. See Section 3.1 for a list of $\sigma$-twisted derivations.

The object we deform in this thesis is the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0. The undeformed algebra is generated by the elements $H$, $E$ and $F$, such that the following relations hold:

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

When the Lie bracket is the commutator bracket in an associative algebra $[A,B] = AB - BA$, a well-known standard representation of these commutation relations by the first order differential operators, acting on the vector space of polynomials or the vector space of formal power series or on some other appropriate vector space of functions in the variable $t$, is given by

$$E \mapsto \partial, \quad H \mapsto -2t\partial, \quad F \mapsto -t^2\partial.$$

Larsson and Silvestrov (2005) generalise these operators such that $\partial$ is replaced by $\partial_\sigma \in \mathcal{D}_\sigma(\mathscr{A})$ (where $\mathcal{D}_\sigma(\mathscr{A})$ is the set of $\sigma$-derivations on $\mathscr{A}$), giving a $\sigma$-twisted derivation on an algebra $\mathscr{A}$. Throughout this thesis we make extensive use of Theorem 1, from Hartwig, Larsson, and Silvestrov (2003), which introduces a $\sigma$-deformed bracket $\langle \cdot,\cdot \rangle_\sigma : \mathscr{A}\cdot\partial_\sigma \times \mathscr{A}\cdot\partial_\sigma \to \mathscr{A}\cdot\partial_\sigma$ and a deformed six-term Jacobi identity, allowing us to examine a quasi-deformation of $\mathfrak{sl}_2(\mathbb{F})$, which was shown by Larsson and Silvestrov (2005) to be a quasi-hom-Lie algebra. We examine the $\sigma$-deformations of the three relations above and the deformed six-term Jacobi identity in the following polynomial algebras: $\mathbb{F}[t]$, $\mathbb{F}[t]/(t^2)$, $\mathbb{F}[t]/(t^3)$, $\mathbb{F}[t]/(t^4)$, $\mathbb{F}[t]/(t^5)$, $\mathbb{F}[t]/(t^n)$ and $\mathbb{F}[t]/((t-t_0)^3)$. For the polynomial algebras $\mathbb{F}[t]/(t^i)$ and $\mathbb{F}[t]/((t-t_0)^3)$, where $i = 3,4,5,n$, we assume that $\mathbb{F}$ contains all the $i$th and third roots of unity, respectively. This thesis thus examines if we can find the explicit $\sigma$-deformed relations and six-term Jacobi identity of quasi-hom-Lie algebras on the mentioned polynomial algebras; which we succeed with, with the exception of finding an explicit deformed six-term Jacobi identity of the polynomial algebra $\mathbb{F}[t]/(t^n)$, which remains an open problem.

To minimise calculation errors, majority of the calculations, such as the deformed six-term Jacobi identities and the solutions to the systems of linear equations, have been performed symbolically using MATLAB or Maple.

# Chapter 2

# Definitions

In this chapter we set the definitions and notation which will be used in this thesis.

Throughout the paper, $\mathbb{F}$ is a field of characteristic zero, $\mathscr{A}$ is a commutative, associative $\mathbb{F}$-algebra with unity 1, and $\circlearrowleft_{x,y,z}$ denotes cyclic summation with respect to $x, y, z$.

**Definition 1** (Humphreys (1972))**.** A vector space $L$ over a field $\mathbb{F}$, with an operation $L \times L \to L$, denoted $(x,y) \mapsto [x,y]$ and called the bracket or commutator of $x$ and $y$, is called a *Lie algebra* over $\mathbb{F}$ if the following axioms are satisfied:

- The bracket operation is bilinear,
- $[x,x] = 0$ for all $x \in L$,
- $\circlearrowleft_{x,y,z} [x,[y,z]] = 0,$ where $x,y,z \in L$.

**Definition 2** (Hartwig, Larsson, and Silvestrov (2003))**.** A *hom-Lie algebra* (L, $\varsigma$) is a non-associative algebra $L$ together with an algebra homomorphism $\varsigma : L \to L$, such that:

- $\langle x,y \rangle_L = -\langle y,x \rangle_L$,
- $\circlearrowleft_{x,y,z} \langle (\mathrm{id} + \varsigma)(x), \langle y,z \rangle_L \rangle_L = 0$,

for all $x,y,z \in L$, where $\langle \cdot,\cdot \rangle_L$ denotes the product in $L$.

**Definition 3** (Larsson and Silvestrov (2004a))**.** A *quasi-Lie algebra* structure on $L$ is a tuple $(L, \langle \cdot,\cdot \rangle, \alpha, \beta, \theta, \omega)$ where

- $L$ be a vector space over the field $\mathbb{F}$,
- $\langle \cdot,\cdot \rangle_L : L \times L \to L$ is a bilinear bracket map called a product or bracket in L,
- $\alpha, \beta : L \to L$ are linear maps,
- $\theta : D_\theta \to \mathscr{L}_{\mathbb{F}}(L)$ and $\omega : D_\omega \to \mathscr{L}_{\mathbb{F}}(L)$ are maps with domains of definition $D_\omega, D_\theta \subseteq L \times L$, and where $\mathscr{L}_{\mathbb{F}}(L)$ is the set of linear maps of the linear space $L$ over the field $\mathbb{F}$,

such that the following conditions hold:

- $\langle x,y\rangle_L = \omega(x,y)\langle y,x\rangle_L, \quad \text{for } (x,y) \in D_\omega,$
- $\circlearrowleft_{x,y,z} \theta(z,x)(\langle \alpha(x), \langle y,z\rangle\rangle + \beta\langle x, \langle y,z\rangle\rangle) = 0,$

for all $(z,x),(y,z),(x,y) \in D_\theta$.

**Definition 4** (Larsson and Silvestrov (2004b))**.** A *quasi-hom-Lie algebra (qhl-algebra)* is a tuple $(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega)$ where

- $L$ is a $\mathbb{F}$-linear space,
- $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear map called a product or bracket in $L$,
- $\alpha, \beta : L \to L$, are linear maps,
- $\omega : D_\omega \to \mathcal{L}_{\mathbb{F}}(L)$ is a map with domain of definition $D_\omega \subseteq L \times L$, and where $\mathcal{L}_{\mathbb{F}}(L)$ denotes the linear space of $\mathbb{F}$-linear maps of the $\mathbb{F}$-linear space $L$,

such that the following conditions hold:

- ($\beta$-twisting) The map $\alpha$ is a $\beta$-twisted algebra homomorphism, that is,
$$\langle \alpha(x), \alpha(y)\rangle_L = \beta \circ \alpha\langle x,y\rangle_L, \quad \forall x,y \in L;$$
- ($\omega$-symmetry) The product satisfies a generalized skew-symmetry condition
$$\langle x,y\rangle_L = \omega(x,y)\langle y,x\rangle_L, \quad \forall (x,y) \in D_\omega;$$
- (qhl-Jacobi identity) The bracket satisfies a generalized Jacobi identity
$$\circlearrowleft_{x,y,z} \omega(z,x)(\langle \alpha(x), \langle y,z\rangle_L\rangle_L + \beta\langle x, \langle y,z\rangle_L\rangle_L) = 0,$$
for all $(z,x),(x,y),(y,z) \in D_\omega$.

**Example 1.** Set $\varsigma$ = id in Definition 2 and we get the definition of a Lie algebra.

**Example 2.** Set $\theta = \omega$ and the condition $\langle \alpha(x), \alpha(y)\rangle = \beta \circ \alpha\langle x,y\rangle$ in Definition 3 and we get a quasi-hom-Lie algebra.

**Example 3.** Set $\beta$ = id and $\omega$ = -id in Definition 4 and we get class of hom-Lie algebras.

**Definition 5** (Hartwig and Silvestrov (2002))**.** A linear operator $\mathcal{D} : \mathscr{A} \to \mathscr{A}$ is called a *derivation* in $\mathscr{A}$ if
$$\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$$
for all $a,b \in \mathscr{A}$.

**Definition 6** (Larsson and Silvestrov (2005))**.** Let $\sigma$ be an algebra endomorphism on $\mathscr{A}$. Let $\mathcal{D}_\sigma(\mathscr{A})$ denote the set of $\sigma$-derivations on $\mathscr{A}$, where $\partial_\sigma \in \mathcal{D}_\sigma(\mathscr{A})$. A *twisted derivation* or *$\sigma$-derivation* on the algebra $\mathscr{A}$ is an $\mathbb{F}$-linear map $\partial_\sigma : \mathscr{A} \to \mathscr{A}$ such that:
$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b),$$
which is a $\sigma$-twisted Leibniz rule.

# Chapter 3

# Twisted derivations

In this section we present a list of different $\sigma$-twisted derivations, which we later apply to each of the examined quasi-deformations on polynomial algebras in Chapters 5-11.

## 3.1 Examples of twisted derivations

In the following cases $\sigma$ takes on different operations, which causes the $\sigma$-derivations and $\sigma$-twisted Leibniz rules to become as follows:

**Example 4** (The Ordinary Differential operator)**.** In this case $\sigma = \mathrm{id}$ and $(\partial a)(t) = a'(t)$, which obeys the Leibniz rule:

$$(\partial(ab))(t) = (\partial_\sigma(a))(t)b(t) + a(t)(\partial_\sigma(b))(t).$$

**Example 5** (The Shifted Difference operator, Hartwig, Larsson, and Silvestrov (2003))**.** In this case $\sigma = s$ where $s(f)(t) := f(t+1)$ and $(\partial_\sigma a)(t) = a(t+1) - a(t)$, which obeys the Leibniz rule:

$$(\partial_\sigma(ab))(t) = (\partial_\sigma a)(t)b(t) + a(t+1)(\partial_\sigma b)(t).$$

**Example 6** (The Jackson q-Derivation operator, Hartwig, Larsson, and Silvestrov (2003))**.** In this case $\sigma(f)(t) := f(qt)$ and $(D_q(a))(t) = \frac{a(qt)-a(t)}{qt-t}$, which obeys the Leibniz rule:

$$(D_q(ab))(t) = (D_q a)(t)b(t) + a(qt)(D_q b)(t).$$

**Example 7** (The Continuous q-Difference operator, Hartwig and Silvestrov (2002))**.** In this case $\sigma(f)(t) := f(qt)$ and $(D_q(a))(t) = a(qt) - a(t)$, which obeys Leibniz rule:

$$(D_q(ab))(t) = (D_q a)(t)b(t) + a(qt)(D_q b)(t).$$

**Example 8** (The Eulerian operator, Hartwig and Silvestrov (2002))**.** In this case $\sigma = \mathrm{id}$ and $(\partial a)(t) = ta'(t)$, which obeys the Leibniz rule:

$$(\partial(ab))(t) = ta'(t)b(t) + a(t)tb'(t).$$

**Example 9** (The Divided Differences operator, Hartwig and Silvestrov (2002))**.** In this case $\sigma = \mathrm{id}$ and $(\partial a)(t) = \frac{a(t)-a(s)}{t-s}$, which obeys the Leibniz rule:

$$(\partial(ab))(t) = (\partial a)(t)b(t) + a(t)(\partial b)(t).$$

**Example 10** (The Nilpotent Imaginary Derivative operator, Smits (1968))**.** In this case $\sigma(f)(t) := \overline{f(t)}$ (complex conjugate) and $(Da)(t) = \mathrm{Im}(a(t))$, which obeys the Leibniz rule:

$$(D(ab))(t) = (Da)(t)b(t) + \overline{a(t)}(Db)(t).$$

# Chapter 4

# Hartwig-Larsson-Silvestrov construction theorem of quasi-hom-Lie algebras of twisted vector fields

This chapter is devoted to the theorem by Hartwig, Larsson, and Silvestrov (2003), which will be used in all the following chapters about quasi-deformations on polynomial algebras.

The set of all $a \in \mathscr{A}$ such that $a \cdot \partial_\sigma = 0$ is called the annihilator $\operatorname{Ann}(\partial_\sigma)$ of $\partial_\sigma$. For the algebra endomorphism $\sigma$ and an element $\delta \in \mathscr{A}$, we assume the following conditions:

$$\sigma(\operatorname{Ann}(\partial_\sigma)) \subseteq \operatorname{Ann}(\partial_\sigma), \tag{4.1}$$

$$\partial_\sigma(\sigma(a)) = \delta\sigma(\partial_\sigma(a)),\ \forall a \in \mathscr{A}. \tag{4.2}$$

Let

$$\mathscr{A} \cdot \partial_\sigma = \{a \cdot \partial_\sigma | a \in \mathscr{A}\}$$

denote the cyclic left $\mathscr{A}$-submodule of $\mathcal{D}_\sigma(\mathscr{A})$ generated by $\partial_\sigma$. The following theorem introduces an $\mathbb{F}$-algebra structure on $\mathscr{A} \cdot \partial_\sigma$, which makes it a quasi-hom-Lie algebra.

**Theorem 1.** *If condition* (4.1) *holds, then the map*

$$\langle \cdot, \cdot \rangle_\sigma : \mathscr{A} \cdot \partial_\sigma \times \mathscr{A} \cdot \partial_\sigma \to \mathscr{A} \cdot \partial_\sigma$$

*defined as*

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle_\sigma = (\sigma(a) \cdot \partial_\sigma) \circ (b \cdot \partial_\sigma) - (\sigma(b) \cdot \partial_\sigma) \circ (a \cdot \partial_\sigma),$$

*where $a, b \in \mathscr{A}$, is a well-defined $\mathbb{F}$-algebra product on the $\mathbb{F}$-linear space $\mathscr{A} \cdot \partial_\sigma$. The map satisfies the following identities:*

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle_\sigma = (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)) \cdot \partial_\sigma, \text{ and} \tag{4.3}$$

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle_\sigma = -\langle b \cdot \partial_\sigma, a \cdot \partial_\sigma \rangle_\sigma, \tag{4.4}$$

*where $a, b, c \in \mathscr{A}$. If condition* (4.2) *holds, then the deformed six-term Jacobi identity holds:*

$$\circlearrowleft_{a,b,c} \left( \langle \sigma(a) \cdot \partial_\sigma, \langle b \cdot \partial_\sigma, c \cdot \partial_\sigma \rangle_\sigma \rangle_\sigma + \delta \cdot \langle a \cdot \partial_\sigma, \langle b \cdot \partial_\sigma, c \cdot \partial_\sigma \rangle_\sigma \rangle_\sigma \right) = 0.$$

See Hartwig, Larsson, and Silvestrov (2003) for proofs.

# Chapter 5

# Quasi-deformations

In this chapter we first make an ansatz regarding when $\sigma$ and $\partial_\sigma$ operates on identity, which we require to be able to study the $\sigma$-twisted relations on the polynomial algebras in the following section and chapters; then we examine the polynomial algebra $\mathbb{F}[t]$.

Let the elements we defined in Chapter 1

$$e := \partial_\sigma, \quad h := -2t\partial_\sigma, \quad f := -t^2\partial_\sigma,$$

span an $\mathbb{F}$-linear subspace $\mathcal{F} := \mathrm{LinSpan}_{\mathbb{F}}\{e,h,f\} = \mathrm{LinSpan}_{\mathbb{F}}\{\partial_\sigma, -2t\partial_\sigma, -t^2\partial_\sigma\}$ of $\mathscr{A}\cdot\partial_\sigma$. If we have an element $t \in \mathscr{A}$, and make the ansatz:

$$\begin{aligned}\partial_\sigma(1) &= d_0 + d_1 t + \ldots + d_k t^k,\\ \sigma(1) &= s_0 + s_1 t + \ldots + s_l t^l.\end{aligned}$$

Since $\sigma(1) = \sigma(1\cdot 1) = \sigma(1)^2$, then $\sigma(1) = 1$ or $\sigma(1) = 0$ when all non-negative integer powers of $t$ are linearly independent over $\mathbb{F}$ (making $s_1 = \ldots = s_l = 0$), hence $s_0 = 1$ or $s_0 = 0$. Then it follows that:

$$\begin{aligned}\partial_\sigma(1) = \partial_\sigma(1\cdot 1) &= \sigma(1)\partial_\sigma(1) + \partial_\sigma(1)1 \text{ (by the Leibniz rule)}\\ &= (\sigma(1)+1)\partial_\sigma(1) = (s_0+1)(d_0 + d_1 t + \ldots + d_k t^k).\end{aligned}$$

Since $\partial_\sigma(1) = d_0 + d_1 t + \ldots + d_k t^k$, if $s_0 = 1$ then $d_0 = \ldots = d_k = 0$. If $s_0 = 0$ then $d_0,\ldots,d_k$ are arbitrary. We thus have 2 cases.

Case 1. Let $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$, then we have the following $\sigma$-deformed relations:

$$\begin{aligned}\langle h,e\rangle &= (\sigma(-2t)\partial_\sigma(1) - \sigma(1)\partial_\sigma(-2t))\partial_\sigma = 2\partial_\sigma(t)\partial_\sigma,\\ \langle h,f\rangle &= (2\sigma(t)\partial_\sigma(t^2) - 2\sigma(t)^2\partial_\sigma(t))\partial_\sigma = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma,\\ \langle e,f\rangle &= (\sigma(1)\partial_\sigma(-t^2) - \sigma(-t^2)\partial_\sigma(1))\partial_\sigma = -(\sigma(t)+t)\partial_\sigma(t)\partial_\sigma.\end{aligned} \tag{5.1}$$

Case 2. Let $\sigma(1) = 0$ and $\partial_\sigma(1) = d_0 + d_1 t + \ldots + d_k t^k$. This gives us $\sigma(t^w) = \sigma(1\cdot t^w) = \sigma(1)\sigma(t^w) = 0, \forall w \in \mathbb{N}$, which gives us the following $\sigma$-deformed relations:

$$\begin{aligned}\langle h,e\rangle &= (\sigma(-2t)\partial_\sigma(1) - \sigma(1)\partial_\sigma(-2t))\partial_\sigma = 0,\\ \langle h,f\rangle &= (2\sigma(t)\partial_\sigma(t^2) - 2\sigma(t)^2\partial_\sigma(t))\partial_\sigma = 0,\\ \langle e,f\rangle &= (\sigma(1)\partial_\sigma(-t^2) - \sigma(-t^2)\partial_\sigma(1))\partial_\sigma = 0.\end{aligned}$$

This case is thus trivial, and in the following chapters we only consider Case 1.

## 5.1 Quasi-hom-Lie algebras of the twisted vector fields for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]$

Let $\mathscr{A}$ be the polynomial algebra $\mathbb{F}[t]$, $\sigma(1) = 1$, $\partial_\sigma(1) = 0$, and $\sigma(t) = q(t)$, $\partial_\sigma(t) = p(t)$, where $q(t), p(t) \in \mathbb{F}[t]$. We have closure of Eqs. (5.1), meaning that the RHS of the relations are linear combinations of the elements $e$, $f$ and $h$, in three specific cases of the polynomials $p(t)$ and $q(t)$:

- Case 1: $q(t) = q_0 + q_1 t, q_1 \neq 0, p(t) = p_0 \neq 0,$
- Case 2: $q(t) = q_0, q_0 \neq 0, p(t) = p_0 + p_1 t,$
- Case 3: $q(t) = 0, p(t) = p_0 + p_1 t.$

With these three cases, the relations in Eq. (5.1) and the deformed six-term Jacobi identity become as follows.

**Case 1:**

$$\begin{aligned}\langle h, e\rangle &= \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2p_0\partial_\sigma = 2p_0 e,\\ \langle h, f\rangle &= \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = 2(q_0 + q_1 t)p_0 t\partial_\sigma = -q_0 p_0 h - 2q_1 p_0 f,\\ \langle e, f\rangle &= \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = -(q_0 + q_1 t + t)p_0\partial_\sigma = -q_0 p_0 e + \frac{q_1 + 1}{2} p_0 h.\end{aligned}$$

For the deformed six-term Jacobi identity to hold, the condition (4.2) must be satisfied: $\partial_\sigma(\sigma(a)) = \delta\sigma(\partial_\sigma(a))$, $\forall a \in \mathscr{A}$. When $a = t^0 = 1$ we have a trivial case of the condition since $\partial_\sigma(1) = 0$. When $a = t$ the left hand side of the condition is:

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(q_0 + q_1 t) = q_0\partial_\sigma(1) + q_1\partial_\sigma(t) = q_1 p_0.$$

The right hand side is:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_0) = \delta p_0\sigma(1) = \delta p_0.$$

Hence:

$$q_1 p_0 = \delta p_0 \implies \delta = q_1.$$

We show that the condition holds for an arbitrary $a = t^k$:

$$\begin{aligned}\partial_\sigma(\sigma(t^{k+1})) &= \partial_\sigma(\sigma(t)^{k+1}) = \partial_\sigma(\sigma(t))\sigma(t)^k + \sigma^2(t)\partial_\sigma(\sigma(t)^k)\\ &= \delta\sigma\partial_\sigma(t)(\sigma(t)^k + \sigma^2(t)\delta\sigma\partial_\sigma(t^k) = \delta\sigma(\partial_\sigma(t)t^k + \sigma(t)\partial_\sigma(t^k)) = \delta\sigma(\partial_\sigma(t^{k+1}))\end{aligned}$$

We thus have a deformed six-term Jacobi identity on $\mathscr{A}\cdot\partial_\sigma = \mathbb{F}\cdot\partial_\sigma$:

$$\circlearrowleft_{x,y,z} (\langle\sigma(x), \langle y, z\rangle\rangle + q_1\langle x, \langle y, z\rangle\rangle) = 0.$$

Setting $\alpha(x) := q_1^{-1}\sigma(x)$ we see that it is the Jacobi identity for a hom-Lie algebra:

$$\circlearrowleft_{x,y,z} (\langle(\alpha + \mathrm{id})(x), \langle y, z\rangle\rangle) = 0.$$

**Case 2:**

$$\langle h,e\rangle = \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2(p_0+p_1t)\partial_\sigma = 2p_0e - p_1h,$$

$$\langle h,f\rangle = \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = (2q_0(p_0+p_1t)t)\partial_\sigma = -q_0p_0h - 2q_0p_1f,$$

$$\langle e,f\rangle = \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = -q_0p_0e + \frac{q_0p_1+p_0}{2}h + p_1f.$$

Using the same argument as in Case 1 we see that the left hand side is:

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(q_0) = q_0\partial_\sigma(1) = 0,$$

and the right hand side is:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_0+p_1t) = \delta(p_0\sigma(1)+p_1\sigma(t)) = \delta(p_0+p_1q_0).$$

Hence:

$$0 = \delta(p_0+p_1q_0) \implies \delta = 0.$$

The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle\rangle) = 0,$$

which is also the deformed six-term Jacobi identity for a hom-Lie algebra.

**Case 3:**

$$\langle h,e\rangle = \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2(p_0+p_1t)\partial_\sigma = 2p_0e - p_1h,$$

$$\langle h,f\rangle = \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = 0,$$

$$\langle e,f\rangle = \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = -(0+t)(p_0+p_1t)\partial_\sigma = \frac{1}{2}p_0h + p_1f.$$

We, again, use the same argument as in Case 1, and see that the left hand side is:

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(0) = 0,$$

and the right hand side is:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_0+p_1t+p_2t^2+\ldots+p_nt^n) = \delta p_0,$$

which gives the condition:

$$0 = \delta p_0.$$

Which means that $\delta$ depends on $p_0$. If $p_0 = 0$, then $\delta$ is arbitrary, which we can set to 0. If $p_0 \neq 0$, then $\delta = 0$. Hence, we get the same deformed six-term Jacobi identity of a hom-Lie algebra as in Case 2:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle\rangle) = 0.$$

### 5.1.1 Examples of twisted derivations

In this section we take the examples of $\sigma$-derivations from Section 3.1 and we will see what the deformed relations (5.1) become with different values of $\partial_\sigma(t)$ and $\sigma(t)$.

**Example 11** (The Ordinary Differential operator)**.** We get the ordinary differential operator, where $\sigma(t) = t$ and $\partial_\sigma(t) = 1$, by setting $q_1 = 1$, $q_0 = 0$ and $p_0 = 1$, $p_1 = 0$, which gives us the relations from Case 1:

$$\langle h, e\rangle = 2p_0 e = 2e,$$

$$\langle h, f\rangle = -q_0 p_0 h - 2q_1 p_0 f = -2f,$$

$$\langle e, f\rangle = -q_0 p_0 e + \frac{q_1+1}{2} p_0 h = h,$$

and the deformed six-term Jacobi identity for a hom-Lie algebra:

$$\circlearrowleft_{x,y,z} (\langle(\sigma + \mathrm{id})(x), \langle y, z\rangle\rangle) = 0.$$

**Example 12** (The Shifted Difference operator)**.** We get the shifted difference operator, where $\sigma(t) = t+1$ and $\partial_\sigma(t) = 1$, by setting $q_0 = q_1 = 1$, and $p_0 = 1$, $p_1 = 0$, which gives us the relations from Case 1:

$$\langle h, e\rangle = 2p_0 e = 2e,$$

$$\langle h, f\rangle = -q_0 p_0 h - 2q_1 p_0 f = -h - 2f,$$

$$\langle e, f\rangle = -q_0 p_0 e + \frac{q_1+1}{2} p_0 h = -e + h,$$

and the deformed six-term Jacobi identity for a hom-Lie algebra:

$$\circlearrowleft_{x,y,z} (\langle(\sigma + \mathrm{id})(x), \langle y, z\rangle\rangle) = 0.$$

**Example 13** (The Jackson q-Derivation operator)**.** We get the Jackson q-operator, where $\sigma(t) = qt$ and $\partial_\sigma(t) = 1$, by setting $q_1 = q$, $q_0 = 0$ and $p_0 = 1$, $p_1 = p_2 = 0$, which gives us the relations from Case 1:

$$\langle h, e\rangle = 2p_0 e = 2e,$$

$$\langle h, f\rangle = -q_0 p_0 h - 2q_1 p_0 f = -2qf,$$

$$\langle e, f\rangle = -q_0 p_0 e + \frac{q_1+1}{2} p_0 h = \frac{q+1}{2} h,$$

and the deformed six-term Jacobi identity for a hom-Lie algebra:

$$\circlearrowleft_{x,y,z} (\langle(\alpha + \mathrm{id})(x), \langle y, z\rangle\rangle) = 0,$$

where $\alpha = q^{-1}\sigma$.

