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ASYMPTOTIC METHODS FOR PRICING EUROPEAN OPTION IN A MARKET MODEL WITH TWO STOCHASTIC VOLATILITIES

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School of Education, Culture and Communication

Abstract

Modern financial engineering is a part of applied mathematics that studies market models. Each model is characterized by several parameters. Some of them are familiar to a wide audience, for example, the price of a risky security, or the risk free interest rate. Other parameters are less known, for example, the volatility of the security. This parameter determines the rate of change of security prices and is determined by several factors. For example, during the periods of stable economic growth the prices are changing slowly, and the volatility is small. During the crisis periods, the volatility significantly increases. Classical market models, in particular, the celebrated Nobel Prize awarded Black–Scholes–Merton model (1973), suppose that the volatility remains constant during the lifetime of a financial instrument. Nowadays, in most cases, this assumption cannot adequately describe reality. We consider a model where both the security price and the volatility are described by random functions of time, or stochastic processes. Moreover, the volatility process is modelled as a sum of two independent stochastic processes. Both of them are mean reverting in the sense that they randomly oscillate around their average values and never escape neither to very small nor to very big values. One is changing slowly and describes low frequency, for example, seasonal effects, another is changing fast and describes various high frequency effects. We formulate the model in the form of a system of a special kind of equations called stochastic differential equations. Our system includes three stochastic processes, four independent factors, and depends on two small parameters. We calculate the price of a particular financial instrument called European call option. This financial contract gives its holder the right (but not the obligation) to buy a predefined number of units of the risky security on a predefined date and pay a predefined price. To solve this problem, we use the classical result of Feynman (1948) and Kac (1949). The price of the instrument is the solution to another kind of problem called boundary value problem for a partial differential equation. The resulting equation cannot be solved analytically. Instead we represent the solution in the form of an expansion in the integer and half-integer powers of the two small parameters mentioned above. We calculate the coefficients of the expansion up to the second order, find their financial sense, perform numerical studies, and validate our results by comparing them to known verified models from the literature. The results of our investigation can be used by both financial institutions and individual investors for optimization of their incomes.

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I dedicate this thesis to the memory of Gregório Canhanga, my Father whom I miss everyday. Nothing would be possible without his dedication and effort. God bless him and accompany him eternally.

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Canhanga, B., Malyarenko, A., Murara, J.-P. and Silvestrov, S. Pricing European Options under Stochastic Volatilities Models. In: Silvestrov, S., Rančić, M. (eds.) *Engineering Mathematics I. Electromagnetics, Fluid mechanics, Material physics and Financial engineering*, Springer, Heidelberg (2016).

II Canhanga, B., Malyarenko, A., Ni, Y. and Silvestrov, S. Perturbation Methods for Pricing European Options in a Model with Two Stochastic Volatilities. In: Manca, R., McClean, S., Skiadas, C. H. (Eds.) *New Trends in Stochastic Modelling and Data Analysis*. ISAST (2015), 199–210.

III Canhanga, B., Malyarenko, A., Murara, J.-P., Ni, Y. and Silvestrov ,S. Numerical Studies on Asymptotic of European Option under Multiscale Stochastic Volatility. In C. H. Skiadas (Ed.) *16th ASMDA Conference Proceedings.* 30 June – 04 July 2015, Piraeus, Greece (2015), 53–66.

IV Canhanga, B., Malyarenko, A., Ni, Y. and Silvestrov, S. Second Order Asymptotic Expansion for Pricing European Options in a Model with Two Stochastic Volatilities. In C. H. Skiadas (Ed.) *16th ASMDA Conference Proceedings.* 30 June –04 July 2015, Piraeus, Greece (2015), 37–52.

V Canhanga, B., Malyarenko, A., Ni, Y., Rančić, M., and Silvestrov, S. Numerical Methods on European Options Second Order Asymptotic Expansions for Multiscale Stochastic Volatility. In S. Sivasundaram (Ed.), 11th International Conference on Mathematical problems in Engineering, Aerospace, and Sciences ICNPAA 2016, La Rochelle, France, 05–08 July 2016.

VI Ni, Y., Canhanga, B., Malyarenko, A., and Silvestrov, S. Approximation Methods of European Option Pricing in Multiscale Stochastic Volatility Model. In S. Sivasundaram (Ed.), 11th International Conference on Mathematical problems in Engineering, Aerospace, and Sciences ICNPAA 2016, La Rochelle, France, 05–08 July 2016.

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Parts of this thesis have been presented in communications given at the following international conferences:

1: 3rd Stochastic Modeling Techniques and Data Analysis International Conference, 11–14 June 2014, Lisbon, Portugal.

2: 16th Applied Stochastic Models and Data Analysis International Conference, 30 June–04 July 2015, Piraeus, Greece.

3: 4th Stochastic Modeling Techniques and Data Analysis International Conference, 01–04 June 2016, Valetta, Malta.

4: 11th International Conference on Mathematical Problems in Engineering Aerospace and Sciences ICNPAA2016, 05–08 July, 2016, La Rochelle, France.

Contents

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December 2016
Betuel Canhanga

Chapter 1

Introduction

1.1 A historical prospective

Modern financial engineering is a part of applied mathematics that studies *market models*. To define a market model, one has to introduce necessary mathematical tools first.

Louis Bachelier solved this problem in his PhD thesis [2]. He developed a mathematical theory of a physical phenomenon observed in 1828 by an English botanist Robert Brown and called *Brownian motion*, a continual swarming motion performed by pollen grains suspended in water. A nice picture of such movement can be seen in [1], where the authors simulate Brownian motion in two dimensions.

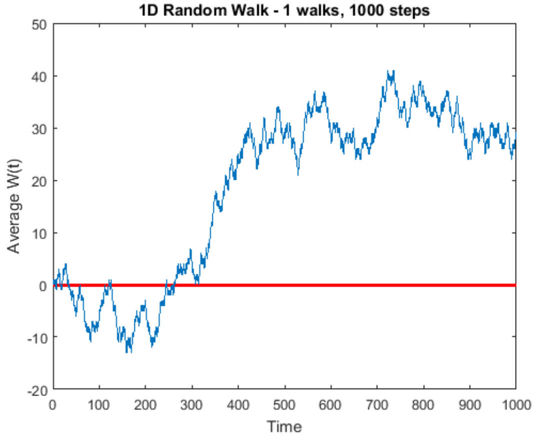

Figure 1.1:

In particular, Bachelier proved that, in Fig. 1.1, the position at time t, $W(t)$, of a single grain performing a one-dimensional Brownian motion starting from 0 at time 0 is governed by the following probabilistic law:

$$\mathsf{P}\{a < W(t) \le b\} = \int_a^b G(t,0,y)\,\mathrm{d}y, \tag{1.1}$$

where

$$G(t,x,y) = \frac{\mathrm{e}^{-(y-x)^2/(2t)}}{\sqrt{2\pi t}}. \tag{1.2}$$

Moreover, Bachelier found the finite-dimensional distributions of the stochastic process $W(t)$:

$$\begin{aligned}\mathsf{P}\{a_1 < W(t_1) \le b_1, \dots, a_n < W(t_n) \le b_n\} = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} G(t_1,0,\xi_1)G(\tau_2,\xi_1,\xi_2)\\ \times \cdots G(\tau_n,\xi_{n-1},\xi_n)\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n\end{aligned} \tag{1.3}$$

for $\tau_i = t_i - t_{i-1}$ and all $0 < t_1 < t_2 < \cdots < t_n$.