**Example 14** (The Continuous q-Difference operator)**.** We get the continuous q-difference operator, where $\sigma(t) = qt$ and $\partial_\sigma(t) = (q-1)t$, by setting $q_1 = q$, $q_0 = 0$ and $p_1 = q-1$, $p_0 = 0$, which is a case that does not have closure of (5.1). Instead we get a deformation of $\mathfrak{sl}_2(\mathbb{F})$ where the brackets of the basis elements $e, f, h$ are not linear combinations of these elements:

$$\begin{aligned}\langle h,e\rangle &= 2\partial_\sigma(t)\partial_\sigma = 2(q-1)t\partial_\sigma = (1-q)h,\\ \langle h,f\rangle &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma = 2(q^2-q)t^3\partial_\sigma,\\ \langle e,f\rangle &= -(\sigma(t)+t)\partial_\sigma(t)\partial_\sigma = (q^2-1)f.\end{aligned}$$

By using the given values of $p_0, p_1, q_0, q_1$ and solving the condition (4.2) for $\delta$ we get the deformed six-term Jacobi identity:

$$\circlearrowleft_{x,y,z}(\langle(\sigma+\mathrm{id})(x), \langle y,z\rangle\rangle) = 0.$$

**Example 15** (The Eulerian operator)**.** We get the Eulerian operator, where $\sigma(t) = t$ and $\partial_\sigma(t) = t$, by setting $q_1 = 1$, $q_0 = 0$ and $p_1 = 1$, $p_0 = 0$, which gives us another case where we do not have closure. We get the relations:

$$\begin{aligned}\langle h,e\rangle &= 2\partial_\sigma(t)\partial_\sigma = -h,\\ \langle h,f\rangle &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma = 2t^3\partial_\sigma,\\ \langle e,f\rangle &= -(\sigma(t)+t)\partial_\sigma(t)\partial_\sigma = -2t^2\partial_\sigma = 2f.\end{aligned}$$

By using the given values of $p_0, p_1, q_0, q_1$ and solving the condition (4.2) for $\delta$ we get the deformed six-term Jacobi identity:

$$\circlearrowleft_{x,y,z}(\langle(\sigma+\mathrm{id})(x), \langle y,z\rangle\rangle) = 0.$$

**Example 16** (The Divided Differences operator)**.** We get the divided differences operator, where $\sigma(t) = t$ and $\partial_\sigma(t) = 1$, by setting $q_1 = 1$, $q_0 = 0$ and $p_0 = 1$, $p_1 = 0$, which gives us the relations from Case 1:

$$\begin{aligned}\langle h,e\rangle &= 2p_0e = 2e,\\ \langle h,f\rangle &= -q_0p_0h - 2q_1p_0f = -2f,\\ \langle e,f\rangle &= -q_0p_0e + \frac{q_1+1}{2}p_0h = h,\end{aligned}$$

and the deformed six-term Jacobi identity for a hom-Lie algebra:

$$\circlearrowleft_{x,y,z}(\langle(\sigma+\mathrm{id})(x), \langle y,z\rangle\rangle) = 0.$$

**Example 17** (Nilpotent Imaginary Derivative operator)**.** Let $p_j, q_j \in \mathbb{C}$, where $j = 0,1,\dots,n$, and $q_j = x_{q_j} + iy_{q_j}$ and $p_j = x_{p_j} + iy_{p_j}$, where $i$ is the imaginary unit. We get the nilpotent imaginary derivative operator, where $\sigma(t) = \overline{q_0} + \overline{q_1}t + \ldots + \overline{q_n}t^n$ and $\partial_\sigma(t) = y_{p_0} + y_{p_1}t + \ldots + y_{p_n}t^n$, by setting $q_0 = \overline{q_0}$, $q_1 = \overline{q_1}, \dots,$ $q_n = \overline{q_n}$ and $p_0 = y_{p_0}$, $p_1 = y_{p_1}, \dots,$ $p_n = y_{p_n}$. This is another case (for most values of $p_j$ and $q_j$) where we do not have closure. The relations are thus:

$$\begin{aligned}\langle h,e\rangle &= 2\partial_\sigma(t)\partial_\sigma = 2(y_{p_0} + y_{p_1}t + \ldots + y_{p_n}t^n)\partial_\sigma,\\ \langle h,f\rangle &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma = 2(\overline{q_0} + \overline{q_1}t + \ldots + \overline{q_n}t^n)(y_{p_0} + y_{p_1}t + \ldots + y_{p_n}t^n)t\partial_\sigma,\\ \langle e,f\rangle &= -(\sigma(t)+t)\partial_\sigma(t)\partial_\sigma = -(\overline{q_0} + \overline{q_1}t + \ldots + \overline{q_n}t^n + t)(y_{p_0} + y_{p_1}t + \ldots + y_{p_n}t^n)\partial_\sigma.\end{aligned}$$

**Example 18** ($\sigma(t)$ as a polynomial of degree $k$)**.** Let $\sigma(t) = S(t) = \sum_{j=0}^{k} s_j t^k$. The $\sigma$-twisted Leibniz rule is defined as:

$$(\partial_\sigma(ab))(t) = \partial_\sigma(a)(t)b(t) + \sigma(a)(t)\partial_\sigma(b)(t),$$

where

$$(\partial_\sigma a)(t) = \frac{\sigma(a)(t) - a(t)}{\sigma(t) - t} = \frac{a(S(t)) - a(t)}{S(t) - t}.$$

We get this $\sigma$-derivation by setting $q_i = s_i$ for $i = 0, 1, \ldots, k$, and $p_0 = 1$, $p_1 = p_2 = \ldots = p_n = 0$, where $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$. The relations (5.1) thus become:

$$\langle h, f \rangle = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma = 2(s_0 t + s_1 t^2 + s_2 t^3 + \ldots + s_k t^{k+1})\partial_\sigma,$$

$$\langle h, e \rangle = 2\partial_\sigma(t)\partial_\sigma = 2e,$$

$$\langle e, f \rangle = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma = -(s_0 + s_1 t + s_2 t^2 + \ldots + s_k t^k + t)\partial_\sigma.$$

# Chapter 6

# Quasi-hom-Lie algebras of twisted vector fields for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]/(t^2)$

We take the algebra $\mathscr{A}$ as $\mathbb{F}[t]/(t^2)$, which means that $t^n = 0$ for $n \geq 2$. The elements are $e = \partial_\sigma$, $h = -2t\partial_\sigma$, $f = -t^2\partial_\sigma = 0$. From Chapter 5, we have that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Set:

$$\partial_\sigma(t) = p_0 + p_1 t, \quad \sigma(t) = q_0 + q_1 t. \tag{6.1}$$

Since $t^2 = 0$, we see that:

$$\sigma(t^2) = \sigma(t)^2 = (q_0 + q_1 t)^2 = q_0^2 + 2q_0 q_1 t = 0 \implies q_0 = 0, \tag{6.2}$$

$$\partial_\sigma(t^2) = \partial_\sigma(t)t + \sigma(t)\partial_\sigma(t) = (t + \sigma(t))\partial_\sigma(t) = 0. \tag{6.3}$$

Using the $\sigma$-twisted relation of Eq. (4.3) of Theorem 1, with the elements $e = \partial_\sigma, h = -2t\partial_\sigma, f = -t^2\partial_\sigma = 0$, along with what we showed in Eq. (6.2) (that $q_0 = 0$), $\partial_\sigma(1) = 0$, $\sigma(1) = 1$, and the $\sigma$-twisted Leibniz rule:

$$\langle h, e\rangle = \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2\partial_\sigma(t)\partial_\sigma = 2p_0 e - p_1 h, \tag{6.4}$$

$$\langle h, f\rangle = \langle -2t\partial_\sigma, 0\partial_\sigma\rangle_\sigma = 0, \tag{6.5}$$

$$\langle e, f\rangle = \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = 0. \tag{6.6}$$

Since $\partial_\sigma(t^2) = 0$, which we showed in Eq. (6.3) was $\partial_\sigma(t^2) = (t + \sigma(t))\partial_\sigma(t)$. Using the equations from Eq. (6.1) and that $q_0 = 0$, we get:

$$\partial_\sigma(t^2) = (t + q_1 t)(p_0 + p_1 t) = (1 + q_1)p_0 t = 0,$$

which gives us two cases, when $p_0 = 0$ and when $q_1 + 1 = 0$.

**Case 1:** Consider when $p_0 = 0$, which gives us the following relations, using what we showed in Eqs. (6.4)-(6.6):

$$\langle h, e\rangle = 2p_0 e - p_1 h = -p_1 h,$$

$$\langle h, f\rangle = 0,$$

$$\langle e, f\rangle = 0.$$

Since we cannot choose values such that $\langle h, f\rangle = -2f$, $\langle h, e\rangle = 2e$ and $\langle e, f\rangle = h$ we cannot recover $\mathfrak{sl}_2(\mathbb{F})$.

The condition for the deformed six-term Jacobi identity is that $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$. Using the equations from Eq. (6.1), the left hand side becomes:

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(q_1 t) = q_1 \partial_\sigma(t) = q_1 p_1 t,$$

and the right hand side becomes:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_1 t) = \delta p_1 \sigma(t) = \delta p_1 q_1 t.$$

Let $\delta = \delta_0 + \delta_1 t$, then the RHS becomes:

$$\delta(p_1 q_1 t) = (\delta_0 + \delta_1 t)(p_1 q_1 t) = \delta_0 p_1 q_1 t.$$

The condition for the six-term Jacobi identity is thus:

$$q_1 p_1 t = \delta_0 p_1 q_1 t.$$

Subtracting the LHS from both sides of the equality, we set the condition equal to zero:

$$\delta_0 p_1 q_1 t - q_1 p_1 t = 0.$$

$\delta_1 = \xi_1$ is arbitrary since it is not included in the equation.

1. If $p_1 q_1 \neq 0$, then $\delta_0 = 1$. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y, z\rangle_\sigma\rangle_\sigma + (1 + \xi_1 t)\cdot\langle x, \langle y, z\rangle_\sigma\rangle_\sigma\right) = 0,$$

which defines a hom-Lie-algebra.

2. If $q_1 = 0$ and/or $p_1 = 0$, then $\delta_0$ is arbitrary set to $\xi_0$, which gives the deformed six-term Jacobi identity:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y, z\rangle_\sigma\rangle_\sigma + (\xi_0 + \xi_1 t)\cdot\langle x, \langle y, z\rangle_\sigma\rangle_\sigma\right) = 0,$$

which defines a hom-Lie-algebra.

**Case 2:** The second case when the equality $\partial_\sigma(t^2) = 0$ is satisfied occurs when $p_0 \neq 0$ and $q_1 + 1 = 0 \implies q_1 = -1$. With this value on $q_1$ we get the relations, from Eqs. (6.4)-(6.6):

$$\langle h, e\rangle = 2p_0 e - p_1 h,$$

$$\langle h, f\rangle = 0,$$

$$\langle e, f\rangle = 0.$$

The condition for the deformed six-term Jacobi identity is, again, $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$. With $p_0 \neq 0$ and $q_1 = -1$, the left hand side is:

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(-t) = -(p_0 + p_1 t),$$

and the right hand side is:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_0 + p_1 t) = \delta(p_0 + p_1\sigma(t)) = \delta(p_0 - p_1 t).$$

Let $\delta = \delta_0 + \delta_1 t$ again, then the RHS is:

$$\delta(p_0 - p_1 t) = \delta_0 p_0 - \delta_0 p_1 t + \delta_1 p_0 t.$$

This gives us that the condition for the deformed six-term Jacobi identity is:

$$-(p_0 + p_1 t) = \delta_0 p_0 - \delta_0 p_1 t + \delta_1 p_0 t.$$

Subtracting the LHS from both sides of the equality, we set the condition equal to zero, which we can use to create a linear system of equations:

$$\begin{cases} \delta_0 p_0 + p_0 = 0 \\ -\delta_0 p_1 + \delta_1 p_0 + p_1 = 0 \end{cases}$$

Solving the first equation for $\delta_0$ gives:

$$\delta_0 = -1.$$

Solving the second equation for $\delta_1$ gives:

$$\delta_1 = -\frac{2p_1}{p_0}.$$

The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y, z\rangle_\sigma\rangle_\sigma + (-1 - \frac{2p_1}{p_0} t) \cdot \langle x, \langle y, z\rangle_\sigma\rangle_\sigma\right) = 0,$$

which defines a quasi-hom-Lie algebra.

## 6.1 Examples of twisted derivations

In this section we take the examples of $\sigma$-derivations from Section 3.1 and we will see what the deformed relations (6.4)-(6.6) become with different parameters defining $\partial_\sigma(t)$ and $\sigma(t)$.

The equalities of Eqs. (6.2) and (6.3) require that $q_0 = 0$, and $p_0 = 0$ or $q_1 = -1$. Hence, the following $\sigma$-derivations are not defined on the quotient ring, since they either have $q_0 \neq 0$ or $q_1 \neq -1$ and $p_0 \neq 0$: *the ordinary differential operator*, *the shifted difference operator*, *the Jackson q-derivation operator*, *the continuous q-difference operator* and *the divided differences operator*.

**Example 19** (The Eulerian operator)**.** We get the Eulerian operator, where $\sigma(t) = t$ and $\partial(t) = t$, by setting $q_1 = 1$, $q_0 = 0$ and $p_1 = 1$, $p_0 = 0$ in the Equalities (6.1). The deformed relations (6.4)-(6.6) thus become:

$$\langle h, e\rangle = 2p_0 e - p_1 h = -h,$$

$$\langle h, f\rangle = 0,$$

$$\langle e, f\rangle = 0.$$

We have the following deformed six-term Jacobi identity from Case 1:

$$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y, z\rangle\rangle + (1+\xi_1 t)\langle x, \langle y, z\rangle\rangle) = 0,$$

where $\xi_1$ is arbitrary, which means we have a hom-Lie algebra for this operator.

**Example 20** (Nilpotent Imaginary Derivative operator)**.** Let $\mathbb{F} = \mathbb{C}$ and $p_j, q_j \in \mathbb{C}$, where $j = 0, 1$, and $q_j = x_{q_j} + iy_{q_j}$ and $p_j = x_{p_j} + iy_{p_j}$, where $i$ is the imaginary unit. We get the nilpotent imaginary derivative operator, where $\sigma(t) = \overline{q_0} + \overline{q_1}t$ and $\partial_\sigma(t) = y_{p_0} + y_{p_1}t$, by setting $q_0 = \overline{q_0}$, $q_1 = \overline{q_1}$ and $p_0 = y_{p_0}$, $p_1 = y_{p_1}$ in the Equalities (6.1). The deformed relations (6.4)-(6.6) thus become:

$$\langle h, e\rangle = 2p_0 e - p_1 h = 2y_{p_0} e - y_{p_1} h,$$

$$\langle h, f\rangle = 0,$$

$$\langle e, f\rangle = 0.$$

The equalities of Eqs. (6.2) and (6.3) require that $\overline{q_0} = 0$, and $y_{p_0} = 0$ or $\overline{q_1} = -1$. If $y_{p_0} = 0$, then we have a deformed six-term Jacobi identity of Case 1:

$$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y, z\rangle_\sigma\rangle_\sigma + (1+\xi_1 t)\cdot\langle x, \langle y, z\rangle_\sigma\rangle_\sigma) = 0,$$

where $\xi_1$ is arbitrary.

If $y_{p_0} \neq 0$ and $\overline{q_1} = -1$, then we have a deformed six-term Jacobi identity of Case 2:

$$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y, z\rangle_\sigma\rangle_\sigma + (-1 - \frac{2y_{p_1}}{y_{p_0}} t)\cdot\langle x, \langle y, z\rangle_\sigma\rangle_\sigma) = 0.$$

**Example 21** ($\sigma(t)$ as a polynomial of degree $k$)**.** Let $\sigma(t) = S(t) = \sum_{j=0}^{k} s_j t^k$ where $k < 2$. The $\sigma$-twisted Leibniz rule is defined as:

$$(\partial_\sigma(ab))(t) = \partial_\sigma(a)(t)b(t) + \sigma(a)(t)\partial_\sigma(b)(t),$$

where

$$(\partial_\sigma a)(t) = \frac{\sigma(a)(t) - a(t)}{\sigma(t) - t} = \frac{a(S(t)) - a(t)}{S(t) - t}.$$

We get this $\sigma$-derivation by setting $q_i = s_i$ for $i = 0, 1$, and $p_0 = 1,\ p_1 = 0$ in the Equalities (6.1). The equalities of Eqs. (6.2) and (6.3) require that $s_0 = 0$ and (since $p_0 = 1$) that $s_1 = -1$. The deformed relations (6.4)-(6.6) thus become:

$$\langle h, e\rangle = 2p_0 e - p_1 h = 2e,$$

$$\langle h, f\rangle = 0,$$

$$\langle e, f\rangle = 0.$$

We have the following deformed six-term Jacobi identity from Case 2:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y, z\rangle\rangle - \langle x, \langle y, z\rangle\rangle\right) = 0.$$

# Chapter 7

# Quasi-hom-Lie algebras of twisted vector fields for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]/(t^3)$

We take the algebra $\mathscr{A}$ as $\mathbb{F}[t]/(t^3)$, meaning that $t^n = 0$ for $n \geq 3$, and the field $\mathbb{F}$ now includes all cubic roots of unity. The elements are $e = \partial_\sigma$, $h = -2t\partial_\sigma$, $f = -t^2\partial_\sigma$. From Chapter 5, we have that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Set:

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2, \quad \sigma(t) = q_0 + q_1 t + q_2 t^2. \tag{7.1}$$

Since $t^3 = 0$, we see that:

$$\sigma(t^3) = \sigma(t)^3 = (q_0 + q_1 t + q_2 t^2)^3 = q_0^3 + 3q_0^2 q_1 t + 3q_0^2 q_2 t^2 + 3q_0 q_1^2 t^2 = 0 \implies q_0 = 0, \tag{7.2}$$

$$\partial_\sigma(t^3) = \sigma(t)^2 \partial_\sigma(t) + \partial_\sigma(t^2)t = \sigma(t)^2\partial_\sigma(t) + \sigma(t)t\partial_\sigma(t) + t^2\partial_\sigma(t) = 0. \tag{7.3}$$

Using the $\sigma$-twisted relation of Eq. (4.3) of Theorem 1, the elements $e = \partial_\sigma, h = -2t\partial_\sigma, f = -t^2\partial_\sigma$, what we showed in Eq. (7.2) (that $q_0 = 0$), $\partial_\sigma(1) = 0$, $\sigma(1) = 1$, and the $\sigma$-twisted Leibniz rule:

$$\langle h, e\rangle = \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2p_0 e - p_1 h - 2p_2 f, \tag{7.4}$$

$$\langle h, f\rangle = \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = -2q_1 p_0 f, \tag{7.5}$$

$$\langle e, f\rangle = \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = \frac{q_1 p_0 + p_0}{2} h + (q_1 p_1 + q_2 p_0 + p_1) f. \tag{7.6}$$

We also have to consider the case when $\partial_\sigma(t^3) = 0$, which we showed in Eq. (7.3) was $\partial_\sigma(t^3) = \sigma(t)^2\partial_\sigma(t) + \sigma(t)t\partial_\sigma(t) + t^2\partial_\sigma(t) = (\sigma(t)^2 + (\sigma(t) + t)t)\partial_\sigma(t)$. By Eq. (7.1) and $q_0 = 0$, we get:

$$\partial_\sigma(t^3) = ((q_1 t + q_2 t^2)^2 + q_1 t + q_2 t^2 + t)t)(p_0 + p_1 t + p_2 t^2) = (q_1^2 + q_1 + 1)p_0 t^2 = 0,$$

which gives us two more cases, when $p_0 = 0$ and when $q_1^2 + q_1 + 1 = 0$.

**Case 1:** Consider when $p_0 = 0$, which gives us the following relations, using what we showed in Eqs. (7.4)-(7.6):

$$\langle h, e\rangle = 2p_0 e - p_1 h - 2p_2 f = -p_1 h - 2p_2 f,$$
$$\langle h, f\rangle = -2q_1 p_0 f = 0,$$
$$\langle e, f\rangle = \frac{q_1 p_0 + p_0}{2} h + (q_1 p_1 + q_2 p_0 + p_1) f = (q_1 p_1 + p_1) f.$$

Since we cannot choose values such that $\langle h, f\rangle = -2f$, $\langle h, e\rangle = 2e$ and $\langle e, f\rangle = h$ we cannot recover $\mathfrak{sl}_2(\mathbb{F})$.

The condition for the deformed six-term Jacobi identity is that $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$. Using the equations (7.1), the left hand side becomes:

$$\begin{aligned}\partial_\sigma(\sigma(t)) = \partial_\sigma(q_1 t + q_2 t^2) = q_1 \partial_\sigma(t) + q_2 \partial_\sigma(t^2) &= q_1 \partial_\sigma(t) + q_2(\sigma(t)\partial_\sigma(t) + \partial_\sigma(t) t) \\ &= q_1 p_1 t + q_1 p_2 t^2 + q_2 q_1 p_1 t^2 + q_2 p_1 t^2,\end{aligned}$$

and the right hand side becomes:

$$\begin{aligned}\delta\sigma(\partial_\sigma(t)) &= \delta\sigma(p_1 t + p_2 t^2) = \delta(p_1\sigma(t) + p_2\sigma(t^2)) \\ &= \delta(p_1(q_1 t + q_2 t^2) + p_2(q_1 t + q_2 t^2)^2) = \delta(p_1 q_1 t + p_1 q_2 t^2 + p_2 q_1^2 t^2).\end{aligned}$$

Let $\delta = \delta_0 + \delta_1 t + \delta_2 t^2$, then the RHS becomes:

$$\begin{aligned}\delta(p_1 q_1 t + p_1 q_2 t^2 + p_2 q_1^2 t^2) &= (\delta_0 + \delta_1 t + \delta_2 t^2)(p_1 q_1 t + p_1 q_2 t^2 + p_2 q_1^2 t^2) \\ &= \delta_0 p_1 q_1 t + \delta_0 p_1 q_2 t^2 + \delta_0 p_2 q_1^2 t^2 + \delta_1 p_1 q_1 t^2.\end{aligned}$$

The condition for the six-term Jacobi identity is thus:

$$q_1 p_1 t + q_1 p_2 t^2 + q_2 q_1 p_1 t^2 + q_2 p_1 t^2 = \delta_0 p_1 q_1 t + \delta_0 p_1 q_2 t^2 + \delta_0 p_2 q_1^2 t^2 + \delta_1 p_1 q_1 t^2.$$

By subtracting the LHS from both sides of the equality we can create a linear system of equations:

$$\begin{cases}\delta_0 q_1 p_1 - q_1 p_1 = 0 \\ \delta_0 p_1 q_2 + \delta_0 p_2 q_1^2 + \delta_1 p_1 q_1 - q_1 p_2 - q_2 q_1 p_1 - q_2 p_1 = 0\end{cases}$$

We set $\delta_2$ arbitrary to $\delta_2 := \xi_2$, since it's not included in the linear system of equations. This linear system of equations can be solved with several different values of $q_1$, $q_2$, $p_1$, $p_2$:

- If $p_1 q_1 \neq 0$, then solving the system of linear equations for $\delta_0$ and $\delta_1$ gives $\delta_0 = 1$ and

  $$p_1 q_2 + p_2 q_1^2 + \delta_1 p_1 q_1 - q_1 p_2 - q_2 q_1 p_1 - q_2 p_1 = 0 \implies \delta_1 = \frac{p_2 + p_1 q_2 - p_2 q_1}{p_1}.$$

  The deformed six-term Jacobi identity is thus:

  $$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y, z\rangle_\sigma\rangle_\sigma + (1 + (\frac{p_2 + p_1 q_2 - p_2 q_1}{p_1}) t + \xi_2 t^2) \cdot \langle x, \langle y, z\rangle_\sigma\rangle_\sigma) = 0,$$

  which is the deformed six-term Jacobi identity for a quasi-hom-Lie-algebra.

- If $q_1 = 0,\ p_1 \neq 0,\ q_2 \neq 0$, then solving the linear system of equations for $\delta_0$ gives $\delta_0 = 1$ and $\delta_1 := \xi_1$ is arbitrary. Since $\xi_1$ and $\xi_2$ are arbitrary we can set them to zero, which gives us the deformed six-term Jacobi identity of a hom-Lie-algebra:

$$\circlearrowleft_{x,y,z} \langle(\sigma+\mathrm{id})(x), \langle y,z\rangle_\sigma\rangle_\sigma = 0.$$

- If $p_1 \neq 0,\ q_1 = q_2 = 0$, then all terms of the equations are 0, hence $\delta_0 = \xi_0$ and $\delta_1 = \xi_1$ are arbitrary. This is a somewhat trivial case since $\sigma(t) = 0$. The deformed six-term Jacobi identity is:

$$\circlearrowleft_{x,y,z} (\langle\sigma(x), \langle y,z\rangle_\sigma\rangle_\sigma + (\xi_0+\xi_1 t+\xi_2 t^2)\cdot\langle x, \langle y,z\rangle_\sigma\rangle_\sigma) = 0.$$

  This gives us the deformed six-term Jacobi identity of a hom-Lie algebra if we set $\xi_0 = 1$ and $\xi_1 = \xi_2 = 0$:

$$\circlearrowleft_{x,y,z} \langle(\sigma+\mathrm{id})(x), \langle y,z\rangle_\sigma\rangle_\sigma = 0.$$

- If $p_1 = 0,\ p_2 \neq 0,\ q_1 \neq 0$, then solving the system of linear equations for $\delta_0$ and $\delta_1$ gives $\delta_0 = \frac{1}{q_1}$ and $\delta_1 := \xi_1$ is arbitrary, which gives the deformed six-term Jacobi identity:

$$\circlearrowleft_{x,y,z} (\langle\sigma(x), \langle y,z\rangle_\sigma\rangle_\sigma + (\frac{1}{q_1}+\xi_1 t+\xi_2 t^2)\cdot\langle x, \langle y,z\rangle_\sigma\rangle_\sigma) = 0,$$

  We can again set $\xi_1$ and $\xi_2$ to zero, and after rescaling $\sigma$ we again get the deformed six-term Jacobi identity of a hom-Lie-algebra:

$$\circlearrowleft_{x,y,z} \langle(\sigma+\mathrm{id})(x), \langle y,z\rangle_\sigma\rangle_\sigma = 0.$$

**Case 2:** The second case when the equality $\partial_\sigma(t^3) = 0$ is correct occurs when $p_0 \neq 0$ and $q_1^2 + q_1 + 1 = 0$, which requires that $\mathbb{F}$ includes the roots of the second degree polynomial. The quadratic equation has its roots at $q_{1,1} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $q_{1,2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. We therefore set $q_1 = \omega^k$, where $\omega = e^{\frac{2\pi}{3}i}$ and $k = 1,2$, which gives us $\omega = e^{\frac{2\pi}{3}i} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = e^{\frac{4\pi}{3}i} = \cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. With this value on $q_1$ we get the relations, from Eqs. (7.4)-(7.6):

$$\langle h,e\rangle = 2p_0 e - p_1 h - 2p_2 f,$$
$$\langle h,f\rangle = -2q_1 p_0 f = -2\omega^k p_0 f,$$
$$\langle e,f\rangle = \frac{q_1 p_0 + p_0}{2}h + (q_1 p_1 + q_2 p_0 + p_1)f = \frac{\omega^k p_0 + p_0}{2}h + (\omega^k p_1 + q_2 p_0 + p_1)f.$$

The condition for the deformed six-term Jacobi identity is $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$. With $p_0 \neq 0$ and $q_1 = \omega^k$, the left hand side is:

$$\begin{aligned}\partial_\sigma(\sigma(t)) = \partial_\sigma(\omega^k t + q_2 t^2) = \omega^k\partial_\sigma(t) + q_2\partial_\sigma(t^2)\omega^k(p_0 + p_1 t + p_2 t^2 + q_2(\sigma(t)\partial_\sigma(t) + \partial_\sigma(t)t)\\ = \omega^k p_0 + \omega^k p_1 t + \omega^k p_2 t^2 + \omega^k q_2 p_0 t + \omega^k q_2 p_1 t^2 + p_0 q_2^2 t^2 + p_0 q_2 t + p_1 q_2 t^2,\end{aligned}$$

and the right hand side is:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_0+p_1t+p_2t^2) = \delta(p_0+p_1\sigma(t)+p_2\sigma(t^2))$$
$$= \delta(p_0+p_1(\omega^kt+q_2t^2)+p_2(\omega^kt+q_2t^2)^2) = \delta(p_0+p_1\omega^kt+p_1q_2t^2+p_2\omega^{2k}t^2).$$

Let $\delta = \delta_0+\delta_1t+\delta_2t^2$, then the RHS is:

$$(\delta_0+\delta_1t+\delta_2t^2)(p_0+p_1\omega^kt+p_1q_2t^2+p_2\omega^{2k}t^2)$$
$$= \delta_0p_0+\delta_0p_1\omega^kt+\delta_0p_1q_2t^2+\delta_0p_1q_2t^2+\delta_0p_2\omega^{2k}t^2+\delta_1p_0t+\delta_1\omega^kp_1t^2+\delta_2p_0t^2.$$

This gives us that the condition for the deformed six-term Jacobi identity is:

$$\omega^kp_0+\omega^kp_1t+\omega^kp_2t^2+\omega^kq_2p_0t+\omega^kq_2p_1t^2+p_0q_2^2t^2+p_0q_2t+p_1q_2t^2$$
$$= \delta_0p_0+\delta_0+p_1\omega^kt+\delta_0p_1q_2t^2+\delta_0p_2q_2t^2+\delta_0p_2\omega^{2k}t^2+\delta_1p_0t+\delta_1\omega^kp_1t^2+\delta_2p_0t^2.$$

By subtracting the LHS from both sides of the equality we can create a linear system of equations:

$$\begin{cases}\delta_0p_0-\omega^kp_0=0\\ \delta_0p_1\omega^k+\delta_1p_0-\omega^kp_1-\omega^kq_2p_0-p_0q_2=0\\ \delta_0p_1q_2+\delta_0p_2\omega^{2k}+\delta_1\omega^kp_1+\delta_2p_0-\omega^kp_2-\omega^kq_2p_1-p_0q_2^2-p_1q_2=0\end{cases}$$

Solving the linear system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives $\delta_0=\omega^k$,

$$\delta_1 = \frac{p_0q_2+\omega^kq_2p_0+\omega^kp_1-p_1\omega^{2k}}{p_0} := \xi_1,$$

$$\delta_2 = \frac{\omega^kp_2+p_1q_2-p_2+p_0q_2^2-\frac{\omega^kp_1(\omega^kp_1+p_0q_2+\omega^kp_0q_2)-p_1^2}{p_0}}{p_0} := \xi_2.$$

The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(\omega^k+\xi_1t+\xi_2t^2)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma)=0,$$

which is a quasi-hom-Lie algebra.

## 7.1 Examples of twisted derivations

In this section we take the examples of $\sigma$-derivations from Section 3.1 and we will see what the deformed relations (7.4)-(7.6) become with different parameters defining $\partial_\sigma(t)$ and $\sigma(t)$.

The equalities of Eqs. (7.2) and (7.3) require that $q_0=0$, and $p_0=0$ or $q_1=\omega^k$. Hence, the following $\sigma$-derivations are not defined on the quotient ring, since they either have $q_0\neq 0$ or $q_1\neq\omega^k$ and $p_0\neq 0$: *the ordinary differential operator*, *the shifted difference operator*, *the Jackson q-derivation operator*, *the continuous q-difference operator* and *the divided differences operator*.

**Example 22** (The Eulerian operator)**.** We get the Eulerian operator, where $\sigma(t) = t$ and $\partial(t) = t$, by setting $q_1 = 1$, $q_0 = q_2 = 0$ and $p_1 = 1$, $p_0 = p_2 = 0$ in the Equalities (7.1). The deformed relations (7.4)-(7.6) thus become:

$$\langle h,e\rangle = 2p_0e - p_1h - 2p_2f = -h,$$

$$\langle h,f\rangle = -2q_1p_0f = 0,$$

$$\langle e,f\rangle = \frac{q_1p_0 + p_0}{2}h + (q_1p_1 + q_2p_0 + p_1)f = 2f.$$

We have the following deformed six-term Jacobi identity from Case 1:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle\rangle + (1+\xi_2t^2)\langle x,\langle y,z\rangle\rangle) = 0,$$

where $\xi_2$ is arbitrary, which means, if we set $\xi_2 = 0$, we have a deformed six-term Jacobi identity for a hom-Lie algebra for these parameters.

**Example 23** (Nilpotent Imaginary Derivative operator)**.** Let $\mathbb{F} = \mathbb{C}$ and $p_j, q_j \in \mathbb{C}$, where $j = 0,1,2$, and $q_j = x_{q_j} + iy_{q_j}$ and $p_j = x_{p_j} + iy_{p_j}$, where $i$ is the imaginary unit. We get the nilpotent imaginary derivative operator, where $\sigma(t) = \overline{q_0} + \overline{q_1}t + \overline{q_2}t^2$ and $\partial_\sigma(t) = y_{p_0} + y_{p_1}t + y_{p_2}t^2$, by setting $q_0 = \overline{q_0}$, $q_1 = \overline{q_1}$, $q_2 = \overline{q_2}$ and $p_0 = y_{p_0}$, $p_1 = y_{p_1}$, $p_2 = y_{p_2}$ in the Equalities (7.1). The deformed relations (7.4)-(7.6) thus become:

$$\langle h,e\rangle = 2p_0e - p_1h - 2p_2f = 2y_{p_0}e - y_{p_1}h - 2y_{p_2}f,$$

$$\langle h,f\rangle = -2q_1p_0f = -2\overline{q_1}y_{p_0}f,$$

$$\langle e,f\rangle = \frac{q_1p_0 + p_0}{2}h + (q_1p_1 + q_2p_0 + p_1)f = \frac{\overline{q_1}y_{p_0} + y_{p_0}}{2}h + (\overline{q_1}y_{p_1} + \overline{q_2}y_{p_0} + y_{p_1})f.$$

The equalities of Eqs. (7.2) and (7.3) require that $\overline{q_0} = 0$, and $y_{p_0} = 0$ or $\overline{q_1} = \omega^k$. If $y_{p_0} = 0$, then we have a deformed six-term Jacobi identity of Case 1:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle\rangle + (1 + \frac{y_{p_2}\overline{q_1} - y_{p_2}\overline{q_1}^2 + y_{p_1}\overline{q_1q_2}}{y_{p_1}\overline{q_1}}t + \xi_2t^2)\langle x,\langle y,z\rangle\rangle) = 0,$$

where $\xi_2$ is arbitrary.

If $y_{p_0} \neq 0$ and $\overline{q_1} = \omega^k$, then we have a deformed six-term Jacobi identity of Case 2:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle\rangle + (\omega^k + \xi_1t + \xi_2t^2)\langle x,\langle y,z\rangle\rangle) = 0,$$

where

$$\xi_1 = \frac{y_{p_0}\overline{q_2} + y_{p_1}\overline{q_1} - y_{p_1}\overline{q_1}^2 + y_{p_0}\overline{q_1q_2}}{y_{p_0}},$$

$$\xi_2 = \frac{y_{p_1}\overline{q_2} + y_{p_2}\overline{q_1} + y_{p_0}\overline{q_2}^2 - y_{p_2}\overline{q_1}^3 - \frac{y_{p_1}\overline{q_1}(y_{p_0}\overline{q_2} + y_{p_1}\overline{q_1} - y_{p_1}\overline{q_1}^2 + y_{p_0}\overline{q_1q_2})}{y_{p_0}}}{y_{p_0}}.$$

**Example 24** ($\sigma(t)$ as a polynomial of degree $k$)**.** Let $\sigma(t) = S(t) = \sum_{j=0}^{k} s_j t^k$ where $k < 3$. The $\sigma$-twisted Leibniz rule is defined as:

$$(\partial_\sigma(ab))(t) = \partial_\sigma(a)(t)b(t) + \sigma(a)(t)\partial_\sigma(b)(t),$$

where

$$(\partial_\sigma a)(t) = \frac{\sigma(a)(t) - a(t)}{\sigma(t) - t} = \frac{a(S(t)) - a(t)}{S(t) - t}.$$

We get this $\sigma$-derivation by setting $q_i = s_i$ for $i = 1,2$, and $p_0 = 1$, $p_1 = p_2 = 0$ in the Equalities (7.1). The equalities of Eqs. (7.2) and (7.3) require that $s_0 = 0$ and (since $p_0 = 1$) that $s_1 = \omega^k$. The deformed relations (7.4)-(7.6) thus become:

$$\langle h, e \rangle = 2p_0 e - p_1 h - 2p_2 f = 2e,$$

$$\langle h, f \rangle = -2q_1 p_0 f = -2\omega^k f,$$

$$\langle e, f \rangle = \frac{q_1 p_0 + p_0}{2} h + (q_1 p_1 + q_2 p_0 + p_1) f = \frac{\omega^k + 1}{2} h + s_2 f.$$

We have the following deformed six-term Jacobi identity from Case 2:

$$\circlearrowleft_{x,y,z} \left( \langle \sigma(x), \langle y, z \rangle \rangle + (\omega^k + (s_2 + s_2\omega^k)t + s_2^2 t^2) \langle x, \langle y, z \rangle \rangle \right) = 0.$$

# Chapter 8

# Quasi-hom-Lie algebras of twisted vector fields for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]/(t^4)$

We take the algebra $\mathscr{A}$ as $\mathbb{F}[t]/(t^4)$, meaning that $t^n = 0$ for $n \geq 4$, and the field $\mathbb{F}$ now includes all fourth roots of unity. The elements are $e = \partial_\sigma$, $h = -2t\partial_\sigma$, $f = -t^2\partial_\sigma$. From Chapter 5, we have that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Set:

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3, \quad \sigma(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3. \tag{8.1}$$

Since $t^4 = 0$ we see that:

$$\begin{aligned}\sigma(t^4) = \sigma(t)^4 = (q_0 + q_1 t + q_2 t^2 + q_3 t^3)^4 = \\ q_0^4 + 4q_0^3 q_1 t + 4q_0^3 q_2 t^2 + 4q_0^3 q_3 t^3 + 6q_0^2 q_1^2 t^2 + 12 q_0^2 q_1 q_2 t^3 + 4q_0 q_1^3 t^3 = 0 \implies q_0 = 0,\end{aligned} \tag{8.2}$$

$$\begin{aligned}\partial_\sigma(t^4) = \partial_\sigma(t^3)t + \sigma(t)^3\partial_\sigma(t) = (\partial_\sigma(t^2)t + \sigma(t)^2\partial_\sigma(t))t + \sigma(t)^3\partial_\sigma(t) \\ = (t^3 + \sigma(t)t^2 + \sigma(t)^2 t + \sigma(t)^3)\partial_\sigma(t) = 0.\end{aligned} \tag{8.3}$$

Using the $\sigma$-twisted relation of Eq. (4.3), with $q_0 = 0$, $\partial_\sigma(1) = 0$, $\sigma(1) = 1$, and the $\sigma$-twisted Leibniz rule, we get the following relations:

$$\langle h, e\rangle = \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2(p_0 + p_1 t + p_2 t^2 + p_3 t^3)\partial_\sigma,$$

$$\langle h, f\rangle = \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = 2(q_1 p_0 t^2 + q_1 p_1 t^3 + q_2 p_0 t^3)\partial_\sigma,$$

$$\begin{aligned}\langle e, f\rangle = \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma \\ = -(p_0 t + p_1 t^2 + p_2 t^3 + q_1 p_0 t + q_1 p_1 t^2 + q_1 p_2 t^3 + q_2 p_0 t^2 + q_2 p_1 t^3 + q_3 p_0 t^3)\partial_\sigma.\end{aligned}$$

Since the relations no longer are linear combinations of $e, f, h$, we define a new generator $g_3 := t^3\partial_\sigma$. The relations become:

$$\langle h, e\rangle = 2(p_0 + p_1 t + p_2 t^2 + p_3 t^3)\partial_\sigma = 2p_0 e - p_1 h - 2p_2 f + 2p_3 g_3, \tag{8.4}$$

$$\langle h,f\rangle = 2(q_1p_0t^2+q_1p_1t^3+q_2p_0t^3)\partial_\sigma = -2q_1p_0f+2(q_1p_1+q_2p_0)g_3, \tag{8.5}$$

$$\begin{aligned}\langle e,f\rangle &= -(p_0t+p_1t^2+p_2t^3+q_1p_0t+q_1p_1t^2+q_1p_2t^3+q_2p_0t^2+q_2p_1t^3+q_3p_0t^3)\partial_\sigma \\ &= \frac{p_0+q_1p_0}{2}h+(p_1+q_1p_1+q_2p_0)f-(p_2+q_1p_2+q_2p_1+q_3p_0)g_3.\end{aligned} \tag{8.6}$$

In the case where $\partial_\sigma(t^4)=0$ in Eq. (8.3) we get:

$$\partial_\sigma(t^4) = (t^3+\sigma(t)t^2+\sigma(t)^2t+\sigma(t)^3)\partial_\sigma(t) = p_0(1+q_1+q_1^2+q_1^3)t^3.$$

This gives us two cases, when $p_0=0$ and when $p_0\neq 0$ and $1+q_1+q_1^2+q_1^3=0$.

**Case 1:** Consider when $p_0=0$, which gives us the following relations, using what we showed in Eqs. (8.4)-(8.6):

$$\langle h,e\rangle = -p_1h-2p_2f+2p_3g_3,$$

$$\langle h,f\rangle = 2q_1p_1g_3,$$

$$\langle e,f\rangle = (p_1+q_1p_1)f-(p_2+q_1p_2+q_2p_1)g_3.$$

The condition for the deformed six-term Jacobi identity is that $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$, which for this case the left hand side becomes:

$$\begin{aligned}\partial_\sigma(\sigma(t)) &= \partial_\sigma(q_1t+q_2t^2+q_3t^3) = q_1\partial_\sigma(t)+q_2\partial_\sigma(t^2)+q_3\partial_\sigma(t^3) \\ &= q_1\partial_\sigma(t)+q_2(\partial_\sigma(t)t+\sigma(t)\partial_\sigma(t))+q_3((\partial_\sigma(t)t+\sigma(t)\partial_\sigma(t))t+\sigma(t)^2\partial_\sigma(t)) \\ &= q_1p_1t+q_1p_2t^2+q_1p_3t^3+q_2p_1t^2+q_2p_2t^3+q_2q_1p_1t^2+q_2q_1p_2t^3+q_2^2p_1t^3+q_3p_1t^3 \\ &\quad +q_3q_1p_1t^3+q_3q_1^2p_1t^3.\end{aligned}$$

The right hand side becomes:

$$\begin{aligned}\delta\sigma(\partial_\sigma(t)) &= \delta\sigma(p_1t+p_2t^2+p_3t^3) = \delta(p_1\sigma(t)+p_2\sigma(t)^2+p_3\sigma(t)^3) \\ &= \delta(p_1q_1t+p_1q_2t^2+p_1q_3t^3+p_2q_1^2t^2+2p_2q_1q_2t^3+p_3q_1^3t^3).\end{aligned}$$

Let $\delta = \delta_0+\delta_1t+\delta_2t^2+\delta_3t^3$, then the RHS becomes:

$$\begin{aligned}&(\delta_0+\delta_1t+\delta_2t^2+\delta_3t^3)(p_1q_1t+p_1q_2t^2+p_1q_3t^3+p_2q_1^2t^2+2p_2q_1q_2t^3+p_3q_1^3t^3) \\ &= \delta_0p_1q_1t+\delta_0p_1q_2t^2+\delta_0p_1q_3t^3+\delta_0p_2q_1^2t^2+\delta_02p_2q_1q_2t^3+\delta_0p_3q_1^3t^3 \\ &\quad +\delta_1p_1q_1t^2+\delta_1p_1q_2t^3+\delta_1p_2q_1^2t^3+\delta_2p_1q_1t^3.\end{aligned}$$

The condition for the deformed six-term Jacobi identity is thus:

$$\begin{aligned}&q_1p_1t+q_1p_2t^2+q_1p_3t^3+q_2p_1t^2+q_2p_2t^3+q_2q_1p_1t^2+q_2q_1p_2t^3+q_2^2p_1t^3+q_3p_1t^3+q_3q_1p_1t^3 \\ &+q_3q_1^2p_1t^3 = \delta_0p_1q_1t+\delta_0p_1q_2t^2+\delta_0p_1q_3t^3+\delta_0p_2q_1^2t^2+\delta_02p_2q_1q_2t^3+\delta_0p_3q_1^3t^3 \\ &\quad +\delta_1p_1q_1t^2+\delta_1p_1q_2t^3+\delta_1p_2q_1^2t^3+\delta_2p_1q_1t^3.\end{aligned}$$

By subtracting the LHS from both sides of the equality we can create a linear system of equations:

$$\begin{cases} \delta_0 p_1 q_1 - q_1 p_1 = 0 \\ \delta_0 p_1 q_2 + \delta_0 p_2 q_1^2 + \delta_1 p_1 q_1 - q_1 p_2 - q_2 p_1 - q_2 q_1 p_1 = 0 \\ \delta_0 p_1 q_3 + \delta_0 2 p_2 q_1 q_2 + \delta_0 p_3 q_1^3 + \delta_1 p_1 q_2 + \delta_1 p_2 q_1^2 + \delta_2 p_1 q_1 \\ \quad -q_1 p_3 - q_2 p_2 - q_2 q_1 p_2 - q_2^2 p_1 - q_3 p_1 - q_3 q_1 p_1 - q_3 q_1^2 p_1 = 0 \end{cases}$$

Since $\delta_3$ is not included in the linear system of equations, it is arbitrary set to $\delta_3 := \xi_3$. The linear system of equations can be solved with several different values of $q_1$, $q_2$, $q_3$, $p_1$, $p_2$, $p_3$. which gives us the following cases:

- If $p_1 q_1 \neq 0$, then solving the linear system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives $\delta_0 = 1$ and

$$\delta_1 = \frac{p_2 + q_2 p_1 - p_2 q_1}{p_1},$$

$$\delta_2 = \frac{q_3 p_1^2 q_1 + q_3 p_1^2 - q_2 p_1 p_2 q_1 - p_3 p_1 q_1^2 + p_3 p_1 + p_2^2 q_1^2 - p_2^2 q_1}{p_1^2} := \xi_2.$$

  The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y,z\rangle_\sigma \rangle_\sigma + \left(1 + \frac{p_2 + q_2 p_1 - p_2 q_1}{p_1} t + \xi_2 t^2 + \xi_3 t^3\right) \cdot \langle x, \langle y,z\rangle_\sigma \rangle_\sigma\right) = 0.$$

- If $p_1 \neq 0$, $q_2 \neq 0$, $q_1 = 0$, then solving the linear system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives $\delta_0 = 1$ ,

$$\delta_1 = \frac{p_2 + q_2 p_1}{p_1},$$

  and $\delta_2 := \xi_2$ is arbitrary. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y,z\rangle_\sigma \rangle_\sigma + \left(1 + \frac{p_2 + q_2 p_1}{p_1} t + \xi_2 t^2 + \xi_3 t^3\right) \cdot \langle x, \langle y,z\rangle_\sigma \rangle_\sigma\right) = 0.$$

- If $p_1 \neq 0$, $q_1 = q_2 = 0$ then solving the linear system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives $\delta_0 = 1$, $\delta_1 := \xi_1$ and $\delta_2 := \xi_2$ are arbitrary. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y,z\rangle_\sigma \rangle_\sigma + (1 + \xi_1 t + \xi_2 t^2 + \xi_3 t^3) \cdot \langle x, \langle y,z\rangle_\sigma \rangle_\sigma\right) = 0.$$

- If $p_1 = 0$, $p_2 \neq 0$, $q_1 \neq 0$ then solving the linear system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives $\delta_0 = q_1^{-1}$,

$$\delta_1 = \frac{q_1 p_3 - q_2 p_2 + q_2 q_1 p_2 - p_3 q_1^2}{p_2 q_1^2} \underset{=}{} \xi_1,$$

  and $\delta_2 := \xi_2$ is arbitrary. This gives us the deformed six-term Jacobi identity:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y,z\rangle_\sigma \rangle_\sigma + (q_1^{-1} + \xi_1 t + \xi_2 t^2 + \xi_3 t^3) \cdot \langle x, \langle y,z\rangle_\sigma \rangle_\sigma\right) = 0.$$

- If $p_1 = p_2 = 0$, $q_1 \neq 0$, $q_2 \neq 0$ then solving the linear system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives $\delta_0 = q_1^{-2}$, $\delta_1 := \xi_1$ and $\delta_2 := \xi_2$ are arbitrary. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y,z\rangle_\sigma \rangle_\sigma + (q_1^{-2} + \xi_1 t + \xi_2 t^2 + \xi_3 t^3) \cdot \langle x, \langle y,z\rangle_\sigma \rangle_\sigma\right) = 0.$$

**Case 2:** The second case when $\partial_\sigma(t^4) = 0$ occurs when $p_0 \neq 0$ and $q_1^3 + q_1^2 + q_1 + 1 = 0$, which requires that $\mathbb{F}$ includes the roots of the third degree polynomial. The third degree polynomial has its roots at $q_{1,1} = -1$, $q_{1,2} = -i$, $q_{1,3} = i$. We set $q_1 = \omega^k$, where $\omega = e^{i\pi/2}$ and $k = 1,2,3$, which gives us $\omega = e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$, $\omega^2 = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$ and $\omega^3 = e^{i3\pi/2} = \cos(3\pi/2) + i\sin(3\pi/2) = -i$. With this value on $q_1$ we get the relations, from Eqs. (8.4)-(8.6):

$$\langle h,e\rangle = 2p_0e - p_1h - 2p_2f + 2p_3g_3,$$
$$\langle h,f\rangle = -2\omega^k p_0 f + 2(\omega^k p_1 + q_2p_0)g_3,$$
$$\langle e,f\rangle = \frac{p_0 + \omega^k p_0}{2}h + (p_1 + \omega^k p_1 + q_2p_0)f - (p_2 + \omega^k p_2 + q_2p_1 + q_3p_0)g_3.$$