Equation (1.1) was also derived later by Einstein in [8] from statistical mechanical considerations. Einstein applied it to the determination of molecular diameters.

Does a stochastic process with finite-dimensional distributions (1.3) exist? Consider the set Ω of continuous paths $W\colon [0,\infty) \to \mathbb{R}$. Let $\mathfrak{F}$ be the smallest σ-field of events B on this set which includes all the *simple events*

$$B = \{\, \omega \in \Omega\colon a < \omega(t) \le b \,\}, \qquad t \ge 0, \quad -\infty \le a < b < \infty.$$

In [29], Wiener proved the existence of a probability measure P for which (1.3) holds. His result attached a precise meaning to Bachelier's statement that the Brownian path is continuous: just define

$$W(t,\omega)\colon [0,\infty) \times \Omega \to \mathbb{R} \text{ by } W(t,\omega) = \omega(t).$$

The further progress in this direction is connected with the following observation. The function (1.2) is the *Green function* of the problem of heat flow:

$$\frac{\partial u}{\partial t} = \mathscr{M} u,$$

where $\mathscr{M} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$. It means that the solution of the boundary value problem

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} - \mathscr{M} u(x,t) &= f(x,t), \qquad x \in \mathbb{R}, \quad t > 0,\\ \lim_{t \downarrow 0} u(x,t) &= \varphi(x), \qquad x \in \mathbb{R}\end{aligned}$$

is given by

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(t-\tau,x,y) f(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau + \int_{-\infty}^{\infty} G(t,x,y)\varphi(y)\,\mathrm{d}y.$$

Consider the following operator:

$$\mathscr{L} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x}. \tag{1.4}$$

For a wide class of functions $\sigma(x) > 0$ and $\mu(x)$ the Green function $G(t,x,y)$ of the equation

$$\frac{\partial u}{\partial t} = \mathscr{L}u$$

has the following properties:

$$G(t,x,y) \geq 0, \qquad \int_{-\infty}^{\infty} G(t,x,y)\,\mathrm{d}y = 1,$$
$$G(t,x,y) = \int_{-\infty}^{\infty} G(t-s,x,z)G(s,z,y)\,\mathrm{d}z, \qquad t > s > 0.$$

Using the above properties, Kolmogorov [19] constructed a wide class of stochastic processes similar to Brownian motion, that correspond to a general case of operator $\mathscr{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$. These processes are called *diffusion processes*. We would like to mention a contribution by Feller [9].

Itô [17] proved that if the coefficients $\sigma(x)$ and $\mu(x)$ satisfy the *Lipschits condition*

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq C|x-y|,$$

then the diffusion process $X(t)$ associated with the operator (1.4) solves the stochastic integral equation

$$X(t) = X(0) + \int_0^t \mu(X(s))\,\mathrm{d}s + \int_0^t \sigma(X(s))\,\mathrm{d}W(s),$$

where the first integral in the right hand side is the ordinary Riemann integral, while the second one is the *Itô integral* constructed by Itô in [16]. It is customary to write the above integral equation in the shorthand notation

$$\mathrm{d}X(t) = \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \qquad X(0) = X_0,$$

and call it a *stochastic differential equation*. We use in the thesis a system composed by this class of equations in order to describe the asset and the market dynamics.

In terms of stochastic differential equations, the Bachelier model reads

$$\mathrm{d}S(t) = S(0)(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)), \qquad S(0) = S_0 > 0,$$

where $S(t)$ is the *stock price*, $\mu \in \mathbb{R}$ is a constant *appreciation rate* of the stock price, and $\sigma > 0$ is a constant *volatility*, see [21]. In contrast, the celebrated *Black–Scholes–Merton model* is given by

$$\mathrm{d}S(t) = S(t)(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)), \qquad S(0) = S_0 > 0.$$

According to [24] *"The difference between the two models is analogous to the difference between linear and compound interest."* In fact, the Black–Scholes–Merton model was proposed by Samuelson in [23]. Schachermayer and Teichmann [24] described this story:

> This model was proposed by P. Samuelson in 1965, after he had — led by an inquiry of J. Savage for the treatise [3] — personally rediscovered the virtually forgotten Bachelier thesis in the library of Harvard University.

Note that item [3] in the bibliography to [24] is our reference [3].

The model is named in honour of Black, Scholes, and Merton by the following reason. Black and Scholes [4] and Merton [20] independently calculated the so-called *no-arbitrage price* of two special *financial derivatives*: the European call and European put options. To explain these terms, we need more definitions.

Let T be a positive real number. Call the interval $[0,T]$ the *trading interval*. Add one more security to the market, a *savings account* or *money market account*, and assume that the *short-term interest rate*, r, is a nonnegative constant over the trading interval. In addition, assume that the money market account, $B(t)$, is continuously compounding in value at the rate r and starts from 1. In mathematical language this reads

$$\mathrm{d}B(t) = rB(t)\,\mathrm{d}t, \qquad B(0) = 1.$$

Let $\mathfrak{F}_t$, $0 \le t \le T$, be the minimal σ-field such that the random variables $\{W(s)\colon 0 \le s \le t\}$ are $\mathfrak{F}_t$-measurable and $\mathfrak{F}_t$ contains all subsets of all events of probability 0. The family $\{\mathfrak{F}_t\colon 0 \le t \le T\}$ is an increasing family of σ-fields, or a *filtration*. We need a technical definition. Let $\mathfrak{B}([0,t])$ be the σ-field of Borel sets on the interval $[0,t]$.

Definition 1. A stochastic process $X(t,\omega)\colon [0,T] \times \Omega \to \mathbb{R}$ is called *predictable*, or *progressively measurable* with respect to the filtration $\mathfrak{F}_t$, if for any t the map $X(s,\omega)\colon [0,t] \times \Omega \to \mathbb{R}$ is $\mathfrak{B}([0,t]) \times \mathfrak{F}_t$-measurable.