The LHS of the deformed six-term Jacobi identity in this case is:

$$\begin{aligned}
\partial_\sigma(\sigma(t)) = \partial_\sigma(\omega^k t + q_2t^2 + q_3t^3) = \omega^k\partial_\sigma(t) + q_2\partial_\sigma(t^2) + q_3\partial_\sigma(t^3)\\
= \omega^k\partial_\sigma(t) + q_2(\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t)) + q_3((\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t))t + \sigma(t)^2\partial_\sigma(t))\\
= \omega^k p_0 + \omega^k p_2t^2 + \omega^k p_3t^3 + p_0q_3t^2 + p_1q_2t^2 + p_1q_3t^3 + p_2q_2t^3 + p_0q_2^2t^2 + p_1q_2^2t^3 + \omega^k p_1t + p_0q_2t\\
+\omega^k p_0q_2t + \omega^k p_0p_3t^2 + \omega^k p_1q_2t^2 + \omega^k p_1q_3t^3 + \omega^k p_2q_2t^3 + 2p_0q_2q_3t^3 + \omega^{2k}p_0q_3t^2\\
+\omega^{2k}p_1q_3t^3 + 2\omega^k p_0q_2q_3t^3
\end{aligned}$$

The RHS of the deformed six-term Jacobi identity in this case is:

$$\begin{aligned}
\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_0 + p_1t + p_2t^2 + p_3t^3) = \delta(p_0 + p_1\sigma(t) + p_2\sigma(t)^2 + p_3\sigma(t)^3)\\
= \delta(p_0 + p_1\omega^k t + p_1q_2t^2 + p_1q_3t^3 + p_2\omega^{2k}t^2 + 2\omega^k p_2q_2t^3 + p_3\omega^{3k}t^3)
\end{aligned}$$

Let $\delta = \delta_0 + \delta_1t + \delta_2t^2 + \delta_3t^3$. The RHS is thus:

$$\begin{aligned}
\delta(p_0 + p_1\omega^k t + p_1q_2t^2 + p_1q_3t^3 + p_2\omega^{2k}t^2 + 2\omega^k p_2q_2t^3 + p_3\omega^{3k}t^3)\\
= (\delta_0 + \delta_1t + \delta_2t^2 + \delta_3t^3)(p_0 + p_1\omega^k t + p_1q_2t^2 + p_1q_3t^3 + p_2\omega^{2k}t^2 + 2\omega^k p_2q_2t^3 + p_3\omega^{3k}t^3)\\
= \delta_0p_0 + \delta_0p_1\omega^k t + \delta_0p_1q_2t^2 + \delta_0p_1q_3t^3 + \delta_0p_2\omega^{2k}t^2 + \delta_0 2\omega^k p_2q_2t^3 + \delta_0p_3\omega^{3k}t^3 +\\
\delta_1p_0t + \delta_1p_1\omega^k t^2 + \delta_1p_1q_2t^3 + \delta_1p_2\omega^{2k}t^3 + \delta_2p_0t^2 + \delta_2p_1\omega^k t^3 + \delta_3p_0t^3
\end{aligned}$$

Subtracting the LHS from both sides of the equality, we set the condition equal to zero, which we can use to create a linear system of equations:

$$\begin{cases}
\delta_0p_0 - \omega^k p_0 = 0\\
\delta_0p_1\omega^k + \delta_1p_0 - \omega^k p_1 - q_2p_0 - q_2\omega^k p_0 = 0\\
\delta_0p_1q_2 + \delta_0p_2\omega^{2k} + \delta_1p_1\omega^k + \delta_2p_0 - \omega^k p_2\\
\quad -p_0q_3 - p_1q_2 - p_0q_2^2 - \omega^k p_0p_3 - \omega^k p_1q_2 - \omega^{2k}p_0q_3 = 0\\
\delta_0p_1q_3 + \delta_0 2\omega^k p_2q_2 + \delta_0p_3\omega^{3k} + \delta_1p_1q_2 + \delta_1p_2\omega^{2k} + \delta_2p_1\omega^k + \delta_3p_0 - \omega^k p_3\\
\quad -p_1q_3 - p_2q_2 - p_1q_2^2 - \omega^k p_1q_3 - \omega^k p_2q_2 - 2p_0q_2q_3 - \omega^{2k}p_1q_3 - 2\omega^k p_0q_2q_3 = 0
\end{cases}$$

Solving the linear system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives $\delta_0 = \omega^k$,

$$\delta_1 = \frac{\omega^k p_1 + q_2p_0 + \omega^k q_2p_0 - \omega^{2k}p_1}{p_0} := \xi_1,$$

$$\delta_2 = \frac{\omega^k p_2 + p_0q_3 + p_1q_2 - \omega^{3k}p_2 + p_0q_2^2 + \omega^{2k}p_0q_3 + \omega^k p_0q_3 - \frac{\omega^k p_1(\omega^k p_1 + p_0q_2 - \omega^{2k}p_1 + \omega^k p_0q_2)}{p_0}}{p_0} := \xi_2,$$

$$\delta_3 = \frac{-p_3p_0^2 + 2p_2p_0p_1 - p_1^3 - q_3\omega^{3k}p_0^2p_1 - p_2\omega^{3k}p_0^2q_2 + \omega^{3k}p_0p_1^2q_2 - p_2\omega^{3k}p_0p_1}{p_0^3}$$
$$+ \frac{\omega^{3k}p_1^3 - 3p_2\omega^{2k}p_0^2q_2 + 2\omega^{2k}p_0p_1^2q_2 - p_2\omega^{2k}p_0p_1 + 2q_3\omega^k p_0^3q_2 - 2\omega^k p_0^2p_1q_2^2}{p_0^3}$$
$$+ \frac{-q_3\omega^k p_0^2p_1 + p_2\omega^k p_0^2q_2 + p_3\omega^k p_0^2 - 2\omega^k p_0p_1^2q_2 + 2q_3p_0^3q_2 + q_3p_0^2p_1 + p_2p_0^2q_2}{p_0^3} := \xi_3.$$

The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z} \left(\langle\sigma(x), \langle y,z\rangle_\sigma\rangle_\sigma + (\omega^k + \xi_1t + \xi_2t^2 + \xi_3t^3)\cdot\langle x, \langle y,z\rangle_\sigma\rangle_\sigma\right) = 0.$$

## 8.1 Examples of twisted derivations

In this section we take the examples of $\sigma$-derivations from Section 3.1 and we will see what the deformed relations (8.4)-(8.6) become with different parameters defining $\partial_\sigma(t)$ and $\sigma(t)$.

The equalities of Eqs. (8.2) and (8.3) require that $q_0 = 0$, and $p_0 = 0$ or $q_1 = \omega^k$. Hence, the following $\sigma$-derivations are not defined on the quotient ring, since they either have $q_0 \neq 0$ or $q_1 \neq \omega^k$ and $p_0 \neq 0$: *the ordinary differential operator*, *the shifted difference operator*, *the Jackson q-derivation operator*, *the continuous q-difference operator* and *the divided differences operator*.

**Example 25** (The Eulerian operator)**.** We get the Eulerian operator, where $\sigma(t) = t$ and $\partial(t) = t$, by setting $q_1 = 1$, $q_0 = q_2 = 0$ and $p_1 = 1$, $p_0 = p_2 = 0$ in the Equalities (7.1). The deformed relations (8.4)-(8.6) thus become:

$$\langle h, e\rangle = 2(p_0 + p_1t + p_2t^2 + p_3t^3)\partial_\sigma = -h,$$
$$\langle h, f\rangle = 2(q_1p_0t^2 + q_1p_1t^3 + q_2p_0t^3)\partial_\sigma = 2g_3,$$
$$\langle e, f\rangle = -(p_0t + p_1t^2 + p_2t^3 + q_1p_0t + q_1p_1t^2 + q_1p_2t^3 + q_2p_0t^2 + q_2p_1t^3 + q_3p_0t^3)\partial_\sigma = 2f.$$

We have the following deformed six-term Jacobi identity from Case 1:

$$\circlearrowleft_{x,y,z} \left(\langle\sigma(x), \langle y,z\rangle\rangle + (1+\xi_3t^3)\langle x, \langle y,z\rangle\rangle\right) = 0,$$

where $\xi_3$ is arbitrary, which means we have a hom-Lie algebra for these parameters.

**Example 26** (The Nilpotent Imaginary Derivative operator)**.** Let $p_j, q_j \in \mathbb{C}$, where $j = 0,1,2,3$, and $q_j = x_{q_j} + iy_{q_j}$ and $p_j = x_{p_j} + iy_{p_j}$, where $i$ is the imaginary unit. We get the nilpotent imaginary derivative operator, where $\sigma(t) = \overline{q_0} + \overline{q_1}t + \overline{q_2}t^2 + \overline{q_3}t^3$ and $\partial_\sigma(t) = y_{p_0} + y_{p_1}t + y_{p_2}t^2 + y_{p_3}t^3$, by setting $q_0 = \overline{q_0}$, $q_1 = \overline{q_1}$, $q_2 = \overline{q_2}$, $q_3 = \overline{q_3}$ and $p_0 = y_{p_0}$, $p_1 = y_{p_1}$, $p_2 = y_{p_2}$, $p_3 = y_{p_3}$ in the Equalities (8.1). The deformed relations (8.4)-(8.6) thus become:

$$\langle h, e\rangle = 2(p_0 + p_1t + p_2t^2 + p_3t^3)\partial_\sigma = 2y_{p_0}e - y_{p_1}h - 2y_{p_2}f + 2y_{p_3}g_3,$$

$$\langle h,f\rangle = 2(q_1p_0t^2+q_1p_1t^3+q_2p_0t^3)\partial_\sigma = -2\overline{q_1}y_{p_0}f+2(\overline{q_1}y_{p_1}\overline{q_2}y_{p_0})g_3,$$

$$\langle e,f\rangle = -(p_0t+p_1t^2+p_2t^3+q_1p_0t+q_1p_1t^2+q_1p_2t^3+q_2p_0t^2+q_2p_1t^3+q_3p_0t^3)\partial_\sigma$$

$$= \frac{\overline{q_1}y_{p_0}+y_{p_0}}{2}h+(\overline{q_1}y_{p_1}+\overline{q_2}y_{p_0}+y_{p_1})f-(y_{p_2}+\overline{q_1}y_{p_2}+\overline{q_2}y_{p_1}+\overline{q_3}y_{p_0})g_3.$$

The equalities of Eqs. (8.2) and (8.3) require that $\overline{q_0}=0$, and $y_{p_0}=0$ or $\overline{q_1}=\omega^k$. If $y_{p_0}=0$, then we have a deformed six-term Jacobi identity of Case 1:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(1+\xi_1t+\xi_2t^2+\xi_3t^3)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma)=0,$$

where

$$\delta_1 = \frac{y_{p_2}+\overline{q_2}y_{p_1}-y_{p_2}q_1}{y_{p_1}},$$

$$\xi_2 = \frac{\overline{q_3}y_{p_1}^2q_1+\overline{q_3}y_{p_1}^2-\overline{q_2}y_{p_1}y_{p_2}q_1-y_{p_3}y_{p_1}q_1^2+y_{p_3}y_{p_1}+y_{p_2}^2q_1^2-y_{p_2}^2q_1}{y_{p_1}^2},$$

and $\xi_3$ is arbitrary.

If $y_{p_0}\neq 0$ and $\overline{q_1}=\omega^k$, then we have a deformed six-term Jacobi identity of Case 2:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(\omega^k+\xi_1t+\xi_2t^2+\xi_3t^3)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma)=0,$$

where

$$\xi_1 = \frac{\omega^ky_{p_1}+\overline{q_2}y_{p_0}+\omega^k\overline{q_2}y_{p_0}-\omega^{2k}y_{p_1}}{y_{p_0}},$$

$$\xi_2 = \frac{\omega^ky_{p_2}+y_{p_0}\overline{q_3}+y_{p_1}\overline{q_2}-\omega^{3k}y_{p_2}+y_{p_0}\overline{q_2}^2+\omega^{2k}y_{p_0}\overline{q_3}+\omega^ky_{p_0}\overline{q_3}}{y_{p_0}}$$
$$-\frac{\omega^ky_{p_1}(\omega^ky_{p_1}+y_{p_0}\overline{q_2}-\omega^{2k}y_{p_1}+\omega^ky_{p_0}\overline{q_2})}{y_{p_0}^2},$$

$$\xi_3 = \frac{-y_{p_3}y_{p_0}^2+2y_{p_2}y_{p_0}y_{p_1}-y_{p_1}^3-\overline{q_3}\omega^{3k}y_{p_0}^2y_{p_1}-y_{p_2}\omega^{3k}y_{p_0}^2\overline{q_2}+\omega^{3k}y_{p_0}y_{p_1}^2\overline{q_2}-y_{p_2}\omega^{3k}y_{p_0}y_{p_1}}{y_{p_0}^3}$$
$$+\frac{\omega^{3k}y_{p_1}^3-3y_{p_2}\omega^{2k}y_{p_0}^2\overline{q_2}+2\omega^{2k}y_{p_0}y_{p_1}^2\overline{q_2}-y_{p_2}\omega^{2k}y_{p_0}y_{p_1}+2\overline{q_3}\omega^ky_{p_0}^3\overline{q_2}-2\omega^ky_{p_0}^2y_{p_1}\overline{q_2}^2}{y_{p_0}^3}$$
$$+\frac{-\overline{q_3}\omega^ky_{p_0}^2y_{p_1}+y_{p_2}\omega^ky_{p_0}^2\overline{q_2}+y_{p_3}\omega^ky_{p_0}^2-2\omega^ky_{p_0}y_{p_1}^2\overline{q_2}+2\overline{q_3}y_{p_0}^3\overline{q_2}+\overline{q_3}y_{p_0}^2y_{p_1}+y_{p_2}y_{p_0}^2\overline{q_2}}{y_{p_0}^3}.$$

**Example 27** ($\sigma(t)$ as a polynomial of degree $k$)**.** Let $\sigma(t)=S(t)=\sum_{j=0}^k s_jt^k$ where $k<4$. The $\sigma$-twisted Leibniz rule is defined as:

$$(\partial_\sigma(ab))(t)=\partial_\sigma(a)(t)b(t)+\sigma(a)(t)\partial_\sigma(b)(t),$$

where

$$(\partial_\sigma a)(t)=\frac{\sigma(a)(t)-a(t)}{\sigma(t)-t}=\frac{a(S(t))-a(t)}{S(t)-t}.$$

We get this $\sigma$-derivation by setting $q_i = s_i$ for $i = 0, 1, \ldots, k$, and $p_0 = 1,\ p_1 = p_2 = p_3 = 0$ in the Equalities (8.1). The equalities from Eqs. (8.2) and (8.3) require that $s_0 = 0$ and (since $p_0 = 1$) that $s_1 = \omega^k$. The deformed relations (8.4)-(8.6) thus become:

$$\langle h, e\rangle = 2(p_0 + p_1 t + p_2 t^2 + p_3 t^3)\partial_\sigma = 2e,$$

$$\langle h, f\rangle = 2(q_1 p_0 t^2 + q_1 p_1 t^3 + q_2 p_0 t^3)\partial_\sigma = -2\omega^k f + 2 s_2 g_3,$$

$$\begin{aligned}\langle e, f\rangle &= -(p_0 t + p_1 t^2 + p_2 t^3 + q_1 p_0 t + q_1 p_1 t^2 + q_1 p_2 t^3 + q_2 p_0 t^2 + q_2 p_1 t^3 + q_3 p_0 t^3)\partial_\sigma \\ &= \frac{1+\omega^k}{2} h + s_2 f - s_3 g_3.\end{aligned}$$

We have the following deformed six-term Jacobi identity from Case 2:

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y, z\rangle\rangle + (\omega^k + \xi_1 t + \xi_2 t^2 + \xi_3 t^3)\langle x, \langle y, z\rangle\rangle\right) = 0,$$

where

$$\begin{aligned}\xi_1 &= s_2 + s_2\omega^k,\\ \xi_2 &= s_3 + s_2^2 + s_3\omega^k + s_3\omega^{2k},\\ \xi_3 &= 2 s_2 s_3 + 2 s_2 s_3 \omega^k.\end{aligned}$$

*Remark* 1. Note that the Eulerian operator, the nilpotent imaginary derivative operator and $\sigma(t)$ as a polynomial of degree $k$ are not linear combinations of $e, f, h$.

# Chapter 9

# Quasi-hom-Lie algebras of twisted vector fields for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]/(t^5)$

We take the algebra $\mathscr{A}$ as $\mathbb{F}[t]/(t^5)$, meaning that $t^n = 0$ for $n \geq 5$, and the field $\mathbb{F}$ now includes all fifth roots of unity. The elements are $e = \partial_\sigma$, $h = -2t\partial_\sigma$, $f = -t^2\partial_\sigma$. From Chapter 5, we have that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Set:

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4, \quad \sigma(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4. \tag{9.1}$$

Since $t^5 = 0$ we see that:

$$\begin{aligned}\sigma(t^5) &= \sigma(t)^5 = (q_0 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4)^5 = \\ &q_0^5 + 5q_0^4 q_1 t + 5q_0^4 q_2 t^2 + 5q_0^4 q_3 t^3 + 5q_0^4 q_4 t^4 + 10q_0^3 q_1^2 t^2 + 20q_0^3 q_1 q_2 t^3 + 20q_0^3 q_3 q_1 t^4 + 10q_0^3 q_2^2 t^4 \\ &+ 10q_0^2 q_1^3 t^3 + 30q_0^2 q_1^2 q_2 t^4 + 5q_0 q_1^4 t^4 = 0 \implies q_0 = 0.\end{aligned} \tag{9.2}$$

$$\partial_\sigma(t^5) = \partial_\sigma(t^4)t + \sigma(t)^4\partial_\sigma(t) = (t^4 + \sigma(t)t^3 + \sigma(t)^2 t^2 + \sigma(t)^3 t + \sigma(t)^4)\partial_\sigma(t) = 0. \tag{9.3}$$

Using the $\sigma$-twisted relation of Eq. (4.3), with $q_0 = 0$, $\partial_\sigma(1) = 0$, $\sigma(1) = 1$, and the $\sigma$-twisted Leibniz rule, we get the following relations:

$$\langle h, e\rangle = \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2(p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4)\partial_\sigma,$$

$$\langle h, f\rangle = \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = 2(q_1 p_0 t^2 + q_1 p_1 t^3 + q_1 p_2 t^4 + q_2 p_0 t^3 + q_2 p_1 t^4 + q_3 p_0 t^4)\partial_\sigma,$$

$$\begin{aligned}\langle e, f\rangle &= \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = -(p_0 t + p_1 t^2 + p_2 t^3 + p_3 t^4 \\ &+ q_1 p_0 t + q_1 p_1 t^2 + q_1 p_2 t^3 + q_1 p_3 t^4 + q_2 p_0 t^2 + q_2 p_1 t^3 + q_2 p_2 t^4 + q_3 p_0 t^3 + q_3 p_1 t^4 + q_4 p_0 t^4)\partial_\sigma.\end{aligned}$$

Since the relations no longer are linear combinations of $e, f, h$, we define the new generators $g_3 := t^3\partial_\sigma$ and $g_4 := t^4\partial_\sigma$. The relations become:

$$\langle h, e\rangle = 2(p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4)\partial_\sigma = 2p_0 e - p_1 h - 2p_2 f + 2p_3 g_3 + 2p_4 g_4, \tag{9.4}$$

$$\langle h,f\rangle = 2(q_1p_0t^2+q_1p_1t^3+q_1p_2t^4+q_2p_0t^3+q_2p_1t^4+q_3p_0t^4)\partial_\sigma$$
$$= -2q_1p_0f+2(q_1p_1+q_2p_0)g_3+2(q_1p_2+q_2p_1+q_3p_0)g_4, \quad (9.5)$$

$$\langle e,f\rangle = -(p_0t+p_1t^2+p_2t^3+p_3t^4+q_1p_0t+q_1p_1t^2+q_1p_2t^3+q_1p_3t^4+q_2p_0t^2$$
$$+q_2p_1t^3+q_2p_2t^4+q_3p_0t^3+q_3p_1t^4+q_4p_0t^4)\partial_\sigma$$
$$= \frac{p_0+q_1p_0}{2}h+(p_1+q_1p_1+q_2p_0)f-(p_2+q_1p_2+q_2p_1+q_3p_0)g_3$$
$$-(p_3+q_1p_3+q_2p_2+q_3p_1+q_4p_0)g_4. \quad (9.6)$$

In the case where $\partial_\sigma(t^5)=0$ in Eq. (9.3) we get:

$$\partial_\sigma(t^5) = (t^4+\sigma(t)t^3+\sigma(t)^2t^2+\sigma(t)^3t+\sigma(t)^4)\partial_\sigma(t) = p_0(1+q_1+q_1^2+q_1^3+q_1^4)t^4. \quad (9.7)$$

This gives us two cases, when $p_0=0$, and when $p_0\neq 0$ and $1+q_1+q_1^2+q_1^3+q_1^4=0$.

**Case 1:** We first consider when $p_0=0$, which gives us the following relations, using what we showed in Eqs. (9.4)-(9.6):

$$\langle h,e\rangle = -p_1h-2p_2f+2p_3g_3+2p_4g_4,$$
$$\langle h,f\rangle = 2q_1p_1g_3+2(q_1p_2+q_2p_1)g_4,$$
$$\langle e,f\rangle = (p_1+q_1p_1)f-(p_2+q_1p_2+q_2p_1)g_3-(p_3+q_1p_3+q_2p_2+q_3p_1)g_4.$$

The condition for the deformed six-term Jacobi identity is that $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$, which for this case the left hand side becomes:

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(q_1t+q_2t^2+q_3t^3+q_4t^4) = q_1\partial_\sigma(t)+q_2\partial_\sigma(t^2)+q_3\partial_\sigma(t^3)+q_4\partial_\sigma(t^4)$$
$$= q_1\partial_\sigma(t)+q_2(\partial_\sigma(t)t+\sigma(t)\partial_\sigma(t))+q_3((\partial_\sigma(t)t+\sigma(t)\partial_\sigma(t))t+\sigma(t)^2\partial_\sigma(t))$$
$$+q_4(((\partial_\sigma(t)t+\sigma(t)\partial_\sigma(t))t+\sigma(t)^2\partial_\sigma(t))t+\sigma(t)^3\partial_\sigma(t))$$
$$= q_1p_1t+q_1p_2t^2+q_1p_3t^3+q_1p_4t^4+q_2p_1t^2+q_2p_2t^3+q_2p_3t^4+q_2q_1p_1t^2+q_2q_1p_2t^3+q_2q_1p_3t^4$$
$$+q_2^2p_1t^3+q_2^2p_2t^4+2q_2q_3p_1t^4+q_3p_1t^3+q_3p_2t^4+q_3q_1p_1t^3+q_3q_1p_2t^4$$
$$+q_3q_1^2p_1t^3+q_3q_1^2p_2t^4+2q_3q_1q_2p_1t^4+q_4q_1t^4+q_4q_1p_1t^4+q_4q_1^2p_1t^4+q_4p_1q_1^3t^4.$$

The right hand side becomes:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_1t+p_2t^2+p_3t^3+p_4t^4) = \delta(p_1\sigma(t)+p_2\sigma(t)^2+p_3\sigma(t)^3+p_4\sigma(t)^4)$$
$$= \delta(p_1(q_1t+q_2t^2+q_3t^3+q_4t^4)+p_2(q_1t+q_2t^2+q_3t^3+q_4t^4)^2+p_3(q_1t+q_2t^2+q_3t^3+q_4t^4)^3$$
$$+p_4(q_1t+q_2t^2+q_3t^3+q_4t^4)^4) = \delta(p_1q_1t+p_1q_2t^2+p_1q_3t^3+p_1q_4t^4+p_2q_1^2t^2+2p_2q_1q_2t^3$$
$$+2p_2q_1q_3t^4+p_2q_2^2t^4+p_3q_1^3t^3+3p_3q_1^2q_2t^4+p_4q_1^4t^4).$$

Let $\delta = \delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3 + \delta_4 t^4$, then the RHS becomes:

$$\begin{aligned}
&(\delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3 + \delta_4 t^4)(p_1 q_1 t + p_1 q_2 t^2 + p_1 q_3 t^3 + p_1 q_4 t^4 + p_2 q_1^2 t^2 + 2 p_2 q_1 q_2 t^3 \\
&\qquad + 2 p_2 q_1 q_3 t^4 + p_2 q_2^2 t^4 + p_3 q_1^3 t^3 + 3 p_3 q_1^2 q_2 t^4 + p_4 q_1^4 t^4) \\
&= \delta_0 p_1 q_1 t + \delta_0 p_1 q_2 t^2 + \delta_0 p_1 q_3 t^3 + \delta_0 p_1 q_4 t^4 + \delta_0 p_2 q_1^2 t^2 + \delta_0 2 p_2 q_1 q_2 t^3 + \delta_0 2 p_2 q_1 q_3 t^4 \\
&\quad + \delta_0 p_2 q_2^2 t^4 + \delta_0 p_3 q_1^3 t^3 + \delta_0 3 p_3 q_1^2 q_2 t^4 + \delta_0 p_4 q_1^4 t^4 + \delta_1 p_1 q_1 t^2 + \delta_1 p_1 q_2 t^3 + \delta_1 p_1 q_3 t^4 \\
&\quad + \delta_1 p_2 q_1^2 t^3 + \delta_1 2 p_2 q_1 q_2 t^4 + \delta_1 p_3 q_1^3 t^4 + \delta_2 p_1 q_1 t^3 + \delta_2 p_1 q_2 t^4 + \delta_2 p_2 q_1^2 t^4 + \delta_3 p_1 q_1 t^4.
\end{aligned}$$

The condition for the deformed six-term Jacobi identity is thus:

$$\begin{aligned}
&q_1 p_1 t + q_1 p_2 t^2 + q_1 p_3 t^3 + q_1 p_4 t^4 + q_2 p_1 t^2 + q_2 p_2 t^3 + q_2 p_3 t^4 + q_2 q_1 p_1 t^2 + q_2 q_1 p_2 t^3 \\
&\quad + q_2 q_1 p_3 t^4 + q_2^2 p_1 t^3 + q_2^2 p_2 t^4 + 2 q_2 q_3 p_1 t^4 + q_3 p_1 t^3 + q_3 p_2 t^4 + q_3 q_1 p_1 t^3 + q_3 q_1 p_2 t^4 \\
&\quad + q_3 q_1^2 p_1 t^3 + q_3 q_1^2 p_2 t^4 + 2 q_3 q_1 q_2 p_1 t^4 + q_4 q_1 t^4 + q_4 q_1 p_1 t^4 + q_4 q_1^2 p_1 t^4 + q_4 p_1 q_1^3 t^4 \\
&= \delta_0 p_1 q_1 t + \delta_0 p_1 q_2 t^2 + \delta_0 p_1 q_3 t^3 + \delta_0 p_1 q_4 t^4 + \delta_0 p_2 q_1^2 t^2 + \delta_0 2 p_2 q_1 q_2 t^3 + \delta_0 2 p_2 q_1 q_3 t^4 \\
&\quad + \delta_0 p_2 q_2^2 t^4 + \delta_0 p_3 q_1^3 t^3 + \delta_0 3 p_3 q_1^2 q_2 t^4 + \delta_0 p_4 q_1^4 t^4 + \delta_1 p_1 q_1 t^2 + \delta_1 p_1 q_2 t^3 + \delta_1 p_1 q_3 t^4 \\
&\quad + \delta_1 p_2 q_1^2 t^3 + \delta_1 2 p_2 q_1 q_2 t^4 + \delta_1 p_3 q_1^3 t^4 + \delta_2 p_1 q_1 t^3 + \delta_2 p_1 q_2 t^4 + \delta_2 p_2 q_1^2 t^4 + \delta_3 p_1 q_1 t^4.
\end{aligned}$$

By subtracting the LHS from both sides of the equality we create a linear system of equations:

$$\begin{cases}
\delta_0 p_1 q_1 - p_1 q_1 = 0 \\
\delta_0 p_1 q_2 + \delta_0 p_2 q_1^2 + \delta_1 p_1 q_1 - q_1 p_2 - q_2 p_1 - q_2 q_1 p_1 = 0 \\
\delta_0 p_1 q_3 + \delta_0 2 p_2 q_1 q_2 + \delta_0 p_3 q_1^3 + \delta_1 p_1 q_2 + \delta_1 p_2 q_1^2 + \delta_2 p_1 q_1 \\
\quad - q_1 p_3 - q_2 p_2 - q_2 q_1 p_2 - q_2^2 p_1 - q_3 p_1 - q_3 q_1 p_1 - q_3 q_1^2 p_1 = 0 \\
\delta_0 p_1 q_4 + \delta_1 p_1 q_3 + \delta_2 p_1 q_2 + \delta_3 p_1 q_1 + \delta_0 p_2 q_2^2 + \delta_2 p_2 q_1^2 + \delta_1 p_3 q_1^3 + \delta_0 p_4 q_1^4 + 3\delta_0 p_3 q_1^2 q_2 \\
\quad + 2\delta_0 p_2 q_1 q_3 + 2\delta_1 p_2 q_1 q_2 - p_1 q_4 - p_2 q_3 - p_3 q_2 - p_4 q_1 - p_2 q_2^2 - p_1 q_1 q_4 - 2 p_1 q_2 q_3 \\
\quad - p_2 q_1 q_3 - p_3 q_1 q_2 - p_1 q_1^2 q_4 - p_2 q_1^2 q_3 - p_1 q_1^3 q_4 - 2 p_1 q_1 q_2 q_3 = 0
\end{cases}$$

Since $\delta_4$ is not included in the linear system of equations, it is arbitrary set to $\delta_4 := \xi_4$. The linar system of equations can be solved with several different values of $q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4$, which gives us the following cases:

- If $p_1 q_1 \neq 0$, then solving the linear system of equations for $\delta_0$, $\delta_1$, $\delta_2$ and $\delta_3$ gives $\delta_0 = 1$,

$$\delta_1 = \frac{p_2 + q_2 p_1 - p_2 q_1}{p_1} := \xi_1,$$

$$\delta_2 = \frac{q_3 p_1^2 q_1 + q_3 p_1^2 - q_2 p_1 p_2 q_1 - p_3 p_1 q_1^2 + p_3 p_1 + p_2^2 q_1^2 - p_2^2 q_1}{p_1^2} := \xi_2,$$

$$\begin{aligned}
\delta_3 = &\frac{q_4 p_1^3 q_1^2 + q_4 p_1^3 q_1 + q_3 p_1^3 q_2 + q_4 p_1^3 - q_3 p_1^2 p_2 q_1^2 - p_1^2 p_2 q_2^2 - p_4 p_1^2 q_1^3 - p_3 p_1^2 q_1^2 q_2}{p_1^3} \\
&+ \frac{-2 p_3 p_1^2 q_1 q_2 + p_3 p_1^2 q_2 p_4 p_1^2 + p_1 p_2^2 q_1^2 q_2 + p_1 p_2^2 q_1 q_2 - p_1 p_2^2 q_2 + 2 p_3 p_1 p_2 q_1^3 - p_3 p_1 p_2 q_1^2}{p_1^3} \\
&+ \frac{-p_3 p_1 p_2 q_1 - p_2^3 q_1^3 + p_2^3 q_1^2}{p_1^3} := \xi_3.
\end{aligned}$$

The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}\left(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(1+\xi_1 t+\xi_2 t^2+\xi_3 t^3+\xi_4 t^4)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma\right)=0.$$

- If $p_1\neq 0$, $q_2\neq 0$, $q_1=0$, then solving the linear system of equations for $\delta_0$, $\delta_1$, $\delta_2$ and $\delta_3$ gives $\delta_0=1$,

$$\delta_1=\frac{p_1q_2+p_2}{p_1}:=\xi_1,$$
$$\delta_2=\frac{p_3+p_1q_3}{p_1}:=\xi_2,$$

  and $\delta_3:=\xi_3$ is arbitrary. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}\left(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(1+\xi_1 t+\xi_2 t^2+\xi_3 t^3+\xi_4 t^4)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma\right)=0.$$

- If $p_1\neq 0$, $q_1=q_2=0$, then solving the linear system of equations for $\delta_0$, $\delta_1$, $\delta_2$ and $\delta_3$ gives $\delta_0=1$,

$$\delta_1=\frac{p_2}{p_1},$$

  and $\delta_2:=\xi_2$ and $\delta_3:=\xi_3$ are arbitrary. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}\left(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(1+\frac{p_2}{p_1}t+\xi_2 t^2+\xi_3 t^3+\xi_4 t^4)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma\right)=0.$$

- If $p_1=0$, $p_2\neq 0$, $q_1\neq 0$, then solving the linear system of equations for $\delta_0$, $\delta_1$, $\delta_2$ and $\delta_3$ gives $\delta_0=q_1^{-1}$,

$$\delta_1=\frac{(q_1-1)(p_2q_2-p_3q_1)}{p_2q_1^2}:=\xi_1,$$
$$\delta_2=\frac{q_3p_2^2q_1^3+q_3p_2^2q_1^2-p_2^2q_1q_2^2-q_3p_2^2q_1+p_2^2q_2^2-p_2p_3q_1^3q_2+p_2p_3q_1^2q_2-p_2p_3q_1q_2}{p_2^2q_1^3}$$
$$+\frac{-p_4p_2q_1^4+p_4p_2q_1^2+p_3^2q_1^4-p_3^2q_1^3}{p_2^2q_1^3}:=\xi_2$$

  and $\delta_3:=\xi_3$ is arbitrary. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}\left(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(q_1^{-1}+\xi_1 t+\xi_2 t^2+\xi_3 t^3+\xi_4 t^4)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma\right)=0.$$

- If $p_1=p_2=0$, $q_1\neq 0, q_2\neq 0$, then solving the linear system of equations for $\delta_0$, $\delta_1$, $\delta_2$ and $\delta_3$ gives $\delta_0=q_1^{-2}$,

$$\delta_1=\frac{p_3q_1q_2-2p_3q_2+p_4q_1-p_4q_1^2}{p_3q_1^3}:=\xi_1,$$

  and $\delta_2:=\xi_2$ and $\delta_3:=\xi_3$ are arbitrary. The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}\left(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma+(q_1^{-2}+\xi_1 t+\xi_2 t^2+\xi_3 t^3+\xi_4 t^4)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma\right)=0.$$

**Case 2:** The second case when $\partial_\sigma(t^5) = 0$ occurs when $p_0 \neq 0$ and $q_1^4 + q_1^3 + q_1^2 + q_1 + 1 = 0$, which requires that $\mathbb{F}$ includes the roots of the fourth degree polynomial. The fourth degree polynomial has its roots at $q_{1,1} = \frac{\sqrt{5}-1+\sqrt{2}\sqrt{\sqrt{5}+5}i}{4}$, $q_{1,2} = \frac{-\sqrt{5}-1+\sqrt{2}\sqrt{5-\sqrt{5}}i}{4}$, $q_{1,3} = \frac{-\sqrt{5}-1-\sqrt{2}\sqrt{5-\sqrt{5}}i}{4}$, $q_{1,4} = \frac{\sqrt{5}-1-\sqrt{2}\sqrt{\sqrt{5}+5}i}{4}$. We set $q_1 = \omega^k$, where $\omega = e^{i2\pi/5}$ and $k = 1,2,3,4$, which gives us $\omega = e^{i2\pi/5} = \cos(2\pi/5) + i\sin(2\pi/5) = q_{1,1}$, $\omega^2 = e^{i4\pi/5} = \cos(4\pi/5) + i\sin(4\pi/5) = q_{1,2}$, $\omega^3 = e^{i6\pi/5} = \cos(6\pi/5) + i\sin(6\pi/5) = q_{1,3}$ and $\omega^4 = e^{i8\pi/5} = \cos(8\pi/5) + i\sin(8\pi/5) = q_{1,4}$. With this value on $q_1$ we get the relations, from Eqs. (9.4)-(9.6):

$$\langle h,e\rangle = 2p_0e - p_1h - 2p_2f + 2p_3g_3 + 2p_4g_4,$$

$$\langle h,f\rangle = -2\omega^k p_0 f + 2(\omega^k p_1 + q_2p_0)g_3 + 2(\omega^k p_2 + q_2p_1 + q_2p_0)g_4,$$

$$\begin{aligned}\langle e,f\rangle = \frac{p_0+\omega^k p_0}{2}h + (p_1 + \omega^k p_1 + q_2p_0)f - (p_2 + \omega^k p_2 + q_2p_1 + q_3p_0)g_3 \\ -(p_3 + \omega^k p_3 + q_2p_2 + q_3p_1 + q_4p_0)g_4.\end{aligned}$$

The LHS of the deformed six-term Jacobi identity in this case is:

$$\begin{aligned}\partial_\sigma(\sigma(t)) &= \partial_\sigma(\omega^k t + q_2t^2 + q_3t^3 + q_4t^4) = \omega^k\partial_\sigma(t) + q_2\partial_\sigma(t^2) + q_3\partial_\sigma(t^3) + q_4\partial_\sigma(t^4) \\ &= \omega^k\partial_\sigma(t) + q_2(\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t)) + q_3((\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t))t + \sigma(t)^2\partial_\sigma(t)) \\ &\quad + q_4(((\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t))t + \sigma(t)^2\partial_\sigma(t))t + \sigma(t)^3\partial_\sigma(t)) \\ &= \omega^k p_0 + \omega^k p_2t^2 + \omega^k p_3t^3 + \omega^k p_4t^4 + p_0q_3t^2 + p_1q_2t^2 + p_0q_4t^3 + p_1q_3t^3 + p_2q_2t^3 + p_1q_4t^4 \\ &\quad + p_2q_3t^4 + p_3q_2t^4 + p_0q_2^2t^2 + p_1q_2^2t^3 + p_0q_3^2t^4 + p_2q_2^2t^4 + \omega^k p_1t + p_0q_2t + \omega^k p_0q_2t + \omega^k p_0q_3t^2 \\ &\quad + \omega^k p_1q_2t^2 + \omega^k p_0q_4t^3 + \omega^k p_1q_3t^3 + \omega^k p_2q_2t^3 + \omega^k p_1q_4t^4 + \omega^k p_2q_3t^4 + \omega^k p_3q_2t^4 \\ &\quad + 2p_0q_2q_3t^3 + 2p_0q_2q_4t^4 + 2p_1q_2q_3t^4 + \omega^{2k}p_0q_3t^2 + 2\omega^k p_0q_3^2t^4 + \omega^{2k}p_0q_4t^3 + \omega^{2k}p_1q_3t^3 \\ &\quad + \omega^{3k}p_0q_4t^3 + \omega^{2k}p_1q_4t^4 + \omega^{2k}p_2q_3t^4 + \omega^{3k}p_1q_4t^4 + p_0q_2^2q_3t^4 + 3\omega^{2k}p_0q_2q_4t^4 \\ &\quad + 2\omega^k p_0q_2q_3t^3 + 2\omega^k p_0q_2q_4t^4 + 2\omega^k p_1q_2q_3t^4.\end{aligned}$$

The RHS of the six-term Jacobi identity is:

$$\begin{aligned}\delta\sigma(\partial_\sigma(t)) &= \delta\sigma(p_0 + p_1t + p_2t^2 + p_3t^3 + p_4t^4) = \delta(p_0 + p_1\sigma(t) + p_2\sigma(t)^2 + p_3\sigma(t)^3 + p_4\sigma(t)^4) \\ &= \delta(p_0 + p_1\omega^k t + p_1q_2t^2 + p_1q_3t^3 + p_1q_4t^4 + p_2\omega^{2k}t^2 + 2p_2\omega^k q_2t^3 + 2p_2\omega^k q_3t^4 + p_2q_2^2t^4 \\ &\quad + p_3\omega^{3k}t^3 + 3p_3\omega^{2k}q_2t^4 + p_4\omega^{4k}t^4).\end{aligned}$$

Let $\delta = \delta_0 + \delta_1t + \delta_2t^2 + \delta_3t^3 + \delta_4t^4$. The RHS is thus:

$$\begin{aligned}&(\delta_0 + \delta_1t + \delta_2t^2 + \delta_3t^3 + \delta_4t^4)(p_0 + p_1\omega^k t + p_1q_2t^2 + p_1q_3t^3 + p_1q_4t^4 + p_2\omega^{2k}t^2 + 2p_2\omega^k q_2t^3 \\ &\quad + 2p_2\omega^k q_3t^4 + p_2q_2^2t^4 + p_3\omega^{3k}t^3 + 3p_3\omega^{2k}q_2t^4 + p_4\omega^{4k}t^4) \\ &= \delta_0p_0 + \delta_2p_0t^2 + \delta_3p_0t^3 + \delta_1p_0t + \delta_0\omega^k p_1t + \delta_1\omega^k p_1t^2 + \delta_2\omega^k p_1t^3 + \delta_3\omega^k p_1t^4 + \delta_0p_1q_2t^2 \\ &\quad + \delta_0p_1q_3t^3 + \delta_1p_1q_2t^3 + \delta_0p_1q_4t^4 + \delta_1p_1q_3t^4 + \delta_2p_1q_2t^4 + \delta_0\omega^{2k}p_2t^2 + \delta_1\omega^{2k}p_2t^3 \\ &\quad + \delta_0\omega^{3k}p_3t^3 + \delta_2\omega^{2k}p_2t^4 + \delta_1\omega^{3k}p_3t^4 + \delta_0\omega^{4k}p_4t^4 + \delta_0p_2q_2^2t^4 + 3\delta_0\omega^{2k}p_3q_2t^4 \\ &\quad + 2\delta_0\omega^k p_2q_2t^3 + 2\delta_0\omega^k p_2q_3t^4 + 2\delta_1\omega^k p_2q_2t^4 + \delta_4p_0t^4.\end{aligned}$$

By subtracting the LHS from both sides of the equality we create a linear system of equations:

$$\begin{cases} \delta_0 p_0 - \omega^k p_0 = 0 \\ \delta_1 p_0 + \delta_0 \omega^k p_1 - \omega^k p_1 - p_0 q_2 - \omega^k p_0 q_2 = 0 \\ \delta_2 p_0 + \delta_0 p_1 q_2 + \delta_0 \omega^{2k} p_2 + \delta_1 \omega^k p_1 - \omega^k p_2 - p_0 q_3 - p_1 q_2 - p_0 q_2^2 - \omega^{2k} p_0 q_3 - \omega^k p_0 q_3 \\ \quad -\omega^k p_1 q_2 = 0 \\ \delta_3 p_0 + \delta_1 \omega^{2k} p_2 + \delta_2 \omega^k p_1 + \delta_0 p_1 q_3 + \delta_1 p_1 q_2 + \delta_0 \omega^{3k} p_3 + 2\delta_0 \omega^k p_2 q_2 - \omega^k p_3 - p_0 q_4 \\ \quad - p_1 q_3 - p_2 q_2 - p_1 q_2^2 - \omega^{2k} p_0 q_4 - \omega^{2k} p_1 q_3 - \omega^{3k} p_0 q_4 - \omega^k p_0 q_4 - \omega^k p_1 q_3 - \omega^k p_2 q_2 \\ \quad -2 p_0 q_2 q_3 - 2\omega^k p_0 q_2 q_3 = 0 \\ \delta_4 p_0 + \delta_2 \omega^{2k} p_2 + \delta_1 \omega^{3k} p_3 + \delta_0 \omega^{4k} p_4 + \delta_0 p_2 q_2^2 + \delta_3 \omega^k p_1 + \delta_0 p_1 q_4 + \delta_1 p_1 q_3 + \delta_2 p_1 q_2 \\ \quad +2\delta_0 \omega^k p_2 q_3 + 2\delta_1 \omega^k p_2 q_2 + 3\delta_0 \omega^{2k} p_3 q_2 - p_1 q_4 - p_2 q_3 - p_3 q_2 - p_0 q_3^2 - p_2 q_2^2 - \omega^k p_4 \\ \quad -2\omega^k p_0 q_3^2 - \omega^{2k} p_1 q_4 - \omega^{2k} p_2 q_3 - \omega^{3k} p_1 q_4 - p_0 q_2^2 q_3 - \omega^k p_1 q_4 - \omega^k p_2 q_3 - \omega^k p_3 q_2 \\ \quad -2 p_0 q_2 q_4 - 2 p_1 q_2 q_3 - 2\omega^k p_0 q_2 q_4 - 2\omega^k p_1 q_2 q_3 - 3\omega^{2k} p_0 q_2 q_4 = 0 \end{cases}$$

Solving the linear system of equations for $\delta_0$, $\delta_1$, $\delta_2$, $\delta_3$ and $\delta_4$ gives $\delta_0 = \omega^k$,

$$\delta_1 = \frac{\omega^k p_1 + q_2 \omega^k p_0 + p_0 q_2 - p_1 \omega^{2k}}{p_0} := \xi_1,$$

$$\delta_2 = \frac{\omega^k p_2 + p_0 q_3 + p_1 q_2 - \omega^{3k} p_2 + p_0 q_2^2 + \omega^{2k} p_0 q_3 + \omega^k p_0 q_3}{p_0} - \frac{\omega^k p_1 (\omega^k p_1 + p_0 q_2 - \omega^{2k} p_1 + \omega^k p_0 q_2)}{p_0^2} := \xi_2,$$

$$\begin{aligned} \delta_3 = {} & \frac{\omega^k p_3 + p_0 q_4 + p_1 q_3 + p_2 q_2 - \omega^{4k} p_3 - 2\omega^k p_1 q_2^2 + \omega^{2k} p_0 q_4 - 3\omega^{2k} p_2 q_2 + \omega^{3k} p_0 q_4 - \omega^{3k} p_1 q_3}{p_0} \\ & + \frac{-\omega^{3k} p_2 q_2 + \omega^k p_0 q_4 - \omega^k p_1 q_3 + \omega^k p_2 q_2 + 2 p_0 q_2 q_3 + \frac{\omega^{3k} p_1^3 - \omega^{4k} p_1^3}{p_0^2} + 2\omega^k p_0 q_2 q_3}{p_0} \\ & + \frac{2\omega^{2k} p_1^2 q_2 + \omega^{3k} p_1^2 q_2 - \omega^{2k} p_1 p_2 - \omega^{3k} p_1 p_2 + 2\omega^{4k} p_1 p_2 - 2\omega^k p_1^2 q_2}{p_0^2} := \xi_3, \end{aligned}$$

$$\delta_4 = \frac{\omega^{4k}p_0^2p_1^2q_3 - q_4\omega^{4k}p_0^3p_1 - \omega^{4k}p_0^3p_2q_3 - p_3\omega^{4k}p_0^3q_2 + 2\omega^{4k}p_0^2p_1p_2q_2 - p_3\omega^{4k}p_0^2p_1}{p_0^4}$$
$$+\frac{-\omega^{4k}p_0p_1^3q_2 + 2\omega^{4k}p_0p_1^2p_2 - \omega^{4k}p_1^4 - \omega^{3k}p_0^3p_2q_3 - 4p_3\omega^{3k}p_0^3q_2 - p_4p_0^3 - p_0^2p_1^2q_2^2}{p_0^4}$$
$$+\frac{7\omega^{3k}p_0^2p_1p_2q_2 - \omega^{3k}p_0^2p_2^2 - 3\omega^{3k}p_0p_1^3q_2 + \omega^{3k}p_0p_1^2p_2 + 3q_4\omega^{2k}p_0^4q_2 - 3\omega^{2k}p_0^3p_1q_2q_3}{p_0^4}$$
$$+\frac{-3\omega^{2k}p_0^3p_2q_2^2 - 2\omega^{2k}p_0^3p_2q_3 + 3\omega^{2k}p_0^2p_1^2q_2^2 + 2\omega^{2k}p_0^2p_1^2q_3 - 4\omega^{2k}p_0^2p_1p_2q_2 + 2p_3p_0^2p_1}{p_0^4}$$
$$+\frac{-p_3\omega^{2k}p_0^2p_1 + 3\omega^{2k}p_0p_1^3q_2 + 2q_4\omega^kp_0^4q_2 + 2\omega^kp_0^4q_3^2 - 2\omega^kp_0^3p_1q_2q_3 - q_4\omega^kp_0^3p_1}{p_0^4}$$
$$+\frac{-3\omega^kp_0^3p_2q_2^2 + \omega^kp_0^3p_2q_3 + p_3\omega^kp_0^3q_2 + p_4\omega^kp_0^3 + \omega^kp_0^2p_1^2q_2^2 - 2\omega^kp_0^2p_1^2q_3 + p_0^2p_2^2}{p_0^4}$$
$$+\frac{-2\omega^kp_0^2p_1p_2q_2 + p_0^4q_2^2q_3 + 2q_4p_0^4q_2 + p_0^4q_3^2 - p_0^3p_1q_2^3 + q_4p_0^3p_1 + p_0^3p_2q_2^2 + p_0^3p_2q_3}{p_0^4}$$
$$+\frac{p_3p_0^3q_2 - 3p_0p_1^2p_2 + p_1^4}{p_0^4} := \xi_4.$$

The deformed six-term Jacobi identity is thus:

$$\circlearrowleft_{x,y,z}\left(\langle\sigma(x),\langle y,z\rangle_\sigma\rangle_\sigma + (\omega^k + \xi_1t + \xi_2t^2 + \xi_3t^3 + \xi_4t^4)\cdot\langle x,\langle y,z\rangle_\sigma\rangle_\sigma\right) = 0.$$

## 9.1 Examples of twisted derivations

In this section we take the examples of $\sigma$-derivations from Section 3.1 and we will see what the deformed relations (9.4)-(9.6) become with different parameters defining $\partial_\sigma(t)$ and $\sigma(t)$.

The equalities of Eqs. (9.2) and (9.3) require that $q_0 = 0$, and $p_0 = 0$ or $q_1 = \omega^k$. Hence, the following $\sigma$-derivations are not defined on the quotient ring, since they either have $q_0 \neq 0$ or $q_1 \neq \omega^k$ and $p_0 \neq 0$: *the ordinary differential operator*, *the shifted difference operator*, *the Jackson q-derivation operator*, *the continuous q-difference operator* and *the divided differences operator*.

**Example 28** (The Eulerian operator)**.** We get the Eulerian operator, where $\sigma(t) = t$ and $\partial(t) = t$, by setting $q_1 = 1$, $q_0 = q_2 = q_3 = q_4 = 0$ and $p_1 = 1$, $p_0 = p_2 = p_3 = p_4 = 0$ in the Equalities (9.1). The deformed relations (9.4)-(9.6) thus become:

$$\langle h,e\rangle = 2(p_0 + p_1t + p_2t^2 + p_3t^3 + p_4t^4)\partial_\sigma = -h,$$

$$\langle h,f\rangle = 2(q_1p_0t^2 + q_1p_1t^3 + q_1p_2t^4 + q_2p_0t^3 + q_2p_1t^4 + q_3p_0t^4)\partial_\sigma = 2g_3,$$
$$\langle e,f\rangle = -(p_0t + p_1t^2 + p_2t^3 + p_3t^4 + q_1p_0t + q_1p_1t^2 + q_1p_2t^3 + q_1p_3t^4 + q_2p_0t^2$$
$$+q_2p_1t^3 + q_2p_2t^4 + q_3p_0t^3 + q_3p_1t^4 + q_4p_0t^4)\partial_\sigma = 2f.$$

We have the following deformed six-term Jacobi identity from Case 1:

$$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y,z \rangle \rangle + (1+\xi_4) \cdot \langle x, \langle y,z \rangle \rangle) = 0,$$

where $\xi_4$ is arbitrary, which means we have a hom-Lie algebra for this operator.