A *trading strategy* in the Black–Scholes–Merton model is a pair (φ_1, φ_2) of predictable stochastic processes. We understand $\varphi_1(t)$ (resp. $\varphi_2(t)$) as the number of units of the risky security $S(t)$ (resp. the money market account $B(t)$) in the *portfolio*

$$V(t) = \varphi_1(t)S(t) + \varphi_2(t)B(t)$$

at time t. We see that $V(t)$ is time t price of the portfolio. That's why $V(t)$ is also called the *value process* or wealth process.

Assume that our portfolio is not too big. Technically, assume that the integrals

$$\int_0^T (\varphi_i(t))^2 \,\mathrm{d}t, \qquad i = 1,2$$

are finite with probability 1. Then, the integrals in the right hand side of the equation

$$V(t) = V(0) + \int_0^t \varphi_1(u)\,\mathrm{d}S(u) + \int_0^t \varphi_2(u)\,\mathrm{d}B(u) \tag{1.5}$$

are correctly defined. The economical sense of Equation (1.5) is as follows. Once the portfolio is created at time 0, there are no cash flows neither inside nor outside of the portfolio. Such a portfolio (as well as the corresponding trading strategy) is called *self-financing*.

A particular type of self-financing portfolios is important. An *arbitrage portfolio* or *free lunch portfolio* is a self-financing portfolio $V(t)$ with

$$V(0) = 0, \qquad \mathsf{P}\{V(T) \geq 0\} = 1, \qquad \mathsf{P}\{V(T) > 0\} > 0.$$

The fundamental axiom of financial mathematics is as follows: **market models that contain arbitrage portfolios are not realistic**.

A *financial derivative* or a *contingent claim* that settles at time T is just an $\mathfrak{F}_t$-measurable random variable, say X. The economical sense of this definition is as follows: at *maturity* T the owner of the claim obtains X money units as her payoff. In what follows, we consider only contingent claims of *European type*, that is, X is contingent to $S(T)$, there is a *deterministic* function f with $X = f(S(T))$.

From the economical point of view, a *European call option* is a financial derivative that gives the owner the *right, but not obligation* to buy at maturity a prescribed number of units (shares) of the risky security $S(T)$ for a predefined *strike price* K. If $S(T) \leq K$, the owner does not use her right. Otherwise, she buys the security and immediately sells it. Mathematically, her payoff is

$$X = \max\{S(T) - K, 0\}$$

per share.

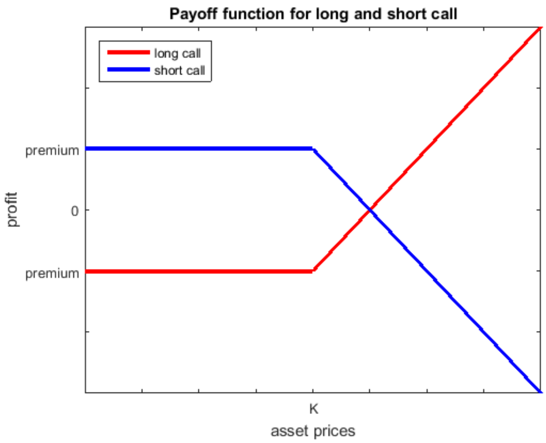

Figure 1.2: Profits for seller and buyer of an call option

In other words, for a European call option, Fig. 1.2 shows that in the case of a long position i.e. buy an option, the buyer pays up-front an amount of money or premium which makes the initial profit function negative and equal to the premium. If at expiration the asset cost less than the strike price, the option will not be executed, the payoff will be equal to zero and the buyer will lose the premium. If the cost of the asset at expiration is greater than the strike price then the buyer of the option will execute the contract and gain the payoff, i.e the difference between the asset price and the strike price. The total gain of the buyer (net profit) is the difference between the payoff and the premium. When the option is worthless the net profit equals to minus premium and it continues negative up to the point where the payoff is bigger than the premium.

For a short position in a European call option, which means to sell an option, as shown in Fig. 1.2, the profit is positive and equal to the premium if the option expire worthless. When the asset price at expiration is greater than the strike price the option will be executed and the holder of a short position will lose the difference between the asset price and the strike price. The net profit will be the difference between the premium and the loss.

Similarly, a *European put option* gives the owner the right, but not obligation to sell the risky security for a price K units per share with payoff

$$X = \max\{K - S(T), 0\}.$$

The holder of long position for European put option, loses the premium when the option expires worthless and gain the payoff, i.e. the difference between the strike price and the asset price if the option expiry worthy. The net profit will

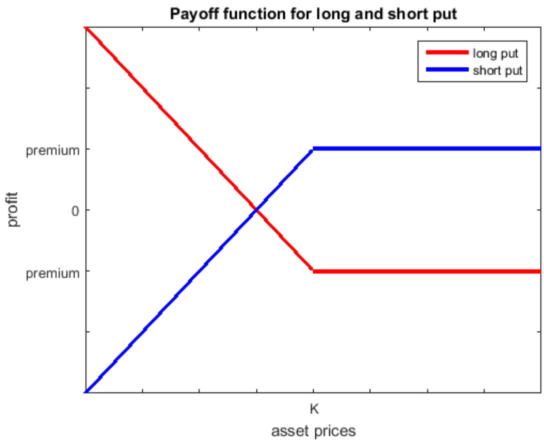

Figure 1.3: Profits for seller and buyer of an put option

be a payoff minus the premium. Figure 1.3 expresses the profit for put options. From the same figure it is possible to see that in the short position the profit is positive from the signature of the contract and remain positive up to the time that the difference between the premium and the payoff becomes negative.

What is the fair (no-arbitrage) price of a contingent claim X? The idea is as follows: construct a self-financing portfolio $V(t)$ with $V(T) = X$. In other words, *replicate* the claim X. The no-arbitrage price of X, C, is equal to $V(0)$.

Indeed, assume that $C < V(0)$. At time 0, buy $1/C$ shares of cheap claim and *sell in short* (i.e., sell a security which you do not own) $1/V(0)$ units of expensive replicating portfolio. The balance is 0, as required. At maturity T, sell the claim, obtain $V(T)/C$ money units, buy the replicating portfolio back, pay $V(T)/V(0)$ money units, and enjoy $(V(T)/C - V(T)/V(0)) > 0$ free lunch. Similarly, when $C > V(0)$, buy cheap, sell in short expensive and enjoy.

A mathematician should immediately ask the following question: *can we replicate any contingent claim*? If so, the market model is called *complete*.

How to check, that the market does not contain arbitrage portfolios and/or is complete? We need a technical tool called *equivalent martingale measure*.