**Example 29** (The Nilpotent Imaginary Derivative operator)**.** Let $p_j, q_j \in \mathbb{C}$, where $j = 0,1,2,3$, and $q_j = x_{q_j} + iy_{q_j}$ and $p_j = x_{p_j} + iy_{p_j}$, where $i$ is the imaginary unit. We get the nilpotent imaginary derivative operator, where $\sigma(t) = \overline{q_0} + \overline{q_1}t + \overline{q_2}t^2 + \overline{q_3}t^3 + \overline{q_4}t^4$ and $\partial_\sigma(t) = y_{p_0} + y_{p_1}t + y_{p_2}t^2 + y_{p_3}t^3$, by setting $q_0 = \overline{q_0}$, $q_1 = \overline{q_1}$, $q_2 = \overline{q_2}$, $q_3 = \overline{q_3}$, $q_4 = \overline{q_4}$ and $p_0 = y_{p_0}$, $p_1 = y_{p_1}$, $p_2 = y_{p_2}$, $p_3 = y_{p_3}$, $p_4 = y_{p_4}$ in the Equalities (9.1). The deformed relations (9.4)-(9.6) thus become:

$$\langle h,e \rangle = 2(p_0 + p_1t + p_2t^2 + p_3t^3 + p_4t^4)\partial_\sigma = 2y_{p_0}e - y_{p_1}h - 2y_{p_2}f + 2y_{p_3}g_3 + 2y_{p_4}g_4,$$

$$\begin{aligned}\langle h,f \rangle &= 2(q_1p_0t^2 + q_1p_1t^3 + q_1p_2t^4 + q_2p_0t^3 + q_2p_1t^4 + q_3p_0t^4)\partial_\sigma \\ &= -2(\overline{q_1}y_{p_0} + \overline{q_1}y_{p_1})f + 2(\overline{q_1}y_{p_1} + \overline{q_2}y_{p_0})g_3 + 2(\overline{q_1}y_{p_2} + \overline{q_2}y_{p_1} + \overline{q_3}y_{p_0})g_4,\end{aligned}$$

$$\begin{aligned}\langle e,f \rangle = -(&p_0t + p_1t^2 + p_2t^3 + p_3t^4 + q_1p_0t + q_1p_1t^2 + q_1p_2t^3 + q_1p_3t^4 + q_2p_0t^2 + q_2p_1t^3 + q_2p_2t^4 \\ &+ q_3p_0t^3 + q_3p_1t^4 + q_4p_0t^4)\partial_\sigma = \frac{\overline{q_1}y_{p_0} + y_{p_0}}{2}h + (\overline{q_1}y_{p_1} + \overline{q_2}y_{p_0} + y_{p_1})f \\ &- (y_{p_2} + \overline{q_1}y_{p_2} + \overline{q_2}y_{p_1} + \overline{q_3}y_{p_0})g_3 - (y_{p_3} + \overline{q_1}y_{p_3} + \overline{q_2}y_{p_2} + \overline{q_3}y_{p_1} + \overline{q_4}y_{p_0})g_4.\end{aligned}$$

The equalities of Eqs. (9.2) and (9.3) require that $\overline{q_0} = 0$, and $y_{p_0} = 0$ or $\overline{q_1} = \omega^k$. If $y_{p_0} = 0$, then we have a deformed six-term Jacobi identity of Case 1:

$$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y,z \rangle_\sigma \rangle_\sigma + (1 + \xi_1t + \xi_2t^2 + \xi_3t^3 + \xi_4t^4) \cdot \langle x, \langle y,z \rangle_\sigma \rangle_\sigma) = 0,$$

where

$$\xi_1 = \frac{y_{p_2} + \overline{q_2}y_{p_1} - y_{p_2}\overline{q_1}}{y_{p_1}},$$

$$\xi_2 = \frac{\overline{q_3}y_{p_1}^2\overline{q_1} + \overline{q_3}y_{p_1}^2 - \overline{q_2}y_{p_1}y_{p_2}\overline{q_1} - y_{p_3}y_{p_1}\overline{q_1}^2 + y_{p_3}y_{p_1} + y_{p_2}^2\overline{q_1}^2 - y_{p_2}^2\overline{q_1}}{y_{p_1}^2},$$

$$\begin{aligned}\xi_3 = &\frac{\overline{q_4}y_{p_1}^3\overline{q_1}^2 + \overline{q_4}y_{p_1}^3\overline{q_1} + \overline{q_3}y_{p_1}^3\overline{q_2} + \overline{q_4}y_{p_1}^3 - \overline{q_3}y_{p_1}^2y_{p_2}\overline{q_1}^2 - y_{p_1}^2y_{p_2}\overline{q_2}^2 - y_{p_4}y_{p_1}^2\overline{q_1}^3 - y_{p_3}y_{p_1}^2\overline{q_1}^2\overline{q_2}}{y_{p_1}^3} \\ &+ \frac{-2y_{p_3}y_{p_1}^2\overline{q_1q_2} + y_{p_3}y_{p_1}^2\overline{q_2}y_{p_4}y_{p_1}^2 + y_{p_1}y_{p_2}^2\overline{q_1}^2\overline{q_2} + y_{p_1}y_{p_2}^2\overline{q_1q_2} - y_{p_1}y_{p_2}^2\overline{q_2} + 2y_{p_3}y_{p_1}y_{p_2}\overline{q_1}^3}{y_{p_1}^3} \\ &\frac{-y_{p_3}y_{p_1}y_{p_2}\overline{q_1}^2 - y_{p_3}y_{p_1}y_{p_2}\overline{q_1} - y_{p_2}^3\overline{q_1}^3 + y_{p_2}^3\overline{q_1}^2}{y_{p_1}^3},\end{aligned}$$

and $\xi_4$ is arbitrary.

If $y_{p_0} \neq 0$ and $\overline{q_1} = \omega^k$, then we have a deformed six-term Jacobi identity of Case 2:

$$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y,z \rangle_\sigma \rangle_\sigma + (\omega^k + \xi_1t + \xi_2t^2 + \xi_3t^3 + \xi_4t^4) \cdot \langle x, \langle y,z \rangle_\sigma \rangle_\sigma) = 0,$$

where

$$\xi_1 = \frac{\omega^k y_{p_1} + \overline{q_2}\omega^k y_{p_0} + y_{p_0}\overline{q_2} - y_{p_1}\omega^{2k}}{y_{p_0}},$$

$$\xi_2 = \frac{\omega^k y_{p_2} + y_{p_0}\overline{q_3} + y_{p_1}\overline{q_2} - \omega^{3k} y_{p_2} + y_{p_0}\overline{q_2}^2 + \omega^{2k} y_{p_0}\overline{q_3} + \omega^k y_{p_0}\overline{q_3}}{y_{p_0}} - \frac{\omega^k y_{p_1}(\omega^k y_{p_1} + y_{p_0}\overline{q_2} - \omega^{2k} y_{p_1} + \omega^k y_{p_0}\overline{q_2})}{y_{p_0}^2},$$

$$\xi_3 = \frac{\omega^k y_{p_3} + y_{p_0}\overline{q_4} + y_{p_1}\overline{q_3} + y_{p_2}\overline{q_2} - \omega^{4k} y_{p_3} - 2\omega^k y_{p_1}\overline{q_2}^2 + \omega^{2k} y_{p_0}\overline{q_4} - 3\omega^{2k} y_{p_2}\overline{q_2}}{y_{p_0}}$$
$$+ \frac{\omega^{3k} y_{p_0}\overline{q_4} - \omega^{3k} y_{p_1}\overline{q_3} - \omega^{3k} y_{p_2}\overline{q_2} + \omega^k y_{p_0}\overline{q_4} - \omega^k y_{p_1}\overline{q_3} + \omega^k y_{p_2}\overline{q_2} + 2y_{p_0}\overline{q_2 q_3}}{y_{p_0}}$$
$$+ \frac{\frac{\omega^{3k} y_{p_1}^3 - \omega^{4k} y_{p_1}^3}{y_{p_0}^2} + 2\omega^k y_{p_0}\overline{q_2 q_3}}{y_{p_0}}$$
$$+ \frac{2\omega^{2k} y_{p_1}^2\overline{q_2} + \omega^{3k} y_{p_1}^2\overline{q_2} - \omega^{2k} y_{p_1} y_{p_2} - \omega^{3k} y_{p_1} y_{p_2} + 2\omega^{4k} y_{p_1} y_{p_2} - 2\omega^k y_{p_1}^2\overline{q_2}}{y_{p_0}^2},$$

$$\xi_4 = \frac{\omega^{4k} y_{p_0}^2 y_{p_1}^2\overline{q_3} - \overline{q_4}\omega^{4k} y_{p_0}^3 y_{p_1} - \omega^{4k} y_{p_0}^3 y_{p_2}\overline{q_3} - y_{p_3}\omega^{4k} y_{p_0}^3\overline{q_2} + 2\omega^{4k} y_{p_0}^2 y_{p_1} y_{p_2}\overline{q_2}}{y_{p_0}^4}$$
$$+ \frac{-y_{p_3}\omega^{4k} y_{p_0}^2 y_{p_1} - \omega^{4k} y_{p_0} y_{p_1}^3\overline{q_2} + 2\omega^{4k} y_{p_0} y_{p_1}^2 y_{p_2} - \omega^{4k} y_{p_1}^4 - \omega^{3k} y_{p_0}^3 y_{p_2}\overline{q_3} - 4y_{p_3}\omega^{3k} y_{p_0}^3\overline{q_2}}{y_{p_0}^4}$$
$$+ \frac{-y_{p_4} y_{p_0}^3 - y_{p_0}^2 y_{p_1}^2\overline{q_2}^2 + 7\omega^{3k} y_{p_0}^2 y_{p_1} y_{p_2}\overline{q_2} - \omega^{3k} y_{p_0}^2 y_{p_2}^2 - 3\omega^{3k} y_{p_0} y_{p_1}^3\overline{q_2} + \omega^{3k} y_{p_0} y_{p_1}^2 y_{p_2}}{y_{p_0}^4}$$
$$+ \frac{3\overline{q_4}\omega^{2k} y_{p_0}^4\overline{q_2} - 3\omega^{2k} y_{p_0}^3 y_{p_1}\overline{q_2 q_3} - 3\omega^{2k} y_{p_0}^3 y_{p_2}\overline{q_2}^2 - 2\omega^{2k} y_{p_0}^3 y_{p_2}\overline{q_3} + 3\omega^{2k} y_{p_0}^2 y_{p_1}^2\overline{q_2}^2}{y_{p_0}^4}$$
$$+ \frac{2\omega^{2k} y_{p_0}^2 y_{p_1}^2\overline{q_3} - 4\omega^{2k} y_{p_0}^2 y_{p_1} y_{p_2}\overline{q_2} + 2y_{p_3} y_{p_0}^2 y_{p_1} - y_{p_3}\omega^{2k} y_{p_0}^2 y_{p_1} + 3\omega^{2k} y_{p_0} y_{p_1}^3\overline{q_2}}{y_{p_0}^4}$$
$$+ \frac{2\overline{q_4}\omega^k y_{p_0}^4\overline{q_2} + 2\omega^k y_{p_0}^4\overline{q_3}^2 - 2\omega^k y_{p_0}^3 y_{p_1}\overline{q_2 q_3} - \overline{q_4}\omega^k y_{p_0}^3 y_{p_1} - 3\omega^k y_{p_0}^3 y_{p_2}\overline{q_2}^2}{y_{p_0}^4}$$
$$+ \frac{y_{p_4}\omega^k y_{p_0}^3 + \omega^k y_{p_0}^2 y_{p_1}^2\overline{q_2}^2 - 2\omega^k y_{p_0}^2 y_{p_1}^2\overline{q_3} + y_{p_0}^2 y_{p_2}^2 - 2\omega^k y_{p_0}^2 y_{p_1} y_{p_2}\overline{q_2} + y_{p_0}^4\overline{q_2}^2\overline{q_3}}{y_{p_0}^4}$$
$$+ \frac{+2\overline{q_4} y_{p_0}^4\overline{q_2} + y_{p_0}^4\overline{q_3}^2 - y_{p_0}^3 y_{p_1}\overline{q_2}^3 + \overline{q_4} y_{p_0}^3 y_{p_1} + y_{p_0}^3 y_{p_2}\overline{q_2}^2 + y_{p_0}^3 y_{p_2}\overline{q_3} + y_{p_3} y_{p_0}^3\overline{q_2}}{y_{p_0}^4}$$
$$+ \frac{-3y_{p_0} y_{p_1}^2 y_{p_2} + y_{p_1}^4 + \omega^k y_{p_0}^3 y_{p_2}\overline{q_3} + y_{p_3}\omega^k y_{p_0}^3\overline{q_2}}{y_{p_0}^4}.$$

**Example 30** ($\sigma(t)$ as a polynomial of degree $k$)**.** Let $\sigma(t) = S(t) = \sum_{j=0}^{k} s_j t^k$ where $k < 5$. The $\sigma$-twisted Leibniz rule is defined as:

$$(\partial_\sigma(ab))(t) = \partial_\sigma(a)(t)b(t) + \sigma(a)(t)\partial_\sigma(b)(t),$$

where

$$(\partial_\sigma a)(t) = \frac{\sigma(a)(t) - a(t)}{\sigma(t) - t} = \frac{a(S(t)) - a(t)}{S(t) - t}.$$

We get this $\sigma$-derivation by setting $q_i = s_i$ for $i = 0, 1, \ldots, k$, and $p_0 = 1$, $p_1 = p_2 = p_3 = p_4 = 0$ in the Equalities (9.1). The equalities from Eqs. (9.2) and (9.3) require that $s_0 = 0$ and (since $p_0 = 1$) that $s_1 = \omega^k$. The deformed relations (9.4)-(9.6) thus become:

$$\langle h, e\rangle = 2(p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4)\partial_\sigma = 2e,$$

$$\langle h, f\rangle = 2(q_1 p_0 t^2 + q_1 p_1 t^3 + q_1 p_2 t^4 + q_2 p_0 t^3 + q_2 p_1 t^4 + q_3 p_0 t^4)\partial_\sigma = -2\omega^k f + 2s_2 g_3 + 2s_3 g_4,$$

$$\langle e, f\rangle = -(p_0 t + p_1 t^2 + p_2 t^3 + p_3 t^4 + q_1 p_0 t + q_1 p_1 t^2 + q_1 p_2 t^3 + q_1 p_3 t^4 + q_2 p_0 t^2$$
$$+ q_2 p_1 t^3 + q_2 p_2 t^4 + q_3 p_0 t^3 + q_3 p_1 t^4 + q_4 p_0 t^4)\partial_\sigma = \frac{1+\omega^k}{2} h + s_2 f - s_3 g_3 - s_4 g_4.$$

We have the following deformed six-term Jacobi identity from Case 1:

$$\circlearrowleft_{x,y,z} (\langle \sigma(x), \langle y, z\rangle\rangle + (\omega^k + \xi_1 t + \xi_2 t^2 + \xi_3 t^3 + \xi_4 t^4) \cdot \langle x, \langle y, z\rangle\rangle) = 0,$$

where

$$\xi_1 = 2s_2\omega^k,$$
$$\xi_2 = 2s_2^2 + s_3 + s_3\omega^k + s_3\omega^{2k},$$
$$\xi_3 = 2s_2 s_3 + 2s_3 s_2\omega^k + s_4 + s_4\omega^k + s_4\omega^{2k} + s_4\omega^{3k},$$
$$\xi_4 = s_2 s_4 + s_3^2 + 2s_3^2\omega^k + s_3 s_2^2 + s_4 s_2 + 2s_4 s_2\omega^k + 2s_4 s_2\omega^{2k} + 2s_2\omega^{2k}.$$

*Remark* 2. Note that the Eulerian operator, the nilpotent imaginary derivative operator and $\sigma(t)$ as a polynomial of degree $k$ are not linear combinations of $e, f, h$.

# Chapter 10

# Quasi-hom-Lie algebras of twisted vector fields for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]/(t^n)$

We take the algebra $\mathscr{A}$ as $\mathbb{F}[t]/(t^n)$, meaning that $t^k = 0$ for all $k \geq n$, where $n$ is an integer greater than 2, and the field $\mathbb{F}$ now includes all $n$th roots of unity. The elements are $e = \partial_\sigma$, $h = -2t\partial_\sigma$, $f = -t^2\partial_\sigma$. From Chapter 5, we have that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Set:

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 + \dots + p_{n-1}t^{n-1}, \quad \sigma(t) = q_0 + q_1 t + q_2 t^2 + \dots + q_{n-1}t^{n-1}. \tag{10.1}$$

**Lemma 1.** *The nth power of $\sigma(t)$ implies that $q_0 = 0$.*

*Proof.* The $n$th power of $\sigma(t)$ gives an expansion where $q_0$ is a coefficient in all terms, since where $q_0$ is not a coefficient, the power $t$ is greater than $n$, hence $t^k = 0$ where $k \geq n$:

$$0 = \sigma(t^n) = \sigma(t)^n = (q_0 + q_1 t + q_2 t^2 + \dots + q_{n-1}t^{n-1})^n = q_0^n + q_0(q_0^{n-2}q_1 t + \dots) \implies q_0 = 0.$$

□

For $\partial_\sigma(t^n)$ we reach the following result:

$$\begin{aligned} 0 = \partial_\sigma(t^n) &= \partial_\sigma(t^{n-1})t + \sigma(t)^{n-1}\partial_\sigma(t) = (\partial_\sigma(t^{n-2})t + \sigma(t)^{n-2}\partial_\sigma(t))t + \sigma(t)^{n-1}\partial_\sigma(t) \\ &= \dots = \big(\dots((\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t))t + \sigma(t)^2\partial_\sigma(t))t + \dots \sigma(t)^{n-2}\partial_\sigma(t)\big)t + \sigma(t)^{n-1}\partial_\sigma(t) \\ &= (t^{n-1} + \sigma(t)t^{n-2} + \dots + \sigma(t)^{n-2}t + \sigma(t)^{n-1})\partial_\sigma(t) = \sum_{i=0}^{n-1}(t^{n-1-i}\sigma(t)^i)\partial_\sigma(t). \end{aligned} \tag{10.2}$$

Using the $\sigma$-twisted relation of Eq. (4.3), with $q_0 = 0$, $\partial_\sigma(1) = 0$, $\sigma(1) = 1$, and the $\sigma$-twisted Leibniz rule, we get the following relations:

$$\langle h, e \rangle = \langle -2t\partial_\sigma, \partial_\sigma \rangle_\sigma = 2\partial_\sigma(t)\partial_\sigma = 2\sum_{i=0}^{n-1}(p_i t^i)\partial_\sigma,$$

$$\langle h,f\rangle = \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma = 2\sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i}(q_i p_j t^{i+j+1})\partial_\sigma,$$

$$\langle e,f\rangle = \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = -(\partial_\sigma(t)t+\sigma(t)\partial_\sigma(t))\partial_\sigma = -(\sum_{i=0}^{n-2}p_i t^{i+1}+\sum_{i=1}^{n-1}\sum_{j=0}^{n-1-i}q_i p_j t^{i+j})\partial_\sigma.$$

Since the relations no longer are linear combinations of $e,f,h$ we define new generators $g_i := t^i\partial_\sigma$ for $i=0,\dots,n-1$. The relations become:

$$\langle h,e\rangle = 2\sum_{i=0}^{n-1}(p_i g_i)\partial_\sigma, \tag{10.3}$$

$$\langle h,f\rangle = 2\sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i}(q_i p_j g_{i+j+1})\partial_\sigma, \tag{10.4}$$

$$\langle e,f\rangle = -(\sum_{i=0}^{n-2}p_i g_{i+1}+\sum_{i=1}^{n-1}\sum_{j=0}^{n-1-i}q_i p_j g_{i+j})\partial_\sigma. \tag{10.5}$$

In the case where $\partial_\sigma(t^n)=0$ in Eq. (10.2) we get:

$$\begin{aligned} 0=\partial_\sigma(t^n) &= (t^{n-1}+\sigma(t)t^{n-2}+\sigma(t)^2t^{n-3}+\cdots+\sigma(t)^{n-2}t+\sigma(t)^{n-1})\partial_\sigma(t)\\ &= (t^{n-1}+q_1t^{n-1}+q_1^2t^{n-1}+\cdots+q_1^{n-2}t^{n-1}+q_1^{n-1}t^{n-1})p_0\\ &= (1+q_1+q_1^2+\cdots+q_1^{n-2}+q_1^{n-1})t^{n-1}p_0. \end{aligned} \tag{10.6}$$

This gives us the two cases when $p_0=0$, and when $p_0\neq 0$ and $1+q_1+q_1^2+\cdots+q_1^{n-1}=0$.

**Case 1:** Consider when $p_0=0$, which gives us the following relations, using what we showed in Eqs. (10.3)-(10.5):

$$\langle h,e\rangle = 2\sum_{i=1}^{n-1}(p_i g_i)\partial_\sigma,$$

$$\langle h,f\rangle = 2\sum_{i=1}^{n-2}\sum_{j=1}^{n-2-i}(q_i p_j g_{i+j+1})\partial_\sigma,$$

$$\langle e,f\rangle = -(\sum_{i=1}^{n-2}p_i g_{i+1}+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1-i}q_i p_j g_{i+j})\partial_\sigma.$$

**Case 2:** The second case when $\partial_\sigma(t^n)=0$ occurs when $p_0\neq 0$ and $1+q_1+q_1^2+\cdots+q_1^{n-1}=0$.

**Lemma 2.** *The solutions to the polynomial* $1+q_1+q_1^2+\cdots+q_1^{n-1}=0$ *is:*

$$\omega^k = e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right)+i\sin\left(\frac{2\pi k}{n}\right),\ \textit{where}\ k=1,2,\dots,n-1.$$

*Remark* 3. Some roots of the polynomial, whenever $n>2$, are complex. Hence, in Case 2 $\mathbb{F}$ has to include these complex values.

*Proof.* Multiply the polynomial with the fraction $\frac{q_1-1}{q_1-1}$:

$$\frac{(q_1-1)(1+q_1+q_1^2+\cdots+q_1^{n-1})}{q_1-1} = \frac{q_1^n-1}{q_1-1} = 0 \implies \frac{q_1^n}{q_1-1} = \frac{1}{q_1-1} \implies q_1^n = 1 \implies q_1 = \sqrt[n]{1}.$$

The solutions to this $n$th root are given by DeMoivre's theorem, Villanueva (2013):

$$q_1 = \sqrt[n]{1} = e^{2\pi ik/n} = \omega^k, \text{ where } k = 1,2,\dots,n-1.$$

□

With this value of $q_1$ we get the following relations of Eqs. (10.3)-(10.5):

$$\langle h,e\rangle = 2\sum_{i=0}^{n-1}(p_i g_i)\partial_\sigma,$$

$$\langle h,f\rangle = 2(\sum_{j=0}^{n-3}\omega^k p_j g_{j+2} + \sum_{i=2}^{n-2}\sum_{j=0}^{n-2-i} q_i p_j g_{i+j+1})\partial_\sigma,$$

$$\langle e,f\rangle = -(\sum_{i=0}^{n-2} p_i g_{i+1} + \sum_{j=0}^{n-2}\omega^k p_j g_{1+j} + \sum_{i=2}^{n-1}\sum_{j=0}^{n-1-i} q_i p_j g_{i+j})\partial_\sigma.$$

The condition for the deformed six-term Jacobi identity for the two cases is that $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$, which for this case the left hand side becomes:

$$\partial_\sigma(\sigma(t)) = \partial_\sigma(q_1 t + q_2 t^2 + \cdots + q_{n-1}t^{n-1}) = q_1\partial_\sigma(t) + q_2\partial_\sigma(t^2) + \cdots + q_{n-1}\partial_\sigma(t^{n-1}).$$

The $\sigma$-derivation of a power $k$ of $t$ is:

$$\begin{aligned}\partial_\sigma(t^k) &= \partial_\sigma(t^{k-1})t + \sigma(t)^{k-1}\partial_\sigma(t) = (\partial_\sigma(t^{k-2})t + \sigma(t)^{k-2}\partial_\sigma(t))t + \sigma(t)^{k-1}\partial_\sigma(t)\\ &= \cdots = (\dots((\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t))t + \sigma(t)^2\partial_\sigma(t))t + \cdots + \sigma(t)^{k-2}\partial_\sigma(t))t + \sigma(t)^{k-2}\partial_\sigma(t)\\ &= \partial_\sigma(t)t^{k-1} + \partial_\sigma(t)\sigma(t)t^{k-2} + \cdots + \partial_\sigma(t)\sigma(t)^{k-2}t + \partial_\sigma(t)\sigma(t)^{k-1} = \sum_{i=0}^{k-1}\partial_\sigma(t)\sigma(t)^i t^{k-1-i}.\end{aligned}$$

The LHS thus becomes:

$$\partial_\sigma(\sigma(t)) = q_1\partial_\sigma(t) + q_2\partial_\sigma(t^2) + \cdots + q_{n-1}\partial_\sigma(t^{n-1}) = \sum_{j=1}^{n-1}\sum_{i=0}^{j-1} q_j\partial_\sigma(t)\sigma(t)^i t^{j-1-i}.$$