Definition 2. A probabilistic measure P^* on the measurable space $(\Omega, \mathfrak{F})$ is called an *equivalent martingale measure* if for any event $A \in \mathfrak{F}$ we have $\mathsf{P}(A) = 0$ if and only if $\mathsf{P}^*(A) = 0$ and all the discounted price processes on the probability space $(\Omega, \mathfrak{F}, \mathsf{P}^*)$ with filtration $\mathfrak{F}_t$ are martingales.

In the case of the Black–Scholes–Merton market, this means that

$$\mathsf{E}^*[S^*(t)] < \infty, \qquad \mathsf{E}^*[S^*(u) \mid \mathfrak{F}_t] = S^*(t)$$

for all $0 \leq t \leq u \leq T$, where E^* denote the mathematical expectation under

$$\mathsf{P}^*, \text{ and } S^*(t) = \frac{1}{B(t)} S(t)$$

is the discounted stock price. The introduced technical tool works as follows.

Theorem 1 (The Fundamental Theorem of Asset pricing)**.** *A market model does not contain arbitrage portfolios if and only if there exists an equivalent martingale measure. A market model is complete if and only if there exist a* unique *equivalent martingale measure.*

Different from the market that we consider in the thesis, the Black–Scholes market is complete. To show this, we need one more technical tool.

Let $\lambda(t)$ be a stochastic process such that the integral $U(t) = \int_0^t \lambda(s)\,\mathrm{d}W(s)$ exists. The stochastic differential equation

$$\mathrm{d}X(t) = X(t)\lambda(t)\,\mathrm{d}W(t), \qquad X(0) = 1$$

has the unique solution

$$X(t) = \exp\left(\int_0^t \lambda(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t |\lambda(u)|^2\,\mathrm{d}u\right).$$

The process $X(t)$ is called the *Doléans-Dade exponential* of the process $U(t)$ after Doléan-Dade [7] and is denoted by $\mathscr{E}(U)$.

Suppose that $\lambda(t)$ is chosen in such a way that $\mathsf{E}[\mathscr{E}(U)(T)] = 1$. Define a probability measure $\tilde{\mathsf{P}}$ on $(\Omega, \mathfrak{F})$ by

$$\tilde{\mathsf{P}}(A) = \int_A \mathscr{E}(U)(T,\omega)\,\mathrm{d}\mathsf{P}(\omega), \qquad A \in \mathfrak{F}.$$

The new measure $\tilde{\mathsf{P}}$ is *equivalent* to P, i.e., $\tilde{\mathsf{P}}$ and P share the same events of probability 0.

Theorem 2 ([13])**.** *The stochastic process*

$$\tilde{W}(t) = W(t) - \int_0^t \lambda(u)\,\mathrm{d}u$$

is a Brownian motion on the probability space $(\Omega, \mathfrak{F}, \tilde{\mathsf{P}})$.

Put $\lambda(t) = \frac{(r-\mu)}{\sigma}$. In economical terms, $-\lambda(t)$ is just the *market price of risk*. Then we have

$$U(t) = \int_0^t \lambda(s)\,\mathrm{d}W(s) = \frac{(r-\mu)}{\sigma} W(t)$$

and

$$\int_0^t |\lambda(u)|^2 \,\mathrm{d}u = \left(\frac{(r-\mu)}{\sigma}\right)^2 t.$$

The equivalent martingale measure P^* has the form

$$\mathsf{P}^*(A) = \int_A \exp\left(\frac{(r-\mu)}{\sigma}W(T) - \frac{1}{2}\left(\frac{(r-\mu)}{\sigma}\right)^2 T\right) \mathrm{d}\mathsf{P}(\omega), \qquad A \in \mathfrak{F}.$$

Under P^*, the discounted stock price $S^*(t)$ satisfies

$$\mathrm{d}S^*(t) = \sigma S^*(t)\,\mathrm{d}W^*(t), \qquad S^*(0) = S_0,$$

where

$$W^*(t) = W(t) - \int_0^t \lambda(u)\,\mathrm{d}u = W(t) - \frac{(r-\mu)}{\sigma}t$$

is a standard Brownian motion on $(\Omega, \mathfrak{F}, \mathsf{P}^*)$. The process $S^*(t)$ is indeed a martingale.

In fact, by transiting to the equivalent martingale measure we eliminated the market risk, hence another name, *risk-neutral measure*.

The time t no-arbitrage price $C(t)$ of any contingent claim X with $\mathsf{E}[|X|] < \infty$ is

$$C(t) = \mathsf{E}^*\left[\frac{X}{B(t)}\right] = \mathsf{E}^*[\mathrm{e}^{-rt}X].$$

In [20], Merton calculated the above expectation directly for the cases of European call and put options. On the other hand, Black and Scholes [4] used another method. They proved the following theorem, see [21, Corollary 3.1.5].

Theorem 3 ([4])**.** *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a measurable function such that under risk neutral probability measure, the expectation $\mathsf{E}^*[|X|]$ is finite, where $X = g(S(T))$. The time-t no-arbitrage price of the contingent claim X that settles at time T is equal to $u(S(t),t)$, where the function $u(s,t)\colon [0,\infty) \times [0,T] \to \mathbb{R}$ solves the* Black–Scholes boundary value problem

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0, \quad u(s,T) = g(s). \tag{1.6}$$

Afterwards, Black and Scholes made a change of variables and obtained a boundary value problem for the heat equation that was solved long before.

If fact, the approach of Black and Scholes is a corollary of the classical *Feynman–Kac formula* named after Feynman [10] and Kac [18]. In the thesis we deal with an asset governed by two stochastic volatilities which leads us into a multidimensional treatment of the above theorem. Bellow we will give the formulation in a multidimensional setting for the Feynman-Kac formula.

In the Black–Scholes–Merton market model, the volatility and the short-term interest rate are constants. Experience shows that in many really existing markets this assumption is unrealistic. To overcome the difficulty, consider the following extension of the above model.

The *standard d-dimensional Brownian motion* is a $\mathbb{R}^d$-valued stochastic process $\mathbf{W}(t) = (W_1(t), \dots, W_d(t))^\top$, where the components are independent standard Brownian motions. Let $\{\mathfrak{F}_t \colon 0 \le t \le T\}$ be a filtration generated by the standard d-dimensional Brownian motion $\mathbf{W}_t$, $0 \le t \le T$. Let $\mathbf{X}(t) = (X_1(t), \dots, X_m(t))^\top$ be a $\mathbb{R}^m$-valued stochastic process. Assume, that it satisfies the stochastic differential equation:

$$\mathrm{d}\mathbf{X}(t) = \boldsymbol{\mu}(t, \mathbf{X}(t))\,\mathrm{d}t + \Sigma(t, \mathbf{X}(t))\,\mathrm{d}\mathbf{W}(t), \qquad \mathbf{X}(0) = \mathbf{X}_0, \tag{1.7}$$

where $\boldsymbol{\mu} \colon [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ is called the *drift*, and where $\Sigma \colon [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m \times d}$ is called the *diffusion*. We say that the corresponding market model has d factors and m variables. If the functions $\mu_i(t, \mathbf{x})$ and $\Sigma_{ij}(t, \mathbf{x})$ are smooth and grow at most linearly at infinity, then the system (1.7) has a unique solution. The solution to (1.7) is a *multidimensional Itô diffusion*, see the classical book by Stroock and Varadhan [28]. An alternative construction of multidimensional Itô diffusions in terms of forward/backward equations was found much earlier by Kolmogorov in [19].

For example, in the Black–Scholes–Merton model we have $d = 1$ factor and $m = 2$ variables $X_1(t) = S(t)$ and $X_2(t) = B(t)$. The drift is

$$\boldsymbol{\mu}(t, \mathbf{X}(t)) = (\mu S(t), rB(t))^\top,$$

and the diffusion is

$$\Sigma(t, \mathbf{X}(t)) = (\sigma S(t), 0)^\top.$$

In more advanced models, volatility and/or short-term interest rate become stochastic. As an example, consider the *Grzelak–Oosterlee–van Veeren model* described in [14] with $d = 3$ correlated factors $Z_i(t)$ and $m = 3$ variables $S(t)$ (stochastic price), $r(t)$ (stochastic interest rate), and $\sigma(t)$ (stochastic volatility):

$$\begin{aligned}
\mathrm{d}S_t &= r_t S_t\,\mathrm{d}t + \sigma_t^p S_t\,\mathrm{d}Z_1(t),\\
\mathrm{d}r_t &= \lambda(\theta_t - r_t)\,\mathrm{d}t + \eta\,\mathrm{d}Z_2(t),\\
\mathrm{d}\sigma_t &= -\kappa(\sigma_t - \overline{\sigma})\,\mathrm{d}t + \gamma\sigma_t^{1-p}\,\mathrm{d}Z_3(t),
\end{aligned}$$

where $\mathrm{d}Z_i(t)\,\mathrm{d}Z_j(t) = \rho_{ij}\,\mathrm{d}t$. We have

$$\begin{aligned}
\mathbf{X}(t) &= (S(t), r(t), \sigma(t))^\top, \\
\boldsymbol{\mu}(t, \mathbf{X}(t)) &= (r(t)S(t), \lambda(\theta(t) - r(t)), -\kappa(\sigma(t) - \overline{\sigma}))^\top, \\
\Sigma(t, \mathbf{X}(t)) &= \begin{pmatrix} \sigma^p(t)S(t) & 0 & 0 \\ \eta\rho_{12} & \eta\sqrt{1-\rho_{12}^2} & 0 \\ \gamma\sigma^{1-p}(t)\rho_{13} & \gamma\sigma^{1-p}(t)a & \gamma\sigma^{1-p}(t)b \end{pmatrix},
\end{aligned}$$

where λ, κ, $\overline{\sigma}$, η, γ, and p are model parameters, and

$$a = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, \qquad b = \left(1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1-\rho_{12}^2}\right)^{1/2}.$$

The Grzelak–Oosterlee–van Veeren model contains many other models as particular cases. For example, when $\lambda = \eta = 0$ and $p = 1/2$, we have the model by Heston [15]. When $\lambda = \eta = 0$ and $p = 1$, we have the Schöbel–Zhu model discussed in [27] and [25]. Finally, when $\lambda = \eta = \kappa = \gamma = p = 0$, we return to the Black–Scholes–Merton model.

In models with stochastic volatility like (1.7), the existence and especially uniqueness of an equivalent martingale measure is a non-trivial issue, see, e.g., [26]. When the above measure exists but is not unique, one needs to specify the *market price of volatility risk*. In economic terms, *market chooses the risk-neutral measure*. This approach will be used in the next section, now we assume that *equation* (1.7) *is already written under one of possible risk-neutral measures*.

Let $\mathbf{X}(t)$ be the solution to (1.7). Define the *infinitesimal generator* $\mathscr{L}$ of the Itô diffusion $\mathbf{X}(t)$ by

$$\mathscr{L}_t = \sum_{i=1}^m \mu_i(t, \mathbf{x})\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m\sum_{k=1}^d \Sigma_{ik}(t, \mathbf{x})\Sigma_{kj}(t, \mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j}.$$

Let $r(s, \mathbf{X}(t))$ be the short-term interest rate, and let $g(\mathbf{x})$ be the payoff of a European contingent claim X settled at time T, and let

$$u(t, \mathbf{x}) = \mathsf{E}\left[\exp\left(-\int_t^T r(s, \mathbf{X}(s))\,\mathrm{d}s\right) g(\mathbf{X}(t)) \mid \mathbf{X}(t) = \mathbf{x}\right] \tag{1.8}$$

be its no-arbitrage price. The celebrated *Feynman–Kac formula* (see, e.g., [22, Theorem 8.2.1]) states the following.

Theorem 4 (Feynman–Kac)**.** *Assume that g is twice continuously differentiable with compact support, r is continuous and bounded from below.*

- *The function $u(t, \mathbf{x})$ satisfies the boundary value problem*

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathscr{L}_t u - ru &= 0, \\ u(T, \mathbf{x}) &= g(\mathbf{x}). \end{aligned} \tag{1.9}$$

- *Conversely, if in addition the solution to* (1.9) *is bounded and continuously differentiable once in t and twice in* $\mathbf{x}$*, then it has the form* (1.8).

It is easy to see that Theorem 3 is indeed a particular case of the Feynman–Kac formula.

The Feynman–Kac formula and Monte Carlo simulation are currently the only available general methods to calculate the right hand side of (1.8).