This sum includes powers greater than $n-1$, meaning many terms of the sum are zero. The expression which includes only non-zero terms is:

$$\begin{aligned}\partial_\sigma(\sigma(t)) = &\sum_{x_1=1}^{n-1}\sum_{i=0}^{n-x_1} q_{x_1}p_i t^{x_1+i-1} + \sum_{x_1=2}^{n-1}\sum_{x_2=1}^{n-x_1}\sum_{i=0}^{n+1-x_1-x_2} q_{x_1}q_{x_2}p_i t^{x_1+x_2+i-2}\\ &+ \sum_{x_1=3}^{n-1}\sum_{x_2=1}^{n-x_1}\sum_{x_3=1}^{n+1-x_1-x_2}\sum_{i=0}^{n+2-\sum_{j=1}^3 x_j} q_{x_1}q_{x_2}q_{x_3}p_i t^{\sum_{j=1}^3 x_j+i-3}\\ &+ \cdots + \sum_{x_1=n-1}^{n-1}\sum_{x_2=1}^{1}\cdots\sum_{x_{n-1}=1}^{1}\sum_{i=0}^{1} q_{x_1}q_{x_2}\dots q_{x_{n-1}}p_i t^{n-1}.\end{aligned}$$

The RHS of the deformed six-term Jacobi identity in this case is:

$$\delta\sigma(\partial_\sigma(t)) = \delta\sigma(p_0 + p_1 t + p_2 t^2 + \cdots + p_{n-1}t^{n-1}) = \delta(p_1\sigma(t) + p_2\sigma(t)^2 + \cdots + p_{n-1}\sigma(t)^{n-1})$$
$$= \delta \sum_{i=0}^{n-1} p_i \sigma(t)^i.$$

This sum includes powers greather than $n-1$, meaning many terms of the sum are zero. The expresson which includes only non-zero terms is:

$$\begin{aligned}\delta\sigma(\partial_\sigma(t)) = \delta\Big(p_0 &+ \sum_{x_1=1}^{n-1} p_1 q_{x_1} t^{x_1} + \sum_{x_1=1}^{n-2}\sum_{x_2=1}^{n-1-x_1} p_2 q_{x_1} q_{x_2} t^{x_1+x_2} \\ &+ \sum_{x_1=1}^{n-3}\sum_{x_2=1}^{n-2-x_1}\sum_{x_3=1}^{n-1-x_1-x_2} p_3 q_{x_1} q_{x_2} q_{x_3} t^{x_1+x_2+x_3} \\ &+ \cdots + \sum_{x_1=1}^{1}\sum_{x_2=1}^{2-x_1}\sum_{x_3=1}^{3-x_1-x_2}\cdots\sum_{x_{n-1}}^{n-1-\sum_{i=0}^{n-2}x_i} p_{n-1} q_{x_1} q_{x_2} q_{x_3}\ldots q_{x_{n-1}} t^{\sum_{i=1}^{n-1}x_i}\Big).\end{aligned}$$

Let $\delta = \delta_0 + \delta_1 t + \cdots + \delta_{n-1}t^{n-1}$, then the RHS becomes:

$$\begin{aligned}(\delta_0 + \delta_1 t + \cdots + \delta_{n-1}t^{n-1})\Big(p_0 &+ \sum_{x_1=1}^{n-1} p_1 q_{x_1} t^{x_1} + \sum_{x_1=1}^{n-2}\sum_{x_2=1}^{n-1-x_1} p_2 q_{x_1} q_{x_2} t^{x_1+x_2} \\ &+ \sum_{x_1=1}^{n-3}\sum_{x_2=1}^{n-2-x_1}\sum_{x_3=1}^{n-1-x_1-x_2} p_3 q_{x_1} q_{x_2} q_{x_3} t^{x_1+x_2+x_3} \\ &+ \cdots + \sum_{x_1=1}^{1}\sum_{x_2=1}^{2-x_1}\sum_{x_3=1}^{3-x_1-x_2}\cdots\sum_{x_{n-1}=1}^{n-1-\sum_{i=0}^{n-2}x_i} p_{n-1} q_{x_1} q_{x_2} q_{x_3}\ldots q_{x_{n-1}} t^{\sum_{i=1}^{n-1}x_i}\Big) \\ = \delta_0\Big(p_0 + \sum_{x_1=1}^{n-1} p_1 q_{x_1} t^{x_1} &+ \sum_{x_1=1}^{n-2}\sum_{x_2=1}^{n-1-x_1} p_2 q_{x_1} q_{x_2} t^{x_1+x_2} + \sum_{x_1=1}^{n-3}\sum_{x_2=1}^{n-2-x_1}\sum_{x_3=1}^{n-1-x_1-x_2} p_3 q_{x_1} q_{x_2} q_{x_3} t^{x_1+x_2+x_3} \\ &+ \cdots + \sum_{x_1=1}^{1}\sum_{x_2=1}^{2-x_1}\sum_{x_3=1}^{3-x_1-x_2}\cdots\sum_{x_{n-1}=1}^{n-1-\sum_{i=0}^{n-2}x_i} p_{n-1} q_{x_1} q_{x_2} q_{x_3}\ldots q_{x_{n-1}} t^{\sum_{i=1}^{n-1}x_i}\Big) \\ + \delta_1\Big(p_0 t + \sum_{x_1=1}^{n-2} p_1 q_{x_1} t^{x_1+1} &+ \sum_{x_1=1}^{n-3}\sum_{x_2=1}^{n-2-x_1} p_2 q_{x_1} q_{x_2} t^{x_1+x_2+1} + \sum_{x_1=1}^{n-4}\sum_{x_2=1}^{n-3-x_1}\sum_{x_3=1}^{n-2-x_1-x_2} p_3 q_{x_1} q_{x_2} q_{x_3} t^{x_1+x_2+x_3+1} \\ &+ \cdots + \sum_{x_1=1}^{1}\sum_{x_2=1}^{2-x_1}\sum_{x_3=1}^{3-x_1-x_2}\cdots\sum_{x_{n-2}=1}^{n-1-\sum_{i=0}^{n-3}x_i} p_{n-2} q_{x_1} q_{x_2} q_{x_3}\ldots q_{x_{n-2}} t^{\sum_{i=1}^{n-2}x_i+1}\Big) +\end{aligned}$$

$$+\delta_2\Big(p_0t^2+\sum_{x_1=1}^{n-3}p_1q_{x_1}t^{x_1+2}+\sum_{x_1=1}^{n-4}\sum_{x_2=1}^{n-3-x_1}p_2q_{x_1}q_{x_2}t^{x_1+x_2+2}+\sum_{x_1=1}^{n-5}\sum_{x_2=1}^{n-4-x_1}\sum_{x_3=1}^{n-3-x_1-x_2}p_3q_{x_1}q_{x_2}q_{x_3}t^{x_1+x_2+x_3+2}$$
$$+\cdots+\sum_{x_1=1}^{1}\sum_{x_2=1}^{2-x_1}\sum_{x_3=1}^{3-x_1-x_2}\cdots\sum_{x_{n-3}=1}^{n-1-\sum_{i=0}^{n-4}x_i}p_{n-3}q_{x_1}q_{x_2}q_{x_3}\dots q_{x_{n-3}}t^{\sum_{i=1}^{n-3}x_i+2}\Big)$$
$$+\cdots+\delta_{n-2}\Big(p_0t^{n-2}+\sum_{x_1=1}^{n-1-(n-2)}p_1q_{x_1}t^{x_1+n-2}\Big)+p_0t^{n-1}.$$

*Remark* 4. Creating a linear system of equations for the condition (4.2), with the calculated RHS and LHS, is not a simple task. Finding the explicit deformed six-term Jacobi identities for this polynomial ring is thus omitted.

## 10.1 Examples of twisted derivations

In this section we take the examples of $\sigma$-derivations from Section 3.1 and we will see what the deformed relations (10.3)-(10.5) become with different parameters defining $\partial_\sigma(t)$ and $\sigma(t)$.

Lemma 1 and the equality of Eq. (10.6) require that $q_0=0$, and $p_0=0$ or $q_1=\omega^k$. Hence, the following $\sigma$-derivations are not applicable, since they either have $q_0\neq 0$ or $q_1\neq\omega^k$ and $p_0\neq 0$: *the ordinary differential operator*, *the shifted difference operator*, *the Jackson q-derivation operator*, *the continuous q-difference operator* and *the divided differences operator*.

**Example 31** (The Eulerian operator)**.** We get the Eulerian operator, where $\sigma(t)=t$ and $\partial(t)=t$, by setting $q_1=1$, $q_0=q_2=\cdots=q_{n-1}=0$ and $p_1=1$, $p_0=p_2=\cdots=p_{n-1}=0$ in the Equalities (10.1). The deformed relations (10.3)-(10.5) thus become:

$$\langle h,e\rangle=2\sum_{i=0}^{n-1}(p_it^i)\partial_\sigma=2g_1,$$

$$\langle h,f\rangle=2\sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i}(q_ip_jt^{i+j+1})\partial_\sigma=2g_3,$$

$$\langle e,f\rangle=-(\sum_{i=0}^{n-2}p_it^{i+1}+\sum_{i=1}^{n-1}\sum_{j=0}^{n-1-i}q_ip_jt^{i+j})\partial_\sigma=-2g_2.$$

**Example 32** (The Nilpotent Imaginary Derivative operator)**.** Let $p_j,q_j\in\mathbb{C}$, where $j=0,1,\dots,n-1$, and $q_j=x_{q_j}+iy_{q_j}$ and $p_j=x_{p_j}+iy_{p_j}$, where $i$ is the imaginary unit. We get the nilpotent imaginary derivative operator, where $\sigma(t)=\overline{q_0}+\overline{q_1}t+\cdots+\overline{q_{n-1}}t^{n-1}$ and $\partial_\sigma(t)=y_{p_0}+y_{p_1}t+\cdots+y_{p_{n-1}}t^{n-1}$, by setting $q_0=\overline{q_0}$, $q_1=\overline{q_1}$, $\dots$, $q_{n-1}=\overline{q_{n-1}}$ and $p_0=y_{p_0}$, $p_1=y_{p_1}$, $\dots$, $p_{n-1}=y_{p_{n-1}}$ in the Equalities (10.1). The deformed relations (10.3)-(10.5) thus become:

$$\langle h,e\rangle=2\sum_{i=0}^{n-1}(p_it^i)\partial_\sigma=2\sum_{i=0}^{n-1}(y_{p_i}g_i)\partial_\sigma,$$

$$\langle h,f\rangle = 2\sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i}(q_ip_jt^{i+j+1})\partial_\sigma = 2\sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i}(\overline{q_i}y_{p_j}g_{i+j+1})\partial_\sigma,$$

$$\langle e,f\rangle = -(\sum_{i=0}^{n-2}p_it^{i+1}+\sum_{i=1}^{n-1}\sum_{j=0}^{n-1-i}q_ip_jt^{i+j})\partial_\sigma = -(\sum_{i=0}^{n-2}y_{p_i}t^{i+1}+\sum_{i=1}^{n-1}\sum_{j=0}^{n-1-i}\overline{q_i}y_{p_j}g_{i+j})\partial_\sigma.$$

Lemma 1 and the equality of Eq. (10.6) require that $\overline{q_0}=0$, and $y_{p_0}=0$ or $\overline{q_1}=\omega^k$.

**Example 33** ($\sigma(t)$ as a polynomial of degree $k$)**.** Let $\sigma(t)=S(t)=\sum_{j=0}^{k}s_jt^k$ where $k<n$. The $\sigma$-twisted Leibniz rule is defined as:

$$(\partial_\sigma(ab))(t)=\partial_\sigma(a)(t)b(t)+\sigma(a)(t)\partial_\sigma(b)(t),$$

where

$$(\partial_\sigma a)(t)=\frac{\sigma(a)(t)-a(t)}{\sigma(t)-t}=\frac{a(S(t))-a(t)}{S(t)-t}.$$

We get this $\sigma$-derivation by setting $q_i=s_i$ for $i=0,1,\dots,k$ and $q_i=0$ for $i=k,k+1,\dots,n-1$, and $p_0=1,\ p_1=p_2=\dots=p_{n-1}=0$ in the Equalities (10.1). The deformed relations (10.3)-(10.5) thus become:

$$\langle h,e\rangle = 2\sum_{i=0}^{n-1}(p_it^i)\partial_\sigma = 2g_0,$$

$$\langle h,f\rangle = 2\sum_{i=1}^{n-2}\sum_{j=0}^{n-2-i}(q_ip_jt^{i+j+1})\partial_\sigma = 2\sum_{i=1}^{\min(k,n-2)}(s_ig_{i+1})\partial_\sigma,$$

$$\langle e,f\rangle = -(\sum_{i=0}^{n-2}p_it^{i+1}+\sum_{i=1}^{n-1}\sum_{j=0}^{n-1-i}q_ip_jt^{i+j})\partial_\sigma = -g_1-\sum_{i=1}^{k}s_ig_i.$$

*Remark* 5. Note that the continuous the Eulerian operator, the nilpotent imaginary derivative operator and $\sigma(t)$ as a polynomial degree of $k$ do not have closure of relations (10.3)-(10.5)

# Chapter 11

# Quasi-hom-Lie algebras of twisted vector fields for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]/((t-t_0)^3)$

We take the algebra $\mathscr{A}$ as $\mathbb{F}[t]/((t-t_0)^3)$, meaning that $(t-t_0)^n = 0$ for $n \geq 3$. The elements are $e = \partial_\sigma$, $h = -2t\partial_\sigma$, $f = -t^2\partial_\sigma$. From Chapter 5, we have that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Set:

$$\partial_\sigma(t) = p_0 + p_1(t-t_0) + p_2(t-t_0)^2, \quad \sigma(t) = q_0 + q_1(t-t_0) + q_2(t-t_0)^2. \tag{11.1}$$

*Remark* 6. It would also be interesting to study when $\partial_\sigma(t-t_0) = p_0 + p_1(t-t_0) + p_2(t-t_0)^2$ and $\sigma(t-t_0) = q_0 + q_1(t-t_0) + q_2(t-t_0)^2$, which could be isomorphic to $\mathbb{F}[t]/(t^3)$.

Since $(t-t_0)^3 = 0$ we can see that:

$$\begin{aligned}\sigma((t-t_0)^3) = \sigma(t-t_0)^3 = (q_0 + q_1(t-2t_0) + q_2(t-2t_0)^2)^3 \\ = q_0^3 + 3q_0^2q_1(t-2t_0) + 3q_0^2q_2(t-2t_0)^2 + 3q_0q_1^2(t-2t_0)^2 = 0 \implies q_0 = 0.\end{aligned}$$

$$\partial_\sigma((t-t_0)^3) = \partial_\sigma(t-t_0)(t-t_0)^2 + \sigma(t-t_0)\partial_\sigma(t-t_0)(t-t_0) + \sigma(t-t_0)^2\partial_\sigma(t-t_0) = 0 \tag{11.2}$$

Using the $\sigma$-twisted relation of Eq. (4.3), with $q_0 = 0$, $\partial_\sigma(1) = 0$, $\sigma(1) = 1$, and the $\sigma$-twisted Leibniz rule, we get the following relations:

$$\langle h, e\rangle = \langle -2t\partial_\sigma, \partial_\sigma\rangle_\sigma = 2(p_0 - p_1t_0 + p_2t_0^2)e - (p_1 - 2p_2t_0)h - 2p_2f, \tag{11.3}$$

$$\begin{aligned}\langle h, f\rangle = \langle -2t\partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = (p_0q_1t_0 - p_0q_2t_0^2 - p_1q_1t_0^2 + p_1q_2t_0^3 + p_2q_1t_0^3 - p_2q_2t_0^4)h \\ -2(p_0q_1 - 2p_0q_2t_0 - 2p_1q_1t_0 + 3p_1q_2t_0^2 + 3p_2q_1t_0^2 - 4p_2q_2t_0^3)f,\end{aligned} \tag{11.4}$$

$$\begin{aligned}\langle e, f\rangle = \langle \partial_\sigma, -t^2\partial_\sigma\rangle_\sigma = -(p_0q_2t_0^2 + p_1q_1t_0^2 - p_1q_2t_0^3 - p_2q_1t_0^3 + p_2q_2t_0^4 - p_0q_1t_0)e \\ + \frac{p_0 + p_2t_0^2 + p_0q_1 - p_1t_0 - 2p_0q_2t_0 - 2p_1q_1t_0 + 3p_1q_2t_0^2 + 3p_2q_1t_0^2 - 4p_2q_2t_0^3}{2}h \\ +(p_1 + p_0q_2 + p_1q_1 - 2p_2t_0 - 3p_1q_2t_0 - 3p_2q_1t_0 + 6p_2q_2t_0^2)f.\end{aligned} \tag{11.5}$$

*Remark* 7. If $t_0 = 0$ we acquire the relations for the polynomial algebra $\mathscr{A} = \mathbb{F}[t]/(t^3)$.

We also have to consider the case when $\partial_\sigma((t-t_0)^3) = 0$, see Eq. (11.2). If we use Eqs. (11.1) and that $q_0 = 0$, we get:

$$\begin{aligned}\partial_\sigma((t-t_0)^3) &= 16p_2q_1^2t_0^4 - 8p_1q_1^2t_0^3 + 4p_0q_1^2t_0^2 - 64p_2q_1q_2t_0^5 + 32p_1q_1q_2t_0^4 - 16p_0q_1q_2t_0^3 \\ &+8p_2q_1t_0^4 - 4p_1q_1t_0^3 + 2p_0q_1t_0^2 + 64p_2q_2^2t_0^6 - 32p_1q_2^2t_0^5 + 16p_0q_2^2t_0^4 - 16p_2q_2t_0^5 + 8p_1q_2t_0^4 \\ &-4p_0q_2t_0^3 + 4p_2t_0^4 - 2p_1t_0^3 + p_0t_0^2 + (-32p_2q_1^2t_0^3 + 12p_1q_1^2t_0^2 - 4p_0q_1^2t_0 + 160p_2q_1q_2t_0^4 - 64p_1q_1q_2t_0^3 \\ &+24p_0q_1q_2t_0^2 - 20p_2q_1t_0^3 + 8p_1q_1t_0^2 - 3p_0q_1t_0 - 192p_2q_2^2t_0^5 + 80p_1q_2^2t_0^4 - 32p_0q_2^2t_0^3 + 48p_2q_2t_0^4 \\ &-20p_1q_2t_0^3 + 8p_0q_2t_0^2 - 12p_2t_0^3 + 5p_1t_0^2 - 2p_0t_0)t + (24p_2q_1^2t_0^2 - 6p_1q_1^2t_0 + p_0q_1^2 - 160p_2q_1q_2t_0^3 \\ &+48p_1q_1q_2t_0^2 - 12p_0q_1q_2t_0 + 18p_2q_1t_0^2 - 5p_1q_1t_0 + p_0q_1 + 240p_2q_2^2t_0^4 - 80p_1q_2^2t_0^3 + 24p_0q_2^2t_0^2 \\ &-56p_2q_2t_0^3 + 18p_1q_2t_0^2 - 5p_0q_2t_0 + 13p_2t_0^2 - 4p_1t_0 + p_0)t^2 = 0.\end{aligned}$$

The coefficients for each exponent of $t$ has to equal zero, so we create a linear system of equations:

$$\begin{cases} 16p_2q_1^2t_0^4 - 8p_1q_1^2t_0^3 + 4p_0q_1^2t_0^2 - 64p_2q_1q_2t_0^5 + 32p_1q_1q_2t_0^4 - 16p_0q_1q_2t_0^3 + 8p_2q_1t_0^4 - 4p_1q_1t_0^3 \\ +2p_0q_1t_0^2 + 64p_2q_2^2t_0^6 - 32p_1q_2^2t_0^5 + 16p_0q_2^2t_0^4 - 16p_2q_2t_0^5 + 8p_1q_2t_0^4 - 4p_0q_2t_0^3 + 4p_2t_0^4 \\ -2p_1t_0^3 + p_0t_0^2 = 0 \\ \\ -32p_2q_1^2t_0^3 + 12p_1q_1^2t_0^2 - 4p_0q_1^2t_0 + 160p_2q_1q_2t_0^4 - 64p_1q_1q_2t_0^3 + 24p_0q_1q_2t_0^2 - 20p_2q_1t_0^3 \\ +8p_1q_1t_0^2 - 3p_0q_1t_0 - 192p_2q_2^2t_0^5 + 80p_1q_2^2t_0^4 - 32p_0q_2^2t_0^3 + 48p_2q_2t_0^4 - 20p_1q_2t_0^3 + 8p_0q_2t_0^2 \\ -12p_2t_0^3 + 5p_1t_0^2 - 2p_0t_0 = 0 \\ \\ 24p_2q_1^2t_0^2 - 6p_1q_1^2t_0 + p_0q_1^2 - 160p_2q_1q_2t_0^3 + 48p_1q_1q_2t_0^2 - 12p_0q_1q_2t_0 + 18p_2q_1t_0^2 - 5p_1q_1t_0 \\ +p_0q_1 + 240p_2q_2^2t_0^4 - 80p_1q_2^2t_0^3 + 24p_0q_2^2t_0^2 - 56p_2q_2t_0^3 + 18p_1q_2t_0^2 - 5p_0q_2t_0 + 13p_2t_0^2 \\ -4p_1t_0 + p_0 = 0 \end{cases}$$

Solving this system symbolically using Maple gives us the following solutions, and tells us that most solutions require that the field $\mathbb{F} = \mathbb{C}$:

Solution 1: We get a trivial solution when $\partial_\sigma(t) = 0$, and $q_1$ and $q_2$ are arbitrary.

Solution 2:

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2p_2t_0^2 \\ 4p_2t_0 \\ p_2 \\ 2q_2t_0 \pm \frac{\sqrt{3}i}{4} - \frac{1}{4} \\ q_2 \end{bmatrix}$$

Solution 3:

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_0 \\ \frac{p_0}{t_0} \\ \frac{p_0}{4t_0^2} \\ 2q_2t_0 \pm \frac{\sqrt{3}i}{4} - \frac{1}{4} \\ q_2 \end{bmatrix}$$

Solution 4:

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -4p_2t_0^2 + 2p_1t_0 \\ p_1 \\ p_2 \\ 0 \\ \pm\frac{\sqrt{3}i}{8t_0} + \frac{1}{8t_0} \end{bmatrix}$$

Solution 5: If $t_0 = 0$, we get the solutions corresponding to Chapter 7.

The condition for the deformed six-term Jacobi identity is that $\partial_\sigma(\sigma(t)) = \delta\sigma(\partial_\sigma(t))$, which for the left hand side becomes:

$$\begin{aligned}\partial_\sigma(\sigma(t)) = q_1\partial_\sigma(t-t_0) + q_2\partial_\sigma((t-t_0)^2) = p_0q_1 + 2p_1q_2t_0^2 + 4p_2q_1t_0^2 - 4p_2q_2t_0^3 + 4p_0q_2^2t_0^2 \\ -8p_1q_2^2t_0^3 + 16p_2q_2^2t_0^4 - p_0q_2t_0 - 2p_1q_1t_0 - 2p_0q_1q_2t_0 + 4p_1q_1q_2t_0^2 - 8p_2q_1q_2t_0^3 \\ +(p_0q_2 + p_1q_1 + p_0q_1q_2 - 3p_1q_2t_0 - 4p_2q_1t_0 - 4p_0q_2^2t_0 + 8p_2q_2t_0^2 + 12p_1q_2^2t_0^2 - 32p_2q_2^2t_0^3 \\ -4p_1q_1q_2t_0 + 12p_2q_1q_2t_0^2)t \\ +(p_1q_2 + p_2q_1 + p_0q_2^2 + 24p_2q_2^2t_0^2 + p_1q_1q_2 - 5p_2q_2t_0 - 6p_1q_2^2t_0 - 6p_2q_1q_2t_0)t^2.\end{aligned}$$

The right hand sides becomes:

$$\begin{aligned}\delta\sigma(\partial_\sigma(t)) = \delta(p_0 + p_1\sigma(t-t_0) + p_2\sigma(t-t_0)^2) \\ = \delta(4p_2q_1^2t_0^2 - 16p_2q_1q_2t_0^3 - 2p_1q_1t_0 + 16p_2q_2^2t_0^4 + 4p_1q_2t_0^2 + p_0 \\ +(-4p_2q_1^2t_0 + 24p_2q_1q_2t_0^2 + 2p_2q_1q_2 + p_1q_1 - 32p_2q_2^2t_0^3 - 8p_2q_2^2t_0 + p_2q_2^2 - 4p_1q_2t_0)t \\ +(p_2q_1^2 - 12p_2q_1q_2t_0 + 24p_2q_2^2t_0^2 + p_1q_2)t^2).\end{aligned}$$

Let $\delta = \delta_0 + \delta_1 t + \delta_2 t^2$, then the right hand side becomes:

$$\begin{aligned}(\delta_0 + \delta_1t + \delta_2t^2)\sigma(\partial_\sigma(t)) = 4\delta_0p_2q_1^2t_0^2 - 16\delta_0p_2q_1q_2t_0^3 - 2\delta_0p_1q_1t_0 + 16\delta_0p_2q_2^2t_0^4 + 4\delta_0p_1q_2t_0^2 \\ +\delta_0p_0 + (4\delta_1p_2q_1^2t_0^2 - 4\delta_0p_2q_1^2t_0 - 16\delta_1p_2q_1q_2t_0^3 + 24\delta_0p_2q_1q_2t_0^2 - 2\delta_1p_1q_1t_0 + \delta_0p_1q_1 \\ +16\delta_1p_2q_2^2t_0^4 - 32\delta_0p_2q_2^2t_0^3 + 4\delta_1p_1q_2t_0^2 - 4\delta_0p_1q_2t_0 + \delta_1p_0)t \\ +(4\delta_2p_2q_1^2t_0^2 - 4\delta_1p_2q_1^2t_0 + \delta_0p_2q_1^2 - 16\delta_2p_2q_1q_2t_0^3 + 24\delta_1p_2q_1q_2t_0^2 - 12\delta_0p_2q_1q_2t_0 - 2\delta_2p_1q_1t_0 \\ +\delta_1p_1q_1 + 16\delta_2p_2q_2^2t_0^4 - 32\delta_1p_2q_2^2t_0^3 + 24\delta_0p_2q_2^2t_0^2 + 4\delta_2p_1q_2t_0^2 - 4\delta_1p_1q_2t_0 + \delta_0p_1q_2 + \delta_2p_0)t^2.\end{aligned}$$

Subtracting the LHS from both sides of the equality, we set the condition equal to zero, which we can use to create a linear system of equations:

$$\begin{cases} 4\delta_0 p_2 q_1^2 t_0^2 - 16\delta_0 p_2 q_1 q_2 t_0^3 - 2\delta_0 p_1 q_1 t_0 + 16\delta_0 p_2 q_2^2 t_0^4 + 4\delta_0 p_1 q_2 t_0^2 + \delta_0 p_0 \\ -(p_0 q_1 + 2p_1 q_2 t_0^2 + 4p_2 q_1 t_0^2 - 4p_2 q_2 t_0^3 + 4p_0 q_2^2 t_0^2 - 8p_1 q_2^2 t_0^3 + 16 p_2 q_2^2 t_0^4 - p_0 q_2 t_0 - 2p_1 q_1 t_0 \\ -2p_0 q_1 q_2 t_0 + 4p_1 q_1 q_2 t_0^2 - 8p_2 q_1 q_2 t_0^3) = 0 \\ \\ 4\delta_1 p_2 q_1^2 t_0^2 - 4\delta_0 p_2 q_1^2 t_0 - 16\delta_1 p_2 q_1 q_2 t_0^3 + 24\delta_0 p_2 q_1 q_2 t_0^2 - 2\delta_1 p_1 q_1 t_0 + \delta_0 p_1 q_1 + 16\delta_1 p_2 q_2^2 t_0^4 \\ -32\delta_0 p_2 q_2^2 t_0^3 + 4\delta_1 p_1 q_2 t_0^2 - 4\delta_0 p_1 q_2 t_0 + \delta_1 p_0 - (p_0 q_2 + p_1 q_1 + p_0 q_1 q_2 - 3p_1 q_2 t_0 - 4p_2 q_1 t_0 \\ -4p_0 q_2^2 t_0 + 8p_2 q_2 t_0^2 + 12 p_1 q_2^2 t_0^2 - 32 p_2 q_2^2 t_0^3 - 4p_1 q_1 q_2 t_0 + 12 p_2 q_1 q_2 t_0^2) = 0 \\ \\ 4\delta_2 p_2 q_1^2 t_0^2 - 4\delta_1 p_2 q_1^2 t_0 + \delta_0 p_2 q_1^2 - 16\delta_2 p_2 q_1 q_2 t_0^3 + 24\delta_1 p_2 q_1 q_2 t_0^2 - 12\delta_0 p_2 q_1 q_2 t_0 - 2\delta_2 p_1 q_1 t_0 \\ +\delta_1 p_1 q_1 + 16\delta_2 p_2 q_2^2 t_0^4 - 32\delta_1 p_2 q_2^2 t_0^3 + 24\delta_0 p_2 q_2^2 t_0^2 + 4\delta_2 p_1 q_2 t_0^2 - 4\delta_1 p_1 q_2 t_0 + \delta_0 p_1 q_2 + \delta_2 p_0 \\ -(p_2 q_1^2 - 12 p_2 q_1 q_2 t_0 + 24 p_2 q_2^2 t_0^2 + p_1 q_2) = 0 \end{cases}$$

Solving the system of equations for $\delta_0$, $\delta_1$ and $\delta_2$ for the 4 different cases gives us the following results.