1.2 The formulation of the problem

In this thesis, we consider a model with three stochastic variables and four factors, i.e. $m = 3$ and $d = 4$. In the original notation by Chiarella and Ziveyi [6] and under the real-world probability measure P it has the form

$$\begin{aligned} \mathrm{d}S &= rS\,\mathrm{d}t + \sqrt{V_1}S\,\mathrm{d}z_1 + \sqrt{V_2}S\,\mathrm{d}z_2, \\ \mathrm{d}V_1 &= (a_1 - b_1V_1)\,\mathrm{d}t + \sigma_1\sqrt{V_1}\,\mathrm{d}z_3, \\ \mathrm{d}V_2 &= (a_2 - b_2V_2)\,\mathrm{d}t + \sigma_2\sqrt{V_2}\,\mathrm{d}z_4, \end{aligned}$$

where the Brownian motion z_1 has correlation ρ_1 with z_3, and z_2 has correlation ρ_2 with z_4. Chiarella and Ziveyi [5] rewrite the above model in the form

$$\begin{aligned} \mathrm{d}S &= \mu S\,\mathrm{d}t + \sqrt{v_1}S\,\mathrm{d}W_1 + \sqrt{v_2}S\,\mathrm{d}W_2, \\ \mathrm{d}v_1 &= \kappa_1(\theta_1 - v_1)\,\mathrm{d}t + \rho_{13}\sigma_1\sqrt{v_1}\,\mathrm{d}W_1 + \sqrt{1-\rho_{13}^2}\sigma_1\sqrt{v_1}\,\mathrm{d}W_3, \\ \mathrm{d}v_2 &= \kappa_2(\theta_2 - v_2)\,\mathrm{d}t + \rho_{24}\sigma_2\sqrt{v_2}\,\mathrm{d}W_2 + \sqrt{1-\rho_{24}^2}\sigma_2\sqrt{v_2}\,\mathrm{d}W_4, \end{aligned}$$

where μ is the instantaneous return per unit time of the underlying asset, θ_i are the long-run means of v_i, κ_i are the speeds of mean-reversion, σ_i are the instantaneous volatilities of v_i. The processes W_i are *independent* Brownian motions.

For each factor, W_i, Chiarella and Ziveyi [5] specify the market price of risk $\lambda_i(t)$ associated with the Brownian instantaneous shocks $\mathrm{d}W_i$. By the Girsanov theorem, the processes

$$W_i^*(t) = W_i(t) + \int_0^t \lambda_i(s)\,\mathrm{d}s$$

are Brownian motions under the risk-neutral probability P^*. Note the $+$ sign in the right hand side. This is because the market price of risk is $\lambda(t)$ of the Girsanov theorem times -1. The market prices are postulated to have the form

$$\lambda_3(t) = \frac{\lambda_1\sqrt{v_1}}{\sigma_1\sqrt{1-\rho_{13}^2}}, \qquad \lambda_4(t) = \frac{\lambda_2\sqrt{v_2}}{\sigma_2\sqrt{1-\rho_{24}^2}}.$$

The model takes the form

$$\begin{aligned}
\mathrm{d}S &= (r-q)S\,\mathrm{d}t + \sqrt{v_1}S\,\mathrm{d}W_1^* + \sqrt{v_2}S\,\mathrm{d}W_2^*,\\
\mathrm{d}v_1 &= [\kappa_1\theta_1 - (\kappa_1+\lambda_1)v_1]\,\mathrm{d}t + \rho_{13}\sigma_1\sqrt{v_1}\,\mathrm{d}W_1^* + \sqrt{1-\rho_{13}^2}\sigma_1\sqrt{v_1}\,\mathrm{d}W_3^*,\\
\mathrm{d}v_2 &= [\kappa_2\theta_2 - (\kappa_2+\lambda_2)v_2]\,\mathrm{d}t + \rho_{24}\sigma_2\sqrt{v_2}\,\mathrm{d}W_2^* + \sqrt{1-\rho_{24}^2}\sigma_2\sqrt{v_2}\,\mathrm{d}W_4^*.
\end{aligned}$$

From Chiarella and Ziveyi model we considered the case where two different mean reverting random variances play a roll on the asset price. One of the variance with a slow rate of return and another with a higher return frequency. This idea comes to cover the fact that most pricing processes are influenced by random factors connected for example to the seasons and also connected to another high frequency random events.

With this in mind we introduced the small parameters $\varepsilon = 1/\kappa_1$ and $\delta = \kappa_2$ into the model by Chiarella and Ziveyi. We also make the volatility of the volatilities to be depending on the rate of reversion and transformed system [5] into

$$\begin{aligned}
\mathrm{d}S &= \mu S\,\mathrm{d}t + \sqrt{V_1}S\,\mathrm{d}W_1 + \sqrt{V_2}S\,\mathrm{d}W_2,\\
\mathrm{d}V_1 &= \frac{1}{\varepsilon}(\theta_1 - V_1)\,\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\xi_1\rho_{13}\sqrt{V_1}\,\mathrm{d}W_1 + \frac{1}{\sqrt{\varepsilon}}\xi_1\sqrt{(1-\rho_{13}^2)V_1}\,\mathrm{d}W_3,\\
\mathrm{d}V_2 &= \delta(\theta_2 - V_2)dt + \sqrt{\delta}\xi_2\rho_{24}\sqrt{V_2}\,\mathrm{d}W_2 + \sqrt{\delta}\xi_2\sqrt{(1-\rho_{24}^2)V_2}\,\mathrm{d}W_4,
\end{aligned} \tag{1.10}$$

where $\frac{1}{\sqrt{\varepsilon}}\xi_1$ and $\sqrt{\delta}\xi_2$ are volatilities for the Heston type variances V_1 and V_2 respectively and $0 < \varepsilon \lll 1$, $0 < \delta \lll 1$.

We would like to price a European type contingent claim with payoff $X = X(S(T))$ written on the time-t, In particular, we want to price a European call option.

1.3 The solution and chapter summaries

In order to define a price of a contingent claim with underlying asset described in the system (1.10) we use Girsanov theorem to transform the stochastic differential system from our problem into another stochastic differential system under risk neutral probability measure

$$
\begin{aligned}
\mathrm{d}S &= (r-q)S\,\mathrm{d}t + \sqrt{V_1}S\,\mathrm{d}W_1^* + \sqrt{V_2}S\,\mathrm{d}W_2^*,\\
\mathrm{d}V_1 &= \left(\frac{1}{\varepsilon}(\theta_1 - V_1) - \frac{1}{\sqrt{\varepsilon}}\xi_1\Lambda_3\sqrt{(1-\rho_{13}^2)V_1}\right)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\xi_1\sqrt{V_1}\rho_{13}\,\mathrm{d}W_1^*\\
&\quad + \frac{1}{\sqrt{\varepsilon}}\xi_1\sqrt{V_1(1-\rho_{13}^2)}\,\mathrm{d}W_3^*,\\
\mathrm{d}V_2 &= \left(\delta(\theta_2 - V_2) - \sqrt{\delta}\xi_2\Lambda_4\sqrt{V_2(1-\rho_{24}^2)}\right)\mathrm{d}t + \sqrt{\delta}\xi_2\sqrt{V_2}\rho_{24}\,\mathrm{d}W_2^*\\
&\quad + \sqrt{\delta}\xi_2\sqrt{V_2(1-\rho_{24}^2)}\,\mathrm{d}W_4^*.
\end{aligned}
\tag{1.11}
$$

The market price of volatilities risk are defined as in [5] and take the following form

$$
\Lambda_3 = \frac{\lambda_1\sqrt{V_1\varepsilon}}{\xi_1\sqrt{1-\rho_{13}^2}}, \quad \Lambda_4 = \frac{\lambda_2\sqrt{V_2}}{\xi_2\sqrt{\delta(1-\rho_{24}^2)}}.
$$

From here, by Feynman–Kac theorem, the price of a European contingent claim $X = X(S(T))$ written on (1.11) can be expressed as the solution of the following boundary value problem

$$
\begin{aligned}
ru - \frac{\partial u}{\partial t} &= (r-q)s\frac{\partial u}{\partial s} + \left[\frac{1}{\varepsilon}(\theta_1 - v_1) - \lambda_1 v_1\right]\frac{\partial u}{\partial v_1} + \left[\delta(\theta_2 - v_2) - \lambda_2 v_2\right]\frac{\partial u}{\partial v_2}\\
&\quad + \frac{1}{2}\left[(v_1+v_2)s^2\frac{\partial^2 u}{\partial s^2} + \frac{\xi_1}{\varepsilon}v_1\frac{\partial^2 u}{\partial v_1^2} + \xi_2\delta v_2\frac{\partial^2 u}{\partial v_2^2}\right]\\
&\quad + \rho_{13}\xi_1\frac{1}{\sqrt{\varepsilon}}sv_1\frac{\partial^2 u}{\partial s\partial v_1} + \xi_2\rho_{24}\sqrt{\delta}sv_2\frac{\partial^2 u}{\partial s\partial v_2}.\\
u(T,s) &= X(s).
\end{aligned}
\tag{1.12}
$$

We solve the above equation using the asymptotic expansion method. From chapter 2 to chapter 5 we deal with a simplified version of the system (1.10) where we assume $\xi_1 = \xi_2 = 1$. Chapters 6 and 7 generalize the analysis for any value of ξ_1, ξ_2.

This thesis contains the introduction *(the first chapter)* and six other chapters which are based on the following papers: paper I: Pricing European Options under Stochastic Volatilities Models, paper II: Perturbation Methods for Pricing European Options in a Model with Two Stochastic Volatilities, paper III: Numerical Studies on Asymptotic of European Option under Multiscale Stochastic Volatility, paper IV: Second Order Asymptotic Expansion for Pricing European Options in a Model with Two Stochastic Volatilities, paper V: Numerical Methods on European Options Second Order Asymptotic Expansions for Multiscale

Stochastic Volatility, paper VI: Approximation Methods of European Option Pricing in Multiscale Stochastic Volatility Model.

Chapter 2

In this chapter we do an extensive survey of models developed as a tentative solution for the Black–Scholes weaknesses. We also describe the evolutionary history of financial modeling after Black–Schools. For each of the model discussed here, we present the main steps to be followed for determining a price of a contingent claim using the underlying asset properties.

After the introduction of stochastic volatilities modeling, the lighting from Constant Elasticity of Volatility model was dropped and models such as Heston, Grzelak–Oosterlee–van Veeren, Hull and White, Schobel–Zhu, Schobel–Zhu–Hull–White among others dominated financial modeling and financial mathematics. In 2009 Christoffersen et al. [6] presented an empirical study "The shape and term structure of the index option smirk: why multi–factor stochastic volatility models work so well". It turns out that, far from one stochastic volatility, models with two stochastic volatilities could capture better the random movements on the asset prices and the random behavior of market risks. At the end of the chapter we use ideas from [6] to construct a model which is the main object of discussion in the thesis.

Chapter 3

In this chapter we treat the problem presented in (1.11) for the case where $\xi_1 = \xi_2 = 1$. The condition $0 < \varepsilon \lll 1$ and $0 < \delta \lll 1$ has the implication that the process V_1 (V_2) is fast (slow) mean-reverting. This can be interpreted as the effects of weekends and the effects of seasons of the year (summer and winter) on the asset price respectively. We assume that the Feller condition holds so that the variance processes are positive. We then generate (1.12) by the transformation of the stochastic differential problem (1.11) under Feynman–Kac theorem. Equation (1.12) can be written as

$$\left(\frac{1}{\varepsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathscr{L}_1 + \mathscr{L}_2 + \sqrt{\delta}\mathscr{M}_1 + \delta\mathscr{M}_2\right)U = 0 \tag{1.13}$$

subject to the terminal value condition $U(T,s,v_1,v_2) = h(S_T)$. The operators in (1.13) are defined as follows:

$$\mathscr{L}_0 = (\theta_1 - v_1)\frac{\partial}{\partial v_1} + \frac{1}{2}v_1\frac{\partial^2}{\partial v_1^2}, \tag{1.14}$$

$$\mathscr{L}_1 = \rho_{13} s v_1 \frac{\partial^2}{\partial s \partial v_1},$$

$$\mathscr{L}_2 = \frac{\partial}{\partial t} + (r-q)s\frac{\partial}{\partial s} + \frac{1}{2}(v_1+v_2)s^2\frac{\partial^2}{\partial s^2} - r - \lambda_1 v_1 \frac{\partial}{\partial v_1} - \lambda_2 v_2 \frac{\partial}{\partial v_2},$$

$$\mathscr{M}_1 = \rho_{24} s v_2 \frac{\partial^2}{\partial s \partial v_2},$$

$$\mathscr{M}_2 = (\theta_2 - v_2)\frac{\partial}{\partial v_2} + \frac{1}{2} v_2 \frac{\partial^2}{\partial v_2^2}.$$

Assuming that the solution of (1.13) can be approximated to

$$\widetilde{U}^{\varepsilon,\delta} \approx U_{0,0} + U_{1,0}\sqrt{\varepsilon} + U_{0,1}\sqrt{\delta} = U_{BS}(\bar{\sigma}) + U_{1,0}^{\varepsilon} + U_{0,1}^{\delta},$$

we solve (1.13) and obtain the approximate price for an European call option. The leading term $U_{BS}(\bar{\sigma})$ is given by the solution

$$\langle \mathscr{L}_2 \rangle U = 0, \quad U(T,s,v_1,v_2) = h(S).$$

Here the notations $\langle \cdot \rangle$ stands for the averaging with respect to invariant distribution Π of the process V_1, i.e.

$$\langle \cdot \rangle = \int \cdot \Pi(dv_1). \tag{1.15}$$

$$\bar{\sigma}(v_2) = \sqrt{\int (v_1+v_2)\,\Pi(dv_1)}. \tag{1.16}$$

The other two terms of the approximation are

$$U_{1,0}^{\varepsilon} = -(T-t)\mathscr{B}^{\varepsilon} U_{BS} \quad \text{and} \quad U_{0,1}^{\delta} = (T-t)\mathscr{A}^{\delta} U_{BS},$$

where

$$\mathscr{B}^{\varepsilon} = \sqrt{\varepsilon}\langle \mathscr{L}_1 \mathscr{L}_0^{-1}(\mathscr{L}_2 - \langle \mathscr{L}_2 \rangle)\rangle \quad \text{and} \quad \mathscr{A}^{\delta} = \frac{\sqrt{\delta}}{2}\langle \mathscr{M}_1 \rangle. \tag{1.17}$$