**Case 1:** When $p_0 = p_1 = p_2 = 0$ the system of equations is zero, and hence $\delta_0 := \xi_0$, $\delta_1 := \xi_1$, $\delta_2 := \xi_2$ are arbitrary, and we have the following relations from Eqs. (11.3)-(11.5):

$$\langle h, e\rangle = 0,$$

$$\langle h, f\rangle = 0,$$

$$\langle e, f\rangle = 0.$$

**Case 2:** Substituting the coefficients in the system of equations according to Solution 2, and we have the following relations from Eqs. (11.3)-(11.5):

$$\langle h, e\rangle = -2(p_2 t_0^2 e + p_2 t_0 h + p_2 f),$$

$$\langle h, f\rangle = \left(\frac{p_2 t_0^3 + \sqrt{3} p_2 t_0^3 i)}{4} + p_2 q_2 t_0^4\right) h + \left(\frac{\sqrt{3} p_2 t_0^2 3i - 3p_2 t_0^2}{2} + 4p_2 q_2 t_0^3\right) f,$$

$$\langle e, f\rangle = \left(\frac{p_2 t_0^3 - \sqrt{3} p_2 t_0^3 i}{4} - p_2 q_2 t_0^4\right) e - \left(\frac{p_2 t_0^2 + \sqrt{3} p_2 t_0^2 3i}{8} + p_2 q_2 t_0^3\right) h + \left(\frac{\sqrt{3} p_2 t_0 i + 7p_2 t_0}{4} - 2p_2 q_2 t_0^2\right) f.$$

Solving the system of linear equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives us:

$$\delta_0 = \frac{(4p_2 t_0^2 - 2p_1 t_0 + p_0)(\sqrt{3} - q_2 t_0 4i - q_2^2 t_0^2 16i + q_1 q_2 t_0 8i + i)i}{8p_2 t_0^2 (2q_1^2 - 8q_1 q_2 t_0 - 4q_1 + 8q_2^2 t_0^2 + 8q_2 t_0 + 1)} := \xi_0,$$

$$
\begin{aligned}
\delta_1 = {} & (\sqrt{3}p_1t_0 + p_1t_0i + p_0q_1^2 2i - p_2t_0^2 4i - \sqrt{43}p_2t_0^2 - p_1q_1^2t_0 2i - p_1q_2t_0^2 4i + p_2q_1t_0^2 8i - p_0q_2^2t_0^2 32i \\
& - p_0q_2^3t_0^3 96i + p_1q_2^2t_0^3 24i - p_0q_2^4t_0^4 128i + p_1q_2^3t_0^4 32i + p_2q_2^2t_0^4 32i + p_1q_2^4t_0^5 128i + p_2q_2^3t_0^5 256i \\
& - \sqrt{23}p_0q_1 - p_0q_1 2i + p_0q_2t_0 4i + \sqrt{23}p_0q_1^2 + \sqrt{83}p_0q_2t_0 + p_0q_1q_2t_0 8i - p_0q_1^2q_2^2t_0^2 64i \\
& + p_1q_1^2q_2^2t_0^3 32i + p_2q_1^2q_2^2t_0^4 128i - \sqrt{23}p_1q_1^2t_0 - \sqrt{83}p_1q_2t_0^2 + \sqrt{83}p_2q_1t_0^2 \\
& - p_0q_1^2q_2t_0 16i + p_0q_1^3q_2t_0 8i - p_2q_1q_2t_0^3 32i + \sqrt{163}p_0q_2^2t_0^2 - \sqrt{243}p_1q_2^2t_0^3 + \sqrt{323}p_2q_2^2t_0^4 \\
& + p_0q_1q_2^2t_0^2 80i - p_1q_1^2q_2t_0^2 8i + p_0q_1q_2^3t_0^3 160i + p_2q_1^2q_2t_0^3 96i - p_1q_1q_2^3t_0^4 128i - p_2q_1q_2^2t_0^4 320i \\
& - p_2q_1^3q_2t_0^3 32i - p_2q_1q_2^3t_0^5 128i - \sqrt{123}p_0q_1q_2t_0 + \sqrt{163}p_1q_1q_2t_0^2 \\
& - \sqrt{163}p_2q_1q_2t_0^3)i/(8p_2t_0^3(2q_1^2 - 8q_1q_2t_0 - 4q_1 + 8q_2^2t_0^2 + 8q_2t_0 + 1)^2) := \xi_1,
\end{aligned}
$$

$$
\begin{aligned}
\delta_2 = {} & -(\sqrt{123}p_0q_1^3 + p_0q_1^3 12i - p_0q_1^4 6i - p_2t_0^2 2i - \sqrt{23}p_2t_0^2 - p_1q_1^2t_0 6i - p_1q_2t_0^2 16i + p_1q_1^4t_0 4i \\
& + p_2q_2t_0^3 24i - p_0q_2^2t_0^2 16i + p_2q_1^2t_0^2 12i + p_0q_2^3t_0^3 448i + p_1q_2^2t_0^3 32i - p_2q_1^3t_0^2 16i + p_0q_2^4t_0^4 1600i \\
& - p_1q_2^3t_0^4 256i - p_2q_2^2t_0^4 64i + p_0q_2^5t_0^5 1792i - p_1q_2^4t_0^5 896i + p_2q_2^3t_0^5 128i + p_0q_2^6t_0^6 1536i + p_2q_2^4t_0^6 640i \\
& - p_1q_2^6t_0^7 1024i - p_2q_2^5t_0^7 2560i + p_0q_2t_0 4i + p_1q_1t_0 4i - \sqrt{73}p_0q_1^2 - p_0q_1^2 7i - \sqrt{63}p_0q_1^4 \\
& + \sqrt{43}p_0q_2t_0 + \sqrt{43}p_1q_1t_0 + p_0q_1q_2t_0 20i + p_0q_1^2q_2^2t_0^2 648i - p_0q_1^3q_2^2t_0^2 512i + p_0q_1^2q_2^3t_0^3 1824i \\
& + p_0q_1^4q_2^2t_0^2 192i - p_1q_1^2q_2^2t_0^3 48i - p_0q_1^3q_2^3t_0^3 960i - p_1q_1^3q_2^2t_0^3 192i + p_0q_1^2q_2^4t_0^4 2432i + p_1q_1^2q_2^3t_0^4 576i \\
& + p_2q_1^2q_2^2t_0^4 96i + p_1q_1^3q_2^3t_0^4 128i + p_2q_1^3q_2^2t_0^4 512i - p_1q_1^2q_2^4t_0^5 768i - p_2q_1^2q_2^3t_0^5 2688i - p_2q_1^4q_2^2t_0^4 128i \\
& + p_2q_1^3q_2^3t_0^5 768i - p_2q_1^2q_2^4t_0^6 1536i - \sqrt{63}p_1q_1^2t_0 - \sqrt{243}p_1q_2t_0^2 + \sqrt{43}p_1q_1^4t_0 + \sqrt{483}p_2q_2t_0^3 \\
& - p_0q_1^2q_2t_0 4i - p_0q_1^3q_2t_0 56i + p_1q_1q_2t_0^2 8i + p_0q_1^4q_2t_0 56i - p_0q_1^5q_2t_0 16i - p_2q_1q_2t_0^3 32i \\
& - \sqrt{723}p_0q_2^2t_0^2 + \sqrt{123}p_2q_1^2t_0^2 - \sqrt{2883}p_0q_2^3t_0^3 - \sqrt{163}p_1q_2^2t_0^3 - \sqrt{163}p_2q_1^3t_0^2 - \sqrt{3203}p_0q_2^4t_0^4 \\
& + \sqrt{1923}p_1q_2^3t_0^4 + \sqrt{1923}p_2q_2^2t_0^4 + \sqrt{3843}p_1q_2^4t_0^5 + \sqrt{1283}p_2q_2^3t_0^5 - \sqrt{3843}p_2q_2^4t_0^6 \\
& - p_0q_1q_2^2t_0^2 240i + p_1q_1^2q_2t_0^2 24i - p_0q_1q_2^3t_0^3 1824i + p_1q_1q_2^2t_0^3 32i - p_1q_1^3q_2t_0^2 32i + p_1q_1^4q_2t_0^2 16i \\
& + p_2q_1^2q_2t_0^3 16i - p_0q_1q_2^4t_0^4 2944i + p_1q_1q_2^3t_0^4 640i + p_2q_1q_2^2t_0^4 64i - p_2q_1^3q_2t_0^3 32i - p_0q_1q_2^5t_0^5 3072i \\
& - p_1q_1q_2^4t_0^5 512i - p_2q_1q_2^3t_0^5 384i + p_1q_1q_2^5t_0^6 1536i + p_2q_1q_2^4t_0^6 4608i + p_2q_1q_2^5t_0^7 1024i \\
& + \sqrt{2883}p_0q_1q_2^2t_0^2 + \sqrt{4803}p_0q_1q_2^3t_0^3 - \sqrt{963}p_1q_1q_2^2t_0^3 - \sqrt{483}p_1q_1^3q_2t_0^2 + \sqrt{1923}p_2q_1^2q_2t_0^3 \\
& - \sqrt{5123}p_1q_1q_2^3t_0^4 - \sqrt{3843}p_2q_1q_2^2t_0^4 + \sqrt{3843}p_2q_1q_2^3t_0^5 + \sqrt{363}p_0q_1q_2t_0 - \sqrt{2643}p_0q_1^2q_2^2t_0^2 \\
& + \sqrt{2403}p_1q_1^2q_2^2t_0^3 - \sqrt{963}p_2q_1^2q_2^2t_0^4 - \sqrt{963}p_0q_1^2q_2t_0 + \sqrt{643}p_0q_1^3q_2t_0 + \sqrt{483}p_1q_1q_2t_0^2 \\
& - \sqrt{1763}p_2q_1q_2t_0^3)i/(16p_2t_0^4(8q_2^2t_0^2 - 4q_1 + 8q_2t_0 + 2q_1^2 - 8q_1q_2t_0 + 1)^3) := \xi_2.
\end{aligned}
$$

**Case 3:** Substituting the coefficients in the system of equations according to Solution 3, with $q_1 = 2q_2t_0 + \frac{\sqrt{3}i}{4} - \frac{1}{4}$, and we have the following relations from Eqs. (11.3)-(11.5):

$$\langle h, e\rangle = \frac{p_0}{2}e - \frac{p_0}{2t_0}h - \frac{p_0}{2t_0^2}f,$$

$$\langle h,f\rangle = (\frac{p_0t_0(4q_2t_0+\sqrt{3}i-1)}{16})h+(\frac{p_0(8q_2t_0+\sqrt{3}i-1)}{8})f,$$

$$\langle e,f\rangle = \frac{p_0t_0(4q_2t_0+\sqrt{3}i-1)}{16}e-\frac{p_0(8q_2t_0+\sqrt{3}i-5)}{16}h+\frac{p_0(7+\sqrt{3}i)}{16t_0}f.$$

Solving the system of linear equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives us:

$$\delta_0 = \frac{(4p_2t_0^2-2p_1t_0+p_0)(\sqrt{3}-q_2t_04i-q_2^2t_0^216i+q_1q_2t_08i+i)i}{4p_0(2q_2t_0-q_1+1)^2} := \xi_0,$$

$$\begin{aligned}\delta_1 = (&p_0q_1-p_1t_0+4p_2t_0^2-\sqrt{3}p_2t_0^24i+2p_1q_2t_0^2-8p_2q_2t_0^3+8p_0q_2^2t_0^2+32p_0q_2^3t_0^3+8p_1q_2^2t_0^3\\ &-32p_1q_2^3t_0^4-64p_2q_2^2t_0^4-\sqrt{3}p_0q_11i+\sqrt{3}p_1t_01i-p_1q_1t_0+\sqrt{3}p_0q_2t_04i+\sqrt{3}p_1q_1t_0i\\ &-4p_0q_1q_2t_0-\sqrt{3}p_1q_2t_0^26i+\sqrt{3}p_2q_2t_0^38i+4p_0q_1^2q_2t_0-4p_1q_1q_2t_0^2+32p_2q_1q_2t_0^3\\ &-24p_0q_1q_2^2t_0^2+16p_1q_1q_2^2t_0^3-16p_2q_1^2q_2t_0^3+32p_2q_1q_2^2t_0^4)/(4p_0t_0(2q_2t_0-q_1+1)^3) := \xi_1,\end{aligned}$$

$$\begin{aligned}\delta_2 = -(&\sqrt{43}p_0q_2t_0-p_2t_0^24i-\sqrt{43}p_2t_0^2+p_1q_1^2t_02i-p_1q_2t_0^28i-p_2q_1t_0^28i+p_2q_2t_0^316i\\ &+p_0q_2^2t_0^224i+p_0q_2^3t_0^332i-p_1q_2^2t_0^316i+p_0q_2^4t_0^4192i+p_1q_2^3t_0^4128i+p_2q_2^2t_0^480i-p_1q_2^4t_0^5128i\\ &-p_2q_2^3t_0^5320i+p_0q_2t_04i+p_1q_1t_04i-\sqrt{33}p_0q_1^2-p_0q_1^23i+\sqrt{43}p_1q_1t_0+p_0q_1q_2t_04i\\ &+p_0q_1^2q_2^2t_0^264i-p_2q_1^2q_2^2t_0^464i+\sqrt{23}p_1q_1^2t_0-\sqrt{243}p_1q_2t_0^2-\sqrt{83}p_2q_1t_0^2+\sqrt{643}p_2q_2t_0^3\\ &+p_0q_1^2q_2t_012i-p_0q_1^3q_2t_08i-p_1q_1q_2t_0^216i-\sqrt{403}p_0q_2^2t_0^2+\sqrt{483}p_1q_2^2t_0^3-\sqrt{483}p_2q_2^2t_0^4\\ &-p_0q_1q_2^2t_0^248i+p_1q_1^2q_2t_0^28i-p_0q_1q_2^3t_0^3192i-p_1q_1q_2^2t_0^364i+p_1q_1q_2^3t_0^464i+p_2q_1q_2^2t_0^4128i\\ &+p_2q_1q_2^3t_0^5128i+\sqrt{203}p_0q_1q_2t_0-\sqrt{163}p_1q_1q_2t_0^2)i/(16p_0t_0^2(2q_2t_0-q_1+1)^4) := \xi_2.\end{aligned}$$

**Case 4:** Substituting the coefficients in the system of equations according to Solution 4, with $q_2 = \frac{\sqrt{3}i}{8t_0}+\frac{1}{8t_0}$, and we have the following relations from Eqs. (11.3)-(11.5):

$$\langle h,e\rangle = (2p_1t_0-6p_2t_0^2)e+(2p_2t_0-p_1)h-2p_2f,$$

$$\langle h,f\rangle = -\frac{t_0^2(p_1-3p_2t_0)(1+\sqrt{3}i)}{8}-\frac{t_0(p_1-4p_2t_0)(1+\sqrt{3}i)}{4},$$

$$\begin{aligned}\langle e,f\rangle = -\frac{t_0^2(p_1-3p_2t_0)(1+\sqrt{3}i)}{8}e+\frac{t_0(7p_1-\sqrt{3}p_1i-20p_2t_0+\sqrt{3}p_2t_04i)}{16}h\\ -\frac{(p_1-2p_2t_0)(\sqrt{3}+7i)i}{8}f.\end{aligned}$$

Solving the system of linear equations for $\delta_0$, $\delta_1$ and $\delta_2$ gives us:

$$\delta_0 = \frac{-(\sqrt{3}-i)(2q_1-4q_2t_0+1)(4p_2t_0^2-2p_1t_0+p_0)i}{16t_0(2q_2t_0-q_1+1)(p_1-2p_2t_0+4p_2q_2t_0^2-2p_2q_1t_0)} := \xi_0,$$

$$\begin{aligned}\delta_1 = {} & (4q_2t_0-1)(\sqrt{3}-1i)(2p_1^2q_2t_0^2+3p_1^2t_0-8p_1p_2q_2^2t_0^4-8p_1p_2q_2t_0^3-14p_1p_2t_0^2-p_0p_1+16p_2^2t_0^3 \\ & +8p_0p_2q_2^2t_0^3+2p_0p_2t_0)i/(16t_0^2(2q_2t_0+1)^2(4p_2q_2t_0^2-2p_2t_0+p_1)^2) := \xi_1,\end{aligned}$$

$$\begin{aligned}\delta_2 = {} & -(\sqrt{3}-i)(-4p_1^3q_2^2t_0^2+18p_1^3q_2t_0-2p_1^3-192p_1^2p_2q_2^4t_0^5-336p_1^2p_2q_2^3t_0^4+80p_1^2p_2q_2^2t_0^3 \\ & -104p_1^2p_2q_2t_0^2+18p_1^2p_2t_0+10p_0p_1^2q_2^2t_0-7p_0p_1^2q_2+256p_1p_2^2q_2^5t_0^7+384p_1p_2^2q_2^4t_0^6 \\ & +1248p_1p_2^2q_2^3t_0^5-320p_1p_2^2q_2^2t_0^4+232p_1p_2^2q_2t_0^3-48p_1p_2^2t_0^2+32p_0p_1p_2q_2^4t_0^4 \\ & +168p_0p_1p_2q_2^3t_0^3-80p_0p_1p_2q_2^2t_0^2+18p_0p_1p_2q_2t_0-128p_2^3q_2^4t_0^7-1280p_2^3q_2^3t_0^6+384p_2^3q_2^2t_0^5 \\ & -192p_2^3q_2t_0^4+40p_2^3t_0^3-384p_0p_2^2q_2^5t_0^6+64p_0p_2^2q_2^4t_0^5-384p_0p_2^2q_2^3t_0^4 \\ & +112p_0p_2^2q_2^2t_0^3-8p_0p_2^2q_2t_0^2)i)/(32t_0^2(2q_2t_0+1)^3(4p_2q_2t_0^2-2p_2t_0+p_1)^3) := \xi_2.\end{aligned}$$

The deformed six-term Jacobi identity for these cases is:

$$\circlearrowleft_{x,y,z}(\langle\sigma(x),\langle y,z\rangle\rangle+(\xi_0+\xi_1t+\xi_2t^2)\cdot\langle x,\langle y,z\rangle\rangle)=0,$$

where $\xi_0$, $\xi_1$, $\xi_2$ are the corresponding solutions to each case.

# Chapter 12

# Conclusions

For all the cases of polynomial algebras we have examined, we found non-trivial $\sigma$-deformed relations and deformed six-term Jacobi identities, with the exception of an explicit deformed six-term Jacobi identity for the polynomial algebra $\mathbb{F}[t]/(t^n)$. For many of the deformed six-term Jacobi identities we also require that $\mathbb{F}$ includes at least some complex values, name the roots of some polynomials, see Lemma 2 and Remark 3 in Chapter 10.

The deformed six-term Jacobi identities we have found all fulfill condition (4.4). Condition (4.3) is obviously fulfilled by 0, since $0 \cdot \partial_\sigma = 0$ and $\sigma(0) \cdot \partial_\sigma = 0$, and is thus an annihilator of $\mathscr{A}$.

There are many other cases of polynomial algebras that can be investigated, such as $\mathbb{F}[t]/((t-t_0)^n)$; and in the polynomial ring $\mathbb{F}[t]/((t-t_0)^3)$, $\sigma(t)$ and $\partial_\sigma(t)$ can be defined differently.

We have also examined several $\sigma$-derivations, and how they act in the different polynomial algebras. The following $\sigma$-derivations are defined on the covered quotient rings, and we have found $\sigma$-deformed relations and deformed six-term Jacobi identities for them: *the Eulerian operator, the nilpotent imaginary derivative operator, and $\sigma(t)$ as a polynomial of degree $k$*. The following $\sigma$-derivations are not defined on quotient rings: *the ordinary differential operator*, *the shifted difference operator*, *the Jackson q-derivation operator*, *the continuous q-difference operator* and *the divided differences operator*. Since they are not defined on the quotient rings with the quasi-hom-Lie algebras deformed six-term Jacobi identity, it could be interesting to see if there are other deformations where they are defined on the quotient rings.

## 12.1 Accomplishment of the requirements of the Swedish National Agency for Higher Education to Master theses

In this section the author confirms that the objectives for Master theses are accomplished, as stated by the Swedish National Agency for Higher Education.

### 12.1.1 Objective 1

To accomplish objective 1, the author *should demonstrate knowledge and understanding in the major field of study, including both broad knowledge in the field and substantially deeper knowledge of certain parts of the area as well as insight into current research and development.*

The author accomplishes this objective as he shows a very broad knowledge of the subject of algebraic structures, and a deeper knowledge of hom-Lie and quasi-hom-Lie algebras. He has also gained insight into current research and development in his studies, as a substantial portion of the thesis is to continue the research performed by Larsson and Silvestrov (2005).

### 12.1.2 Objective 2

To accomplish objective 2, the author *should demonstrate deeper methodological knowledge in the major field of study.*

The author accomplishes this objective as he shows a deep knowledge of the methodology in the field of study. The thesis shows that he knows how to advance a problem and formulate a strategy solve it.

### 12.1.3 Objective 3

To accomplish objective 3, the author *should demonstrate the ability to critically and systematically integrate knowledge and to analyse, assess and deal with complex phenomena, issues and situations even with limited information.*

The author accomplishes this objective throughout the thesis as the subject of quasi-hom-Lie algebras are new and there is not educative literature on the subject; hence the author has had to gain understanding from limited information in research papers, and extrapolate the knowledge accessible to the specific objects covered in this thesis.

### 12.1.4 Objective 4

To accomplish objective 4, the author *should demonstrate the ability to critically, independently and creatively identify and formulate issues and to plan and carry out advanced tasks within specified time frames, thereby contributing to the development of knowledge and to evaluate this work.*

The author has accomplished this objective through highly independent studies. Assistance by the supervisor has occured a few times, otherwise weekly updates have been submitted to the supervisor and studies have continued independently. The studies has also been carried out within the specified time frames.

### 12.1.5 Objective 5

To accomlish objective 5, the author *should demonstrate ability in both national and international contexts, orally and in writing to present and discuss their conclusions and the know-*

*ledge and arguments behind them, in dialogue with different groups.*

The author has demonstrated his ability to discuss his conclusions and knowledge in writing throughout this thesis; and orally in conversations with supervisor and reviewer, as well as during the oral thesis presentation. The language of the written thesis and oral presentation has been English, while Swedish was the main language in conversations with supervisor and reviewer.

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