Operators $\mathscr{B}^{\varepsilon}$ and $\mathscr{A}^{\delta}$ can be simplified into

$$\mathscr{B}^{\varepsilon} = -\Upsilon^{\varepsilon} D_1 D_2, \quad \text{and} \quad \mathscr{A}^{\delta} = \Theta^{\delta} D_1 \frac{\partial}{\partial v_2}$$

for

$$\Upsilon^{\varepsilon} = -\frac{1}{2}\sqrt{\varepsilon}\rho_{13}\theta_1, \; \Theta^{\delta} = \frac{1}{2}\sqrt{\delta}\rho_{24} v_2 \quad \text{and} \quad D_i = s^i \frac{\partial^i}{\partial s^i}.$$

The main result of this chapter is the derivation of the explicit formulae to compute approximate prices for European options on assets governed by (1.10). The use of two stochastic volatilities corresponds to an important improvement from Black–Scholes pricing model.

Chapter 4

This chapter provides analytical and numerical studies on investigating the accuracy of the approximation formulae given by the first order asymptotic expansion. We show that the accuracy of the obtained approximation is plausible. We compare the approximated European option prices obtained by asymptotic expansion method to the prices computed from the same model under [5] approach.

We end the chapter with a recommendation for more extensive studies that have to be done, by considering a wider selection of stocks and options, in order to confirm the efficiency and accuracy of our method. Further numerical studies can be carried out to analyze how the parameters affect the approximation accuracy and also to study how much improvement one can get by using a second-order asymptotic expansion.

Our main goal was to validate the approximation procedure presented in chapter 3. The comparison of results obtained from asymptotic expansion and those obtained by applying Fourier and Laplace transform ([5] approach) gives indications that for the model (1.10), one can accurately price European options using asymptotic expansion approach.

Chapter 5

In chapter 5 we consider the same market conditions described in chapter 3 and by double asymptotic expansion (singular and regular perturbation). We approximate the price of European call option by

$$U^{\varepsilon,\delta} \approx U_{BS} + \sqrt{\varepsilon}U_{1,0} + \sqrt{\delta}U_{0,1} + \sqrt{\delta\varepsilon}U_{1,1} + \varepsilon U_{2,0} + \delta U_{0,2} \tag{1.18}$$

for U_{BS}, $U_{1,0}$ and $U_{0,1}$ given as in chapter 3 and

$$U_{1,1} = -\frac{\tau^2}{3}\mathscr{B}^{\varepsilon}\Theta^{\delta}\frac{\partial}{\partial v_2}U_{BS}.$$

The second order fast correction factor $U_{2,0}$ is

$$U_{2,0} = -0.5\phi(v_1)D_2U_{BS} + C_{2,0}(\tau,S,v_2). \tag{1.19}$$

The function $\phi(v_1)$ in (1.19) is a solution of the following Poisson problem

$$\mathscr{L}_0\phi(v_1) = f^2(v_1,v_2) - \bar{\sigma}^2(v_2)$$

and the integration constant $C_{2,0}(\tau,S,v_2)$ is defined by

$$\begin{aligned} C_{2,0}(\tau,S,v_2) = &\frac{\tau}{4}\left(\langle\phi(v_1)(v_1+v_2)\rangle - \langle\phi(v_1)\rangle\langle(v_1+v_2)\rangle\right)D_2^2U_{BS} \\ &+\frac{\tau^2}{2}\left(\Upsilon^{\varepsilon}\right)^2 D_1^2D_2^2U_{BS}. \end{aligned}$$

The second order slow correction factor is

$$U_{0,2} = \frac{\tau^2}{6}\left(\frac{\partial g}{\partial v_2}\right)^2 \left[(\rho_{24}v_2)^2 D_1 D_2 \left(\frac{\partial^2}{\partial g^2} + \frac{1}{g}\frac{\partial}{\partial g}\right) + \frac{\tau}{6}v_2\frac{\partial^2}{\partial g^2}\right] U_{BS}$$
$$+ \left(\frac{1}{2}v_2\left(\frac{\partial g}{\partial v_2}\right)^2 - \frac{\tau}{2}\frac{\partial^2 g}{\partial v_2^2} + \frac{\tau}{2}(\theta_2 - v_2)\frac{\partial g}{\partial v_2}\right)\frac{\partial}{\partial g}U_{BS}.$$

The main result of this chapter is the semi-analytical form solution for the second order asymptotic expansion (1.18).

Chapter 6

The huge calculation involved in [5] and the recommendations from chapter 4 are motivations to this chapter. Here we compute European options prices for the same model with the same approach as in chapter 4 but, considering higher order on the perturbation.

We consider the model (1.11) for general values of ξ_1 and ξ_2. This generalized version of model (1.11) allows the model parameters $\theta_1,\ \theta_2,\ \rho_{13},\ \rho_{24}$ to take values from a wider range while the model still fulfill the Feller condition.

We use the second order asymptotic expansion and present a close form solution for European option, provide experimental and numerical studies on investigating the accuracy of the approximation formulae given by second order asymptotic expansion and compare the obtained results with results presented by [5].

Along the chapter we do the validation for the approximation process of option pricing presented in chapter 5. The numerical studies indicate that the second order asymptotic expansion although does not increase much the accuracy of the approximation, gives better results than the ones given by the first order asymptotic expansion.

Chapter 7

In this chapter we present three different solutions to European option pricing problem from the model presented in chapter 3. The first solution *(as in chapter 3)* is an implementation of the first order asymptotical expansion, approach which was introduced by [11, 12] in a different multiscale stochastic volatility model in which the variance processes act together as one diffusion factor. Our new model has an non-trivial asymptotic analysis and the resulting asymptotic expansion formula contain different leading and correction terms from the one in [12]. In Section 3 we present the second solution which is a Monte-Carlo simulation scheme. Considering the reducing discretization error and variance of the estimator, we obtain a numerical approximation. We also consider a straightforward adaption from [5] where they use method of characteristic equations, Fourier transform for the asset process and bivariate Laplace transforms for the variance process.

From chapter 3 to chapter 6 we computed an approximate solution for European options prices and compared our results to those presented under a methodology used by [5]. In this chapter, after comparing the results from asymptotic expansion method, Fourier – Laplace transforms approach and Monte Carlo Method we obtained almost the same options prices. This certifies the validation of the computation and numerical studies that we presented in chapters 4 and 6.

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